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Modular Levy copulas join up for series representation

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Abstract

Lévy copulas opened as the generic concept to describe dependence structures of multidimensional Lévy processes. In this paper we contribute a refinement of an existing construction pattern due to Tankov that uses Lévy copulas for spectrally positive processes to build Lévy copulas for general processes. This pattern involves a joining function which we mainly cover in the present work. We deduce a probabilistic interpretation of the joining function and identify separate structures for jump sign and jump size dependence. This distinction renders natural the simulation of multidimensional Lévy processes by series representation. We quantify graphically the effect of hierarchial joinders using simplified sample algorithms for path generation.

1 Introduction

Recently, a few authors set in place the notion of a Lévy copula extending the non-gaussian association between static variates to dynamic Lévy processes; we refer to Tankov (2004),Tankov (2006) and chapter 5 in Cont and Tankov (2004). Identifying the key role of the Lévy measure in dependence modelling the authors constructed copulas for this characteristic and derived algorithms for simulation from Lévy processes with specified dependence. This necessitates the development of comprehensive copula models to describe a wide range of dependence between process components. The literature on parsimonious dependence structures truly speaks in favor of hierarchical structures, see Embrecht et al. (2001) and Savu and Trede (2006) for hierarchical ordinary copulas. Here we take a Lévy copula model due to Tankov Tankov (2004) as grounds for bringing hierarchical patterns to jump sign dependence.

In section 2 we recall the fundamental properties of a Lévy process and state substantial formulae for describing its law. We proceed to investigate the dependence structures of multidimensional processes in particular and adopt the concept of Lévy copulas and Lévy copulas for spectrally positive processes as introduced in Tankov (2004). We show that Lévy copulas achieve to formulate general dependence patterns in multivariate Lévy processes.

In section 3 we reproduce a theorem due to Tankov, that proposes to construct copulas for general Lévy process by assembling copulas for spectrally positive Lévy processes via a joining function. We derive an instance of a constant joining function that covers hierarchical structures. Further we analyze separately the conditional distributions of signs and absolute sizes of the jumps.

In section 4 we discuss multidimensional process simulation by means of series representation. We question how a dissociative perspective on jump sign and jump size dependence may feed the generation of Lévy paths. We finally exhibit the impact of hierarchical joining functions by means of graphical results using simplified sample algorithms.

2 Multivariate Lévy processes and copulas

In this section we state the essentials on Lévy processes and review the achievements in forwarding copula models to the dynamical framework of jump processes.

A Lévy process is a $\mathbb{R}^d$-valued cadlag stochastic process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{R}^d)$ with stationary and independent increments such that $X_0 = 0$. In particular, jumps of the form $X_t - X_{t-} = \Delta X_t$ may occur sudden and at random but countable times. The characteristic function of an $\mathbb{R}^d$-valued Lévy process is of the form

$$\begin{aligned} E[e^{iz.X_t}] &= e^{t\psi(z)}, \quad where \\ \psi(z) &= -\frac{1}{2}z.Az + i\gamma.z + \int_{\mathbb{R}^d} \left(e^{iz.x} - 1 - iz.x1_{|x|\leq 1}\right)\nu(dx), \end{aligned} \tag{1}$$

where A is a symmetric nonnegative-definite square matrix, $\gamma \in \mathbb{R}^d$ and ν a positive measure on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$. Formula (1) is referred to as the Lévy Khinchin representation, whereas subtraction of $iz.x1_{|x|\leq 1}$ is herein after referred to as compensation of small jumps. We call ψ the characteristic exponent of Lévy process $(X_t)_{t\geq 0}$ and identify A as the covariance matrix of the Brownian motion part of the Lévy process and γ as *some* drift vector (see annotation below). Measure ν is of particular importance to the Lévy calculus and thus merits the tag *Lévy measure*. In virtue of the Lévy-Khinchin representation the triplet (γ, A, ν) uniquely determines the distribution of process $(X_t)_{t\geq 0}$. It is hence called the characteristic triplet of Lévy process X_t.

Given that ν satisfies $\int_{\mathbb{R}^d}(|x| \wedge 1)\nu(dx) < \infty$, meaning that the jump process is of finite variation, the characteristic exponent can be reduced to

$$\psi(z) = -\frac{1}{2}z.Az + ib.z + \int_{\mathbb{R}^d} \left(e^{iz.x} - 1\right)\nu(dx).$$

Note here that the drift vector changes with different compensation but the characteristic triplet corresponds conventionally to unit compensation of small jumps. We proceed with a brief review on increasing functions.

Let $F : \bar{\mathbb{R}}^d \to \bar{\mathbb{R}}$ be a d-place function on the extended space. For $a, b \in \bar{\mathbb{R}}^d$ with $a \le b$ and $\overline{(a,b]} \in \bar{\mathbb{R}}^d$, we here define the F-volume of $(a,b]$ to be

$$V_F((a,b]) = \sum_{u \in \{a_1,b_1\} \times \ldots \times \{a_d,b_d\}} (-1)^{N(u)} F(u), \tag{2}$$

where $N(u) = \sharp\{k : u_k = a_k\}$, i.e. the sum of the signed values of F over the vertices of $(a,b]$. F is called d-increasing if the volume $V_F((a,b])$ of any d-box $\overline{(a,b]} \in \bar{\mathbb{R}}^d$ is nonnegative. For $I \in \{1,...,d\}$ we further denote the I-margin of F by

$$F^I((u_i)_{i \in I}) = \lim_{c \to \infty} \sum_{(u_j)_{j \in \bar{I}} \in \{-c,\infty\}^{|\bar{I}|}} F(u_1,...,u_d) \prod_{j \in \bar{I}} sign(u_j)$$

with cardinality $|I|$ and complement $\bar{I} = \{1,...,d\} \backslash I$. .

For the rest of this paper we assume a Lévy process $(X_t)_{t \ge 0}$ with characteristic triplet $(0,0,\nu)$, that is, we eclipse the Brownian motion part from considerations. It is certainly feasible to focus on the jump component of a Lévy process due to independence of the components. We now provide the notion of tail integrals associated to the Lévy measure.

