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Abstract

Technical applications often imply mathematical models based on differential algebraic equations (DAEs). The systems of DAEs may include parameters, which exhibit some uncertainty. We arrange a stochastic model to determine the sensitivity of a solution with respect to the parameters in case of linear systems. The strategy of generalised polynomial chaos yields a larger linear system of DAEs, which describes the problem. We prove sufficient conditions such that the alternative DAE model inherits the index of the original DAE system. Furthermore, the choice of consistent initial values is discussed. We present numerical simulations of the stochastic model using the polynomial chaos.

Keywords: consistent initial values; differential algebraic equations; Galerkin method; index; polynomial chaos; random parameters; stochastic model; uncertainty quantification.

AMS Subject Classification: 65L80; 65C20; 34F05.

1 Introduction

Mathematical modelling of large dynamical systems often yields differential algebraic equations (DAEs) like in simulation of electric circuits, see [4, 6], or in multibody dynamics, see [3], for example. Typically, many technical parameters are present in the system of DAEs. Some parameters may be crucial for the

behaviour of the corresponding solution. Thus information about the sensitivity, i.e., the dependence of the solution on the parameters, is required. Partial derivatives with respect to the parameters yield a local analysis of the dependence.

In contrast, we apply a stochastic model to determine global information about the sensitivity. Thereby, we assume that the crucial parameters exhibit some uncertainties, which are described by random distributions. Modelling the parameters as random variables changes the solutions of the DAEs into stochastic processes. For uncertainty quantification, key data like expected value and variance can be computed by Monte-Carlo simulations or more sophisticated variants like quasi Monte-Carlo methods. Often a huge number of simulations is required for a sufficiently accurate approximation.

We use an alternative strategy to solve the stochastic model, which is based on a polynomial expansion of stochastic processes. Wiener [11] formulated the homogeneous polynomial chaos for stochastic parameters with Gaussian distributions. Cameron and Martin [2] upgraded this approach to arbitrary random fields of second order. The generalised polynomial chaos applies an expansion in case of stochastic parameters with optional distributions, see [1, 7]. Each approach results in a larger coupled system, which has to be solved only once to achieve numerical approximations of the stochastic model.

We consider linear systems of DAEs in this paper. Due to the linear structure, the strategy of polynomial chaos is significantly more efficient than methods of Monte-Carlo type. The aim of the paper is to analyse the index of the constructed systems with respect to the index of the original systems. We specify sufficient conditions to guarantee that the DAEs of the polynomial chaos inherit the index of the underlying DAEs. The choice of consistent initial values for the larger systems is investigated. Moreover, we perform numerical simulations of a system of DAEs reflecting the theoretical examinations.

The paper is organised as follows. We formulate the stochastic model of DAEs for three types of dependence on parameters in Sect. 2. The strategy of the generalised polynomial chaos is outlined. We analyse the index for each case in Sect. 3, where corresponding statements are proved. Consistency of initial values is addressed. Sect. 4 includes numerical simulations of an illustrative example.

2 Problem Definition

We consider a linear system of DAEs written in the form

$$A\dot{\mathbf{x}} + B\mathbf{x} = \mathbf{s}(t) \tag{1}$$

with unknown solution $\mathbf{x} : [t_0, t_1] \to \mathbb{R}^n$, constant matrices $A, B \in \mathbb{R}^{n\times n}$ and time-dependent input signals $\mathbf{s} : [t_0, t_1] \to \mathbb{R}^n$. It holds $\det(A) = 0$. Consistent initial values $\mathbf{x}(t_0) = \mathbf{x}_0$ are predetermined. The solution $\mathbf{x}$ is smooth in differential components and at least continuous in algebraic components. We assume a regular matrix pencil ($\det(A + \lambda B) \not\equiv 0$), which agrees to existence and uniqueness of a solution.

2.1 Parameter-dependent system

We assume that the system (1) includes q parameters $\mathbf{p} = (p_1, \ldots, p_q) \subseteq P$ within some set $P \subset \mathbb{R}^q$ of relevant values. Hence the solution becomes parameter-dependent: $\mathbf{x}(t) = \mathbf{x}(\mathbf{p}; t)$. We will examine the sensitivity of the solution with respect to the parameters.

We observe three cases:

1. The input signals include the parameters, i.e., the system reads

$$A\dot{\mathbf{x}} + B\mathbf{x} = \mathbf{s}(\mathbf{p}; t). \tag{2}$$

   Each choice of the parameters implies a different right-hand side.

2. The matrix A is parameter-dependent and thus it holds

$$A(\mathbf{p})\dot{\mathbf{x}} + B\mathbf{x} = \mathbf{s}(t). \tag{3}$$

   In the following, we assume the structure

$$A(\mathbf{p}) = A_0 + \sum_{j=1}^{q} \eta_j(p_j) A_j \tag{4}$$

   with constant matrices $A_0, A_1, \ldots, A_q \in \mathbb{R}^{n\times n}$ and continuous functions $\eta_j : \mathbb{R} \to \mathbb{R}$. The choice $\eta_j(p_j) \equiv p_j$ is often feasible. Let $\det(A(\mathbf{p})) = 0$ for all $\mathbf{p} \in P$, i.e., the system represents a DAE for all parameters. Without loss of generality, we consider the parameters as a perturbation of some mean values, i.e., $\mathbf{p} \to \bar{\mathbf{p}} + \mathbf{p}$. Consequently, the matrix A_0 includes the constant part $\bar{\mathbf{p}}$, whereas the matrices A_j for $j > 0$ describe perturbations with respect to $\mathbf{p}$. The matrices A_j exhibit non-zero entries just in positions, where A_0 owns non-zero values. However, the matrices A_j for $j > 0$ are sparser than A_0 in general.

3. The parameters influence the matrix B, i.e.,

$$A\dot{\mathbf{x}} + B(\mathbf{p})\mathbf{x} = \mathbf{s}(t). \tag{5}$$

Analogue to the second case, we consider the pattern

$$B(\mathbf{p}) = B_0 + \sum_{j=1}^{q} \eta_j(p_j) B_j \tag{6}$$

with constant matrices $B_0, B_1, \ldots, B_q \in \mathbb{R}^{n\times n}$ and scalar functions η_j. Let the matrices B_j for $j > 0$ represent perturbations of some mean values again.

A further generalisation consists in the presence of parameters within all terms of the DAE (1) simultaneously.

2.2 Stochastic modelling

Now we assume some uncertainty within the parameters. Hence we model the parameters as random variables on a probability space $(\Omega, \mathcal{A}, P)$. Given a random variable $f : \Omega \to \mathbb{R}$ with $f \in L^1(\Omega)$, the expected value of f reads

$$\mathbb{E}(f) := \langle f \rangle := \int_\Omega f(\omega)\, \mathrm{d}P(\omega).$$

For random variables $f, g \in L^2(\Omega)$, the corresponding inner product is defined by

$$\langle fg \rangle := \int_\Omega f(\omega) g(\omega)\, \mathrm{d}P(\omega). \tag{7}$$

Concerning vector-valued random variables $\mathbf{f}, \mathbf{g} : \Omega \to \mathbb{R}^n$, the integrals are considered component-wise.

