Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Lehrstuhl für Angewandte Mathematik
und Numerische Mathematik

Preprint BUW-AMNA 07/03

Kai Tappe

Launching a canon into multivariate Levy processes

August 27, 2007

http://www.math.uni-wuppertal.de/org/Num/

Launching a canon into multivariate Lévy processes

Kai Tappe

August 27, 2007

Abstract

Lévy copulas opened as the generic concept to describe dependence structures of multidimensional Lévy processes. In this paper we contribute an inverse approach for parsimonious copula modelling. Rather than defining a copula directly we construct the association between components in an implicit manner by variate conditioning. This pattern renders natural the simulation of multidimensional Lévy processes by series representation. We quantify graphically the effect of such a definition using simplified sample algorithms for path generation.

1 Introduction

Mathematical finance has witnessed the rising of multidimensional considerations. Modern structures and applications require increasingly to search for models describing multivariate events. The ancient Gaussian setting may no longer maintain to suite modelling needs in structured finance (cf. Embrecht et al., 2002). In static frameworks the notion of an ordinary copula was found to formulate general dependence structures. In the scope of this work we presume knowledge of the copula concept for the coupling of random variables (see Nelson, 2006, for an introduction). The scene of multivariate dynamics yet resorts to Brownian motion and a Gaussian concept of dependence in most financial applications. Only recently, a few authors set in place the notion of a Lévy copula extending the non-gaussian association between variates to a dynamical framework (see Tankov, 2004, 2006; Kallsen and Tankov, 2004). As vehicle, Tankov et al. instantiated multidimensional Lévy models thus settling jump processes into mathematical finance. Identifying the key role of the Lévy measure in dependence modelling the authors constructed copulas for this characteristic and derived algorithms for simulation from Lévy processes with specified dependence.

In section 2 we recall the fundamental properties of a Lévy process and state substantial formulae for describing its law. We proceed to investigate the dependence structures of multidimensional processes in particular and adopt the concept of Lévy copulas as introduced in Tankov (2004). We show that Lévy copulas achieve to formulate general dependence patterns of multivariate Lévy processes.

In section 3 we associate a conditional measure to the law of a Lévy process. This measure gives us reasons for an implicit modelling approach that employs bivariate Lévy copulas and multivariate ordinary copulas. We further exhibit the use of a such conception in matters of multidimensional process simulation by means of series representation.

2 Multivariate Lévy processes and copulas

In this section we state the essentials on Lévy processes and review the achievements in forwarding copula models to the dynamical framework of jump processes.

As the processes encountered in this article eventually have discontinuities, let us begin with clarifying the basic notion of a cadlag function. A function $f : [0, T] \to R^d$ is said to be *cadlag* if it is right-continuous with left limits, i.e. for each $t \in [0, T]$ the limits

$$f(t-) = \lim_{s \to t, s < t} f(s) \quad f(t+) = \lim_{s \to t, s > t} f(s)$$

exist and $f(t) = f(t+)$. In the sense of t as a time variable, cadlag then means that the jumps of f are not predictable. The discontinuity is seen as a sudden event (cf. Cont and Tankov, 2004).

A Lévy process is a $\mathbb{R}^d$-valued cadlag stochastic process $(X_t)_{t \geq 0}$ on a standard probability space $(\Omega, \mathcal{F}, \mathbb{R}^d)$ with stationary and independent increments such that $X_0 = 0$. In particular, jumps of the form $X_t - X_{t-} = \Delta X_t$ may occur sudden and at random but countable times.

2.1 Jump measure and characteristic triplet

In this paragraph we state representation formulae for Lévy processes and elaborate characteristics for their laws.

To every cadlag process $(X_t)_{t\geq 0}$ we can associate a random measure J_X on $\mathbb{R}^d \times [0,\infty)$ containing all the information about the discontinuities. Denoting jump times T_n and jump sizes ΔX_{T_n}, measure J_X tells us when the jumps occur and how big they are (cf. Cont and Tankov, 2004):

$$J_X(\omega,\cdot) = \sum_{n\geq 1} \delta_{(T_n(\omega),\Delta X_{T_n}(\omega))} = \sum_{t\in[0,T]}^{\Delta X_t \neq 0} \delta_{(t,\Delta X_t)}.$$

We call J_X the *jump measure* of process $(X_t)_{t\geq 0}$.

To every Lévy process $(X_t)_{t\geq 0}$ in particular we moreover associate the measure ν on $\mathbb{R}^d$ defined by

$$\nu(A) = E[\sharp\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}], \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Other than J_X, ν measures the expected number, per unit time, of jumps whose size belongs to a given Borel-set A. ν is called the *Lévy measure.* It happens that J_X is a Poisson random measure with intensity measure $\nu(dx)dt$ (see Tankov, 2006, for details on Poisson random measures).

Theorem 2.1 (Lévy-Ito decomposition) *Let $(X_t)_{t\geq 0}$ be a Lévy Process on $\mathbb{R}^d$ with Lévy measure ν. Then there exists a vector γ and a d-dimensional Brownian motion $(B_t)_{t\geq 0}$ with covariance matrix A and a measure J_X on $\mathbb{R}^d \times [0,\infty)$ with intensity measure $\nu(dx)dt$, such that the sample paths of X can be represented as follows:*

$$\begin{aligned} X_t &= \gamma t + B_t + X_t^l + \lim_{\varepsilon \downarrow 0} \widetilde{X}_t^\varepsilon, \quad \textit{where} \qquad (1) \\ X_t^l &= \int_{|x|\geq 1, s\in[0,t]} x J_X(ds \times dx) \quad \textit{and} \\ \widetilde{X}_t^\varepsilon &= \int_{\varepsilon \leq |x| \leq 1, s\in[0,t]} x\{J_X(ds\times dx) - ds \times \nu(dx)\}. \end{aligned}$$

The terms in (1) are independent and the convergence in the last term is almost sure and uniform in t on $[0,T]$. Theorem 2.1 asserts that the path of any Lévy process decomposes into a continuous Gaussian process with drift

$\gamma t + B_t$ and a pure jump process defined by intensity measure ν. Given that ν has finite variation, i.e. $\int_{\mathbb{R}^d}(|x| \wedge 1)\nu(dx) < \infty$, the above representation can be simplified to

$$X_t = bt + B_t + \int_{\mathbb{R}^d, s\in[0,t]} xJ_X(ds \times dx),$$

where $b \in \mathbb{R}^d$ is some other drift vector (see Cont and Tankov, 2004, on different drifts of a Lévy process).