Definition 2.1 *Let X be a $\mathbb{R}^d$-valued Lévy process with Lévy measure ν. The tail integral of ν is the function $U : (\mathbb{R}\backslash 0)^d \to \mathbb{R}$ defined by*

$$U(x_1,...,x_d) = \nu\left(\prod_{j=1}^d \mathcal{I}(x_j)\right) \prod_{j=1}^d sign(x_j),$$

where

$$\mathcal{I}(x) = \begin{cases} [x,\infty), & x \ge 0; \\ (-\infty,x), & x < 0. \end{cases}$$

Note that $(-1)^d U$ is d-increasing and left-continuous. We know from elementary statistics that any probability measure can be characterized by its distribution function. In a similar way, any Lévy measure corresponds to the set of its marginal tail integrals. Next, the definition of a Lévy copula compares to that of an ordinary copula apart from the domain.

Definition 2.2 *A function $F : \bar{\mathbb{R}}^d \to \bar{\mathbb{R}}$ is called a Lévy copula, if*

1. *$F(u_1, ..., u_d) \neq \infty$ for $(u_1, ..., u_d) \neq (\infty, .., \infty)$*

2. *$F(u_1, ..., u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, ..., d\}$*

3. *F is d-increasing*

4. *$F^i(u) = u$ for any $i \in \{1, ..., d\}$, $u \in \mathbb{R}$.*

We clearly recognize groundedness on the axes, the d-increasing property and uniformity of the margins.

We can confine ourselves to the notion of Lévy copulas for spectrally positive Lévy processes when limiting considerations to Lévy processes with only positive jumps in each component, i.e. with characteristic Lévy measures ν that have restricted support $[0, \infty)^d \setminus \{0\}$.

Definition 2.3 *A function $F : \bar{\mathbb{R}}_+^d \to \bar{\mathbb{R}}_+$ is called a Lévy copula, if*

1. *$F(u_1, ..., u_d) \neq \infty$ for $(u_1, ..., u_d) \neq (\infty, .., \infty)$*

2. *$F(u_1, ..., u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, ..., d\}$*

3. *F is d-increasing*

4. *$F^i(u) = u$ for any $i \in \{1, ..., d\}$, $u \in \bar{\mathbb{R}}_+$.*

In view of distinction, we shall denote copulas for spectrally positive Lévy processes by F^+. The obvious linkage between the two definitions is given by the following result due to Tankov: if F^+ is a Lévy copula on $\bar{\mathbb{R}}_+^d$, then it can be extended to a Lévy copula F on $(-\infty, \infty]^d$ by

$$F(u_1, ..., u_d) = \begin{cases} F^+(u_1, ..., u_d), & (u_1, ..., u_d) \in [0, \infty]^d; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

As a consequence, findings in the general case transfer to the spectrally positive case.

Example 2.1 *Let $\theta > 0$. Then*

$$F^+(u_1, ..., u_d) = \left(\sum_{i=1}^{d} u_i^{-\theta} \right)^{-1/\theta} \tag{4}$$

is a Lévy copula on $[0, \infty]^d$. It can be extended to a Lévy copula F on $(-\infty, \infty]^d$ by

$$F(u_1, ..., u_d) = \left(\sum_{i=1}^{d} |u_i|^{-\theta} \right)^{-1/\theta} 1_{u_i \geq 0, i=1,...,d}. \tag{5}$$

We refer to 4 as Clayton Lévy copula and to 5 as extended Clayton Lévy copula.

The fundamental result now deals with the association between a Lévy process' marginal tail integrals through Lévy copulas. First to be announced in Tankov Tankov (2004), it draws on a direct adaptation of Sklar's findings for the case of multidimensional distribution functions.

Theorem 2.1 *Let ν be a Lévy measure on $\mathbb{R}^d \backslash \{0\}$. Then there exists a Lévy copula F such that the tail integrals of ν satisfy*

$$U^I((x_i)_{i\in I}) = F^I((U_i(x_i))_{i\in I})$$

for any non-empty $I \subset \{1, ..., d\}$ and any $(x_i)_{i\in I} \in (\mathbb{R}\backslash 0)^d$. Conversely, if F is a d-dimensional Lévy copula and $\nu_1, ..., \nu_d$ are Lévy measures on $\mathbb{R}\backslash\{0\}$ with tail integrals $U_1, ..., U_d$, then there exists a unique Lévy measure on $(\mathbb{R}\backslash\{0\})^d$ with one-dimensional marginal tail integrals $U_1, ..., U_d$.

The first part assigns Lévy copulas the capability to represent all types of dependence between the jumps of a Lévy process. The second part renders possible to construct multidimensional Lévy models by specifying separately jump dependence structure and one-dimensional laws for the components. We call any such F the Lévy copula of X.

We close this section with a probabilistic interpretation of Lévy copulas which goes also back to Tankov. Let thereto be F a Lévy copula on $(-\infty, \infty]^d$ satisfying

$$\lim_{(x_i)_{i\in I}\to\infty} F(u_1, ..., x_d) = F(u_1, ..., u_d)|_{(x_i)_{i\in I}=\infty} \tag{6}$$

for all $I \subset \{1, ..., d\}$. This Lévy copula defines a positive measure μ on $\mathbb{R}^d$ with Lebesgue margins such that for each $a, b \in \mathbb{R}^d$ with $a \leq b$,

$$V_F(|a, b|) = \mu((a, b]).$$

Defining $f : (u_1, ..., u_d) \mapsto (U_1^{-1}(u_1), ..., U_d^{-1}(u_d))$, the relation between Lévy measure ν and measure μ is

$$\nu(A) = \mu(\{u \in \mathbb{R}^d : f(u) \in A\}.$$

One can then show that there exists a family, indexed by $\xi \in \mathbb{R}$, of positive Radon measures $K(\xi, dx_2, ..., dx_d)$ on $\mathbb{R}^{d-1}$, such that $\xi \mapsto K(\xi, dx_2, ..., dx_d)$ is Borel measurable and