The parameters $\mathbf{p} \in P$ exhibit the form

$$p_j = \xi_j(\omega) \qquad \text{for } \; j = 1, \ldots, q$$

with random variables $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_q)$. We assume some classical distribution of the random variables like uniform type or Gaussian type, for example. Given functions f, g depending on the random variables, the inner product (7) can be calculated by multidimensional real integration

$$\langle fg \rangle = \int_{\mathbb{R}^q} f(\boldsymbol{\xi}) g(\boldsymbol{\xi}) \rho(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} \tag{8}$$

using the density function ρ of the distribution. According to the assumptions in the previous section, we suppose $\langle \eta_j(\xi_j) \rangle = 0$ for all j, since non-zero expected

values can be shifted to the constant part of the matrices A, B in (1). For example, the sum (4) exhibits the form

$$A(\boldsymbol{\xi}) = A_0 + \sum_{j=1}^{q} \eta_j(\xi_j) A_j$$

now, where the matrix A_0 includes the expected values. Imposing a standardisation like $\mathrm{Var}(\eta_j(\xi_j)) = \langle \eta_j(\xi_j)^2 \rangle = 1$ for all j implies that the matrices A_j for $j > 0$ determine the magnitude of the stochastic perturbation.

Applying this stochastic modelling, the solution of the linear system (1) becomes a random process:

$$\mathbf{X} : \Omega \times [t_0, t_1] \to \mathbb{R}^n, \quad (\omega, t) \mapsto \mathbf{X}(\omega, t) = \mathbf{X}(\boldsymbol{\xi}(\omega), t).$$

We are interested in the expected value and the variance of this process, for example, to quantify the uncertainty with respect to the parameters.

2.3 Generalised polynomial chaos

Let $(\Phi_i)_{i \in \mathbb{N}_0}$ with $\Phi_i : \mathbb{R}^q \to \mathbb{R}$ be a complete system of orthogonal polynomials with respect to the inner product (8), i.e., it holds $\langle \Phi_i \Phi_j \rangle = \langle \Phi_i^2 \rangle \delta_{ij}$ using the Kronecker symbol δ. For example, uniform and Gaussian distribution imply the Legendre and the Hermite polynomials, respectively. We assume $\Phi_0 \equiv 1$ and $\Phi_i \equiv \xi_i$ for $i = 1, \ldots, q$. The technique of generalised polynomial chaos, see [7], applies the expansion

$$\mathbf{X}(\boldsymbol{\xi}(\omega), t) = \sum_{i=0}^{\infty} \mathbf{v}_i(t) \Phi_i(\boldsymbol{\xi}(\omega)) \tag{9}$$

for the random process including the coefficient functions $\mathbf{v} : [t_0, t_1] \to \mathbb{R}^n$. Considering the paths $\mathbf{X}(\omega, \cdot)$ as solution of (1), it is required that differential components are smooth and algebraic components are continuous with respect to time. For simplicity, we demand smoothness in all components. We apply the concept of Sobolev spaces, see for example [9], to define a norm component-wise by

$$\|X\|^2 := \left\langle \|X\|^2_{H^1([t_0,t_1])} \right\rangle = \left\langle \|X\|^2_{L^2([t_0,t_1])} + \|\dot{X}\|^2_{L^2([t_0,t_1])} \right\rangle. \tag{10}$$

The series (9) converges with respect to the norm (10) in each component if $\|X_j\| < \infty$ holds for each $j = 1, \ldots, n$ ($\mathbf{X} = (X_1, \ldots, X_n)^\top$).

To obtain a numerical approximation, the expansion (9) has to be truncated, i.e.,

$$\mathbf{X}(\boldsymbol{\xi}(\omega), t) \doteq \sum_{i=0}^{m} \mathbf{v}_i(t) \Phi_i(\boldsymbol{\xi}(\omega)) \tag{11}$$

for some sufficiently large $m \in \mathbb{N}$. Given a numerical solution, approximations for expected value and variance are achieved via

$$\mathbb{E}(\mathbf{X}(\cdot,t)) \doteq \mathbf{v}_0(t), \qquad \mathrm{Var}(\mathbf{X}(\cdot,t)) \doteq \sum_{i=1}^{m} \langle \Phi_i^2 \rangle (\mathbf{v}_i(t))^2, \tag{12}$$

where the squares are computed in each component separately.

Inserting the representation (11) in the linear DAE (1) yields

$$\sum_{i=0}^{m} \Phi_i(\boldsymbol{\xi}) A \dot{\mathbf{v}}_i(t) + \sum_{i=0}^{m} \Phi_i(\boldsymbol{\xi}) B \mathbf{v}_i(t) = \mathbf{s}(t)$$

for the cases $A = A(\boldsymbol{\xi})$, $B = B(\boldsymbol{\xi})$ or $\mathbf{s}(t) = \mathbf{s}(\boldsymbol{\xi}; t)$.

To determine the unknown coefficient functions $\mathbf{v}_i$, we perform a Galerkin approach, i.e., the residual has to be orthogonal to the space of applied polynomials with respect to the inner product (8). Thereby, the orthogonality of the basis polynomials yields a decoupling to some extend. According to the three types of dependence on parameters introduced in Sect. 2.1, we obtain three cases. Corresponding to the system (2), it follows

$$\langle \Phi_l^2 \rangle A \dot{\mathbf{v}}_l(t) + \langle \Phi_l^2 \rangle B \mathbf{v}_l(t) = \langle \Phi_l \mathbf{s}(\boldsymbol{\xi}; t) \rangle$$

and thus

$$A \dot{\mathbf{v}}_l(t) + B \mathbf{v}_l(t) = \frac{1}{\langle \Phi_l^2 \rangle} \langle \Phi_l \mathbf{s}(\boldsymbol{\xi}; t) \rangle \tag{13}$$

for $l = 0, 1, \ldots, m$. The system (3) with (4) implies

$$\langle \Phi_l^2 \rangle A_0 \dot{\mathbf{v}}_l(t) + \left(\sum_{i=0}^{m} \sum_{j=1}^{q} \langle \eta_j(\xi_j) \Phi_i \Phi_l \rangle A_j \dot{\mathbf{v}}_i(t) \right) + \langle \Phi_l^2 \rangle B \mathbf{v}_l(t) = \langle \Phi_l \mathbf{s}(t) \rangle$$

or, equivalently, using $\langle \Phi_0^2 \rangle = 1$

$$A_0 \dot{\mathbf{v}}_l(t) + \left(\sum_{i=0}^{m} \sum_{j=1}^{q} \frac{\langle \eta_j(\xi_j) \Phi_i \Phi_l \rangle}{\langle \Phi_l^2 \rangle} A_j \dot{\mathbf{v}}_i(t) \right) + B \mathbf{v}_l(t) = \delta_{0l} \mathbf{s}(t) \tag{14}$$

for $l = 0, 1, \ldots, m$. Likewise, the third case (5) with (6) leads to

$$\langle \Phi_l^2 \rangle A \dot{\mathbf{v}}_l(t) + \langle \Phi_l^2 \rangle B_0 \mathbf{v}_l + \left(\sum_{i=0}^{m} \sum_{j=1}^{q} \langle \eta_j(\xi_j) \Phi_i \Phi_l \rangle B_j \mathbf{v}_i(t) \right) = \langle \Phi_l \mathbf{s}(t) \rangle,$$

which results in

$$A \dot{\mathbf{v}}_l(t) + B_0 \mathbf{v}_l + \left(\sum_{i=0}^{m} \sum_{j=1}^{q} \frac{\langle \eta_j(\xi_j) \Phi_i \Phi_l \rangle}{\langle \Phi_l^2 \rangle} B_j \mathbf{v}_i(t) \right) = \delta_{0l} \mathbf{s}(t) \tag{15}$$

for $l = 0, 1, \ldots, m$. The systems include $(m+1)n$ equations for $(m+1)n$ unknown coefficient functions in each case.