Using the Lévy-Ito decomposition and the infinite divisibility of the distribution of X_t for any $t \geq 0$ one can derive in a straight forward manner the second fundamental result:

Theorem 2.2 (Lévy Khinchine representation) *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure ν and drift γ and covariance matrix A of the corresponding Brownian motion part. Then its characteristic function has the following form*

$$\begin{aligned} E[e^{iz.X_t}] &= e^{t\psi(z)}, \quad \textit{where} \\ \psi(z) &= -\frac{1}{2}z.Az + i\gamma.z + \int_{\mathbb{R}^d} \left(e^{iz.x} - 1 - iz.x1_{|x|\leq 1}\right)\nu(dx). \end{aligned} \tag{2}$$

We call ψ the characteristic exponent of Lévy process $(X_t)_{t\geq 0}$. Again, if $\int_{\mathbb{R}^d}(|x| \wedge 1)\nu(dx) < \infty$ formula (2) can be reduced to

$$\psi(z) = -\frac{1}{2}z.Az + ib.z + \int_{\mathbb{R}^d} \left(e^{iz.x} - 1\right)\nu(dx).$$

These representation formulae entail that the distribution of Lévy process $(X_t)_{t\geq 0}$ is sufficiently described by triplet (γ, A, ν), which is hence called the characteristic triplet.

2.2 Lévy copulas

In this paragraph we reproduce the idea of using copula methods to describe multivariate dependence in Lévy models.

To begin with, we give a brief review on increasing functions. Denoting the extended real line by $\overline{\mathbb{R}}$, let $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ be a d-place function. For

$a, b \in \overline{\mathbb{R}}^d$ with $a \leq b$ and $\overline{(a, b]} \in \overline{\mathbb{R}}^d$, we here define the F-volume of $(a, b]$ to be

$$V_F((a, b]) = \sum_{u \in \{a_1, b_1\} \times \ldots \times \{a_d, b_d\}} (-1)^{N(u)} F(u),$$

where $N(u) = \sharp\{k : u_k = a_k\}$, i.e. the sum of the signed values of F over the vertices of $(a, b]$. F is called d-increasing if the volume $V_F((a, b])$ of any d-box $\overline{(a, b]} \in \overline{\mathbb{R}}^d$ is nonnegative. For $I \in \{1, ..., d\}$ we further denote the I-margin of F by

$$F^I((u_i)_{i \in I}) = \sum_{(u_j)_{j \in I^c} \in \{-\infty, \infty\}^{|I^c|}} F(u_1, ..., u_d) \prod_{j \in I^c} sign(u_j)$$

with cardinality $|I|$ and complement $I^c = \{1, ..., d\} \backslash I$.

For the rest of this paper we assume a Lévy process $(X_t)_{t \geq 0}$ with characteristic triplet $(0, 0, \nu)$, i.e. we eclipse the Brownian motion part from considerations. It is certainly feasible to focus on the jump component of a Lévy process due to independence of the components.

Definition 2.1 *Let $(X_t)_{t \geq 0}$ be a $\mathbb{R}^d$-valued Lévy process with Lévy measure ν. The tail integral of ν is the function $U : (\overline{\mathbb{R}} \backslash 0)^d \to \overline{\mathbb{R}}$ defined by*

$$U(x_1, ..., x_d) = \nu \left(\prod_{j=1}^{d} \mathcal{I}(x_j) \right) \prod_{j=1}^{d} sign(x_j),$$

where

$$\mathcal{I}(x) = \begin{cases} [x, \infty), & x \geq 0; \\ (-\infty, x], & x < 0. \end{cases}$$

From elementary statistics it is a known fact that any probability measure can be characterized by its distribution function. In a similar way, any Lévy measure corresponds to the set of its marginal tail integrals. The basic concept of a Lévy copula is defined in the following

Definition 2.2 *A function $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ is called a Lévy copula, if*

1. *$F(u_1, ..., u_d) \neq \infty$ for $(u_1, ..., u_d) \neq (\infty, .., \infty)$*
2. *$F(u_1, ..., u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, ..., d\}$*
3. *F is d-increasing*

4. $F^i(u) = u$ *for any* $i \in \{1, ..., d\}$, $u \in \mathbb{R}$.

One clearly recognizes groundedness on the axes, the d-increasing property and uniformity of the margins. The fundamental result due to Tankov (2004) relates tail integrals of a Lévy process to its margins.

Theorem 2.3 *Let* ν *be a Lévy measure on* $\mathbb{R}^d \backslash \{0\}$*. Then there exists a Lévy copula* F *such that the tail integrals of* ν *satisfy*

$$U^I((u_i)_{i \in I}) = F^I((U_i(x_i))_{i \in I})$$

for any non-empty $I \subset \{1, ..., d\}$ *and any* $(x_i)_{i \in I} \in (\mathbb{R}\backslash 0)^d$*. Conversely, if* F *is a d-dimensional Lévy copula and* $\nu_1, ..., \nu_d$ *are Lévy measures on* $\mathbb{R}\backslash\{0\}$ *with tail integrals integrals* $U_1, ..., U_d$*, then there exists a unique Lévy measure* $(\mathbb{R}\backslash\{0\})^d$ *with one-dimensional marginal tail integrals* $U_1, ..., U_d$*.*

Theorem 2.3 gives us to understand that any dependence structure between jumps of a Lévy process X with Lévy measure ν is described by a Lévy copula. In turn it is possible to construct multidimensional Lévy models by specifying separately jump dependence structure and one-dimensional laws for the components.

2.3 Generic copula models

In this paragraph we overview some established copula models. Herein and in the following we use the notation

$$S = \{x \in \mathbb{R}^d : sign(x_1) = ... = sign(x_d)\}.$$

One can show that the components $X^1, ..., X^d$ of an $\mathbb{R}^d$-valued Lévy process X are independent if and only if their continuous martingale parts are independent and the Lévy measure ν is given by

$$\nu(B) = \sum_{i=1}^{d} v_i B_i, \quad \text{for all } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

where for every i, ν_i denotes the i-th margin of ν and $B_i = \{x \in \mathbb{R} : (0, ..., 0, x, 0, ..., 0) \in B\}$. This finding involves that the marginal tail integrals

$U^I((x_i)_{i\in I})$ vanish for all $I \subset \{1, ..., d\}$ and all $(x_i)_{i\in I} \in (\mathbb{R} \setminus \{0\})^{|I|}$. Then the independence copula is given by

$$F_\perp(u_1, ..., u_d) = \sum_{i=1}^{d} x_i \prod_{j\neq i} 1_\infty(x_j).$$

Jumps of a $\mathbb{R}^d$-valued Lévy process X are considered completely dependent if there exists a strictly ordered set D of S such that $\nu(\mathbb{R}^d \setminus D) = 0$. Similar to the ordinary case, the complete dependence Lévy copula is given by

$$F_\|(x_1, ..., x_d) = \min(|x_1|, ..., |x_d|)1_K(x_1, ..., x_d) \prod_{i=1}^{d} sign(x_i).$$

