Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Lehrstuhl für Angewandte Mathematik
und Numerische Mathematik

Lehrstuhl für Optimierung und Approximation

Preprint BUW-AMNA-OPAP 10/01

Kasra Mohaghegh, Roland Pulch and Jan ter Maten

Model Order Reduction using Singularly Perturbed Systems

January 2010

http://www.math.uni-wuppertal.de

Model Order Reduction using Singularly Perturbed Systems

Kasra Mohaghegh[a,1], Roland Pulch[a], Jan ter Maten[b]

[a] *Lehrstuhl für Angewandte Mathematik und Numerische Mathematik, Fachbereich Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, Gaußstr. 20, D-42119 Wuppertal, Germany.*

[b] *NXP Semiconductors, High Tech Campus 46, PostBox WDA-2, Room 2.210 NL-5656 AE Eindhoven, The Netherlands.*

Abstract

Nowadays electronic circuits comprise about a hundred million components on slightly more than one square centimeter. The model order reduction (MOR) techniques are among the most powerful tools to conquer this complexity and scale, although the nonlinear MOR is still an open field of research. On the one hand, the MOR techniques are well developed for linear ordinary differential equations (ODEs). On the other hand, we deal with differential algebraic equations (DAEs), which result from models based on network approaches. There are the direct and the indirect strategy to convert a DAE into an ODE. We apply the direct approach, where an artificial parameter is introduced in the linear system of DAEs. It follows a singularly perturbed problem. On compact domains, uniform convergence of the transfer function of the regularized system towards the transfer function of the system of DAEs is proved in the general linear case. A substitute model of a transmission line yields a test example for this approach.

Key words: differential algebraic equations, semi-explicit systems, direct approach, ε-embedding, linear model order reduction, parametric model reduction

---

Email addresses: `mohaghegh@math.uni-wuppertal.de` (Kasra Mohaghegh), `pulch@math.uni-wuppertal.de` (Roland Pulch), `jan.ter.maten@nxp.com` (Jan ter Maten)

[1] Corresponding author, Tel. +49 202 439 4773, Fax +49 202 439 3668

1. Introduction

The tendency to analyze and design systems of ever increasing complexity is becoming more and more a dominating factor in progress of chip design. Along with this tendency, the complexity of the mathematical models increases both in structure and dimension. Complex models are more difficult to analyze, and it is also harder to develop control algorithms. Therefore model order reduction (MOR) is of utmost importance. For the linear case, quite a number of approaches are well-established and resolve large systems of ordinary differential equations (ODEs) efficiently, see [1]. Nonlinear problems can be approximated by a linearization, see [11].

We want to generalize according techniques to the case of linear systems of differential algebraic equations (DAEs). On the one hand, a high-index DAE problem can be converted into a low-index system by analytic differentiations, see [3]. A transformation to index zero yields an equivalent system of ODEs. On the other hand, a regularization is directly feasible in case of semi-explicit systems of DAEs. Thereby, we obtain a singularly perturbed problem of ODEs with an artificial parameter ε, which approximates the original DAEs. Thus according MOR techniques can be applied to the ODE system. An MOR approach for DAEs is achieved by considering the limit $\varepsilon \to 0$.

In [10] we considered the regularization for semi-explicit systems of DAEs via an ε-embedding, i.e., the direct approach. The MOR techniques apply a transfer function defined in frequency domain. We proved the convergence of the transfer function of the regularized system to the transfer function of the original system of DAEs.

In this work we extend the strategy to general linear systems of DAEs. We show that a regularization via an ε-embedding is feasible using the Kronecker canonical form. However, this approach exhibits disadvantages in corresponding numerical methods. Hence we apply the singular value decomposition to achieve an alternative regularization. In each approach we prove the pointwise convergence of the transfer functions. Moreover, it follows the uniform convergence on compact domains. The theoretical properties allow for using MOR techniques within two scenarios. Firstly, we can reduce the regularized system of ODEs for small parameter ε, which yields an approximation to the original system of DAEs. Secondly, a parametric model reduction is considered and the limit $\varepsilon \to 0$ results in an approach for DAEs, where the quality of the approximation still has to be investigated.

The paper is organized as follows. In Sect. 2, we briefly review the analysis of the input-output behavior of linear dynamical systems in frequency domain to apply MOR. The semi-explicit systems of DAEs and the regularization technique are introduced in Sect. 3. We extend this approach to general linear systems of DAEs in Sect. 4. Finally, the results of numerical simulations of an illustrative example are presented.

2. Linear Dynamical Systems

A continuous time-invariant (lumped) multi-input multi-output linear dynamical system can be derived from an RLC circuit by applying modified nodal analysis (MNA), see [7]. The system is of the form

$$\left\{ \begin{array}{rcl} C\frac{\mathrm{d}x(t)}{\mathrm{d}t} & = & -Gx(t) + Bu(t) \\ w(t) & = & Lx(t) + Du(t) \end{array} \right. \tag{1}$$

with initial condition $x(0) = x_0$. Here t is the time variable, $x(t) \in \mathbb{R}^n$ is referred as inner state (and the corresponding n-dimensional space is called state space), $u(t) \in \mathbb{R}^m$ is an input, $w(t) \in \mathbb{R}^p$ is an output. The dimensionality n of the state vector is called the order of the system. The number of inputs and outputs is m and p, respectively, and $G \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $L \in \mathbb{R}^{p\times n}$, $C \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{p\times m}$ are the state space matrices. Without loss out generality we assume $D = 0$. The matrices C and G in (1) are allowed to be singular, and we only assume that the pencil $G+sC$ is regular, i.e., the matrix $G+sC$ is singular only for a finite number of values $s \in \mathbb{C}$. For more details on existence and uniqueness of a solution, see [8]. Basically, MOR techniques aim to derive a system