For the original system (1), we assume a consistent initial value $\mathbf{x}(t_0) = \mathbf{x}_0$. In the polynomial chaos approach, we like to perform the simple choice

$$\mathbf{v}_0(t_0) = \mathbf{x}_0, \qquad \mathbf{v}_i(t_0) = \mathbf{0} \quad \text{for all } \; i > 0 \tag{16}$$

due to the meaning of the coefficient functions. However, these initial values are not consistent in general, since the parameters influence the algebraic relations. Likewise, the initial value $\mathbf{x}_0$ is not always consistent in a Monte-Carlo simulation of the stochastic model, since the realisations of the parameters imply different consistency conditions.

3 Index Analysis

In this section, we analyse the structure of the linear systems of DAEs from the polynomial chaos in comparison to the original systems. Although different index concepts exist, see for example [5], the definitions of the index coincide in the linear case. Furthermore, the possibility to impose the initial conditions (16) is investigated.

3.1 Parameters in right-hand side

We discuss the DAE system (13), which consists of separate subsystems for the coefficient functions. The assumption of a regular matrix pencil for A, B yields nonsingular transformation matrices $P, Q \in \mathbb{R}^{n\times n}$ with

$$PAQ = \begin{pmatrix} I_{n_1} & 0 \\ 0 & N \end{pmatrix}, \qquad PBQ = \begin{pmatrix} C & 0 \\ 0 & I_{n_2} \end{pmatrix}, \tag{17}$$

where $C \in \mathbb{R}^{n_1\times n_1}$ is regular and $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix, see [5]. Let $k > 0$ be the nilpotency index, i.e., it holds $N^{k-1} \neq 0, N^k = 0$. Remark that the nilpotency index of (2) does not depend on the structure of the right-hand side. Applying the transformation (17) to (13) yields

$$PAQ\dot{\mathbf{w}}_l(t) + PBQ\mathbf{w}_l(t) = \frac{1}{\langle \Phi_l^2 \rangle} P\langle \Phi_l \mathbf{s}(\boldsymbol{\xi}; t)\rangle$$

for $l = 0, 1, \ldots, m$ with $\mathbf{w}_l := Q^{-1}\mathbf{v}_l$. Let $d := (m+1)n$ and

$$\mathbf{v} := (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m) \in \mathbb{R}^d, \quad \mathbf{w} := (\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m) \in \mathbb{R}^d.$$

Accordingly, we write the complete system as

$$(I_{m+1} \otimes PAQ)\dot{\mathbf{w}}(t) + (I_{m+1} \otimes PBQ)\mathbf{w}(t) = \tilde{\mathbf{s}}(t) \tag{18}$$

employing the notation of Kronecker products. The transformed right-hand side $\tilde{\mathbf{s}} : [t_0, t_1] \to \mathbb{R}^d$ can be non-zero in all components, since the same holds for the original right-hand side in (13). Using a specific permutation matrix $\hat{P} \in \mathbb{R}^{d \times d}$, we construct the matrices

$$\hat{P}(I_{m+1} \otimes PAQ)\hat{P}^\top = \begin{pmatrix} I_{(m+1)n_1} & 0 \\ 0 & I_{m+1} \otimes N \end{pmatrix}$$

and

$$\hat{P}(I_{m+1} \otimes PBQ)\hat{P}^\top = \begin{pmatrix} I_{m+1} \otimes C & 0 \\ 0 & I_{(m+1)n_2} \end{pmatrix}.$$

Since $I_{m+1} \otimes C$ is regular and $I_{m+1} \otimes N$ is nilpotent with index k, we achieve the following result.

Theorem 1 *If the linear DAE system (2) exhibits index k, then the corresponding linear DAE system (13) obtained by the generalised polynomial chaos is also of index k independent from the choice of the right-hand side in (2).*

We see that the DAEs of the polynomial chaos inherit the index of the original system for arbitrary right-hand side in this case. However, the right-hand side influences the conditions for consistent initial values. The following discussion is based on the system (18). Recall that the matrices exhibit block diagonal structure, i.e., the subsystems are decoupled.

Without loss of generality, we assume the partitioning

$$\mathbf{s}(\boldsymbol{\xi}; t) \equiv \mathbf{s}_0(t) + \mathbf{s}_1(\boldsymbol{\xi}; t)$$

with $\langle \Phi_0 \mathbf{s}_1(\boldsymbol{\xi}; t) \rangle = 0$. Let $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$ be a consistent initial value corresponding to the DAE (1) with input signal $\mathbf{s}_0$. Then the choice $\mathbf{v}_0(t_0) = \mathbf{x}_0$ is feasible. Since $\langle \Phi_l \mathbf{s}_1(\boldsymbol{\xi}; t) \rangle \neq 0$ may occur for each $l > 0$, the initial values $\mathbf{v}_l(t_0) = \mathbf{0}$ are not consistent in general. However, we can make the choice (16) consistent by modelling the right-hand side as

$$\mathbf{s}(\boldsymbol{\xi}; t) \equiv \mathbf{s}_0(t) + \left(\frac{3}{\varepsilon^2}(t - t_0)^2 - \frac{2}{\varepsilon^3}(t - t_0)^3 \right) \mathbf{s}_1(\boldsymbol{\xi}; t) \tag{19}$$

with a small $\varepsilon > 0$, i.e., a smooth transition is achieved in $[t_0, t_0 + \varepsilon]$. If the influence of the parameters is not significant within a small time interval at the beginning, this modification is reasonable.

3.2 Parameters in first matrix

Now we consider the linear DAE system (14) corresponding to the underlying system (3). We assume $\det B \neq 0$ in the analysis, which guarantees the uniqueness of steady state solutions with respect to autonomous systems (3). The analysis follows an approach applied in [10]. Using the assumption, the system (14) is equivalent to

$$B^{-1}A_0\dot{\mathbf{v}}_l(t) + \left(\sum_{i=0}^{m}\sum_{j=1}^{q}\frac{\langle\eta_j(\xi_j)\Phi_i\Phi_l\rangle}{\langle\Phi_l^2\rangle}B^{-1}A_j\dot{\mathbf{v}}_i(t)\right) + \mathbf{v}_l(t) = \delta_{0l}B^{-1}\mathbf{s}(t) \tag{20}$$

for $l = 0, 1, \ldots, m$. The complete system can be written in the form

$$(I_{m+1} \otimes B^{-1}A_0 + E)\dot{\mathbf{v}}(t) + \mathbf{v}(t) = \tilde{\mathbf{s}}(t)$$

with a constant matrix

$$E := \sum_{j=1}^{q} S_j \otimes B^{-1}A_j, \tag{21}$$

where

$$S_j = (\sigma_{li}^j) \in \mathbb{R}^{(m+1)\times(m+1)}, \quad \sigma_{li}^j := \frac{\langle\eta_j(\xi_j)\Phi_i\Phi_l\rangle}{\langle\Phi_l^2\rangle} \tag{22}$$

for $l, i = 0, 1, \ldots, m$ and $j = 1, \ldots, q$. The matrix E includes the stochastic perturbations. Using the constant

$$C := \max\left\{\frac{|\langle\eta_j(\xi_j)\Phi_i\Phi_l\rangle|}{\langle\Phi_l^2\rangle} : \ i, l = 0, 1, \ldots, m; \ j = 1, \ldots, n\right\}, \tag{23}$$

the magnitude of E can be estimated in a matrix norm

$$\|E\|_\infty \leq \sum_{j=1}^{q}\|S_j \otimes B^{-1}A_j\|_\infty \leq (m+1)\cdot q \cdot C \cdot \|B^{-1}\|_\infty \cdot \max_{j=1,\ldots,q}\|A_j\|_\infty.$$

Thus the norm of E can be bounded by choosing sufficiently small entries in the matrices A_j.