Aside from these extremes, parametric copula models that transition between complete dependence and independence in a smooth fashion are desirable. Exemplary for parametric copula models we echo the Archimedean construction as found in Tankov (2004). Let $\Phi : [-1, 1] \to [-\infty, \infty]$ be a strictly increasing continuous function with $\Phi(1) = \infty, \Phi(0) = 0, \Phi(-1) = -\infty$, having nonnegative derivatives of order up to d on $(-1, 0)$ and $(0, 1)$, and satisfying

$$\frac{\partial^d \Phi(e^x)}{\partial x^d} \geq 0, \quad \frac{\partial^d \Phi(-e^x)}{\partial x^d} \leq 0.$$

Let $\widetilde{\Phi}(u) = 2^{d-2}\{\Phi(u) - \Phi(-u)\}$ for $u \in [-1, 1]$. Then

$$F(u_1, ..., u_d) = \Phi\left(\prod_{i=1}^{d} \widetilde{\Phi}^{-1}(u_i)\right)$$

defines an Archimedean Lévy copula.

Example 2.1 *Let* $\Phi(x) = \eta(-\log(|x|))^{-1/\theta}1_{x\geq 0} - (1-\eta)(-\log(|x|))^{-1/\theta}1_{x<0}$ *with* $\theta > 0$ *and* $\eta \in (0, 1)$. *Then*

$$\begin{aligned} \widetilde{\Phi}(u) &= 2^{d-2}\{-\log(|x|)\}^{-1/\theta} sign(x), \text{ and} \\ \widetilde{\Phi}^{-1}(u) &= e^{-|2^{2-d}u|^{-\theta}} sign(u). \end{aligned}$$

This produces the two parameter Clayton family of Lévy copulas

$$F(u_1, ..., u_d) = 2^{2-d}\left(\sum_{i=1}^{d} |u_i|^{-\theta}\right)^{-1/\theta} (\eta 1_{u_1,...,u_d\geq 0} - (1-\eta)1_{u_1,...,u_d<0}).$$

In the following we analyze the role of the parameters on the basis of the two dimensional version

$$F(u,v) = \left(|u|^{-\theta} + |v|^{-\theta}\right)^{-1/\theta} (\eta 1_{uv \geq 0} - (1-\eta) 1_{uv<0}). \tag{3}$$

To this end we plotted the density for various parameter configurations.

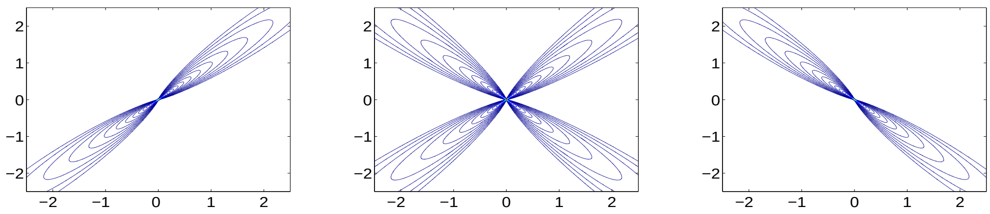

Figure 1: Contour plot of Clayton Lévy density with $\theta = 5$ and various η. From left to right: $\eta = 1$, $\eta = 0.5$, $\eta = 0$

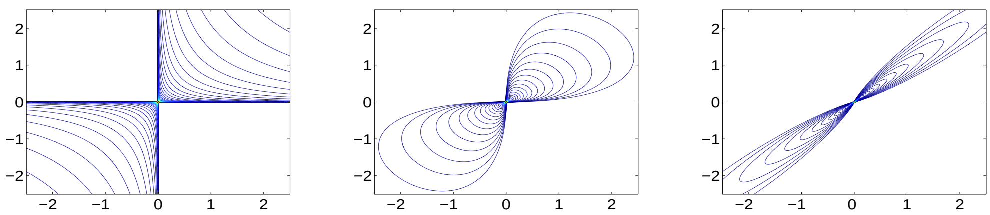

Figure 2: Contour plot of Clayton Lévy density with $\eta = 1$ and various θ. From left to right: $\theta = 0.1$, $\theta = 1$, $\theta = 5$

From figure 1 it is obvious that the parameter η is responsible for sign dependence of the jumps: if close to 1 jumps tend to have the same directions, if approaching zero jumps tend to have opposite directions. Figure 2 shows that the parameter θ determines the dependence of absolute values of jumps: the higher the value the more associated the jump sizes. Remarkably, all plots feature a symmetric jump behavior.

3 Canonical dependence and direct simulation

In this section we cover a probabilistic interpretation of the Lévy measure and how it avails an efficient algorithm for simulating Lévy processes. We introduce en route a new class of Lévy copulas, which is viable for the purpose of path generation.

Apart from the trivial cases of Brownian motion or subordinated Brownian motion, incremental methods for simulation of Lévy processes are not available in the multidimensional setting (see Cont and Tankov, 2004, for subordination). One typically resorts to compound Poisson approximations or series representations. In the scope of this work we content ourselves with simulation by series representation as to Tankov (2006), which has feeded a probabilistic interpretation of the Lévy measure and which has motivated our new copula design.

3.1 Conditional probability measure

In this paragraph we show existence of a probability measure related to the Lévy measure. Further we propose to use ordinary copula models for its distributional description. This formulates a new class of Lévy copulas.

