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Fachbereich Mathematik und Naturwissenschaften

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Roland Pulch

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Roland Pulch

Bergische Universität Wuppertal, Fachbereich Mathematik und Naturwissenschaften, Lehrstuhl für Angewandte Mathematik und Numerische Mathematik, Gaußstr. 20, D-42119 Wuppertal, Germany.
Email: pulch@math.uni-wuppertal.de

Abstract

A model based on multirate partial differential algebraic equations yields an efficient numerical simulation of electric circuits in radio frequency applications. Considering frequency modulation, free parameters of the model are determined appropriately by a minimisation strategy. We apply the multirate approach to simulate a modified version of a Colpitts oscillator, which exhibits frequency modulation at widely separated time scales.

1 Introduction

Mathematical modelling of electric circuits leads to implicit systems of ordinary differential equations (ODEs) or systems of differential algebraic equations (DAEs), see [2]. A numerical integration of the systems becomes inefficient in case of modulated radio frequency (RF) signals with widely separated time scales. A multidimensional model enables an efficient representation of RF signals. Consequently, a system of multirate partial differential algebraic equations (MPDAEs) is introduced by Brachtendorf et al. [1]. In case of frequency modulated signals, the determination of an adequate local frequency function is crucial for the efficiency of the multidimensional model, see Narayan and Roychowdhury [3].

Phase conditions are applied to identify the local frequency function in [4]. Alternatively, properties based on minimisation determine these free parameters such that the multidimensional representation becomes efficient. In case of quasiperiodic signals, the minimisation strategy refers to biperiodic boundary value problems, see [5]. In case of envelope modulated signals, a corresponding technique for initial-boundary value problems is constructed in [6]. A variational calculus using a transformation formula yields a necessary condition for the multivariate representation in each case. Numerical methods have to include the additional condition for achieving the optimal solution of the MPDAE system.

In this article, we consider a Colpitts oscillator, where one capacitance of the circuit is controlled by a time-dependent input signal. Consequently, the output voltages become frequency modulated signals with widely separated time scales. We apply the multirate model following [6] to perform a numerical simulation of this LC-oscillator.

2 Multirate Model

Implicit systems of ODEs or systems of DAEs are used to describe the transient behaviour of electric circuits, see [2]. We refer to such a system as DAEs in the following and apply the general form (also valid for implicit ODEs)

$$\mathbf{F}\left(\tfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(t), \mathbf{x}(t), t\right) = \mathbf{0}, \tag{1}$$

where $\mathbf{x} : \mathbb{R} \to \mathbb{R}^k$ and $\mathbf{F} : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$. The solution $\mathbf{x} \in C^1$ includes unknown node voltages and branch currents. The function $\mathbf{F}$ depends on slowly varying input signals in its third argument. In addition, we assume that the solution $\mathbf{x}$ features a fast time scale with frequency modulation. Following [3], the DAE system (1) is transformed into the system of warped MPDAEs

$$\mathbf{F}\left(\tfrac{\partial\hat{\mathbf{x}}}{\partial t_1}(t_1, t_2) + \nu(t_1)\tfrac{\partial\hat{\mathbf{x}}}{\partial t_2}(t_1, t_2), \hat{\mathbf{x}}(t_1, t_2), t_1\right) = \mathbf{0} \tag{2}$$

with $\hat{\mathbf{x}} : \mathbb{R}^2 \to \mathbb{R}^k$ and $\nu : \mathbb{R} \to \mathbb{R}$. The representation $\hat{\mathbf{x}} \in C^1$ is called the multivariate function (MVF) of $\mathbf{x}$ and depends on two independent variables assigned to the time scales (t_1: slow, t_2: fast). Furthermore, the frequency modulation is described by the local frequency function ν, which is unknown a priori. We define the warping function

$$\Psi(t_1) := \int_0^{t_1} \nu(s)\, \mathrm{d}s. \tag{3}$$

Given a solution of the MPDAEs (2), the reconstruction scheme

$$\mathbf{x}(t) := \hat{\mathbf{x}}(t, \Psi(t)) \tag{4}$$

yields a solution of the DAEs (1). Initial-boundary value problems of the system (2) read

$$\hat{\mathbf{x}}(t_1, t_2) = \hat{\mathbf{x}}(t_1, t_2 + 1), \quad \hat{\mathbf{x}}(0, t_2) = \mathbf{h}(t_2) \quad \text{for all } t_1 \geq 0 \text{ and } t_2 \in \mathbb{R} \tag{5}$$