$$\left\{ \begin{array}{rcl} \tilde{C}\frac{\mathrm{d}\tilde{x}(t)}{\mathrm{d}t} & = & -\tilde{G}\tilde{x}(t) + \tilde{B}u(t), \quad \tilde{x}(t) \in \mathbb{R}^q \\ \tilde{w}(t) & = & \tilde{L}\tilde{x}(t) + \tilde{D}u(t), \quad \tilde{x}(0) = \tilde{x}_0, \quad \tilde{w}(t) \in \mathbb{R}^p, \end{array} \right. \tag{2}$$

of order q with $q \ll n$ that can replace the original high-order system (1) in the sense that the input-output behavior of both systems nearly agrees. A common way is to identify a subspace of dimension $q \ll n$, that captures the dominant information of the dynamics and to project (1) onto this subspace, spanned by some basis vectors $\{v_1, \ldots, v_q\}$. The linear system of the form (1) is often referred to as the representation of the system in time domain, or in the state space. Equivalently, one can also represent the system in frequency domain via the Laplace transform. Recall that for a vector-valued function $f(t)$, the Laplace transform is defined component-wise by

$$(\mathbb{L}(f))(s) := \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t, \qquad s \in \mathbb{C}. \tag{3}$$

The physically meaningful values of the complex variable s are $s = \mathrm{i}\omega$, where the real parameter $\omega \geq 0$ is referred to as the frequency. Taking the Laplace transformation of the system (1), we obtain the frequency domain formulation

$$\left\{ \begin{array}{rcl} sCX(s) & = & -GX(s) + BU(s) \\ W(s) & = & LX(s), \end{array} \right. \tag{4}$$

where $X(s)$, $Y(s)$ and $U(s)$ represents the Laplace transform of $x(t)$, $y(t)$ and $u(t)$, respectively. For simplicity, we assume that we have initial conditions $x(0) = x_0 = 0$ and $u(0) = 0$. Eliminating the variable $X(s)$ in (4), we see that the input $U(s)$ and the output $Y(s)$ in the frequency domain are related by the following $p \times m$ matrix-valued rational function

$$H(s) = L \cdot (G + s \cdot C)^{-1} \cdot B. \tag{5}$$

$H(s)$ is known as the *transfer function* or *Laplace-domain impulse response* of the linear system (1).

3. Semi-Explicit Systems of DAEs

Systems of DAEs result in the mathematical modeling of a wide variety of problems like mechanical engineering, electric circuit design and others. We consider a semi-explicit system

$$\begin{array}{rcll} y'(t) & = & f(y(t), z(t)), & \qquad y : \mathbb{R} \to \mathbb{R}^k \\ 0 & = & g(y(t), z(t)), & \qquad z : \mathbb{R} \to \mathbb{R}^l \end{array} \tag{6}$$

with differential and perturbation index 1 or 2. For the construction of numerical methods to solve initial value problems of (6), the direct as well as the indirect approach can be used. The direct approach applies an ε-*embedding* of the DAEs (6), i.e., the system changes into

$$\begin{array}{rcl} y'(t) & = & f(y(t), z(t)) \\ \varepsilon z'(t) & = & g(y(t), z(t)) \end{array} \quad \Leftrightarrow \quad \begin{array}{rcl} y'(t) & = & f(y(t), z(t)) \\ z'(t) & = & \frac{1}{\varepsilon} g(y(t), z(t)) \end{array} \tag{7}$$

with a real parameter $\varepsilon \neq 0$. Techniques for ODEs can be employed for the singularly perturbed system (7). The limit $\varepsilon \to 0$ yields an approach for solving the DAEs (6). The applicability and quality of the resulting method still has to be investigated.

Alternatively, the indirect approach is based on the *state space form* of the DAEs (6) with index 1, i.e.,

$$y'(t) = f(y(t), \Phi(y(t))) \tag{8}$$

with $z(t) = \Phi(y(t))$. To evaluate the function Φ, the nonlinear system

$$g(y(t), \Phi(y(t))) = 0 \tag{9}$$

is solved for given value $y(t)$. Consequently, the system (8) represents ODEs for the differential variables y and methods for ODEs can be applied. In each evaluation of the right-hand side in (8), a nonlinear system (9) has to be solved. More details on techniques based on the ε-embedding and the state space form can be found in [8].

Although some MOR methods for DAEs already exist, several techniques are restricted to ODEs or exhibit better properties in the ODE case in comparison to the DAE case. The direct or the indirect approach enables the usage of MOR schemes for the ODEs (7) or (8), where an approximation with respect to the original DAEs (6) follows. The aim is to obtain suggestions for MOR schemes via these strategies, where the quality of the resulting approximations still has to be analyzed in each method. For the moment we restrict to semi-explicit DAE systems of the type (1). According to (6), the solution x and the matrix C exhibit the partitioning:

$$x = \begin{pmatrix} y \\ z \end{pmatrix}, \qquad C = \begin{pmatrix} I_{k\times k} & 0 \\ 0 & 0_{l\times l} \end{pmatrix}. \tag{10}$$

The order of the system is $n = k + l$, where k and l are the dimensions of the differential part and the algebraic part (constraints), respectively, determined by the semi-explicit system (6). Following the direct approach, the ε-embedding changes the system (1) into

$$\left\{ \begin{array}{rcl} C(\varepsilon)\frac{\mathrm{d}x(t)}{\mathrm{d}t} & = & -Gx(t) + Bu(t), \qquad x(0) = x_0, \\ w(t) & = & Lx(t), \end{array} \right. \tag{11}$$

where

$$C(\varepsilon) = \begin{pmatrix} I_{k\times k} & 0 \\ 0 & \varepsilon I_{l\times l} \end{pmatrix} \qquad \text{for } \ \varepsilon \in \mathbb{R}$$

with the same inner state and input/output as before. Concerning the relation between the original system (1) and the regularized system (11) with respect to the transfer function, we achieve the following statement. Without loss of generality, the induced matrix norm of the Euclidean vector norm is applied. For a proof and more details on Theorem 1, see [10].