Let F be a Lévy copula on $\overline{\mathbb{R}}^d$ that satisfies the following continuity condition at infinity

$$\lim_{(u_i)_{i\in I}\to\infty} F(u_1, ..., u_d) = F(u_1, ..., u_d)|_{(u_i)_{i\in I}=\infty} \tag{4}$$

for all $I \subset \{1, ..., d\}$. This Lévy copula defines a positive measure μ on $\mathbb{R}^d$ with Lebesgue margins such that for each $a, b \in \mathbb{R}^d$ with $a \leq b$,

$$V_F((a, b]) = \mu((a, b]).$$

Defining $f : (u_1, ..., u_d) \mapsto (U_1^{-1}(u_1), ..., U_d^{-1}(u_d))$, the relation between Lévy measure ν and measure μ is

$$\nu(A) = \mu(\{u \in \mathbb{R}^d : f(u) \in A\}. \tag{5}$$

One can then show that there exists a family, indexed by $\xi \in \mathbb{R}$, of positive Radon measures $K(\xi, \cdot)$ on $\mathbb{R}^{d-1}$, such that $\xi \mapsto K(\xi, dx_2, ..., dx_d)$ is Borel measurable and

$$\mu(dx_1, dx_2, ..., dx_d) = \lambda(dx_1) \otimes K(x1, dx_2, ..., dx_d). \tag{6}$$

$\{K(\xi,\cdot)\}_{\xi\in\mathbb{R}}$ is called the *family of conditional probability distributions* associated to the Lévy copula F. Denoting

$$K_\xi(x_2,...,x_d) = K(\xi,(-\infty,x2]\times ...\times(-\infty,x_d]),$$

Tankov showed that there exist a nullset N such that for every $\xi\in\mathbb{R}\setminus N$, K_ξ is a probability distribution function satisfying

$$K_\xi(x_2,...,x_d) = sign(\xi)\frac{\partial}{\partial\xi}V_F((\xi\wedge 0,\xi\vee 0]\times(-\infty,x_2]\times ...\times(-\infty,x_d]) \quad (7)$$

in every point $(x_2,...,x_d)$ where K_ξ is continuous.

These findings now motivate the basic idea to reverse the approach to modelling Lévy copulas: instead of designing the generic Lévy copula F from which K can be derived, we propose to define an implicit dependence structure by modelling conditional probability distribution K_ξ in the first place. From there the multivariate Lévy measure obtains via interrelations (5) and (6). The crux therein is to keep the jump dependence structure separated from the marginal process evolution, which is certainly not fulfilled per se. The following result establishes a sufficient (and necessary) design of qualified conditional measures.

Theorem 3.1 *Let $\nu_i, i=1,...,d$ be marginal Lévy measures with corresponding tail integrals U_i and $f:(u_1,...,u_d)\mapsto(U_1^{-1}(u_1),...,U_d^{-1}(u_d))$. Further let K be a conditional measure on $\mathbb{R}^{d-1}$ such that $\nu=\mu(f)$ in the sense of* (5) *and* (6) *is a Lévy measure on $\mathbb{R}^d$ with margins ν_i. Then there exist Lévy copulas $F_i:\overline{\mathbb{R}}^2\to\overline{\mathbb{R}}, i=2,...,d$ and a family, indexed by $\xi\in\mathbb{R}$, of ordinary copula functions $C_\xi:[0,1]^{d-1}\to[0,1]$ with*

$$K_\xi(x_2,...,x_d) = C_\xi\left(\left\{sign(\xi)\frac{\partial}{\partial\xi}V_{F_i}((0\wedge\xi,0\vee\xi]\times(-\infty,x_i])\right\}_{i=2}^{d}\right) \quad (8)$$

Conversely, if $F_i:\overline{\mathbb{R}}^2\to\overline{\mathbb{R}}, i=2,...,d$ Lévy copulas, $C_\xi:[0,1]^{d-1}\to[0,1],\xi\in\mathbb{R}$ ordinary copula functions and conditional measure K on $\mathbb{R}^{d-1}$ defined as in (8)*, then $\nu=\mu(f)$ in the sense of* (5) *and* (6) *is a Lévy measure on $\mathbb{R}^d$ with margins ν_i.*

Remark 3.1 *Equation* (8) *involves the functions $G^i_\xi(x) := sign(\xi)\frac{\partial}{\partial\xi}V_{F_i}((0\wedge\xi,0\vee\xi]\times(-\infty,x_i])$, which by* (7) *and the arguments used therefor really are distribution functions.*

Proof of Theorem 3.1: First part. By Sklar's theorem there exist for all $\xi \in \mathbb{R}$ a $(d-1)$-dimensional ordinary copula C_ξ and univariate marginal distribution functions $G^i_\xi(x_2), i = 2, ..., d$ such that

$$K_\xi(x_2, ..., x_d) = C_\xi(G^2_\xi(x_2), ..., G^d_\xi(x_d)). \tag{9}$$

Our goal is to represent the G^i_ξ's as in remark 3.1. For this purpose, let $i \in \{2, ..., d\}$ and consider the bivariate tail integral $U^{1,i}_\nu$, to which there exists by theorem 2.3 a Lévy copula F^i on $\mathbb{R}^2$ so that $U^{1,i}_\nu(x_1, x_i) = F^i(U_1(x_1), U_i(x_i))$. Due to the construction of Lévy measure ν on the other hand, it holds for $x_1 < 0, x_i \geq 0$, say,

$$\begin{aligned} U^{1,i}_\nu(x_1, x_i) &= -\mu(\{u \in \mathbb{R}^d : u_1 \in (U_1(x_1), 0], u_i \in (0, U_i(x_i)]\}) \\ &= -\int_{U_1(x_1)}^0 \int_0^{U_i(x_i)} \int_{\mathbb{R}^{d-2}} K(\xi, dx_2, ...dx_d)d\xi \\ &= \int_0^{U_1(x_1)} K_\xi(\infty, .., U_i(x_i).., \infty) - K_\xi(\infty, .., 0, .., \infty)d\xi. \end{aligned}$$

By (9), remark 3.1 and uniformity at the margins of an ordinary copula the bivariate margin can be written as

$$U^{1,i}_\nu(x_1, x_i) = \int_0^{U_1(x_1)} G^i_\xi(U_i(x_i)) - G^i_\xi(0)dx_1. \tag{10}$$

Equating the results gives

$$\int_0^{U_1(x_1)} G^i_\xi(U_i(x_i)) - G^i_\xi(0)dx_1 \quad = \quad F^i(U_1(x_1), U_i(x_i)),$$

where one can certainly represent the expression on the right hand side in terms of volume functions as follows

$$\begin{aligned} F^i(U_1(x_1), U_i(x_i)) &= -V_{F^i}((U_1(x_1), 0] \times (-\infty, U_i(x_i)]) \\ &\quad +V_{F^i}((U_1(x_1), 0] \times (-\infty, 0]). \end{aligned}$$

Differentiation then yields $G^i_\xi(x) := -\frac{\partial}{\partial \xi} V_{F^i}((\xi, 0] \times (-\infty, x])$, whereas the general case $x_1, x_i \in \mathbb{R}$ can be derived analogously.