$$\mu(dx_1, dx_2, ..., dx_d) = \lambda(dx_1) \otimes K(x1, dx_2, ..., dx_d).$$

$K(\xi, dx_2, ..., dx_d)$ is called the *family of conditional probability distributions associated to the Lévy copula F*. Denoting

$$F_\xi(x_2, ..., x_d) = K(\xi, (-\infty, x2], ..., (-\infty, x_d]),$$

Tankov showed that there exist a nullset N such that for every $\xi \in \mathbb{R} \setminus N$, F_ξ is a probability distribution function satisfying

$$F_\xi(x_2, ..., x_d) = sign(\xi) \frac{\partial}{\partial \xi} V_F((\xi \wedge 0, \xi \vee 0] \times (-\infty, x_2] \times ... \times (-\infty, x_d]) \quad (7)$$

in every point $(x_2, ..., x_d)$ where F_ξ is continuous. The conditional distribution function F_ξ^+ associated to Lévy copula F^+ on $[0, \infty]^d$ takes the form

$$F_\xi^+(x_2, ..., x_d) = \frac{\partial}{\partial \xi} F^+(\xi, x_2, ..., x_d). \quad (8)$$

3 Modular Lévy copulas

In this paragraph we renew Tankov's idea of constructing general Lévy copulas by joined copula modules. We discuss distributional features of copulas of this type and promote a hierarchical joinder.

With the notion of Lévy copulas for spectrally positive processes at hand one can think of modelling the jump dependence structure of a $\mathbb{R}^d$-valued Lévy process in each of the 2^d corners separately. Tankov Tankov (2004) accesses a such modular design pattern in his

Theorem 3.1 *For each $\alpha = \{\alpha_1, ..., \alpha_d\} \in \{-1, 1\}^d$ let $g^\alpha(u) : [0, \infty] \to [0, 1]$ be a nonnegative, increasing function satisfying*

$$\sum_{\alpha \in \{-1,1\}^d \text{ with } \alpha_k = -1} g^\alpha(u) = 1 \text{ and } \sum_{\alpha \in \{-1,1\}^d \text{ with } \alpha_k = 1} g^\alpha(u) = 1 \quad (9)$$

for all $u \in [0, \infty]$ and all $k \in \{1, ..., d\}$. Moreover, let F^α be positive Lévy copulas that satisfy the following continuity property at infinity: for all $I \subset \{1, ..., d\}, (u_i)_{i \in \bar{I}} \in [0, \infty]^{|\bar{I}|}$ we have

$$\lim_{(u_i)_{i \in I} \to (\infty, ..., \infty)} F^\alpha(u_1, ..., u_d) = F^\alpha(v_1, ..., v_d),$$

where $v_i = u_i$ for $i \in I$ and $v_i = \infty$ otherwise. Then

$$F(u_1, ..., u_d) = F^\alpha(|u_1| g^\alpha(|u_1|), ..., |u_d| g^\alpha(|u_d|)) \prod_{i=1}^{d} sign(u_i)$$

with $\alpha = (sign(u_1), ..., sign(u_d))$ *defines a Lévy copula.*

In this section and in the following we shall refer to Lévy copulas $F^\alpha, \alpha \in \{-1,1\}^d$ as modules and to the set of functions $\{g^\alpha, \alpha \in \{-1,1\}^d\}$ as joinder, for F so-defined we shall thence use the term *modular Lévy copula.* It is left as a side note that normalization of the joinder is to meet uniformity at the margins of the modular Lévy copula; see Tankov Tankov (2004) for the derivation.

Example 3.1 *Let the module copula* F^+ *be given by 4 choosing* $d = 2$*. Further define the joinder by*

$$g^\alpha(u) = \begin{cases} 1, & \textit{for } \alpha_1 = \alpha_2; \\ 0, & \textit{otherwise.} \end{cases}$$

for all $u \in [0, \infty]$*. This produces the modular Lévy copula* F *on* $(-\infty, \infty]^2$ *with*

$$F(u_1, u_2) = \Big((|u_1|g^\alpha)^{-\theta} + (|u_2|g^\alpha)^{-\theta} \Big)^{-1/\theta} sign(u_1 u_2),$$

as it were a co-moving Clayton Lévy copula.

Here we employed a piecewise constant joinder and equal modules. In view of a parsimonious copula model (and notational ease) we shall resort to equal modules $F^\alpha \equiv F^+$ in the following, the derivations still hold in the general case. As to the joinder we shall keep to constant functions $g^\alpha(u) \equiv g^\alpha \in \mathbb{R}^+$, too, the case of a non-constant joinder will be revisited in future works.

3.1 Hierarchical joinder

In this paragraph we propose a hierarchically structured joinder and we illustrate how to meet normalization.

Our general conception of a hierarchical joinder is put in constant joining functions $g^\alpha \in \mathbb{R}^+$ corresponding to the chance of jump sign vector α, where chance is induced hierarchically by the interdependence of pairs of jump signs. We promote the idea in the three-dimensional case. Let therefore

$$m_1 : \begin{pmatrix} b_1 & a_1 \\ a_1 & b_1 \end{pmatrix}, \qquad m_2 : \begin{pmatrix} b_2 & a_2 \\ a_2 & b_2 \end{pmatrix}$$

be pair dependence matrices of jump signs. The position in the matrices corresponds to the conventional ordering of quadrants, that is, we understand entries a_i as factors of equal signs and entries b_i as factors of opposite signs for $i = 1, 2$. Assume now that the bivariate sign vector (α_1, α_2) corresponds to matrix m_1, whereas both (α_1, α_3) and (α_2, α_3) are associated to matrix m_2. This induces a hierarchical association of the kind shown in figure 1. What we want is to compute the chance g^α of trivariate sign vector $(\alpha_1, \alpha_2, \alpha_3)$ by pairwise multiplication of the matrix coefficients, for instance, the chance of $(1, -1, 1)$ would be $b_1 \cdot a_2 \cdot b_2$. Normalization of the joinder is yet to be achieved. To this, let us identify the sign vector $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \{-1, 1\}^3$ with vertices of an axially parallel cube in three-dimensional space centering in the origin. This is plotted in figure 1, where we inserted the temporary chances of the vertices resulting from the above multiplication.