Theorem 1. *For fixed* $s \in \mathbb{C}$ *with* $\det(G + sC) \neq 0$ *and* $\varepsilon \in \mathbb{R}$ *satisfying*

$$|s| \cdot |\varepsilon| \leq \frac{c}{\|(G + sC)^{-1}\|_2} \tag{12}$$

for some $c \in (0, 1)$*, the transfer functions* $H(s)$ *and* $H_\varepsilon(s)$ *of the systems* (1) *and* (11) *exist and it holds*

$$\|H(s) - H_\varepsilon(s)\|_2 \leq \|L\|_2 \cdot \|B\|_2 \cdot K(s) \cdot |s| \cdot |\varepsilon|$$

with the constant

$$K(s) = \frac{1}{1 - c} \left\|(G + sC)^{-1}\right\|_2^2 .$$

This theorem demonstrates the pointwise convergence of the transfer function for each s. Moreover, it can be shown that uniform convergence is given in each compact subset of $\mathbb{C}$.

For MOR of the DAE system (1), we have two ways to handle the artificial parameter ε, which results in two different scenarios. In the first scenario, we fix a small value of the parameter ε. Thus we use one of the standard techniques for the reduction of the corresponding system of ODEs. Finally, MOR techniques yield a reduced ODE (with small ε inside). The system of ODEs with small ε represents a regularized DAE. Any reduction scheme for ODEs is feasible. Figure 1 indicates the steps. Recent research shows that the poor man's truncated balance realization (PMTBR), see [12], can be applied efficiently if the matrix C in (1) is regular, which is indeed our case in (11). The approximation from the reduced ODE yields the approximation of the original DAE. In the second scenario, the parameter ε is considered as an independent variable (value not predetermined). We can use methods of the parametric MOR for reducing the corresponding ODE system. The applied parametric MOR is based on [4, 9] in this case. The limit $\varepsilon \to 0$ yields an approximation corresponding to the original DAEs (1). The existence of the approximation in this limit still has to be analyzed. Figure 2 illustrates the strategy. Theorem 1 provides the theoretical background for the both

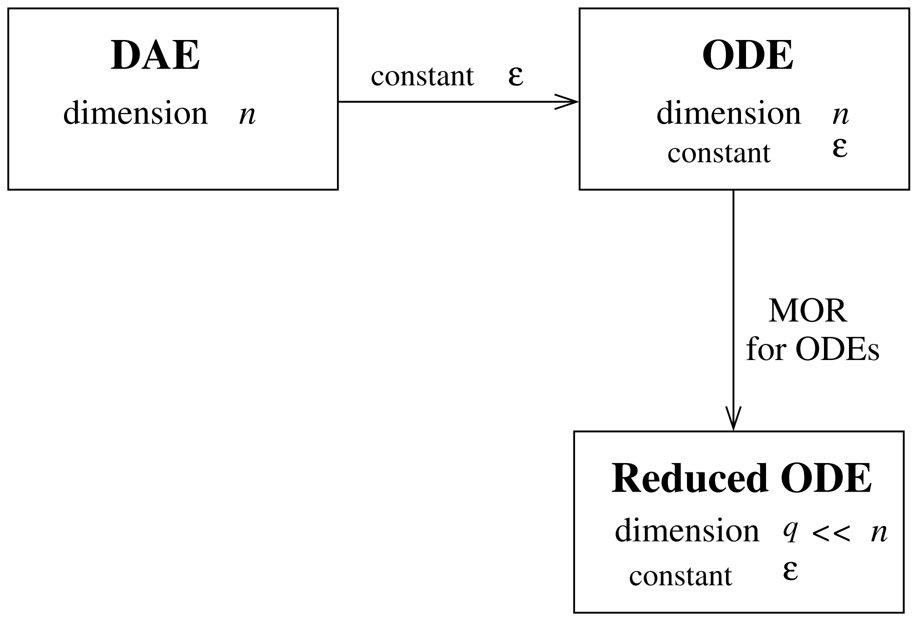

Figure 1: The approach of the ε-embedding for MOR in the first scenario.

scenarios. We apply an MOR scheme based on an approximation of the transfer function to the system of ODEs. Let $\tilde{H}_\varepsilon(s)$ be a corresponding approximation of $H_\varepsilon(s)$. It follows

$$\|H(s) - \tilde{H}_\varepsilon(s)\|_2 \leq \|H(s) - H_\varepsilon(s)\|_2 + \|H_\varepsilon(s) - \tilde{H}_\varepsilon(s)\|_2 \tag{13}$$

for each $s \in \mathbb{C}$ with $\det(G + sC) \neq 0$. Due to Theorem 1, the first term on the right-hand side of (13) becomes small for sufficiently small parameter ε. However, ε should be chosen larger than the machine precision on a computer. The second term on the right-hand side of (13) becomes small if an MOR method for ODEs works successfully. In the following , we present a generalization of Theorem 1.

4. General Linear Systems

The simplest and best understood DAEs are linear equations of the form (1). We will focus on this kind of problems to generalize the direct approach (ε-embedding) from Section 3. Thus we consider an arbitrary singular matrix C now.

