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DAE-Index and Convergence Analysis of Lumped Electric Circuits Refined by 3-D Magnetoquasistatic Conductor Models

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Abstract In this paper the field/circuit coupling is reconsidered for (non-linear) lumped electric circuits refined by 3-D magnetoquasistatic conductor models, where the circuit is described by modified nodal analysis and the field is discretized in the terms of the finite integration technique. The coupling of the two systems of differential-algebraic equations is discussed and two numerical approaches are proposed, the weak (co-simulation) and strong coupling (monolithic). The DAE-index of the subproblems and of the full problem are analyzed and the convergence properties of the co-simulation are studied by inspecting the dependencies of algebraic constraints on solutions of previous iterations. Finally computational results of a simple half rectifier circuit are exemplarily given to prove the concepts.

1 Introduction

Many basic elements in circuit analysis are described by (non-)linear relations, disregarding field effects. Sometimes, they are replaced by more complex but more realistic *companion models* to meet reality. These give, however, only a partial insight into field effects. In contrast *refined models* directly rely upon Maxwell's equations. We restrict our analysis to the refinement of two conductor types, which are embedded into electric circuits and exhibit proximity and skin effects with eddy currents.

The coupled problem is a system of differential-algebraic equations (DAEs) originating from Kirchhof's Laws and the discrete Maxwell Equations. It can be directly addressed by solving one *monolithic* system using a field- or circuit-oriented approach. In the field approach, commonly the circuit is described using loop/branch techniques and is solved within the field simulator. This approach is quite successful and well understood [1], but it is neither efficient for coupling with very large circuits nor usable within modern circuit simulators that are based on modified nodal

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analysis (MNA). The circuit-oriented approach relies on MNA. Although intensive research has been carried out [2], companion models are still widespread.

Obviously the strongly coupled approaches above will not have the advantages of both simulators. Therefore *co-simulation* becomes beneficial [3]. It allows the use of different simulators for each subproblem and it provides a natural support for diversifying integration methods and time-stepping rates (*multirate*) towards the subproblems. The coupling is given mathematically by a dynamic iteration scheme.

The paper is organized as follows: In Sections 2 and 3 we recall the circuit and field settings, in Section 4 we analyze the DAE-index of the field-system, in Section 5 we introduce the monolithic coupling and its co-simulation with index and convergence analysis, in Sections 6 and 7 we give an example and final conclusions.

2 Lumped Electric Circuit

Electric circuits are given by elements connected by Kirchhoff's Laws and their description is given here in terms of MNA, which yields DAEs since it is based on redundant coordinates. In the charge-flux oriented formulation, [4], the system reads

$$\begin{aligned} A_{\mathrm{C}}\tfrac{\mathrm{d}}{\mathrm{d}t}q + A_{\mathrm{R}}r(A_{\mathrm{R}}^{T}e,t) + A_{\mathrm{L}}i_{\mathrm{L}} + A_{\mathrm{V}}i_{\mathrm{V}} + A_{\mathrm{I}}i(t) + A_{\lambda}i_{\lambda}(A_{\lambda}^{T}e,t) = 0,\\ \tfrac{\mathrm{d}}{\mathrm{d}t}\Phi - A_{\mathrm{L}}^{T}e = 0, \qquad A_{\mathrm{V}}^{T}e - v(t) = 0,\\ q - q_{\mathrm{C}}(A_{\mathrm{C}}^{T}e,t) = 0, \quad \Phi - \Phi_{\mathrm{L}}(i_{\mathrm{L}},t) = 0, \end{aligned} \tag{1}$$

with incidence matrices A, node potentials e, independent and controlled current and voltage sources i, i_λ and v, currents through voltage and flux controlled elements i_V and i_L, charges and fluxes q and Φ, functions of charges, fluxes and resistances q_C, Φ_L and r (with positive definite derivatives), respectively.