Second part. In order to show that $\nu = \mu(f)$ has margins $\nu_i, i = 1, ..., d$ it suffices to consider its marginal tail integrals $U^i_\nu, i = 1, ..., d$. The goal is to prove equality between the input tail integrals U_i and the implicit tail integrals U_ν^i. Same as before, it holds for $x_i < 0$, say,

$$U_\nu^i(x_i) = -\mu(\{u \in \mathbb{R}^d : u_1 \in (U_i(x_i), 0]\}).$$

Since K_ξ is a probability measure on $\mathbb{R}^{d-1}$ the first tail integral obtains as

$$\begin{aligned} U_\nu^1(x_1) &= -\mu(\{u \in \mathbb{R}^d : u_i \in (U_1(x_1), 0)]\}) \\ &= \int_0^{U_1(x_1)} \int_{\mathbb{R}^{d-1}} K(\xi, dx_2, ...dx_d) d\xi \\ &= U_1(x_1). \end{aligned}$$

Using the same arguments as in the previous case one obtains for the i-th tail, $i \in \{2, ..., d\}$,

$$\begin{aligned} U_\nu^i(x_i) &= -\mu(\{u \in \mathbb{R}^d : u_i \in (U_i(x_i), 0]\}) \\ &= -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ... \int_{U_i(x_i)}^{0} ... \int_{-\infty}^{\infty} K(\xi, dx_2, ..., dx_d) d\xi. \\ &= \int_{-\infty}^{\infty} \left(K_\xi(\infty, .., U_i(x_i), .., \infty) - K_\xi(\infty, .., 0, .., \infty)\right) d\xi \qquad (11) \end{aligned}$$

By (9), remark 3.1 and uniformity at the margins of an ordinary copula it results that

$$U_\nu^i(x_i) = \int_{-\infty}^{\infty} G_\xi^i(U_i(x_i)) - G_\xi^i(0) d\xi.$$

Since $G_\xi^i(y_i)$ is a perfect derivative, the desired result follows from uniformity at the margins of a Lévy copula

$$\begin{aligned} \int_{\mathbf{R}} (G_\xi(U_i(x_i)) - G_\xi(0)) sign(x) d\xi &= -V_{F_i}((-\infty, \infty] \times (U_i(x_i), 0]) \\ &= F_i(\infty, U_i(x_i)) - F_i(-\infty, U_i(x_i)) \\ &= F_i^2(U_i(x_i)) \\ &= U_i(x_i), \end{aligned}$$

whereas the general case $x_1, x_i \in \mathbb{R}$ can be derived analogously. It is not yet concluded that ν defined via conditional probability measure K really is a

Lévy measure satisfying the integrability condition $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$. But this is automatically fulfilled if its one-dimensional margins are Lévy measures

$$\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) \leq \int_{\mathbb{R}^d} \sum_{i=1}^{d}(x_i^2 \wedge 1)\nu(dx) \leq \sum_{i=1}^{d} \int_{\mathbb{R}}(x_i^2 \wedge 1)\nu_i(dx_i) < \infty.$$

□

Theorem 3.1 distinguishes the first margin, which we shall hereafter call the *canon*. We announce conventionally the first variable the canon, albeit the choice is arbitrary. It is striking that theorem 3.1 induces a dependence structure between the marginal Lévy measures without mentioning explicitly the Lévy copula. There still exists a corresponding Lévy copula by theorem 2.3, but it is only given implicitly through the designed measure ν. In this paragraph and in the following we shall refer to the implicit design pattern as *canonization* and to the implicit Lévy copula as the *canonical Lévy copula*. The following lemma should give first reasons for our naming.

Lemma 3.1 *Let $C_\xi, \xi \in \mathbb{R}$ be arbitrary ordinary copulas, $F_i = F_\|, i = 2, ..., d$ and ν as in the previous theorem. Then the marginal Lévy copulas satisfy $F^{i,j} = F_\|$, $\forall i, j$, i.e. the bivariate margins are completely dependent.*

Proof: Let $F_i = F_\| = \min\{|x_1|, |x_i|\}1_S(x_1, x_i)sign(x_1)sign(x_i)$. Then

$$G^i_{x_1}(x_i) = 1_{x_i \geq x_1 \geq 0} + 1_{x_1 < 0} - 1_{0 > x_1 \geq x_i}. \tag{12}$$

Again it suffices to consider the tail integrals. We want to show that the implicit bivariate tail integral $U^{i,j}_\nu$ can be represented as univariate tails U_i, U_j coupled by complete dependence copula $F_\|$. For this purpose, we renew the argumentation from the proof of the second part of theorem 3.1 and obtain in a similar manner

$$\begin{aligned} U^{i,j}_\nu(x_i, x_j) &= \int_{-\infty}^{\infty} [F_{x_1}(\infty, U_i(x_i), U_j(x_j), \infty) - F_{x_1}(\infty, ., U_i(x_i), 0, ., \infty) \\ &\quad - F_{x_1}(\infty, ., 0, U_j(x_j), ., \infty) + F_{x_1}(\infty, ., 0, 0, ., \infty)]dx_1 \end{aligned}$$

Assuming $x_1 < 0$ one gets by means of (8), (12) and the properties of ordinary copula functions

$$
\begin{aligned}
F_{x_1}(\infty,..,U_i(x_i),..,U_j(x_j),..,\infty) &= C^{i,j}(\{1_{U_i(x_i)\geq x_1\geq 0}+1_{x_1<0}-1_{0>x_1\geq U_i(x_i)}\}_{i=i,j}) \\
&= (1_{x_1<0}-1_{0>x_1\geq U_i(x_i)})\cdot(1_{x_1<0}-1_{0>x_1\geq U_i(x_i)}) \\
&= 1_{x_1\leq U_i(x_i),x_1\leq U_j(x_j)} \\
F_{x_1}(\infty,..,U_i(x_i),..,0,..,\infty) &= C^{i,j}(1_{U_i(x_i)\geq x_1\geq 0}+1_{x_1<0}-1_{0>x_1\geq U_i(x_i)},1_{x_1<0}) \\
&= (1_{x_1<0}-1_{0>x_1\geq U_i(x_i)})\cdot 1_{x_1<0} \\
&= 1_{x_1\leq U_i(x_i)} \\
F_{x_1}(\infty,..,0,..,U_j(x_j),..,\infty) &= C^{i,j}(1_{x_1<0},1_{U_i(x_i)\geq x_1\geq 0}+1_{x_1<0}-1_{0>x_1\geq U_i(x_i)}) \\
&= 1_{x_1<0}\cdot(1_{x_1<0}-1_{0>x_1\geq U_j(x_j)}) \\
&= 1_{x_1\leq U_j(x_j)} \\
F_{x_1}(\infty,..,0,..,0,..,\infty) &= C^{i,j}(1_{x_1<0},1_{x_1<0}) \\
&= 1.
\end{aligned}
$$