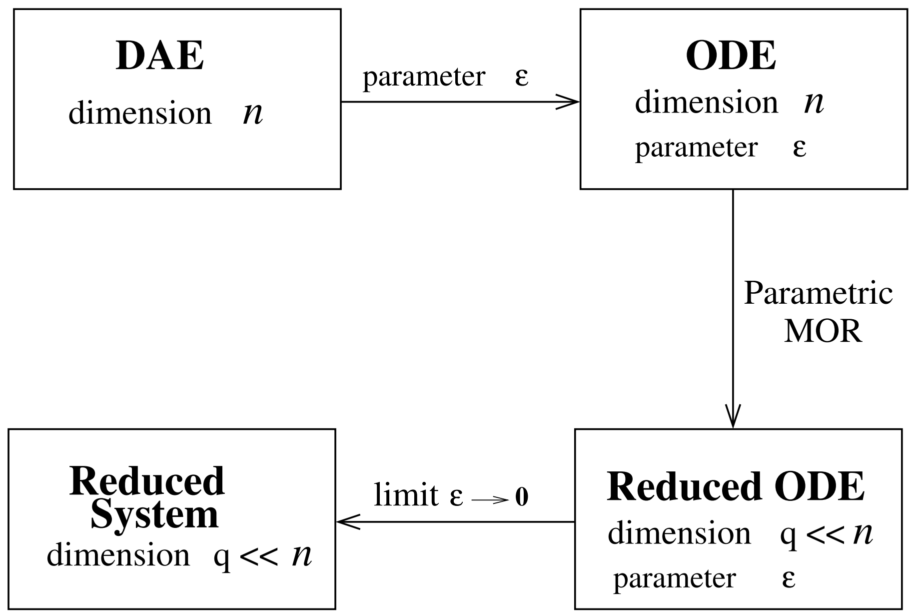

Figure 2: The approach of the ε-embedding for MOR in the second scenario.

4.1. Transformation to Kronecker Form

Here investigations are closely related to the theory of matrix pencils, see [14] . This field provides the proposition that the linear DAE (1) is uniquely solvable if and only if the matrix pencil $\{C, G\}$ is regular, i.e. the polynomial $\det(\lambda C + G)$ does not vanish identically. We consider constant coefficient matrices $C, G \in \mathbb{R}^{n\times n}$ and the C^r-mapping $u : [t_0, t_1] \to \mathbb{R}^m$ represents a time-dependent source term. Due to the regular matrix pencil, the matrices G and C can be transformed simultaneously to the *Kronecker canonical form*, see [8],

$$PCQ = \begin{pmatrix} I_{n-m} & 0 \\ 0 & N \end{pmatrix}, \quad PGQ = \begin{pmatrix} M & 0 \\ 0 & I_m \end{pmatrix} \tag{14}$$

with the regular matrices $P, Q \in \mathbb{R}^{n\times n}$. It holds $M \in \mathbb{R}^{(n-m)\times(n-m)}$ and $N \in \mathbb{R}^{m\times m}$ is a nilpotent matrix with the nilpotency index ν, i.e., $N^\nu = 0$ but $N^{\nu-1} \neq 0$. If C is regular, then we have $\nu = 0$. Since we suppose that the matrix C in (1) is singular, the matrix N in (14) exhibits the following structure

$$N = \begin{pmatrix} 0 & * & \dots & * \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & * \\ 0 & \dots & \dots & 0 \end{pmatrix}. \tag{15}$$

(More precisely, just one diagonal may be occupied.) The special structure of N in (15) allows us to add the variable ϵ on the diagonal of the matrix N, which corresponds to a regularization of the product PCQ. Without this transformation the matrix C does not have any special pattern in general to add this variable ε to avoid the singularity. The transformation of the pencil $\{C, G\}$ to its Kronecker canonical form corresponds to a decoupling of the DAE (1) into

$$\dot{y} + My = \eta(t) \tag{16a}$$

$$N\dot{z} + z = \delta(t) \tag{16b}$$

with

$$x = Q \cdot \begin{pmatrix} y \\ z \end{pmatrix}, \quad P \cdot B \cdot u(t) = \begin{pmatrix} \eta(s) \\ \delta(t) \end{pmatrix}. \tag{16c}$$

Now (16a) is already an explicit system of ODEs for y. We can obtain an ODE for z from (16b), see [8]. It follows an equivalent ODE and the differential index is ν.

We define $v := Q^{-1}x$. Applying the Kronecker canonical form (14) changes the linear DAEs (1) into the transformed linear system

$$\left\{ \begin{array}{rcl} PCQ\frac{\mathrm{d}v(t)}{\mathrm{d}t} & = & -PGQv(t) + PBu(t) \\ w(t) & = & LQv(t). \end{array} \right. \tag{17}$$

According to (14), we define $\hat{C} := PCQ$. Remark that both C and N are singular matrices. The representation of the system in the frequency domain via the Laplace transform (3) results to

$$H(s) = LQ \cdot (PGQ + s\,\underbrace{PCQ}_{\hat{C}})^{-1} \cdot PB. \tag{18}$$

Following the direct approach, the ε-embedding changes the system (17) into

$$\left\{\begin{array}{rcl} \hat{C}_\varepsilon \frac{\mathrm{d}v(t)}{\mathrm{d}t} & = & -PGQv(t) + PBu(t) \\ w(t) & = & LQv(t) \end{array}\right. \tag{19}$$

with

$$\hat{C}_\varepsilon := \begin{pmatrix} I_{n-m} & 0 \\ 0 & \varepsilon I_m + N \end{pmatrix} \quad \text{for} \ \ \varepsilon \neq 0 \tag{20}$$

and the same inner state and input/output as before. For $\varepsilon \neq 0$, the matrix $\hat{C}_\varepsilon$ is regular in (20) and the transfer function of (19) reads

$$H_\varepsilon(s) = LQ \cdot (PGQ + s\hat{C}_\varepsilon)^{-1} \cdot PB. \tag{21}$$

Concerning the relation between the original system (17) and the regularized system (19) with respect to the transfer function, we achieve the following statement. Without loss of generality, the induced matrix norm of the Euclidean vector norm is applied again.