The next theorem demonstrates a sufficient condition to preserve the index of the original DAE system.

Theorem 2 *Given the linear DAE system (3) with* $\det B \neq 0$*, let* k *be the index of this system for* $A(\mathbf{p}) = A_0$*. If*

$$\operatorname{kern}((B^{-1}A_0)^k) \subseteq \operatorname{kern}(A_j) \quad \textit{for all} \ \ j = 1, \ldots, q. \tag{24}$$

is fulfilled, then the corresponding linear DAE system (14) of the polynomial chaos exhibits index k *for sufficiently small* $\|E\|$ *from (21).*

Proof:

The Jordan form of the matrix $B^{-1}A_0$ reads

$$B^{-1}A_0 = TJT^{-1}, \qquad J = \begin{pmatrix} R & 0 \\ 0 & N \end{pmatrix} \tag{25}$$

with regular matrix $R \in \mathbb{R}^{n_1 \times n_1}$ and nilpotent matrix $N \in \mathbb{R}^{n_2 \times n_2}$ of index k.

Let $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{R}^n$ be the column vectors of the transformation matrix T in (25). Thus $\mathbf{u}_j$ for $j = n_1 + 1, \ldots, n$ correspond to the generalised eigenspace of the eigenvalue zero. It follows $(B^{-1}A_0)^k \mathbf{u}_j = \mathbf{0}$ for $j = n_1 + 1, \ldots, n$. We construct the set of vectors

$$\mathbf{e}_i \otimes \mathbf{u}_j \in \mathbb{R}^d \quad \text{for} \quad i = 1, \ldots, m+1 \quad \text{and} \quad j = 1, \ldots, n \tag{26}$$

using the canonical unit vectors $\mathbf{e}_i \in \mathbb{R}^{m+1}$. Due to the structure of the matrix (21), the condition (24) implies

$$E(\mathbf{e}_i \otimes \mathbf{u}_j) = \sum_{l=1}^{q} (S_l \mathbf{e}_j) \otimes (B^{-1}(A_l \mathbf{u}_j)) = \mathbf{0}$$

for $j = n_1 + 1, \ldots, n$ and thus

$$(I_{m+1} \otimes B^{-1}A_0 + E)(\mathbf{e}_i \otimes \mathbf{u}_j) = \mathbf{e}_i \otimes B^{-1}A_0 \mathbf{u}_j$$

for $j = n_1 + 1, \ldots, n$. Successively, it follows

$$(I_{m+1} \otimes B^{-1}A_0 + E)^r (\mathbf{e}_i \otimes \mathbf{u}_j) = \mathbf{e}_i \otimes (B^{-1}A_0)^r \mathbf{u}_j$$

for $r = 1, \ldots, k$. Consequently, the structure of the generalised eigenspace (for $\lambda = 0$) of the enlarged system is just a multiple reproduction with respect to the original system. It follows that a transformation matrix $\hat{T} \in \mathbb{R}^d$ (including a permutation) exists with

$$\hat{T}^{-1}(I_{m+1} \otimes B^{-1}A_0 + E)\hat{T} = \begin{pmatrix} \hat{R} & 0 \\ H_2 & I_{m+1} \otimes N \end{pmatrix},$$

where the matrix $\hat{R} \in \mathbb{R}^{(m+1)n_1 \times (m+1)n_1}$ exhibits the structure

$$\hat{R} = I_{m+1} \otimes R + H_1.$$

The matrices H_1, H_2 depend on E and thus their norms can be bounded by a constant times $\|E\|$. It follows that the matrix $\hat{R}$ is regular for sufficiently small $\|E\|$.

If $H_2 = 0$ holds, we immediately conclude the nilpotency index k. For $H_2 \neq 0$, the block structure implies that we can solve the first part of dimension $(m+1)n_1$ separately. The result is inserted in the second part of dimension $(m+1)n_2$. It follows that the DAE (14) exhibits the differential index k provided that the solution of the first part and the right-hand side are sufficiently smooth. Since the differential index is equal to the nilpotency index in the linear case, the proof is completed. □

The result of Theorem 2 can also be achieved for singular matrix B but regular matrix pencil in case of index $k = 1$. The corresponding analysis applies the transformation (17).

The condition (24) is sufficient and not necessary. It implies that the structure of the generalised eigenspaces (for $\lambda = 0$) coincide for the discussed DAE systems. Furthermore, examples exist, where the requirement (24) is violated but the index remains the same. However, the structure of the generalised eigenspace changes, since $\operatorname{rank}(I_{m+1} \otimes B^{-1}A_0 + E) > (m+1)\operatorname{rank}(B^{-1}A_0)$ is given in the example.

The sufficient condition (24) becomes harder to satisfy the higher the index k is. However, mathematical modelling aims at achieving systems with low index. For example, models of electric circuits exhibit an index $k \leq 2$ in general. In the special case of index $k = 1$, the condition (24) is equivalent to

$$\operatorname{kern}(A_0) \subseteq \operatorname{kern}(A_j) \quad \text{for all } \; j = 1, \ldots, q. \tag{27}$$

The structure of the matrices A_j with respect to A_0 often fulfil this constraint, since most entries of the matrices A_j are equal to zero.

Now we address the choice of consistent initial values. In comparison to the previous case in Sect. 3.1, more detailed results can be achieved, since most parts of the right-hand side in (14) are equal to zero. Concerning the specification (16), the following sufficient condition holds.

Theorem 3 *Let the system (3) with matrix $A(\mathbf{p}) = A_0$ and $\det(B) \neq 0$ be of index k and $\mathbf{x}(t_0) = \mathbf{x_0}$ a consistent initial value. The initial values (16) are consistent with respect to the system (14) provided that (24) and*

$$\operatorname{kern}((B^{-1}A_0)^\top)^k) \subseteq \operatorname{kern}((B^{-1}A_j)^\top) \quad \textit{for all } \; j = 1, \ldots, q \tag{28}$$

holds.