The numerical properties of (1) are well known, e.g. the DAE-index has been discussed by decomposing the unknown $(e, i_{\mathrm{V}}, i_{\mathrm{L}}, q, \Phi)$ into algebraic and differential parts using a projector Q_C onto the kernel of A_C^T, i.e.,

$$Q_C \ker A_C^T = \ker A_C^T \quad \text{and} \quad A_C^T Q_C = 0,$$

and its complementary projection $P_C = I - Q_C$. We assume in the terms above

C1 No loops of capacitors and voltage sources, i.e., $\ker(A_C, A_R, A_V)^T = \{0\}$.
C2 No cutsets of inductors and/or current sources, i.e., $\ker Q_C^T A_V = \{0\}$.
C3 Voltage controlled current sources parallel to capacitors, i.e., $Q_C^T A_\lambda i_\lambda = 0$.

This splits the unknown into $y := (P_C e, j_L)^T$ and $z := (Q_C e, j_V, q, \phi)^T$, such that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}y = f_1(y, z, i_\lambda), \qquad 0 = g_1(y,z), \tag{2}$$

is an index-1 description of (1) since the derivative $\frac{\partial}{\partial z}g_1$ can be shown to be non-singular assuming **C1-C3**. This motivates the following important property [5]:

Theorem 1. *Let us consider a lumped electric circuit in form* (1) *that respects* ***C3****, then the flux/charged oriented MNA leads to an index-1 DAE iff* ***C1-C2*** *hold, it leads otherwise to an index-2 DAE.*

3 Electromagnetic Field

The electromagnetic field is described by Maxwell's equations. We assume that they are spatially discretized by either the finite integration technique or the finite elements method using lowest-order Whitney elements and staggered grids [6, 7]. This gives their discrete counterpart, also called the Maxwell's Grid Equations (MGE)

$$C\widehat{e} = -\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\widehat{b}}\ , \qquad \tilde{C}\widehat{h} = \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\widehat{d}} + \widehat{\widehat{j}}\ , \qquad \tilde{S}\widehat{\widehat{d}} = q\ , \qquad S\widehat{\widehat{b}} = 0\ , \tag{3}$$

with discrete curl operators C and $\tilde{C}$, divergence operators S and $\tilde{S}$, electric and magnetic field strength $\widehat{e}$ and $\widehat{h}$, current density, discrete magnetic and electric flux $\widehat{\widehat{j}}$, $\widehat{\widehat{b}}$ and $\widehat{\widehat{d}}$, respectively. For linear materials, MGE are accomplished by

$$\widehat{\widehat{b}} = M_{\mu}\widehat{h}\ , \qquad \widehat{\widehat{d}} = M_{\varepsilon}\widehat{e}\ , \qquad \widehat{\widehat{j}} = M_{\sigma}\widehat{e}\ , \tag{4}$$

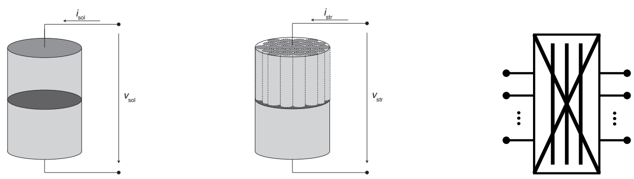
with symmetric positive (semi-)definite matrices M_{μ}, M_{ε} and M_{σ} for the permeabilities, permittivities and conductivities, respectively; only M_{σ} is generally singular due to non-conducting regions [8]. We obtain from (3,4) the curl-curl equation

$$M_{\varepsilon}\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\widehat{a} + M_{\sigma}\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{a} + K_{\nu}\widehat{a} = \widehat{\widehat{j}}_{\mathrm{src}}\ , \tag{5}$$

with $\widehat{a}$ denoting the magnetic vector potential (MVP) and $K_{\nu} := \tilde{C}M_{\nu}C$.