Lemma 2. *Let* $A,\ \tilde{A} \in \mathbb{R}^{n\times n},\ \ \det(A) \neq 0$ *and* $\Delta A := A - \tilde{A}$ *where* ΔA *is sufficiently small. Then it holds*

$$\|A^{-1} - \tilde{A}^{-1}\|_2 \leq \frac{\|A^{-1}\|_2^2 \cdot \|\Delta A\|_2}{1 - \|A^{-1}\|_2 \cdot \|\Delta A\|_2}.$$

Proof. The definition of the matrix norm leads to

$$\|A^{-1} - \tilde{A}^{-1}\|_2 = \max_{\|x\|_2=1} \left\| A^{-1}x - \tilde{A}^{-1}x \right\|_2 .$$

Suppose $y := A^{-1}x,\ \tilde{y} := \tilde{A}^{-1}x$, then the sensitivity analysis of linear systems yields

$$\frac{\|\Delta y\|_2}{\|y\|_2} \leq \frac{\kappa(A)}{1 - \kappa(A)\frac{\|\Delta A\|_2}{\|A\|_2}} \left(\frac{\|\Delta A\|_2}{\|A\|_2} + \underbrace{\frac{\|\Delta x\|_2}{\|x\|_2}}_{0} \right) \tag{22}$$

where the quantity

$$\kappa(A) \equiv \left\|A^{-1}\right\|_2 \|A\|_2 \tag{23}$$

is the relative condition number. So by substituting the value of $\kappa(A)$ we have:

$$\|y - \tilde{y}\|_2 \leq \frac{\|A^{-1}\|_2 \cdot \|\Delta A\|_2 \cdot \|A^{-1}\|_2 \, \|x\|_2}{1 - \|A^{-1}\|_2 \cdot \|\Delta A\|_2}$$

then

$$\|A^{-1} - \tilde{A}^{-1}\|_2 \leq \frac{\|A^{-1}\|_2^2 \cdot \|\Delta A\|_2}{1 - \|A^{-1}\|_2 \cdot \|\Delta A\|_2}.$$

and we obtain the desired formula. $\square$

For example, we conclude from Lemma 2 that

$$\lim_{\Delta A \to 0} \tilde{A}^{-1} = A^{-1}, \tag{24}$$

which is an obvious relation.

Theorem 3. *For fixed $s \in \mathbb{C}$ with $\det(PGQ + s\hat{C}) \neq 0$ and $\varepsilon \in \mathbb{R}$ satisfying*

$$|s| \cdot |\varepsilon| \leq \frac{c}{\|(PGQ + s\hat{C})^{-1}\|_2} \tag{25}$$

for some $c \in (0, 1)$, the transfer functions $H(s)$ from (18) and $H_\varepsilon(s)$ from (21) exist and it holds

$$\|H(s) - H_\varepsilon(s)\|_2 \leq \|L\|_2 \cdot \|B\|_2 \cdot \|P\|_2 \cdot \|Q\|_2 \cdot K(s) \cdot |s| \cdot |\varepsilon|$$

with the constant

$$K(s) = \frac{1}{1-c} \left\|(PGQ + s\hat{C})^{-1}\right\|_2^2.$$

Proof. The condition (25) guarantees that the matrices $PGQ + s\hat{C}_\varepsilon$ are regular. The definition of the transfer functions implies

$$\begin{aligned} &\|H(s) - H_\varepsilon(s)\|_2 \\ &\leq \|L\|_2 \cdot \|Q\|_2 \cdot \left\|(PGQ + s\hat{C})^{-1} - (PGQ + s\hat{C}_\varepsilon)^{-1}\right\|_2 \cdot \|P\|_2 \cdot \|B\|_2 . \end{aligned}$$

Applying Lemma 2, the term in the above right-hand side satisfies the estimate

$$\begin{aligned} &\left\|(PGQ + s\hat{C})^{-1} - (PGQ + s\hat{C}_\varepsilon)^{-1}\right\|_2 \\ &\leq \frac{1}{1-c} \left\|(PGQ + s\hat{C})^{-1}\right\|_2^2 \cdot \left\|(PGQ + s\hat{C}) - (PGQ + s\hat{C}_\varepsilon)\right\|_2 . \end{aligned}$$

Using basic calculations, it follows

$$\left\|(PGQ + s\hat{C}) - (PGQ + s\hat{C}_\varepsilon)\right\|_2 = |s| \cdot \left\|\hat{C} - \hat{C}_\varepsilon\right\|_2 = |s| \cdot |\varepsilon|.$$

Thus the proof is completed. $\square$

We conclude from Theorem 3 that

$$\lim_{\varepsilon \to 0} H_\varepsilon(s) = H(s) \tag{26}$$

for each $s \in \mathbb{C}$ with $G + sC$ regular. Uniform convergence is given in a compact set $s \in S \subset \mathbb{C}$ again.

Although the above theorem guarantees the convergence of the transfer function of the regularized system to the transfer function of the original system in the limit case, we encounter two drawbacks. The first one is related to the numerical calculation of the Kronecker canonical form. The numerical computation of the Kronecker canonical form might be unstable, see [5], due to a possible ill-conditioning of the matrices P and Q. The second issue is that the upper bound in Theorem 3 includes the norms of P and Q now, since we have applied a transformation before the regularization. If these norms are large, we obtain a pessimistic estimate. The numerical difficulties lead us to seek an alternative just for the numerical calculation although from the theoretical point of view the above approach is feasible.