Proof:

Considering the Jordan form (25) of $B^{-1}A_0$, the system (20), which is equivalent

to (14), can be transformed to

$$T^{-1}B^{-1}A_0T\dot{\mathbf{w}}_l(t) + \left(\sum_{i=0}^{m}\sum_{j=1}^{q}\sigma_{li}^{j}T^{-1}B^{-1}A_jT\dot{\mathbf{w}}_i(t)\right) + \mathbf{w}_l(t) = \delta_{0l}\tilde{\mathbf{s}}(t) \qquad (29)$$

for $l = 0, 1, \ldots, m$ with $\mathbf{w}_l := T^{-1}\mathbf{v}_l$ and $\tilde{\mathbf{s}} := T^{-1}B^{-1}\mathbf{s}$. The inspected initial values (16) imply $\mathbf{w}_0(t_0) = T^{-1}\mathbf{x}_0$ and $\mathbf{w}_l(t_0) = \mathbf{0}$ for $l > 0$. The conditions (28) and (24) correspond to left and right (generalised) eigenvectors, respectively. Both requirements yield the structure

$$T^{-1}B^{-1}A_jT = \begin{pmatrix} D_j & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } \; j = 1, \ldots, q$$

with matrices $D_j \in \mathbb{R}^{n_1 \times n_1}$. Thus the stochastic perturbation influences just the regular parts (corresponding to non-zero eigenvalues) of the subsystems (29). In the regular parts, the initial values can be chosen arbitrarily. Since the right-hand side is equal to zero in the systems for $l = 1, \ldots, m$, the choice $\mathbf{w}_l(t_0) = \mathbf{0}$ is feasible for $l > 0$. The selection $\mathbf{w}_0(t_0) = T^{-1}\mathbf{x}_0$ is consistent due to the consistency of the starting value $\mathbf{x}_0$ with respect to the original system (3). Hence the initial values (16) represent a consistent choice. □

Again the sufficient condition (28) becomes strong in case of high index k. For index $k = 1$, the requirement reduces to the condition

$$\operatorname{kern}(A_0^\top) \subseteq \operatorname{kern}(A_j^\top) \quad \text{for all } \; j = 1, \ldots, q, \qquad (30)$$

which is often valid due to the structure of the matrices. Moreover, the relations (27) and (30) are equivalent for symmetric matrices.

3.3 Parameters in second matrix

Finally, we analyse the linear DAE system (15) corresponding to the DAEs (5). The complete system can be written in the form

$$(I_{m+1} \otimes A)\dot{\mathbf{v}}(t) + (I_{m+1} \otimes B_0 + F)\mathbf{v}(t) = \tilde{\mathbf{s}}(t)$$

with the constant matrix

$$F := \sum_{j=1}^{q} S_j \otimes B_j \qquad (31)$$

using the matrices S_j from (22). Accordingly, we obtain the estimate

$$\|F\|_\infty \leq (m+1) \cdot q \cdot C \cdot \max_{j=1,\ldots,q} \|B_j\|_\infty,$$

where the constant (23) is involved. Again the magnitude of F becomes arbitrarily small for sufficiently small matrices B_j. In this case, we assume the regularity of the matrix B_0. Consequently, the matrix $I_{m+1} \otimes B_0 + F$ is nonsingular for sufficiently small $\|F\|$. Thus we discuss the equivalent system

$$(I_{m+1} \otimes B_0 + F)^{-1}(I_{m+1} \otimes A)\dot{\mathbf{v}}(t) + \mathbf{v}(t) = \tilde{\mathbf{r}}(t) \tag{32}$$

with $\tilde{\mathbf{r}} := (I_{m+1} \otimes B_0 + F)^{-1}\tilde{\mathbf{s}}$.

Theorem 4 *Consider a linear DAE system (5) for $B(\mathbf{p}) = B_0$ and $\det(B_0) \neq 0$ with index k. If the conditions*

$$\operatorname{kern}((B_0^{-1}A)^{k-1}) \subseteq \operatorname{kern}(B_j) \quad \textit{for all } \; j = 1, \ldots, q \tag{33}$$

are satisfied, then the polynomial chaos approach produces a linear DAE system (15) of the same index k provided that $\|F\|$ from (31) is sufficiently small.

Proof:

Since B_0 is regular, we have the relation

$$(I_{m+1} \otimes B_0 + F)^{-1} = (I_d + (I_{m+1} \otimes B_0^{-1})F)^{-1}(I_{m+1} \otimes B_0^{-1}).$$

If $\|F\|_\infty < \|B_0^{-1}\|_\infty^{-1} \leq \|B_0\|_\infty$ holds, then the Neumann inverse exists, i.e.,

$$(I_d + (I_{m+1} \otimes B_0^{-1})F)^{-1} = \sum_{l=0}^{\infty}(-1)^l((I_{m+1} \otimes B_0^{-1})F)^l.$$

It follows

$$\begin{aligned}
\tilde{A} &:= (I_{m+1} \otimes B_0 + F)^{-1}(I_{m+1} \otimes A) \\
&= \sum_{l=0}^{\infty}(-1)^l((I_{m+1} \otimes B_0^{-1})F)^l(I_{m+1} \otimes B_0^{-1}A) \\
&= I_{m+1} \otimes B_0^{-1}A + \sum_{l=1}^{\infty}(-1)^l((I_{m+1} \otimes B_0^{-1})F)^l(I_{m+1} \otimes B_0^{-1}A).
\end{aligned}$$

The structure of this matrix determines the index. We observe the Jordan form (25) with respect to the matrix $B_0^{-1}A$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{R}^n$ be the column vectors of the transformation matrix T again. Thereby, we assume that $\mathbf{u}_j$ for $j = n_1 + 1, \ldots, n$ span the generalised eigenspace corresponding to eigenvalue zero. It follows $\mathbf{u}_j \in \operatorname{kern}((B_0^{-1}A)^k)$ for these vectors. We use again the set of vectors (26). For sufficiently small $\|F\|$, we have $\tilde{A}(\mathbf{e}_i \otimes \mathbf{u}_j) \neq 0$ for $i = 1, \ldots, m+1$ and $j = 1, \ldots, n_1$. Furthermore, the matrix (31) implies

$$F(\mathbf{e}_i \otimes \mathbf{u}_j) = \sum_{l=1}^{q}(S_l \otimes B_l)(\mathbf{e}_i \otimes \mathbf{u}_j) = \sum_{l=1}^{q}(S_l\mathbf{e}_i) \otimes (B_l\mathbf{u}_j) \quad \text{for all } \; i, j.$$

Remark that each summand for $l > 0$ in the series representing $\tilde{A}$ exhibits the form

$$(-1)^l((I_{m+1} \otimes B_0^{-1})F)^{l-1}(I_{m+1} \otimes B_0^{-1})F(I_{m+1} \otimes B_0^{-1}A),$$

where the multiplication with $I_{m+1} \otimes B_0^{-1}A$ is followed by a multiplication with F. Since $B_0^{-1}A\mathbf{u}_j \in \operatorname{kern}((B_0^{-1}A)^{k-1})$ holds for $j = n_1 + 1, \ldots, n$, we conclude using the condition (33)

$$F(I_{m+1} \otimes B_0^{-1}A)(\mathbf{e}_i \otimes \mathbf{u}_j) = \mathbf{0}$$

and thus

$$\tilde{A}^r(\mathbf{e}_i \otimes \mathbf{u}_j) = \mathbf{e}_i \otimes (B_0^{-1}A)^r\mathbf{u}_j$$

for $r = 1, \ldots, k$. Hence the statement follows as in Theorem 2. □

The condition (33) is hard to satisfy for large k, since the matrices B_0, B_j are not related to the matrix A. Fortunately, this requirement is always satisfied for index $k = 1$. In case of index $k = 2$, the condition (33) reduces to

$$\operatorname{kern}(A) \subseteq \operatorname{kern}(B_j) \quad \text{for all } \; j = 1, \ldots, q.$$

If the given sufficient conditions are not satisfied, then the actual index can be determined for each particular system by an analysis of the generalised eigenspace corresponding to eigenvalue zero.