4 Field Models as Refined Network Elements

Conductor models for connecting field and circuit parts are well-known. Most common are solid and stranded conductors (Fig. 1). We use the given symbol for a (mul-



(a) Solid **(b)** Stranded **(c)** Symbol

Fig. 1: Conductor models **(a)**, **(b)** and device symbol **(c)** that embeds both into the circuit.

tiport) device that consists of (multiple) conductors of both types which are tightly coupled by the field. The latter is described by the curl-curl equation and excited by the source current density $\widehat{\widehat{j}}_{\mathrm{src}}$ due to the connected circuit [9]. Typically voltages of solids (v_{sol}) and currents of stranded conductors (i_{str}) are considered to be given and thus the excitation reads

$$\widehat{\widehat{j}}_{\mathrm{src}} = M_\sigma Q_{\mathrm{sol}} v_{\mathrm{sol}} + Q_{\mathrm{str}} i_{\mathrm{str}} \ . \tag{6}$$

Here $Q = [Q_{\mathrm{sol}}, Q_{\mathrm{str}}]$ denotes the coupling matrix. Each column corresponds to a conductor model and imposes currents/voltages onto edges of the grid. The unknown currents i_{sol} and voltages v_{str} are obtained by the additional equations

$$i_{sol} = G_{\mathrm{sol}} v_{\mathrm{sol}} - Q_{\mathrm{sol}}^T M_\sigma \tfrac{\mathrm{d}}{\mathrm{d}t} \widehat{a} \ , \qquad v_{\mathrm{str}} = R_{\mathrm{str}} i_{\mathrm{str}} + Q_{\mathrm{str}}^T \tfrac{\mathrm{d}}{\mathrm{d}t} \widehat{a} \ , \tag{7}$$

with the diagonal matrices of solid conductances G_{sol} and stranded resistances $R_{\mathrm{str}} = G_{\mathrm{str}}^{-1}$. Let the model above fulfill the following assumptions

F1 The field is magnetoquasistatic, i.e., $\max |\tfrac{\mathrm{d}}{\mathrm{d}t} \widehat{\widehat{d}}| \ll \max |\widehat{\widehat{j}}_{\mathrm{src}}|$.
F2 The solution is unique, i.e., $[M_\sigma, K_\nu] := \det(\lambda M_\sigma + K_\nu) \neq 0$ for a λ.
F3 The models are spatially disjunct, i.e., $Q_{(i)}^T \cdot Q_{(j)} = 0$, for all $i \neq j$.
F4 The excitation is consistent, i.e., $\ker(C Q_{\mathrm{sol}}) = \{0\}$, $\ker(C M_{\sigma,\mathrm{aniso}}^+ Q_{\mathrm{str}}) = \{0\}$.

The first assumption **F1** implies the neglect of $M_\varepsilon \frac{\mathrm{d}^2}{\mathrm{d}t^2} \widehat{a}$ in the curl-curl equation (5), such that (5) becomes a first order DAE, which is generally not uniquely solvable due to the non-trivial nullspaces of M_σ and C. Thus the integration requires the selection of one solution within the equivalent class defined by $\widehat{\widehat{b}} = C\widehat{a}$, therefore we need the *gauging* assumption **F2**, [10]. From **F3** we obtain the equivalent form

$$M_{\sigma,\mathrm{fillin}} \tfrac{\mathrm{d}}{\mathrm{d}t} \widehat{a} + K_\nu \widehat{a} = M_\sigma Q_{\mathrm{sol}} v_{\mathrm{sol}} + Q_{\mathrm{str}} G_{\mathrm{str}} v_{\mathrm{str}} := \widehat{\widehat{j}}^{\,*}_{\mathrm{src}} \ , \tag{8a}$$

$$Q_{\mathrm{sol}}^T K_\nu \widehat{a} = i_{\mathrm{sol}} \ , \tag{8b}$$

$$G_{\mathrm{str}} Q_{\mathrm{str}}^T M_{\sigma,\mathrm{aniso}}^+ K_\nu \widehat{a} = i_{\mathrm{str}} \ , \tag{8c}$$

where $M_{\sigma,\mathrm{aniso}}^+$ is the pseudoinverse of the anisotropic conductivity matrix for stranded conductors and $M_{\sigma,\mathrm{fillin}} := M_\sigma + Q_{\mathrm{str}} G_{\mathrm{str}} Q_{\mathrm{str}}^T$ is a combined (dense) matrix for both conductor types.