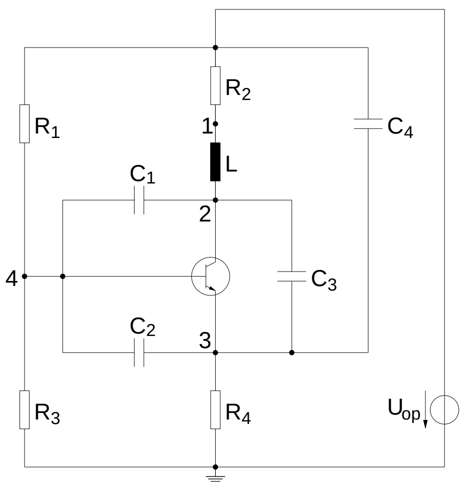

Figure 1: Circuit diagram of Colpitts oscillator.

with a predetermined periodic function $\mathbf{h}$. To identify the free parameters ν appropriately, an additional condition is required. Properties based on a specific minimisation generate simple MVFs, see [5]. For problems (5), the requirement is

$$p(t_1; \hat{\mathbf{x}}) := \sum_{l=1}^{k} w_l \int_0^1 \left(\frac{\partial \hat{x}_l}{\partial t_1}(t_1, t_2) \right)^2 \mathrm{d}t_2 \quad \rightarrow \quad \text{min.} \qquad \text{for each } \; t_1 \geq 0 \tag{6}$$

with the components $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_k)^\top$ using weights $w_1, \ldots, w_k \geq 0$. A variational calculus implies a necessary condition for an optimal solution, see [6]. This constraint demands the orthogonality of $\frac{\partial \hat{\mathbf{x}}}{\partial t_1}$ and $\frac{\partial \hat{\mathbf{x}}}{\partial t_2}$ with respect to the inner product corresponding to the integral norm applied in (6).

3 Simulation of a Colpitts Oscillator

The considered circuit of a Colpitts oscillator is illustrated in Fig. 1. The mathematical model of this circuit represents an implicit system of four ODEs, see [2]. The autonomous oscillator involves an inherent fast oscillation of about 8 kHz. In contrast, the capacitance C_3 is controlled by a slow harmonic oscillation with frequency 1 Hz (period $T_1 = 1$ s) here, see Fig. 2 (left). We apply the multirate approach according to the two time scales. A simulation of the biperiodic problem with phase conditions corresponding to (2) is discussed for this example in [4], where the used equations and technical parameters can also be found.

Alternatively, we simulate the initial-boundary value problem (5) of the sys-

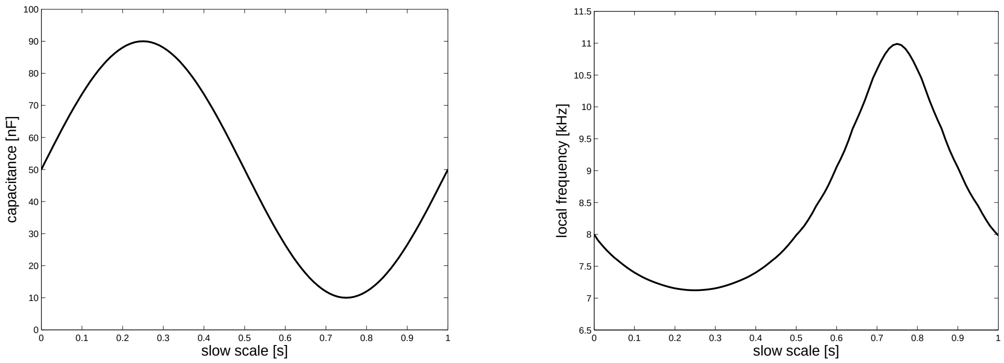

Figure 2: Controlled capacitance (left) and computed optimal local frequency function (right) of Colpitts oscillator.