This yields immediately the characteristic integrand for negative values of x_1 as follows

$$
1_{x_1\leq U_i(x_i),x_1\leq U_j(x_j)}-1_{x_1\leq U_i(x_i)}-1_{x_1\leq U_j(x_j)}+1 \quad = \quad 1_{x_1\geq U_i(x_i),x_1\geq U_j(x_j)}.
$$

In a similar way we obtain the integrand $1_{x_1\leq U_i(x_i),x_1\leq U_j(x_j)}$ for positive values of x_1. With that it holds

$$
\begin{aligned}
U_\nu^{i,j}(x_i,x_j) &= \int_{-\infty}^{0} 1_{x_1\geq U_i(x_i),x_1\geq U_j(x_j)}dx_1+\int_0^{\infty} 1_{x_1\leq U_i(x_i),x_1\leq U_j(x_j)}dx_1 \\
&= \min\{|U_i(x_i)|,|U_j(x_j)|\}1_S(x_i,x_j).
\end{aligned}
$$

□

Lemma 3.1 gives us to understand that canonization confers complete dependence upon any pair dependence no matter what the association between non-canon variables may be. This is a very strong result and it assists essentially the interpretation of the projected variable as the system's rule, the canon.

3.2 Series representation

In this paragraph we address a series representation for multidimensional Lévy processes and we highlight the advantage of canonization.

Series representations go back to Rosinski and others, who proved almost sure convergence of series of random variables to Lévy processes with specified characteristic triplets. Tankov extended their findings to jump dependence modelling with Lévy copulas. In view of canonical Lévy copulas we somewhat modify theorem 5.6. in Tankov (2006) to suit our distinct approach. Again, the focus is set on the finite variational case.

Theorem 3.2 (Series representation) *Let v_i be marginal Lévy measures with tail integrals $U_i, i = 1, ..., d$ and $K(x_1, dx_2, ..., dx_d)$ be a conditional probability measure on $\mathbb{R}^{d-1}$, such that $\nu = \mu(f)$ with μ and f as before is a Lévy measure preserving the margins. Further let $\{V_k\}$ be a sequence of independent r.v., uniformly distributed on $[0, 1]$. Introduce d random sequences $\{\Gamma_k^1\}, ..., \{\Gamma_k^d\}$, independent from $\{V_k\}$ such that*

1. *$N = \sum_{k=1}^{\infty} \delta_{\{\Gamma_k^1\}}$ is a Poisson random measure on $\mathbb{R}$ with intensity measure λ*
2. *Conditionally on $\{\Gamma_k^1\}$, the random vector $(\{\Gamma_k^2\}, ..., \{\Gamma_k^d\})$ is independent from $\{\Gamma_j^i\}$ with $j \neq k$ and all i and is distributed on $\mathbb{R}^{d-1}$ with law $K(\Gamma_k^1, dx_2, ..., dx_d)$.*

Then $\{X_t\}_{0 \leq t \leq 1}$ defined by

$$X_t^i = \sum_{k=1}^{\infty} U_i^{-1}(\Gamma_k^i) 1_{[0,t]}(V_k), \; i = 1, ..., d \tag{13}$$

is a Lévy process on the time interval $[0, 1]$ with characteristic function

$$E[e^{iu.X_t}] = \exp\left(t \int_{\mathbb{R}^d} (e^{iu.x} - 1)\nu(dx)\right).$$

The proof is essentially the same as brought up in Tankov (2004) for the case of a generic Lévy copula. The only difference is that the copula is given implicitly without mention, yet it does exist by theorem 2.3 and 3.1 and so the proof applies. One still misses reference to some Lévy copula F in the representation formula, if anything conditional measure K is brought up. Therefore, an implicit design via canonization (15) should better representations in terms of simplicity. To this, we instance in turn how K may

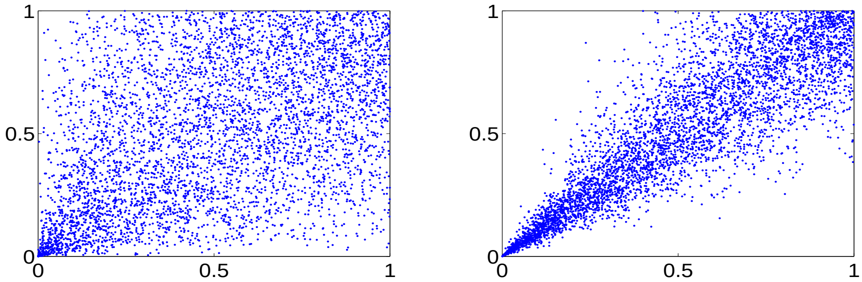

Figure 3: Scatter plot of ordinary Clayton copula samples with $\kappa = 1$ (right) and $\kappa = 5$ (left).

arise using either the explicit or the implicit design pattern. So assume a three-dimensional generic Lévy copula of Clayton type first

$$F(u_1, u_2, u_3) = \left(|u_1|^{-\theta} + |u_2|^{-\theta} + |u_3|^{-\theta}\right)^{-1/\theta} (\eta 1_{u_1 u_2 u_3 \geq 0} - (1-\eta) 1_{u_1 u_2 u_3 < 0}).$$

By (8) the probability distribution function of measure K conditioned on ξ then obtains as

$$\begin{aligned} F_\xi(x_2, x_3) &= \left(\xi^{-\theta} + x_2^{-\theta} + x_3^{-\theta}\right)^{-1/\theta - 1} \xi^{-\theta-1} \eta \\ &+ \left(\xi^{-\theta} + 0 + x_3^{-\theta}\right)^{-1/\theta-1} \xi^{-\theta-1} (1-\eta) \\ &+ \left(\xi^{-\theta} + x_2^{-\theta} + 0\right)^{-1/\theta-1} \xi^{-\theta-1} (1-\eta) \\ &+ \left(|\xi|^{-\theta} + 0 + 0\right)^{-1/\theta} \xi^{-\theta-1} \eta, \end{aligned}$$

where we assumed $x_2, x_3 > 0$. It appears that the distribution function is far from being generic itself. One can imagine that formulae become very tedious when increasing the dimension. In contrast, let now be F the two-dimensional Clayton Lévy copula defined by (3) and C the two-place ordinary Clayton copula

$$C(u, v) = (u^{-\theta} + v^{-\theta})^{1/\theta}, \tag{14}$$

and define the conditional measure K canonically via (8) with $F_i = F$ for $i = 2, 3$. We provide figure 3 for illustration of samples from ordinary Clayton copula (14). As to Tankov (2004), a straight forward computation yields

$$\begin{aligned} G_\xi(x_i) &:= sign(\xi)\frac{\partial}{\partial\xi}V_F((0\wedge\xi,0\vee\xi]\times(-\infty,x_i]) \\ &= \left\{(1-\eta)+\left(1+\left|\frac{\xi}{x_i}\right|^{\theta}\right)^{-1-1/\theta}(\eta-1_{x_i<0})\right\}1_{\xi\geq 0} \\ &+ \left\{\eta+\left(1+\left|\frac{\xi}{x_i}\right|^{\theta}\right)^{-1-1/\theta}(1_{x_i\geq 0}-\eta)\right\}1_{\xi<0}, \end{aligned}$$

producing the distribution function of conditional measure K as follows

$$K_\xi(x_2,x_3)=(G_\xi(x_2)^{-\theta}+G_\xi(x_3)^{-\theta})^{1/\theta}. \tag{15}$$