4.2. Transformation via Singular Value Decomposition

In the following we will introduce an alternative to the Kronecker canonical form which has no side effect for the numerical implementation. We apply the *singular value decomposition* (SVD), see [13], to the matrix C in the system (1). For an arbitrary matrix $M \in \mathbb{R}^{m \times n}$, it exists a factorization of the form

$$UMV^\top = \Sigma, \tag{27}$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices. The matrix $\Sigma \in \mathbb{R}^{m \times n}$ is diagonal with nonnegative real entries, which are the singular values. The factorization (27) is called a singular value decomposition of M. Applying the SVD form (27) transforms the linear system of DAEs (1) into

$$\left\{ \begin{array}{rcl} UCV^\top V \frac{\mathrm{d}x(t)}{\mathrm{d}t} & = & -UGV^\top Vx(t) + UBu(t) \\ w(t) & = & LV^\top Vx(t). \end{array} \right. \tag{28}$$

We define $z := Vx$. It follows

$$\left\{ \begin{array}{rcl} \begin{pmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{pmatrix} \frac{\mathrm{d}z(t)}{\mathrm{d}t} & = & -UGV^\top z(t) + UBu(t) \\ w(t) & = & LV^\top z(t), \end{array} \right. \tag{29}$$

where the diagonal matrix $\tilde{\Sigma} \in \mathbb{R}^{r\times r}$ $(r < n)$ contains the positive singular values. Thus $\tilde{\Sigma}$ is regular. The ε-embedding changes the system into

$$\left\{ \begin{array}{rcl} \underbrace{\begin{pmatrix} \tilde{\Sigma} & 0 \\ 0 & \varepsilon I_{n-r} \end{pmatrix}}_{C_\varepsilon} \frac{\mathrm{d}z(t)}{\mathrm{d}t} & = & -UGV^\top z(t) + UBu(t) \\ w(t) & = & LV^\top z(t). \end{array} \right. \tag{30}$$

The introduced matrix C_ε is regular. We obtain the same result as for the transformation to Kronecker canonical form.

Theorem 4. *For fixed $s \in \mathbb{C}$ with $\det(UGV^\top + C_\varepsilon) \neq 0$ and $\varepsilon \in \mathbb{R}$ satisfying*

$$|s| \cdot |\varepsilon| \leq \frac{c}{\|(UGV^\top + C_\varepsilon)^{-1}\|_2} \tag{31}$$

for some $c \in (0,1)$, the transfer functions $H(s)$ and $H_\varepsilon(s)$ of the systems (29) *and* (30) *exist and it holds*

$$\|H(s) - H_\varepsilon(s)\|_2 \leq \|L\|_2 \cdot \|B\|_2 \cdot K(s) \cdot |s| \cdot |\varepsilon|$$

with the constant

$$K(s) = \frac{1}{1-c} \left\| \left(UGV^\top + sUCV^\top\right)^{-1} \right\|_2^2 .$$

The steps of the proof for the previous theorem are like in the proof of Theorem 1, see [10], since the matrices after applying the SVD exhibit a semi-explicit structure. Remark that the transfer function of (29) coincides with the function of (1).

The two drawbacks of the transformation to Kronecker canonical form are omitted by the SVD. Firstly, stable numerical methods exist to compute the SVD efficiently. Secondly, we obtain the same upper bound as for the semi-explicit systems. The reason is that the orthogonal matrices feature the optimal property

$$\|U\|_2 = \left\|U^\top\right\|_2 = \|V\|_2 = \left\|V^\top\right\|_2 = 1.$$

Hence the SVD can be used as a brilliant alternative for decoupling the dynamical system. Afterwards, we can apply the ε-embedding.

5. Test Example and Numerical Results

We consider the substitute model of a transmission line (TL), see [6], which consists of N cells. Each cell includes a capacitor, an inductor and two resistors, see Figure 3. This TL model represents a scalable benchmark problem (both in differential part and algebraic part), because we can select the number N of cells. The state variables $x \in \mathbb{R}^{2N+3}$ consist of the volt-

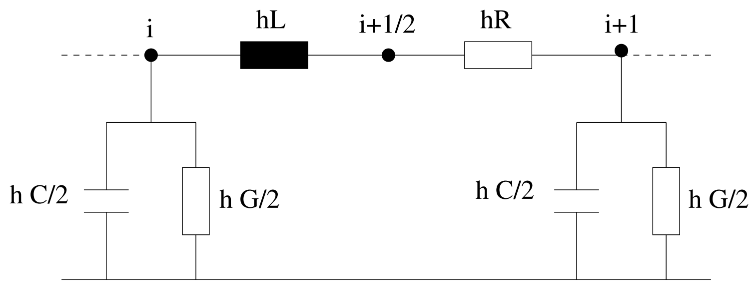

Figure 3: One cell of RLC transmission line.

ages at the nodes, the current traversing the inductor L and currents at the boundaries of the circuit. The used physical parameters are

$$C = 10^{-14}\ \mathrm{F/m},\ L = 10^{-8}\ \mathrm{H},\ R = 0.1\ \Omega/\mathrm{m},\ G = 10\ \mathrm{S/m}.$$

We apply modified nodal analysis, see [7], to the circuit and then the state x contains the unknowns:

$$\begin{array}{cc} (V_0, V_1, \dots, V_N), & (I_{\frac{1}{2}}, I_{\frac{3}{2}}, \dots, I_{N-\frac{1}{2}}), \\ (V_{\frac{1}{2}}, V_{\frac{3}{2}}, \dots, V_{N-\frac{1}{2}}), & (I_0, I_N). \end{array}$$