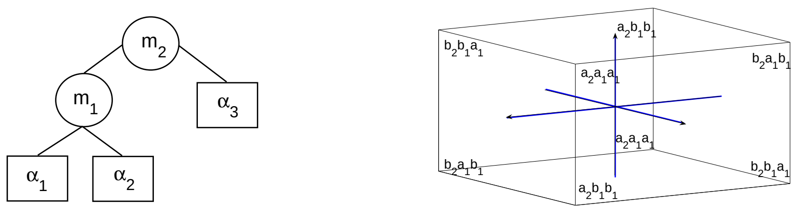

Figure 1: Left plot: Tree-diagram of hierarchic structured of sign dependence. Right plot: Cube with raw vertex labelling according to dependence matrices.

We renew condition (9)

$$\sum_{\alpha \in \{-1,1\}^3 \text{ with } \alpha_k = -1} g^\alpha = 1 \text{ and } \sum_{\alpha \in \{-1,1\}^3 \text{ with } \alpha_k = 1} g^\alpha = 1$$

and understand it as a must for a uniform joinder on each face of the cube. One can easily see that the faces have corresponding vertex labels. Then with

$$\Sigma = a_2 a_1 a_1 + a_2 b_1 b_1 + b_2 a_1 b_1 + b_2 b_1 a_1$$

normalization is straight forward and we deduce an appropriate joinder as given in the following table.

α_1	1	1	1	1	-1	-1	-1	-1
α_2	1	1	-1	-1	1	1	-1	-1
α_3	1	-1	1	-1	1	-1	1	-1
g^α	$\frac{a_2a_1a_1}{\Sigma}$	$\frac{a_2b_1b_1}{\Sigma}$	$\frac{b_2a_1b_1}{\Sigma}$	$\frac{b_2b_1a_1}{\Sigma}$	$\frac{b_2b_1a_1}{\Sigma}$	$\frac{b_2a_1b_1}{\Sigma}$	$\frac{a_2b_1b_1}{\Sigma}$	$\frac{a_2a_1a_1}{\Sigma}$

The above pattern is very simple, yet it allows for a variety of associations between jump signs. One can easily think of other nestings than shown in figure 1 but in the scope of this paper we content ourselves with the given structure.

Example 3.2 *Let F^+ be a Lévy copula of Clayton type 4 choosing $d = 3$. Further let $a_1 = \frac{1}{4}, b_1 = \frac{3}{4}, a_2 = \frac{3}{4}, b_2 = \frac{1}{4}$ producing the hierarchical joinder*

α_1	1	1	1	1	-1	-1	-1	-1
α_2	1	1	-1	-1	1	1	-1	-1
α_3	1	-1	1	-1	1	-1	1	-1
g^α	$\frac{3}{36}$	$\frac{27}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{27}{36}$	$\frac{3}{36}$.

By means of Theorem 3.1 these define a modular Lévy copula F on $(-\infty, \infty]^3$ as follows

$$F(u_1, u_2, u_3) = \begin{cases} \left(\sum_{i=1}^3 (|u_i|\frac{27}{36})^{-\theta}\right)^{-1/\theta} \prod_{i=1}^3 sig(u_i), & \alpha_1 = \alpha_2, \alpha_1 \neq \alpha_3; \\ \left(\sum_{i=1}^3 (|u_i|\frac{3}{36})^{-\theta}\right)^{-1/\theta} \prod_{i=1}^3 sig(u_i), & otherwise. \end{cases} \tag{10}$$

We have contour plotted the bivariate densities of copula 10 in figure 2 using module parameter $\theta \in \{1.5, 5\}$. Each plot shows a cross section of the trivariate density from different perspectives but at the same level 1 of the hidden variable. One can clearly observe that the (1,2)-margin has greater mass in the 3rd quadrant than in the others, meaning that a co-movement of the 1st and 2nd component in negative direction is more likely than jumps in other directions. In contrast, the (1,3)-margin as well as the (2,3)-margin have greater mass in the 4th quadrant than in the others, meaning that negative jumps in the 3rd component are most likely to accompany positive jumps in the 1rd and 2nd component, respectively. These findings are feasible due to $a_1 < b_1, a_2 > b_2$ and $\alpha_i = 1$ for either hidden margin i. In virtue of a single module copula F^+, the shape of the contour lines is constant over quadrants in all three plots. As a result of a higher value for module parameter θ the lower right plot exhibits rather clustered contours, meaning that the level curves are more centered (at 1) and thus dependence of absolute jump sizes is stronger.

The idea brought up in this paragraph can be readily forwarded to the higher dimensional case. This is revisited in a future work.

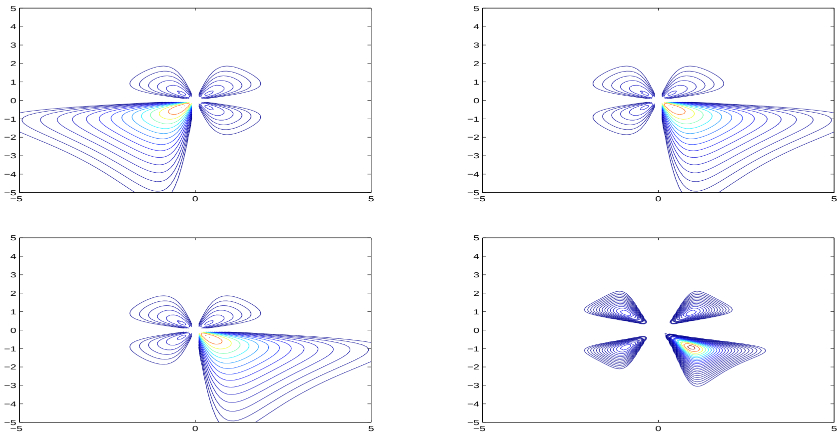

Figure 2: Contour plots of hierarchical Clayton density (10). Upper left: (1,2)-marginal density. Upper right: (1,3)-marginal density. Lower left: (2,3)-marginal density. Each using $\theta = 1.5$. Lower right: (2,3)-marginal density using $\theta = 5$.