Lemma 1. *Let the field problem consist of solid and stranded conductors which fulfill **F1-F3**, then the curl-curl equation* (8a) *is (algebraic) index-1 for given voltages and has only non-trivial components in the differential part.*

Proof. The matrix pencil of (8a) is regular due to **F2**, so the symmetric positive semi-definiteness of $M_{\sigma,\mathrm{fillin}}$ implies that (8a) is index-1 and there is the Kronecker form

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \widehat{a}_1(t) + U_1 K_\nu V_1 \ \widehat{a}_1(t) = U_1 \ \widehat{\widehat{j}}^{\,*}_{\mathrm{src}} \ , \tag{9a}$$

$$\widehat{a}_2(t) = U_2 \ \widehat{\widehat{j}}^{\,*}_{\mathrm{src}} \ , \tag{9b}$$

that splits the MVP $\widehat{a} = V_1\widehat{a}_1 + V_2\widehat{a}_2$ into differential and algebraic components by using the regular matrices $U^T = \left(U_1^T \;, U_2^T\right)$ and $V = (V_1 \;, V_2)$. From

$$U_2 M_{\sigma,\mathrm{fillin}} = U_2 \left(M_\sigma + Q_{\mathrm{str}} G_{\mathrm{str}} Q_{\mathrm{str}}^T\right) = 0$$

follows that both $U_2 M_\sigma$ and $U_2 Q_{\mathrm{str}} G_{\mathrm{str}}$ vanish because the images of M_σ and Q_{str} are distinct, since **F3** is assumed. Hence we finally conclude that $\widehat{a}_2 = U_2 \,\widehat{\widehat{j}} = 0$. □

Let us now study the full system (8) in the abstract form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{a} = f_{2a}(\widehat{a}, v_\lambda), \qquad 0 = f_{2b}(\widehat{a}, v_\lambda), \qquad 0 = g_2(\widehat{a}, i_\lambda), \tag{10}$$

where the voltages $v_\lambda = (v_{\mathrm{sol}}, v_{\mathrm{str}})^T$ and the currents $i_\lambda = (i_{\mathrm{sol}}, i_{\mathrm{str}})^T$ are combined in vectors. The algebraic evaluation f_{2b} is trivial in our case because of Lemma 1 and the algebraic function g_2 consists of

$$0 = g_{\mathrm{sol}}(\widehat{a}, i_{\mathrm{sol}}), \qquad 0 = g_{\mathrm{str}}(\widehat{a}, i_{\mathrm{str}}).$$

System (10) establishes a relation between currents (i_{sol}, i_{str}) and voltages (v_{sol}, v_{str}) and we can choose which quantity is treated as unknown for each conductor type in the field system, since then the other quantity is defined by the coupled electric circuit. Therefore we will distinguish between the possible sets in the following.

Theorem 2. *Let the field problem consist of solid and stranded conductors which fulfill **F1**-**F4**, then the system* (8) *has (differential) index-2 and only for the case of given voltages* v_{sol} *and* v_{str} *it has index-1.*

Proof. In the case of given voltages the currents i_λ are obtained by evaluations of the algebraic equation g_2. Thus one differentiation with respect to time yields an ODEs, hence we have (differential) index-1. In all other cases the arguments are analogue to the case of given i_{sol} and v_{str}. Now the function f_{2a} in (10) depends on the unknown v_{sol} and one time derivative yields the additional *hidden constraint*:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} g_{\mathrm{sol}}(\widehat{a}, i_{\mathrm{sol}}) = \frac{\partial}{\partial \widehat{a}} g_{\mathrm{sol}} \cdot f_2(\widehat{a}, v_{\mathrm{sol}}) + \frac{\mathrm{d}}{\mathrm{d}t} i_{\mathrm{sol}} =: h_{\mathrm{sol}}(\widehat{a}, v_{\mathrm{sol}}, \frac{\mathrm{d}}{\mathrm{d}t} i_{\mathrm{sol}}),$$

and since the conductivity matrices M_σ and $M_{\sigma,\mathrm{aniso}}$ reflect **F3** ($M_{\sigma,\mathrm{aniso}} Q_{\mathrm{sol}} = 0$), another differentiation of this constraint gives