The discussion of consistency conditions becomes more complicated in this case, since the stochastic perturbation (31) is present in the inverse matrix as well as in the right-hand side of the system (32). Nevertheless, we can make the initial values (16) consistent again by using a construction like (19) for the matrix $B(\boldsymbol{\xi})$ at the beginning of the time interval.

4 Illustrative Example

In this section, we present numerical simulations of a test example corresponding to the above discussions.

4.1 Model of test example

We consider the electric circuit of a transistor amplifier, which is illustrated in Fig. 1. The corresponding mathematical model is introduced in [5]. A nonlinear system of DAEs

$$A\dot{\mathbf{x}} + \mathbf{f}(\mathbf{x}) = \mathbf{s}(t) \tag{34}$$

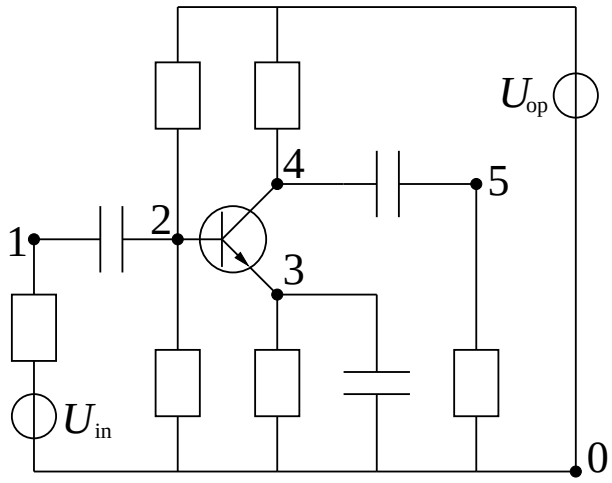

Figure 1: Circuit of transistor amplifier.

results, where the unknowns are the five voltages $\mathbf{x} := (U_1, U_2, U_3, U_4, U_5)^\top$. The involved matrix reads

$$A = \begin{pmatrix} -C_1 & C_1 & & & \\ C_1 & -C_1 & & & \\ & & -C_2 & & \\ & & & -C_3 & C_3 \\ & & & C_3 & -C_3 \end{pmatrix} \tag{35}$$

and the predetermined functions are

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} -\frac{U_1}{R_0} \\ -U_2(\frac{1}{R_1} + \frac{1}{R_2}) - 0.01g(U_2 - U_3) \\ -\frac{U_3}{R_3} + g(U_2 - U_3) \\ -\frac{U_4}{R_4} - 0.99g(U_2 - U_3) \\ -\frac{U_5}{R_5} \end{pmatrix}, \quad \mathbf{s}(t) = \begin{pmatrix} -\frac{U_{in}(t)}{R_0} \\ -\frac{U_{op}}{R_2} \\ 0 \\ -\frac{U_{op}}{R_4} \\ 0 \end{pmatrix}.$$

The current-voltage relation of the bipolar transistor is described by the nonlinear function

$$g(U_2 - U_3) = \alpha \left(\exp\left((U_2 - U_3)/\beta \right) - 1 \right). \tag{36}$$

According to [5], the technical parameters are set to $R_0 = 1000\ \Omega$, $R_i = 9000\ \Omega$ for $i = 1, \ldots, 5$, $\alpha = 10^{-6}$ V/A, $\beta = 0.026$ V, $U_{op} = 6$ V, $C_1 = 1\ \mu$F, $C_2 = 2\ \mu$F, $C_3 = 3\ \mu$F.

Applying the constant input signal $U_{in} \equiv 0$, the autonomous system (34) exhibits a steady state solution $\hat{\mathbf{x}} = (\hat{U}_1, \ldots, \hat{U}_5)^\top$. To achieve a linear DAE system of the form (1), we perform a linearisation of (34) around this state. Using the

abbreviations $\hat{h} := \exp(\beta^{-1}(\hat{U}_2 - \hat{U}_3))$ and

$$a := \alpha\beta^{-1}\hat{h}, \quad b := \alpha(\hat{h} + \beta^{-1}\hat{h}(\hat{U}_3 - \hat{U}_2) - 1),$$

the linearisation of the relation (36) is given by the function

$$\hat{g}(U_2 - U_3) := \alpha(\hat{h} + \beta^{-1}\hat{h}((U_2 - \hat{U}_2) - (U_3 - \hat{U}_3)) - 1) = a(U_2 - U_3) + b.$$

Consequently, we obtain a linear system (1) with the matrix and right-hand side

$$B := \begin{pmatrix} -\frac{1}{R_0} & & & & \\ & -\frac{1}{R_1} - \frac{1}{R_2} - 0.01a & 0.01a & & \\ & a & -\frac{1}{R_3} - a & & \\ & -0.99a & 0.99a & -\frac{1}{R_4} & \\ & & & & -\frac{1}{R_5} \end{pmatrix},$$

$$\hat{\mathbf{s}}(t) := \left(-\tfrac{U_{in}(t)}{R_0}, \quad -\tfrac{U_{op}}{R_2} + 0.01b, \quad -b, \quad -\tfrac{U_{op}}{R_4} + 0.99b, \quad 0\right)^\top.$$

The linear and nonlinear DAE exhibit nilpotency index 1 and differential index 1, respectively, see [5].

In the following, we use the input signal

$$U_{in}(t) = U_{amp} \sin\left(\tfrac{2\pi}{T}t\right)$$

with $U_{amp} = 0.4$ V and $T = 0.01$ s, which oscillates around the previously used state $U_{in} \equiv 0$. This input forces a periodic solution.

We construct a stochastic model by considering the three capacitances as random variables. We apply the relation

$$\tilde{C}_j(\xi_j) = C_j(1 + \gamma_j\xi_j) \quad \text{for} \;\; j = 1, 2, 3$$

with real constants $\gamma_j \in (0, 1)$ and random variables, which exhibit independent uniform distributions $\xi_j \in [-1, 1]$. Thus the corresponding random-dependent matrix is partitioned via

$$A(\boldsymbol{\xi}) = A_0 + \xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3,$$

where A_0 is equal to (35) and

$$A_1 := C_1\gamma_1 \begin{pmatrix} -1 & 1 & & & \\ 1 & -1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \quad A_2 := C_2\gamma_2 \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & -1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$

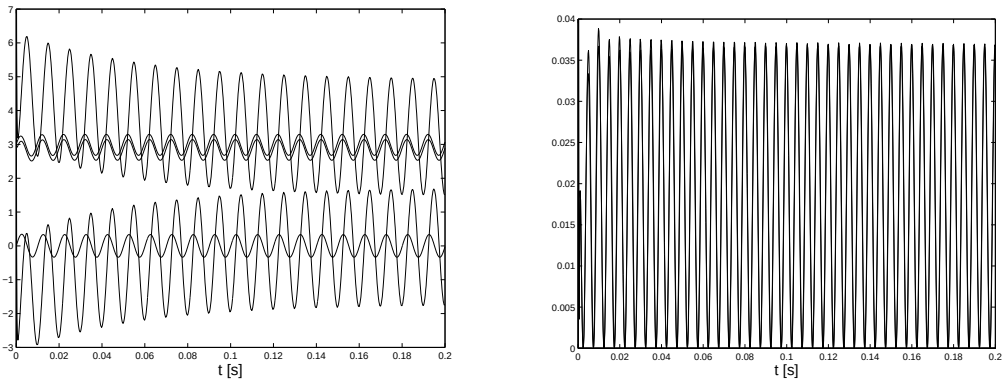

Figure 2: Expected values (left) and variances (right) computed by polynomial chaos in case of stochastic parameter C_2.

$$A_3 := C_3\gamma_3 \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & -1 & 1 \\ & & & 1 & -1 \end{pmatrix}.$$

We use the technique of the generalised polynomial chaos based on Legendre polynomials due to the uniform distributions. Hence we obtain a linear DAE system (14). The orthogonality of the basis polynomials implies a tridiagonal structure in the matrices (22).