3.2 Conditional distributions with modular design

In this paragraph we find conditional probability distributions associated to joinder and module. The findings herein do not depend on a hierarchic joinder.

So far we intentionally avoided to speak of probabilities g^α and rather used the term chance. Regarding example 3.2, joinder $g^\alpha = g^{\{\alpha_1,\alpha_2,\alpha_3\}}$ only comes as a probability distribution when conditioning on either of the values $\alpha_i, i \in \{1,2,3\}$. For instance $g^{\{1,\alpha_2,\alpha_3\}}$ is a probability distribution function on $\{-1,1\}^2$. In view of the conditional distribution function (8) associated to the modular Lévy copula F one would intuitively have the vector $(sign(x_2), sign(x_3))$ be distributed according to g^α given $\alpha_1 = sign(x_1)$. We show that this is actually the case.

Let us look to this end at the conditional probability of, say, opposing jumps $x_2 < 0, x_3 \geq 0$ given $x_1 = \xi$. Denoting $\mathbb{P}_\xi(\cdot) := \mathbb{P}(\cdot|x_1 = \xi)$ one can easily derive

$$\begin{aligned}\mathbb{P}_\xi(x_2 < 0, x_3 \geq 0) &= F_\xi(0,\infty) - F_\xi(-\infty,\infty) - F_\xi(0,0) + F_\xi(-\infty,0) \\ &= \frac{\partial F^+}{\partial u_1}(|\xi|g^{\xi,-1,1},\infty,\infty)g^{(\xi,-1,1)}, \qquad (11)\end{aligned}$$

by way of the following identities

$$
\begin{aligned}
F_\xi(0,\infty) &= \frac{\partial F^+}{\partial u_1}(|\xi|g^{\xi,1,1},0,\infty)g^{(\xi,1,1)} + \frac{\partial F^+}{\partial u_1}(|\xi|g^{\xi,1,-1},0,\infty)g^{(\xi,1,-1)} \\
&+ \frac{\partial F^+}{\partial u_1}(|\xi|g^{\xi,-1,1},\infty,\infty)g^{(\xi,-1,1)} + \frac{\partial F^+}{\partial u_1}(|\xi|g^{\xi,-1,-1},\infty,\infty)g^{(\xi,-1,-1)} \\
F_\xi(0,0) &= \frac{\partial F^+}{\partial u_1}(|\xi|g^{\xi,1,1},0,\infty)g^{(\xi,1,1)} + \frac{\partial F^+}{\partial u_1}(|\xi|g^{\xi,1,-1},0,\infty)g^{(\xi,1,-1)} \\
&+ \frac{\partial F^+}{\partial u_1}(|\xi|g^{\xi,-1,1},\infty,0)g^{(\xi,-1,1)} + \frac{\partial F^+}{\partial u_1}(|\xi|g^{\xi,-1,-1},\infty,\infty)g^{(\xi,-1,-1)} \\
F_\xi(-\infty,a) &= 0 \quad \forall a \in \overline{\mathbb{R}}.
\end{aligned}
$$

Due to uniformity of F^+ at the margins, as part of definition 2.3, we get

$$
\begin{aligned}
\frac{\partial F^+}{\partial u_1}(|\xi|g^{\xi,-1,1},\infty,\infty)g^{(\xi,-1,1)} &= \frac{\partial F^1_+}{\partial u_1}(|\xi|g^{\xi,-1,1})g^{(\xi,-1,1)} \\
&= g^{(\xi,-1,1)}.
\end{aligned}
$$

The proving does by no means depend on a specific sign vector and so the result can be generalized to

$$\mathbb{P}_\xi(sign(x_2)=\alpha_2, sign(x_3)=\alpha_3) = g^{\{\xi,\alpha_2,\alpha_3\}}.$$

Same in higher dimensions, it comes with condition (9) that g^α always is a distribution function given $\alpha_i \in \{-1,1\}$ for one $i \in \{1,...,d\}$, no matter the joinder nor the modules.

Theorem 3.2 *Let F be a modular Lévy copula on $\mathbb{R}^d$ with joinder $g^\alpha \in \mathbb{R}^+$ and modules $F^\alpha = F^+$. Further let $\xi = x_1$ be a given realization. Then it holds*

$$\mathbb{P}_\xi\left(sign((X)_{i=2}^d)=\widehat{\alpha}\right) = g^{\{\xi,\widehat{\alpha}\}}, \tag{12}$$

and we call $g^{\{\xi,\widehat{\alpha}\}}, \widehat{\alpha} \in \{-1,1\}^{d-1}$ the first conditional probability function *associated to g^α.*

Proof: The proof is essentially the same as in the three-dimensional case. Suppose $(\alpha)_{i=2}^d = \mathbf{1}$, the other cases being derived analogously. It holds

$$
\begin{aligned}
\mathbb{P}_\xi\left(sign((X)_{i=2}^d)=\widehat{\alpha}\right) &= \sum_{(x2,...,x_d)\in\{0,\infty\}^{d-1}} (-1)^{N(x_2,...,x_d)} F_\xi(x_2,...,x_d) \\
&= \frac{\partial F^+}{\partial u_1}(|\xi|g^{\{\xi,\widehat{\alpha}\}},\infty,...,\infty)g^{\{\xi,\widehat{\alpha}\}}
\end{aligned}
$$

due to cancellation of terms, where we denote $N(x2,...,xd) = \sharp\{k : x_k = 0\}$. Uniformity of F^+ at the margins then gives the desired result. It justifies the interpretation of joinder g^α as chances of pure jump signs.