As two more unknowns than the equations appear, we have $3N+3$ unknowns and $3N+1$ equations. Thus two boundary conditions are necessary. Equations for the main nodes and the intermediate nodes in each cells are

$$\begin{array}{cccccccccl} \frac{h}{2}C\dot{V}_0 & + & \frac{h}{2}GV_0 & + & I_{\frac{1}{2}} & - & I_0 & = 0, & & \\ hC\dot{V}_i & + & hGV_i & + & I_{i+\frac{1}{2}} & - & I_{i-\frac{1}{2}} & = 0, & i = 1, \dots, N-1, \\ \frac{h}{2}C\dot{V}_N & + & \frac{h}{2}GV_N & + & I_N & - & I_{N-\frac{1}{2}} & = 0, & & \end{array}$$

$$
\begin{array}{rcll}
-I_{i+\frac{1}{2}} & + & \frac{V_{i+1/2}-V_{i+1}}{hR} & = 0, \\
hL\dot{I}_{i+1/2} & + & (V_{i+1/2} - V_i) & = 0, \quad i = 0, 1, \ldots, N-1.
\end{array}
$$

We apply the boundary conditions

$$
\begin{array}{rcll}
I_0 & - & u(t) & = 0, \\
L\dot{I}_N & + & V_N & = 0
\end{array}
$$

with the independent current source u.

For the first simulation the variable ε is fixed to 10^{-14} and 10^{-7}, respectively, and the PMTBR is used as a reduction scheme for the ODE system. Figure 4 shows the transfer function both for the DAE and the ODE (including ε) and the reduced ODE with fixed ε. The number in parentheses shows the order of the systems. For all runs we fixed the number of cells to $N = 300$, which results in the order $n = 903$ of the original system of DAEs (1).

In Figure 5 the dashed line illustrates the absolute error between the DAE system and the ODE (with ε). The result of Theorem 1 is satisfied as the two systems demonstrate a perfect match for low frequencies and the error increases just for higher frequencies and then is smoothly decreasing for larger frequencies. The strait line in Figure 5 shows the absolute error between the original DAE system and the reduced ODE with fixed ε. It is clear that due to the small parameter $\varepsilon = 10^{-14}$ we have a nearly perfect agreement between the original DAE and regularized DAE (i.e. the ODE with ε). The error is below -180 decibel, but as we increase the parameter to $\varepsilon = 10^{-7}$ the error also increases and we do not have an acceptable error, see Figure 5. In both cases the reduced model is of order less than 10 and is able to approximate the ODE system well. We always calculate the error between the reduced model and the original DAE case. The relatively large parameter $\varepsilon = 10^{-7}$ causes also a higher error of the reduction scheme, since the error of the regularization is included.

The second scenario with parametric MOR is studied now. We apply the parametric MOR following [9]. The limit $\varepsilon \rightarrow 0$ gives the result for the reduced DAE, see Figure 6. The value in parentheses shows the order of the systems. We simulate again the TL model with $N = 300$ cells. We plot the transfer function for the parameters $\varepsilon = 0,\ 10^{-10},\ 10^{-14}$, see Figure 6. The error plot for the parametric reduction scheme is shown in Figure 7. The error plot shows that we an overall perfect match for the case of $\varepsilon = 0, 10^{-14}$

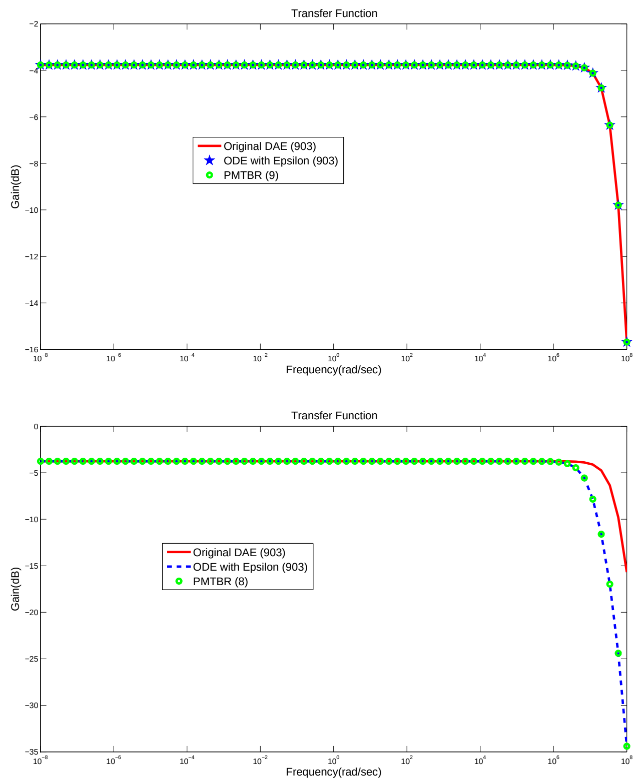

Figure 4: Original transfer function for DAE and ODE and reduced transfer function for $\varepsilon = 10^{-7}$ (down) and $\varepsilon = 10^{-14}$ (up).

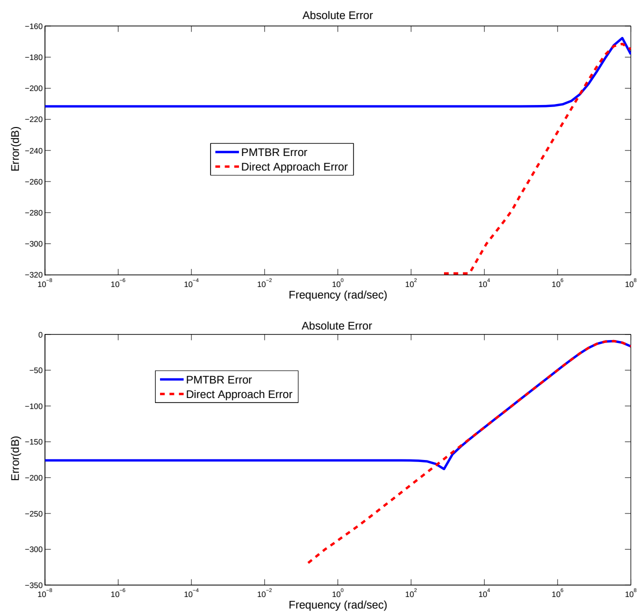

Figure 5: Absolute error plot for the ε-embedding and PMTBR methods in case of the parameters $\varepsilon = 10^{-7}$ (down) $\varepsilon = 10^{-14}$ (up).