The conditions (27) and (30) are fulfilled. Theorem 2 implies that the system (14) inherits the index $k = 1$ from the original system (3) provided that the constants γ_i are sufficiently small. Moreover, a consistent initial value of (3) yields a consistent choice (16) for (14) by Theorem 3.

4.2 Simulation of the linear system

We perform numerical simulations of the linearised system corresponding to the transistor amplifier. A consistent initial value $\mathbf{x}_0 \in \mathbb{R}^5$ is calculated. For different parameters, the solution of the deterministic system reaches a periodic oscillation after some transient phase. We consider two separate cases. Firstly, just the parameter C_2 is modelled stochastically with $\gamma_2 = 0.2$ ($\gamma_1 = \gamma_3 = 0$). Secondly, the parameter C_3 is the only one replaced by a random variable with $\gamma_3 = 0.2$ ($\gamma_1 = \gamma_2 = 0$). Thus uncertainties of 20% are modelled.

Using the Legendre polynomials up to degree $m = 2$ in (11) already yields sufficiently accurate numerical solutions in both cases. We integrate the DAE sys-

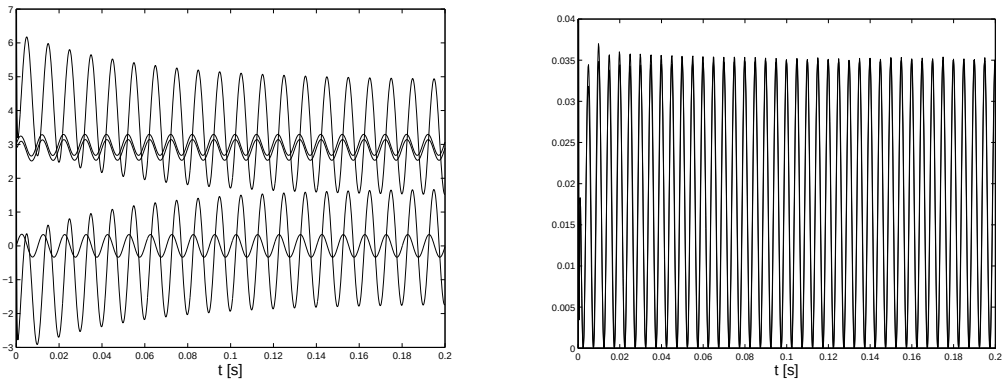

Figure 3: Expected values (left) and variances (right) determined by Monte-Carlo simulation in case of stochastic parameter C_2.

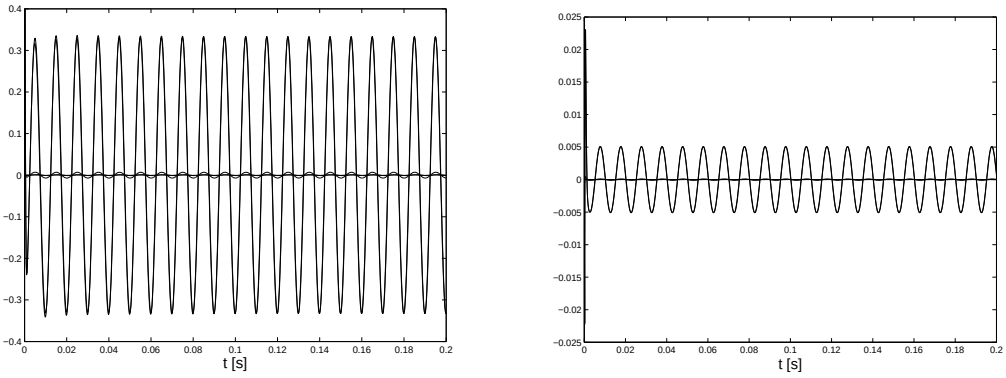

Figure 4: Coefficient functions corresponding to basis polynomials of degree 1 (left) and degree 2 (right) in case of stochastic parameter C_2.

tems (14) by trapezoidal rule. Initial values are given via (16). The formula (12) yields the approximations for expected values and variances.

In the first case, the resulting expected values and variances are shown for all five components in Fig. 2. Thereby, the variance of three components is nearly zero. Thus the parameter C_2 hardly influences these components. The expected value corresponds to the coefficient functions $\mathbf{v}_0$. Fig. 4 illustrates the other functions $\mathbf{v}_1, \mathbf{v}_2$, which also represent periodic oscillations. For comparison, the results of a Monte-Carlo simulation with 1000 samples are given in Fig. 3. We observe a good agreement to the results of the polynomial chaos.

For the second case, the numerical results obtained by the polynomial chaos are depicted in Fig. 5. The expected value exhibits the same behaviour as in the first case. Remark that the displayed time intervals are $[0, 0.2]$ and $[0, 0.4]$ in

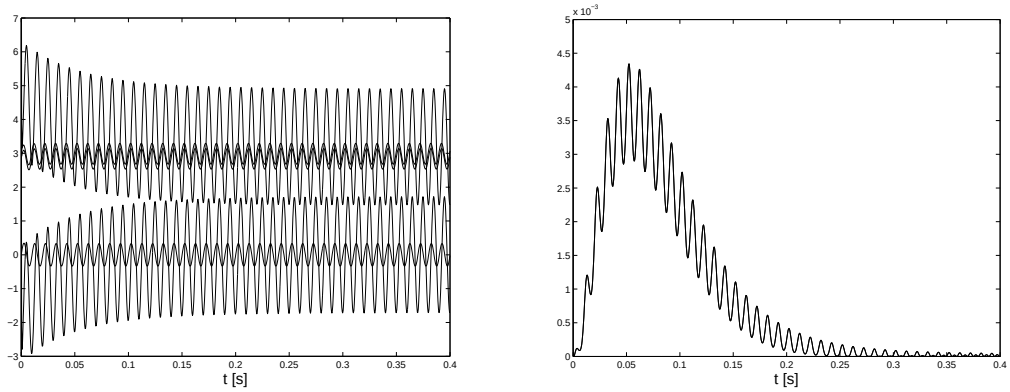

Figure 5: Expected values (left) and variances (right) determined by polynomial chaos in case of stochastic parameter C_3.

the two simulations. In contrast, the variance increases at the beginning for two components and decreases nearly to zero later. Hence the parameter C_3 affects the solution just within the transient phase. Afterwards, the influence of this parameter is damped out.