Knowing the conditional jump sign distribution we now interest ourselves in the absolute jumps size distribution of $(X_2,...,X_d)$ given their signs $(\alpha_2,...,\alpha_d)$. Again we start of with an discussion of the three-dimensional case assuming $(\alpha_2,\alpha_3) = (-1,1)$. By elementary statistics it holds

$$\mathbb{P}_\xi(|X_2| \leq x_2, |X_3| \leq x_3|\alpha_2,\alpha_3) = \mathbb{P}_\xi(0 \geq X_2 \geq -x_2, 0 \leq X_3 \leq x_3)/g^{(\xi,-1,1)}.$$

Similarly to (11) we derive an unconditional probability as follows

$$\begin{aligned}\mathbb{P}_\xi(0 \geq X_2 \geq -x_2, 0 \leq X_3 \leq x_3) &= F_\xi(0,x_3) - F_\xi(x_2,x_3) - F_\xi(0,0) + F_\xi(x_2,0)\\ &= \frac{\partial F^+}{\partial u_1}(|\xi|g^{(\xi,-1,1)}, x_2 g^{(\xi,-1,1)}, x_3 g^{(\xi,-1,1)})g^{(\xi,-1,1)}.\end{aligned}$$

Then the absolute jump size probability conditional on the jump signs obtains as

$$\begin{aligned}\mathbb{P}_\xi(|X_2| \leq x_2, |X_3| \leq x_3|\alpha_2,\alpha_3) &= \frac{\partial F^+}{\partial u_1}(|\xi|g^{\xi,-1,1}, x_2 g^{(\xi,-1,1)}, x_3 g^{(\xi,-1,1)})\\ &= F^+_{|\xi|g^{\xi,-1,1}}(x_2 g^{(\xi,-1,1)}, x_3 g^{(\xi,-1,1)})\end{aligned}$$

The result is all but unexpected due to construction. The arguments used in the three-dimensional case apply just as well for arbitrary dimensions.

Theorem 3.3 *Let F be a modular Lévy copula on $\mathbb{R}^d$ with joinder $g^\alpha \in \mathbb{R}^+$ and modules $F^\alpha = F^+$. Further let $\xi = x_1$ be a given realization. Then it holds*

$$\mathbb{P}_\xi(|X_i| \leq x_i, i = 2,...,d|\widehat{\alpha}) = \frac{\partial F^+}{\partial u_1}(|\xi|g^{\xi,\widehat{\alpha}}, xg^{(\xi,\widehat{\alpha})}). \tag{13}$$

Proof: The proof is essentially the same as in the three-dimensional case. Suppose $(\alpha)_{i=2}^d = \mathbf{1}$, the other cases being derived analogously. By formula (8) we have

$$\begin{aligned}\mathbb{P}_\xi\left(0 \leq X_i \leq x_i, i = 2,...,d\right) &= \sum_{u_i \in \{0,x_i\}} (-1)^{N(u_2,...,u_d)} F_\xi(u_2,...,u_d)\\ &= \frac{\partial F^+}{\partial u_1}(|\xi|g^{\{\xi,\widehat{\alpha}\}}, x_2,...,x_d)g^{\{\xi,\widehat{\alpha}\}}\end{aligned}$$

due to cancellation of terms. Together with formula (12) for the stand-alone probability of the sign vector we conclude

$$\begin{aligned}\mathbb{P}_\xi(|X_i| \le x_i, i = 2, ..., d|\widehat{\alpha}) &= \frac{\mathbb{P}_\xi\left(0 \le X_i \le x_i, i = 2, ..., d\right)}{\mathbb{P}_\xi\left(sign(\widehat{X}) = \widehat{\alpha}\right)} \\ &= \frac{\partial F^+}{\partial u_1}(|\xi|g^{\{\xi,\widehat{\alpha}\}}, xg^{\{\xi,\widehat{\alpha}\}}).\end{aligned}$$

Theorem 3.3 has the distribution of the transformed absolute values of jumps $(|X_2|g^\alpha, ..., |X_d|g^\alpha)$ conditional on realized jump $X_1 = \xi$ and jump signs α be that associated to the module F^+ in the sense of (8) with conditioning argument $|\xi|g^\alpha$. This feeds the idea of a modular design in a way that jump sizes are close upon separated from jump signs.

4 Simulation by series representation

In this section we introduce a conditional sampling algorithm for modular Lévy copulas. We show further how this method avails naturally the simulation of multidimensional Lévy processes by series representation.

Series representations go back to Rosinski and others, who proved almost sure convergence of series of random variables to Lévy processes with specified characteristic triplets. Tankov Tankov (2004) extended their findings to jump dependence modelling with Lévy copulas. We reproduce his findings, while the focus is set on the finite variational case only.

Theorem 4.1 (Series representation) *Let ν be a Lévy measure on $\mathbb{R}^d$ (finite variation) with marginal tail integrals $U_i, i = 1, ..., d$ and Lévy copula $F(u_1, ..., u_d)$ and let $K(\xi, dx_2, ..., dx_d)$ be the corresponding conditional probability distribution. Let $\{V_i\}$ be a sequence of independent r.v., uniformly sitributed on $[0, 1]$. Introduce d random sequences $\{\Gamma_i^1\}, ..., \{\Gamma_i^d\}$, independent from $\{V_i\}$ such that*

1. *$N = \sum_{i=1}^\infty \delta_{\{\Gamma_i^1\}}$ is a Poisson random measure on $\mathbb{R}$ with intensity measure λ*
2. *Conditionally on $\{\Gamma_i^1\}$, the random vector $(\{\Gamma_i^2\}, ..., \{\Gamma_i^d\})$ is independent from $\{\Gamma_i^k\}$ with $j \neq i$ and all k and is distributed on $\mathbb{R}^{d-1}$ with law $K(\Gamma_i^1, dx_2, ..., dx_d)$.*