$$\frac{\partial}{\partial v_{\mathrm{sol}}} h_{\mathrm{sol}} = Q_{\mathrm{sol}}^T K_\nu M_{\sigma,\mathrm{fillin}}^+ M_\sigma Q_{\mathrm{sol}} = Q_{\mathrm{sol}}^T K_\nu Q_{\mathrm{sol}} = Q_{\mathrm{sol}}^T C^T M_\nu C Q_{\mathrm{sol}} \;,$$

which is non-singular due to **F4**; thus it is index-2. □

We conclude, if some voltage is considered unknown, then (8) is an index-2 Hessenberg system (with index-1 evaluations), which is less ill-conditioned than the general case. Additionally the index-2 variables enter the computation linearly and without time-dependence. Therefore the differential components are not affected by the derivatives of perturbations, [11], and thus the numerical difficulties correspond to index-1.

5 Coupling

We assign the voltages v_λ to differences of node potentials e and map the controlled current sources i_λ to the currents through the conductors in (1)

$$v_\lambda = A_\lambda^T e\,, \qquad\qquad i_\lambda = (i_{\mathrm{sol}}, i_{\mathrm{str}})^T. \tag{11}$$

We finally obtain the *monolithic* system consisting of (1), (8) and (11).

Theorem 3. *Let us consider an electric circuit in the form* (1) *with* ***C1-C2****, which is monolithically coupled via* (11) *to a field model* (8) *of solid and stranded conductors fulfilling* ***F1-F4****, then the full system is (differential) index-1.*

Proof. The algebraic part of the MVP is insignificant for solid and stranded conductors according to Lemma 1. Hence after embedding the field into the circuit system the separated unknowns of the full system read

$$y := (P_C e, j_L, \widehat{a}_1)^T, \qquad\qquad z := (Q_C e, j_V, q, \phi, i_\lambda)^T. \tag{12}$$

The critical partial derivative of the algebraic equation $\frac{\partial}{\partial z} g$ consisting of g_1 and g_2 is non-singular, since the first is regular due to **C1-C2** and the second is just an evaluation of a differential variable ($\widehat{a}_1$). Thus we have index-1. $\square$

The assumption **C3** is not required in the monolithic coupling because the algebraic part of the MVP was shown to vanish for the excitement of solid and stranded conductors.

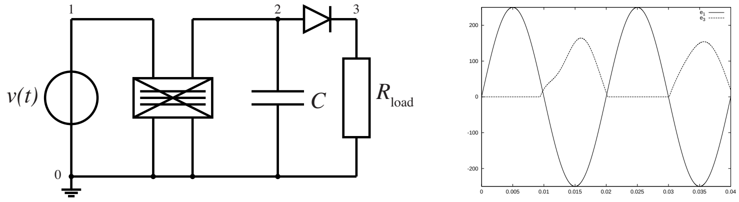
Alternatively the subproblems could be treated separately by a *waveform relaxation scheme* (of Jacobi or Gauß-Seidel type). When applying these schemes to DAEs one has to pay attention to algebraic constraints to avoid numerical instabilities, [12]. We suggest the following Gauß-Seidel scheme

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{dt}}\widehat{a}^{(1)} &= f_2(\widehat{a}^{(1)}, v^{(0)}), \quad v^{(0)} := A_\lambda^T(y^{(0)} + z^{(0)}), \quad & \tfrac{\mathrm{d}}{\mathrm{d}t} y^{(1)} &= f_1(y^{(1)}, z^{(1)}, i_\lambda^{(1)}),\\ 0 &= g_2(\widehat{a}^{(1)}, i_\lambda^{(1)}), & 0 &= g_1(y^{(1)}, z^{(1)}, i_\lambda^{(1)}), \end{aligned}$$

where the field is coupled via the current sources i_λ to the circuit. The scheme above shows directly that there is no dependence in algebraic constraints (g_1, g_2) on old algebraic iterates (i_λ, z) and hence the convergence is guaranteed [13] and we obtain:

Lemma 2. *Let us consider an electric circuit in the form* (1) *that respects* ***C1-C2*** *is coupled via* (11) *to a field model* (8) *of solid and stranded conductors fulfilling* ***F1-F4****, then both subproblems are index-1 and the waveform-relaxation above will converge.*

The additional assumption **C3** can eliminate the i_λ-dependence of the algebraic equation g_1 and allows us to exchanges the computational order of the subproblems and thus we may compute the circuit first without losing the convergence guarantee.