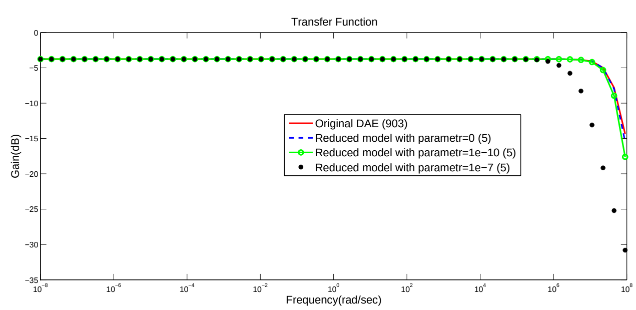

Figure 6: Original transfer function reduced with parametric scheme (second scenario) with different values for parameter ε.

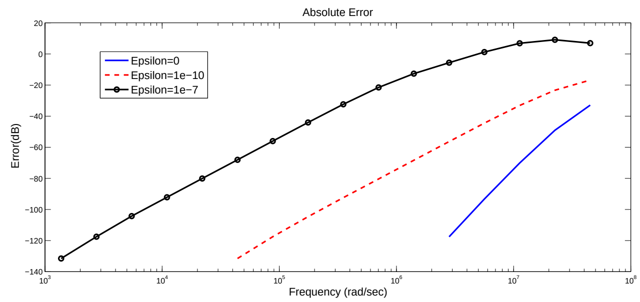

Figure 7: Absolute error plot for the ε-embedding, reduction carry out by parametric algorithm by [9], $\varepsilon = 0, 10^{-7}, 10^{-10}$.

and as the value for the parameter ε increases, the accuracy of the method and of the reduction algorithm decrease. It is also important to mention that the order of the reduced system in this case is nearly half the previous one.

6. Conclusions

Firstly, the ε-embedding, i.e., the direct approach, is applied to a general linear system of DAEs via a singular value decomposition. Thereby, we obtain an approximating system of ODEs. Secondly, MOR techniques perform the reduction of the system of ODEs for the two scenarios of a fixed ε or a variable parameter (parametric scheme). The presented approach enables the usage of MOR methods for ODEs. Most of the linear reduction schemes are designed and adopted for ODEs such as poor man's truncated balanced realization or spectral zeros preservation of Antoulas [2]. The test example of the linear transmission line model has been simulated successfully in both scenarios. The next step is to test the method on examples from industrial applications. Further investigations are necessary to apply the approach to non-linear systems.

Acknowledgement

This work has been supported by the Marie Curie Research Training Network COMSON (COupled Multiscale Simulation and Optimization in Nanoelectronics). The authors are indebted to Dr. Michael Striebel (University of Wuppertal) for helpful discussions.

References

[1] A. C. Antoulas. *Approximation of large-scale Dynamical Systems, advances in design and control*. SIAM, 2005.

[2] A. C. Antoulas. A new result on passivity preserving model reduction. *System & Control Letters*, vol 54:361–374, April 2005.

[3] U. M. Ascher and L. R. Petzold. *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*. SIAM, 1998.

[4] L. Daniel, O. C. Siong, L. S. Chay, K. H. Lee, and J. White. A multiparameter moment-matching model-reduction approach for generating geometrically parameterized interconnect performance models. *IEEE Trans. CAD*, 23(5):678–693, 2004.

[5] P. V. Dooren. The generalized eigenstructure problem in linear system theory. *IEEE Trans. Aut. Contr.*, AC-26:111–129, 1981.

[6] M. Günther. *Ladungsorientierte Rosenbrock-Wanner-Methoden zur numerischen Simulation digitaler Schaltungen*. VDI, Düsseldorf, 1995.

[7] M. Günther and U. Feldmann. CAD based electric circuit modeling in industry I: mathematical structure and index of network equations. volume 8 of *Surv. Math. Ind.*, pages 97–129, 1999.

[8] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems*. Springer, Berlin, 2nd edition, 1996.

[9] Y. Li, Z. Bai, Y. Su, and X. Zeng. Parameterized model order reduction via a two-directional Arnoldi process. In *IEEE/ACM international conference on computer-aided design*, pages 868–873, 2007.

[10] K. Mohaghegh, R. Pulch, M. Striebel, and J. ter Maten. Model order reduction for semi-explicit systems of differential algebraic equations. In *Proceedings MATHMOD 09 Vienna*, 2009.

[11] K. Mohaghegh, M. Striebel, J. ter Maten, and R. Pulch. Nonlinear model order reduction based on trajectory piecewise linear approach: comparing different linear cores. In *Scientific Computing in Electrical Engineering SCEE 2008*. Springer, 2009.

[12] J. Phillips and L. M. Silveira. Poor's man TBR: a simple model reduction scheme. In *DATE*, volume 2, pages 938–943, 2004.

[13] J. Stoer and R. Bulirsch. *Introduction to Numerical Analysis*. Springer, New York, 2nd edition, 1993.

[14] M. Striebel. *Hierarchical mixed multirating for distributed itegration of DAE network equations in chip design*. PhD thesis, Bergischen Universität Wuppertal, 2006.