This design pattern produces true relief for the high dimensional case because an increasing number of variables is only visible in the ordinary copula. For example, in the d-dimensional case the exchangeable ordinary Clayton copula $C(u) = (\sum_{i=2}^d u_i^{-\theta})^{1/\theta}$ implies a canonical Lévy copula by its conditional distribution function

$$F_\xi(x_2,...,x_d)=(\sum_{i=2}^{d}G_\xi(x_i)^{-\theta})^{1/\theta}$$

3.3 Simulation from Lévy processes

In this paragraph we exhibit the use of canonical Lévy copulas for path simulation by series representation. We take univariate Lévy measures $\nu_i, i = 1,...,d$ to be given with invertible tail integrals U_i and we suppose that the dependence structure is specified implicitly by conditional measure K as in theorem 3.1, where we make the simplifying assumption $C_\xi = C$.

By means of theorem 3.2 simulation of a Lévy path is straight forward. As regards actual implementation Tankov (2004) makes plausible the use of Poisson arrivals $\{\Gamma_k^1\}$ and random series truncation

$$X_t^i=\sum_{k:\Gamma_k^1<\tau}U_i^{-1}(\Gamma_k^i)1_{[0,t]}(V_k),\ i=1,...,d.$$

with some $\tau>0$. We shall not dwell on implementation details but sampling from conditional probability measure $K(\Gamma_i^1,\cdot)$, as it were the crucial part.

With K assumed a composition of $(d-1)$-dimensional ordinary copula C and univariate distribution functions $G^i_{\Gamma^1_k}$ sampling from it unfolds as easy as follows:

Algorithm 3.1 (Simulating from conditional measure K) *Samples $(x_2, ..., x_d)$ from conditional measure $K(x_1, \cdot)$ given a canon realization x_1, an ordinary copula C and Lévy copulas F_i, $i = 2, ..., d$.*

- Generate sample $(u_2, ..., u_d)$ from ordinary copula C
- Set $x_i = (G^i_{x_1})^{-1}(u_i)$ for $i = 2, ..., d$

The conditional sample vector is given by $(x_2, ..., x_d)$.

As far as sampling from ordinary copula C is concerned, we refer to the literature (see Embrecht et al., 2002, for the standard sampling approach and Aas et al.,2006, Whelan,2004 for sophisticated algorithms). As regards inversion of conditional distribution function G^i_ξ associated to bivariate Lévy copula F_i, we note that analytic solutions are available in some cases. With F_i equal to Clayton Lévy copula (3) for example, the inverse is

$$\begin{aligned}
G_\xi^{-1}(u) &= A(\xi,u)|\xi|\left\{B(\xi,u)^{-\frac{\theta}{\theta+1}}-1\right\}^{-1/\theta}\\
\text{with } A(\xi,u) &= sign(u-1+\eta)1_{\xi\geq 0}+sign(u-\eta)1_{\xi<0}\\
\text{and } A(\xi,u) &= \left\{\frac{u-1+\eta}{\eta}1_{u\geq 1-\eta}+\frac{1-\eta-u}{1-\eta}1_{u<1-\eta}\right\}1_{\xi\geq 0}\\
&+ \left\{\frac{u-\eta}{1-\eta}1_{u\geq\eta}+\frac{\eta-u}{\eta}1_{u<\eta}\right\}1_{\xi<0}.
\end{aligned}$$

Lemma 3.1 has given a first analytic impression of the canonization principle, let us now treat the subject from a numerical point of view on the basis of canonical Lévy copula (15). To this end, we assume Poisson arrivals $\{\Gamma^1_k\}_{k=1}^{N(\tau)}$, where $N(\tau) = max\{k : \Gamma^1_k < \tau\}$, and we sample $\{(\Gamma^2_k, \Gamma^3_k)\}_{k=1}^{N(\tau)}$ by repeated application of algorithm 3.1. Figures 4-7 then show marginal behaviors with various parameter configurations for ordinary and Lévy copulas of Clayton type, where we applied inverse tail integrals U_i^{-1} to the Γ^i_k's. We take the impact of parametrization in the ordinary case to be known from the literature and figure 3.

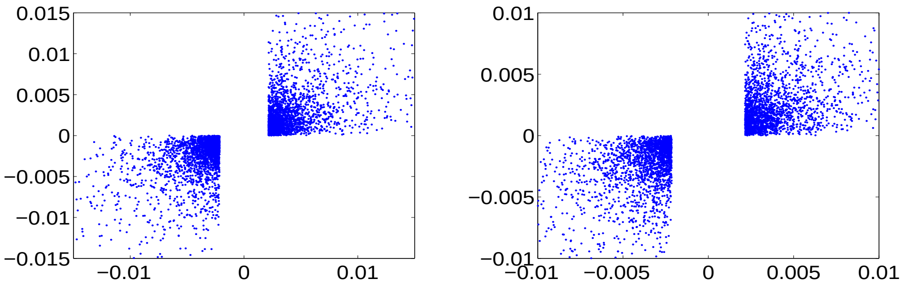

Figure 4: Scatter plot of (1,2)-margin from canonical Clayton Lévy copula with $\eta = 1$, $\theta = 1$ varying $\kappa = 1$ (left) and $\kappa = 5$ (right).