(a) Half rectifier: $v_{\mathrm{eff}} = 250\,\mathrm{V}$, $f = 50\,\mathrm{Hz}$, $R_{\mathrm{load}} = 100\,\Omega$ and Shockley diode $\mathrm{I_s} = 1\mu\mathrm{A}$

(b) Voltage in nodes 1 and 3, obtained by `mono`, $H = 5\mu\mathrm{s}$

Fig. 2: Refined half rectifier circuit and its input and computed output voltages.

6 Numerical Experiments

The numerical experiments have been obtained with code that is implemented within the *COMSON DP* (`http://www.comson.org`) using field models constructed by *EM Studio* from *CST* (`http://www.cst.com`). It is capable of both, the monolithic and the co-simulation of non-linear circuits refined by (linear) conductor models. The discussed schemes have been supplied to the DP, where the time integration is kept simple by using Backward Euler.

The example of Fig. 2 is a refined half rectifier with a transformer consisting of two stranded conductors and a solid core. The co-simulation without iterations of time frames (`cosim1`) is slightly faster than the monolithic simulation (`mono`) using the same time steps H and it yields comparable results if the accuracy requirement is quite low. For decreasing step sizes `cosim1` does not linearly improve its accuracy as `mono` and `cosim3` (3 iterations) do (Fig. 3), but the latter has an increased computational effort due to the additional iterations.

Adaptive time-integrators in the co-simulation apply the same step size to the circuit and the field, as long as they do not have multirate potential itself. This is in line with the fact that the field reflects the dynamics of the coupled circuit nodes.

7 Conclusions

The field problem is essentially an index-1 DAE, the monolithic coupled system is still index-1 and the convergence of the proposed co-simulation is guaranteed, as documented by the computation of a refined rectifier circuit. The co-simulation uses problem-specific software packages and exploits multirate potentials if available in the circuit, but does not have all possible benefits. Applying a step size and iteration control will improve efficiency, and the use of more complex equivalent circuits (e.g. additional inductivities) might require fewer field updates [14, 15].

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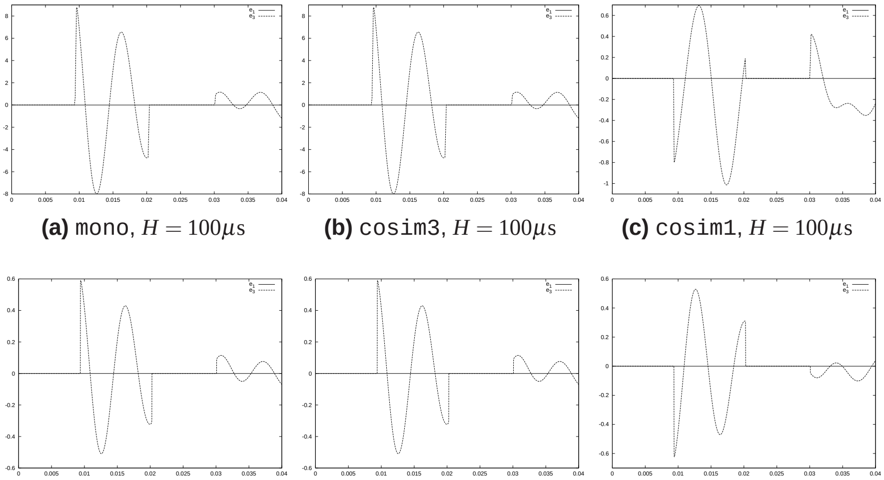

(a) `mono`, $H = 100\mu$s **(b)** `cosim3`, $H = 100\mu$s **(c)** `cosim1`, $H = 100\mu$s

(d) `mono`, $H = 10\mu$s **(e)** `cosim3`, $H = 10\mu$s **(f)** `cosim1`, $H = 10\mu$s

Fig. 3: Errors in the voltages compared to the results of `mono`, $H = 5 \cdot 10^{-6}$ from Fig. 2b

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