Then $\{X_t\}_{0 \le t \le 1}$ with

$$X_t^i = \sum_{k=1}^\infty U_k^{-1}(\Gamma_k^i)1_{[0,t]}(V_k), \; i = 1, ..., d$$

is a Lévy process on the time interval $[0,1]$ *with characteristic function*

$$E[e^{iu.X_t}] = \exp\left(t\int_{\mathbb{R}^d}(e^{iu.x}-1)\nu(dx)\right).$$

With this result simulation is straight forward. As regards actual implementation Tankov Tankov (2004) makes plausible the use of Poisson arrivals $\{\Gamma_k^1\}$ and random series truncation $X_t^i = \sum_{k:\Gamma_k^1<\tau} U_i^{-1}(\Gamma_k^i)1_{[0,t]}(V_k),\ i = 1,...,d.$ with some $\tau > 0$. We shall not dwell on implementation details but sampling from conditional probability measure $K(\Gamma_i^1,\cdot)$, as it were the crucial part. It is true that we know explicitly the distribution of K given Γ_i^1 by formula (7) but its complexity may in some cases interfere with simple random number generation. So is the case of modular Lévy copulas in high dimensions. One can show (which we do not here) that the conditional distribution function results as

$$F_\xi(x_2,...,x_d) = \sum \frac{\partial F^+}{\partial u_1}(|\xi|g^c,|c_2|g^c,...,|c_d|g^c)g^c \prod_{i:c_i=x_i} sign(c_i), \tag{14}$$

where the sum is taken over all $c \in \xi \times \{-\infty, x_2\} \times ... \times \{-\infty, x_d\}$. A distribution function of such form would surely handicap us in using conditional sampling algorithms, for instance. However, our former results on jump size and jump sign distributions find an obvious remedy.

4.1 Conditional sampling algorithms

In this paragraph we propose a conditional sampling algorithm for number generation from conditional measure $K(\xi,\cdot)$. Given a realization ξ we pick the corner first and simulate absolute jump sizes subsequently.

All of last section's results base a given value for the first component. This goes along with the conditional perspective of point *2.* in Theorem 4.1. A simple conditioning argument yields

$$\begin{aligned} F_\xi(x_2,...,x_d) &= \mathbb{P}_\xi(X_i \le x_i, i=2,...,d) && (15)\\ &= \mathbb{P}_\xi(|X_i| \le |x_i|, i=2,...,d|\alpha)\cdot \mathbb{P}_\xi(\alpha) && (16)\end{aligned}$$

with both $\mathbb{P}_\xi(|X_i| \le |x_i|, i=2,...,d|\alpha)$ and $\mathbb{P}_\xi(\alpha)$ being known from Theorems 3.3 and 3.2, respectively.

Algorithm 4.1 (Simulating from conditional measure K)
Samples $(x_2,...,x_d)$ *from conditional measure* $K(x_1,\star)$ *given a realization* x_1*, a Lévy copula* F^+ *on* $[0,\infty]^d$ *and constant joinder* g^α*.*

- Pick corner α according to the first conditional probability function associated to g^α
- Simulate absolute jump sizes $(y_2, ..., y_d)$ from the conditional distribution associated to F^+ with conditioning argument $|x_1|g^\alpha$
- Set $x_i = \alpha_i y_i / g^\alpha, i = 2, ..., d$

The conditional sample vector is given by $(x_2, ..., x_d)$.

Of course, we can not redress simulation from a conditional distribution associated to a Lévy copula by conditioning arguments. Yet we do produce relief in a way that the distribution is now associated to a Lévy copula from a spectrally positive process, which is more manageable. Algorithm 4.1 now fits well into Tankov's Tankov (2006) general algorithm for path generation of multidimensional Lévy processes. We promote a slight modification of the original algorithm in order to particularize Lévy copulas of the present type.

Algorithm 4.2 (Simulation of multidimensional Lévy process with cornerwise dependent components by series representation)
Generates trajectory X_t of multidimensional Lévy process by series representation. The dependence is given by a modular Lévy copula with joinder g^α and modules $F^\alpha = F^+$. Let a number τ be fixed depending on the required precision and computational capacity.

- Initialize $k = 0, \Gamma_0^1 = 0$
- Repeat while $|\Gamma_k^1| < \tau$
  - Set $k = k + 1$
  - Simulate exponential(2) T_k and set $\Gamma_k^1 = -(|\Gamma_{k-1}^1| + T_k)$
  - Simulate $(\Gamma_k^2, ..., \Gamma_k^d)$ from distribution $K(\Gamma_k^1, \star)$ by algorithm 4.1
  - Simulate V_k uniform on $[0, 1]$

The trajectory is then given by $X_t^i = \sum_{k:\Gamma_k^1 < \tau} U_i^{-1}(\Gamma_k^i) 1_{[0,t]}(V_k),\ i = 1, ..., d.$

4.2 Numerical results

In this paragraph we present numerical results from an implementation of algorithm 4.2. For the upcoming discussion, we take example 3.2 as our base instance of a modular Lévy copula and suppose α-stable margins.

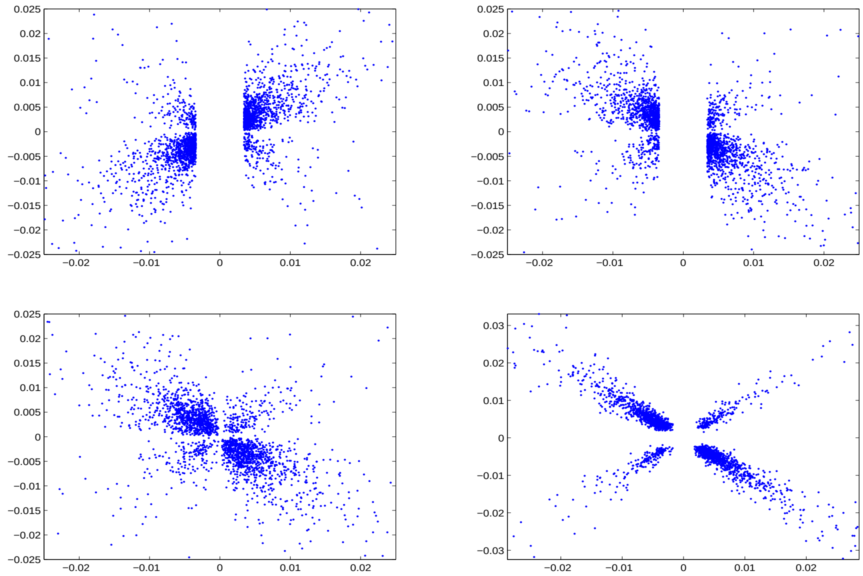

Figure 3: Scatter plots of marginal jump activity of Lévy process with dependence structure (10) and α-stable margins. Upper left: (1,2)-marginal jumps. Upper right: (1,3)-marginal jumps. Lower left: (2,3)-marginal jumps. Each using $\theta = 1.5$. Lower right: (2,3)-marginal jumps using $\theta = 10$.