4.3 Simulation of the nonlinear system

As an outlook, we solve the nonlinear system (34) of the transistor amplifier with stochastic parameters. Although the analysis considers linear systems in this paper, the nonlinear case is often given in the applications.

We consider the same stochastic parameters as in the linear case. Using the strategy of the generalised polynomial chaos for (34), we obtain the larger coupled system

$$A_0\dot{\mathbf{v}}_l(t) + \left(\sum_{i=0}^{m}\sum_{j=1}^{q}\sigma_{li}^{j}A_j\dot{\mathbf{v}}_i(t)\right) + \frac{1}{\langle\Phi_l^2\rangle}\left\langle\Phi_l\,\mathbf{f}\left(\sum_{i=0}^{m}\mathbf{v}_i(t)\Phi_i\right)\right\rangle = \delta_{0l}\mathbf{s}(t)$$

for $l = 0, 1, \ldots, m$. The inner products (8) are evaluated numerically by Gaussian quadrature. Hence the computational effort becomes significantly higher in comparison to the linear system. If this quadrature can not be omitted, then the approach of stochastic collocation, see [8], seems to be more efficient than the polynomial chaos.

Again we use $m = 2$ in the simulations. The integration in time is done by trapezoidal rule. Fig. 6 illustrates the expected values and variances achieved by the polynomial chaos in case of stochastic parameter C_2. The behaviour and

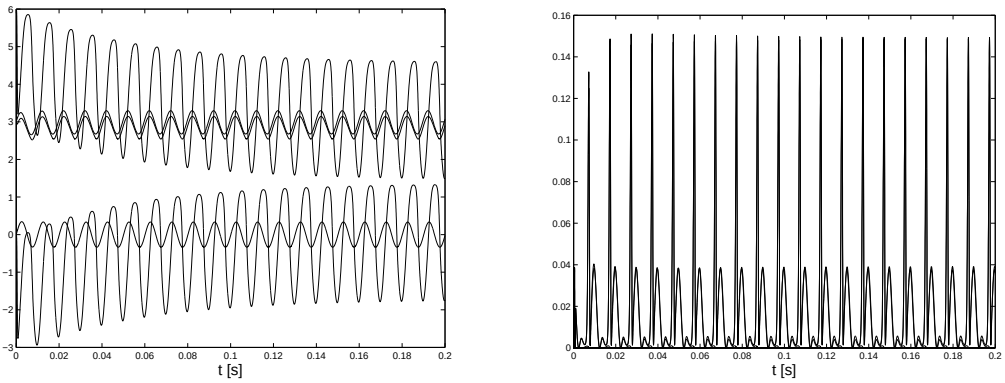

Figure 6: Expected values (left) and variances (right) determined by polynomial chaos with respect to stochastic parameter C_2 in case of nonlinear system.

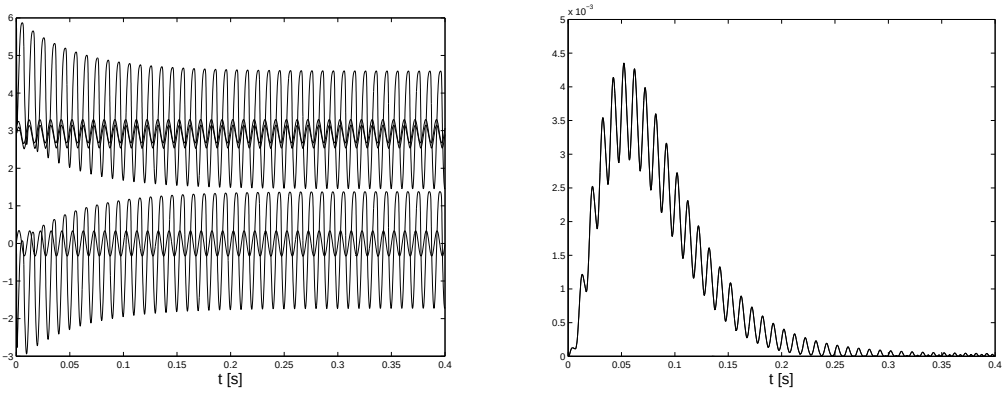

Figure 7: Expected values (left) and variances (right) determined by polynomial chaos with respect to stochastic parameter C_3 in case of nonlinear system.

magnitudes of the expected values agree to the linear case. However, the shape of the oscillations has changed due to the nonlinearity. The variances reach fastly a periodic oscillation again, where the magnitudes and the form are different in comparison to the linear case. Considering the stochastic parameter C_3, the results are shown in Fig. 7. Now the outcome agrees qualitatively and quantitatively to the linear case except for the shape of oscillations in the expected values.

5 Conclusions

We constructed a stochastic model for analysing the sensitivity of solutions for linear DAE systems. The technique of generalised polynomial chaos yields larger coupled systems of linear DAEs. Sufficient conditions have been introduced such that the alternative systems inherit the index of the original systems. These requirements are not necessary in general. Nevertheless, the conditions are often satisfied in the applications due to the structure of parameters within matrices. It follows that the alternative systems exhibit the same index as the underlying DAEs in all considered cases provided that the stochastic perturbation is sufficiently small and that the repective condition on matrices is fulfilled. Further investigations are required to analyse the index properties in case of nonlinear systems of DAEs, where different index concepts exist.

References

[1] F. Augustin, A. Gilg, M. Paffrath, P. Rentrop, U. Wever: *Polynomial chaos for the approximation of uncertainties: chances and limits.* Euro. Jnl. of Applied Mathematics 19 (2008), pp. 149–190.

[2] R. Cameron, W. Martin: *The orthogonal development of nonlinear functionals in series of Fourier-Hermite functionals.* Ann. of Math. 48 (1947) 2, pp. 385–392.

[3] E. Eich-Soellner, C. Führer: *Numerical Methods in Multibody Dynamics.* Teubner, Stuttgart 1998.

[4] M. Günther, U. Feldmann: *CAD based electric circuit modeling in industry I: mathematical structure and index of network equations.* Surv. Math. Ind. 8 (1999), pp. 97–129.

[5] E. Hairer, G. Wanner: *Solving Ordinary Differential Equations. Vol. 2: Stiff and Differential-Algebraic Equations.* 2nd ed., Springer, Berlin 1996.

[6] W. Kampowsky, P. Rentrop, W. Schmitt: *Classification and numerical simulation of electric circuits.* Surv. Math. Ind. 2 (1992), pp. 23–65.

[7] G.E. Karniadakis, C.H. Su, D. Xiu, D. Lucor, C. Schwab, R.A. Todor: *Generalized polynomial chaos solution for differential equations with random inputs.* Research Report No. 2005-01, ETH Zürich (2005).

[8] A. Loeven, J. Witteveen, H. Bijl: *Probabilistic collocation: An efficient non-intrusive approach for arbitrarily distributed parametric uncertainties.* In: M.B. Bragg (Ed.): Proceedings of 45th AIAA Aerospace Sciences Meeting and Exhibit 2007. pp. 1–14.

[9] M. Renardy, R. Rogers: *An Introduction to Partial Differential Equations.* 2nd ed., Springer, New York 2004.

[10] J. Stoer, R. Bulirsch: *Introduction to Numerical Analysis.* 3rd ed., Springer, New York 2002.

[11] N. Wiener: *The homogeneous chaos.* Amer. J. Math. 60 (1938), pp. 897–936.