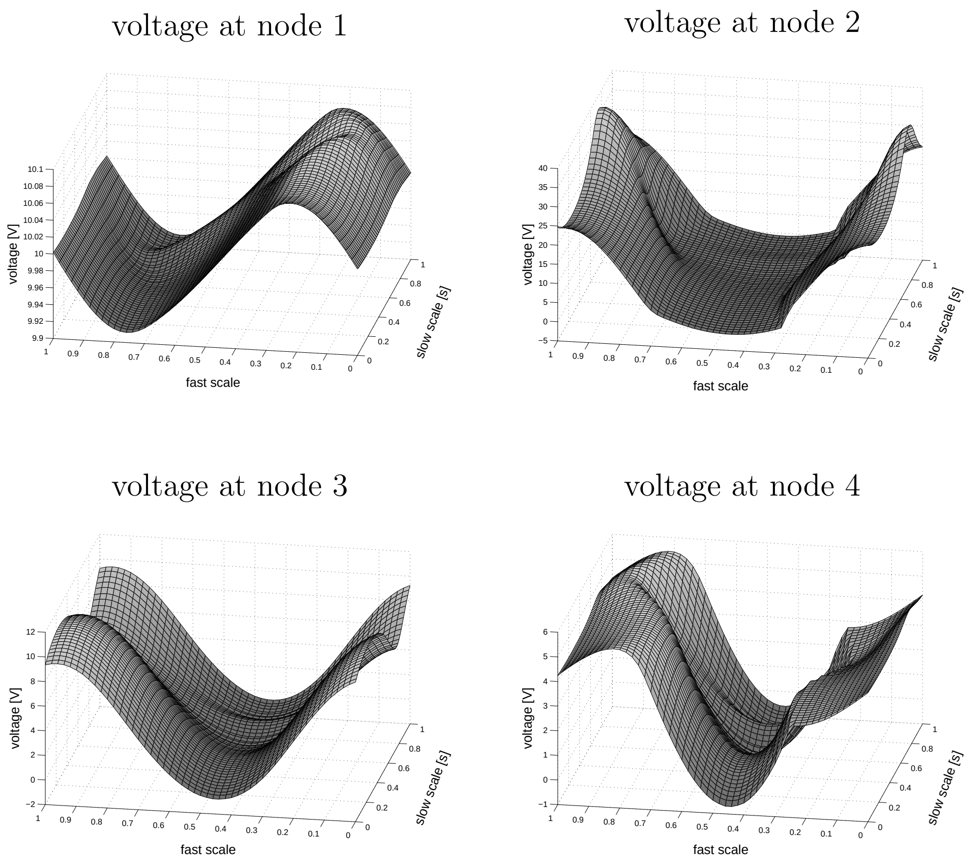

Figure 3: Computed optimal MVFs of Colpitts oscillator.

Table 1: Mean values of function p in (6) for perturbed solutions according to (7).

α	-10^{-3}	-10^{-4}	-10^{-5}	0	10^{-5}	10^{-4}	10^{-3}
$< p(\cdot;\hat{\mathbf{x}}_\alpha) >$	23284	296	50.5	49.3	50.7	299	23304

tem (2). A periodic solution of the autonomous oscillator yields the initial conditions. To identify the local frequency function, we apply the necessary condition following from the minimisation (6) with $w_l = 1$ for all l. A method of lines achieves a numerical solution of (2),(5) including the orthogonality constraint. Fig. 2 (right) shows the resulting optimal local frequency function. The behaviour of the local frequencies is physically reasonable with respect to an LC-oscillator. The MVFs corresponding to the four node voltages of the circuit are illustrated in Fig. 3. The obtained MVFs exhibit a relatively simple structure and thus, cause an efficient representation.

To confirm the optimality of the numerical solution regarding the criterion (6), we provide some perturbed solutions by the underlying transformation formula, see [6]. The computed local frequency ν_* is transformed to a new frequency function, which represents just one example out of an infinite number of feasible transformations. We apply the family

$$\nu_\alpha(t_1) := \nu_*(t_1) + \alpha < \nu_* > \sin\left(\tfrac{2\pi}{T_1} t_1\right) \tag{7}$$

with a parameter $\alpha \in \mathbb{R}$ and the integral mean value

$$< \nu_* > := \frac{1}{T_1} \int_0^{T_1} \nu_*(s)\, \mathrm{d}s. \tag{8}$$

Let $\hat{\mathbf{x}}_\alpha$ be the corresponding MVF. Table 1 demonstrates the integral mean values of the function (6) using small values α. Accordingly, the minimum is obtained at $\alpha = 0$, i.e., the above computed solution.

4 Conclusions

The concept of MPDAEs enables an alternative strategy for simulating electric circuits in RF applications. Techniques based on minimisation are feasible for the determination of the free modelling parameters in case of frequency modulated signals. A corresponding version of a Colpitts oscillator has been simulated efficiently using an initial-boundary value problem of the multirate system, where the necessary condition from a specific minimisation is included.

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