Let us first analyze embedded algorithm 4.1 by way of plotting the actual jumps taken by X into figure 4.2. As to Theorem 4.1 these can be computed from the output sample by the inverse tail integral $U_i^{-1}(\Gamma_k^i)$. The upper left plot shows the (1,2)-margin of jump activity. We can clearly observe that jumps in the first and second component tend to have the same direction, opposing jumps are still possible. This is due to the piecewise constant but non-flat joinder. As regards the upper right and lower left plot, which show jump activity from a (1,3)-marginal and (2,3)-marginal perspective, the outcome is the same but with a stress towards opposing jump directions. These parallel just as well the hierarchical structure of sign dependence.

Concerning the association between absolute jumps sizes we detect a somewhat loose pattern which corresponds to the low value 1.5 for dependence parameter θ. When multiplying the module parameter, the realizations become more in-line. This corresponds to stronger dependence of absolute jump sizes. We leave as a side note that truncation of the series appears in the plots as well.

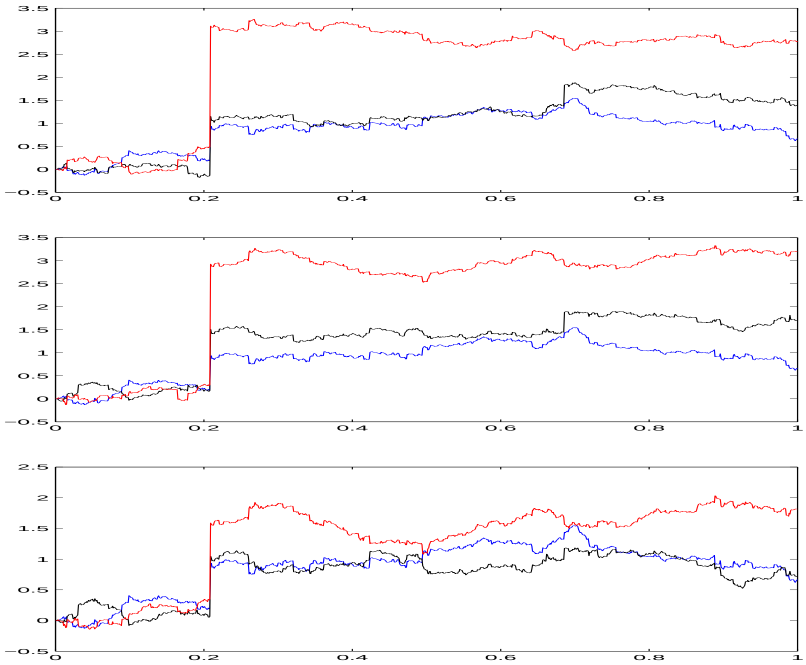

Figure 4: Trajectories of 3-dimensional Lévy paths (1st:blue, 2nd:green, 3rd:red) with dependence specified by modular Lévy copula (10). Upper: hierarchical joinder, module parameter $\theta = 1.5$. Middle: flat joinder, module parameter $\theta = 1.5$. Lower: flat joinder, module parameter $\theta = 5$.

With the conditional sampling approach made plausible we now turn to the analysis of simulated Lévy paths. We refer to figure 4 in the following. The upper plot shows an instance of a trivariate Lévy process with α-stable margins and dependence structure specified by our base copula with $\theta = 1.5$. In particular, jump sign dependence is modelled hierarchically. The latter is evident as the first and second component proceed in a similar manner, while the development of the third component seems to divert. For the plot in the middle we have eased up on emphasizing certain corners by a stressed jump sign dependence and applied a flat joinder $g^{\alpha} = \frac{1}{4}, \alpha \in \{-1, 1\}^3$. The resulting paths thence develop in a more independent fashion. We want to accent that it is not the location of the graphs, which is mainly due to realized jump sizes, but the direction of jumps which speaks here in favor of a loose sign dependence. We have provided the lower graph to clarify things. This time we applied a flat joinder and high jump size dependence

$\theta = 10$. One immediately observes that the absolute height of the jumps are almost the same, the jump directions are yet dissociated. This produces mirror images in some intervals.

With reasons we may thus take our numerical findings to support the analytical derivations as in the three-dimensional case. Both the hierarchical structure for jump sign dependence and Clayton type association between absolute jump sizes reappear in the graphical illustrations. In virtue of algorithms 4.1 and 4.2, which are given for any modular Lévy copula in arbitrary dimension, we expect the results found here to become evident even in a more complex setting.

5 Conclusion

This study refined the modular Lévy copulas due to Tankov and introduced a hierarchical joinder for module assembling. It proved correct that jump sign dependence and jump size dependence is related naturally to the joinder and modules, respectively.

Simulation of Lévy copulas from joined modules by series representation was found to involve the probabilistic interpretations of both the joinder and the modules. This feeded the derivation of a simplified algorithm for path generation. The impact of choices for module copula as well as joining functions on multidimensional process evolution was recognized with the aid of graphical illustrations.

Future research includes generalization to a non-constant joinder and extension to other parsimonious copula ingredients. The algorithmic performance in high dimension is of true interest because appreciable savings in numerical computations are awaited.

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