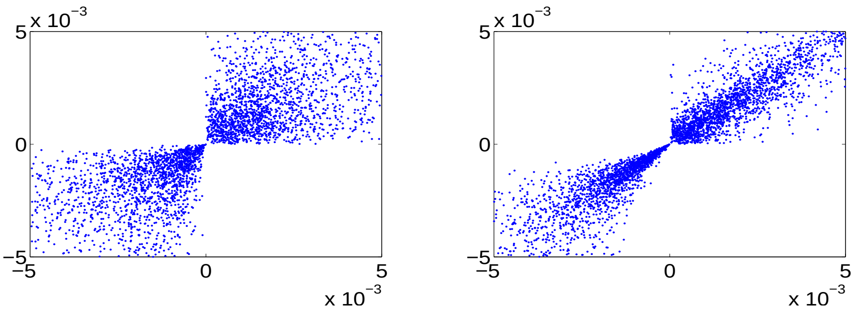

Figure 5: Scatter plot of (2,3)-margin from canonical Clayton Lévy copula with $\eta = 1$ and $\theta = 1$ varying $\kappa = 1$ (left) and $\kappa = 5$ (right).

From figure 4 we can see that varying dependence parameter κ of ordinary copula C has little if any effect on the (1,2)-margin. This is feasible because the bivariate structure of marginal dependence involving the canon variable is totally described by Lévy copula F_i. Different choices for κ do impact the (2,3)-margin, which can be seen from figure 5 and a narrowing of the samples when passing over to strong ordinary dependence. Moreover it seems that tail dependence of the ordinary Clayton copula forwards to the bivariate Lévy margin in a way that jumps are more associated in the third quadrant than in the first. This gives us to understand that the implicit model approach can produce global dependence patterns which are not necessarily symmetric.

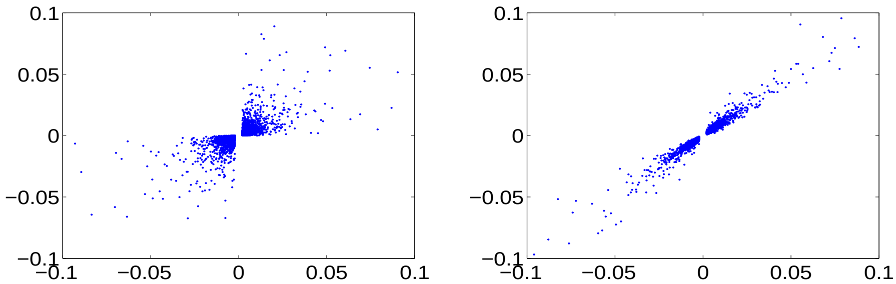

Figure 6: Scatter plot of (1,2)-margin from canonical Clayton Lévy copula with $\eta = 1$, $\kappa = 1$ varying $\theta = 1$ (left) and $\theta = 5$ (right).

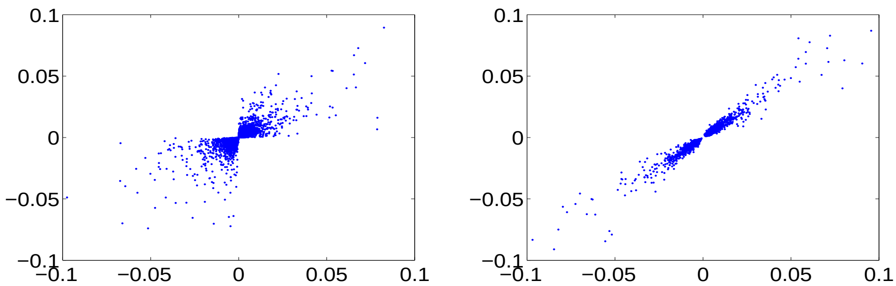

Figure 7: Scatter plot of (2,3)-margin from canonical Clayton Lévy copula with $\eta = 1$ and $\kappa = 1$ varying $\theta = 1$ (left) and $\theta = 5$ (right).

We analyze the role of canon dependence parameter θ by means of figures 6 and 7. It becomes obvious that the (1,2)-margin is certainly effected by an increase in dependence as was expected. The absolute jump sizes are more associated when using a high value of θ. Moreover, dependence in the (2,3)-margin seems to be modified in almost the same way. Although we did not alter the conditional ordinary copula of the 2nd and 3rd component explicitly, association is still more visible in these components when choosing $\theta = 5$. This is feasible due to the concept of canonization and parallels our former findings on the density of the Clayton Lévy copula. It is left as a side remark that we used perfect sign dependence $\eta = 1$ for comprehension, one can imagine other choices with the aid of figure 1. We also note that truncation of small jumps is visible in the plots.

Algorithm 3.1 now fits well into the general algorithm for path generation of a multidimensional Lévy process $(X_t)_{0 \le t \le 1}$ with specified dependence. We promote a slight modification of the original algorithm in order to particularize Lévy copulas of the present type (cf. Tankov, 2006).

Algorithm 3.2 (Simulation of multidimensional Lévy process with dependent components by series representation)
Generates trajectory $X_t, 0 \leq t \leq 1$ of multidimensional Lévy process by series representation. The dependence structure is given by conditional measure K and a number τ is fixed depending on the required precision and computational capacity.

- Initialize $k = 0, \Gamma_0^1 = 0$
- Repeat while $|\Gamma_k^1| < \tau$
  - Set $k = k + 1$
  - Simulate exponential(2) T_k and set $\Gamma_k^1 = -(|\Gamma_{k-1}^1| + T_k)$
  - Simulate $(\Gamma_k^2, ..., \Gamma_k^d)$ from distribution $K(\Gamma_k^1, \cdot)$ by algorithm 3.1
  - Simulate V_k uniform on $[0, 1]$

The trajectory is then given by $X_t^i = \sum_{k=1}^{i} U_i^{-1}(\Gamma_k^i) 1_{[0,t]}(V_k),\ i = 1, ..., d.$

While the previous graphical analysis set focus on distinct jump samples, we here test the parametric effects on multidimensional path generation by means of figur 8. To this end, we perform algorithm 3.2 on the basis of three dimensional Clayton Lévy copula (15) with $\eta = 1$, $\theta = 1$ and $\kappa = 1$ as starting configuration. We have graphed the tracejtories for each component $X^i, i = 1, 2, 3$ in the upper left plot. It shows that canonization is effective to some extent as the 2nd and 3rd component, although they are nearly conditional independent, move into a canonical direction. The upper right plot shows that when increasing conditional dependence to $\kappa = 5$, even small jumps become synchronous in these components while canonization is at the same level. In the lower left plot, we increased canon dependence to $\theta = 5$ while resetting conditional dependence $\kappa = 1$. It is visible that canonization is decisive for process evolution as the non-canon components come up to the canon. These observations parallel our former findings on the jump samples. Same as before, we employed perfect jump sign dependence $\eta = 1$ in these plots. The lower right plot now illustrates the impact of a moderate jump sign dependence $\eta = 0.5$ on path generation, where jump size

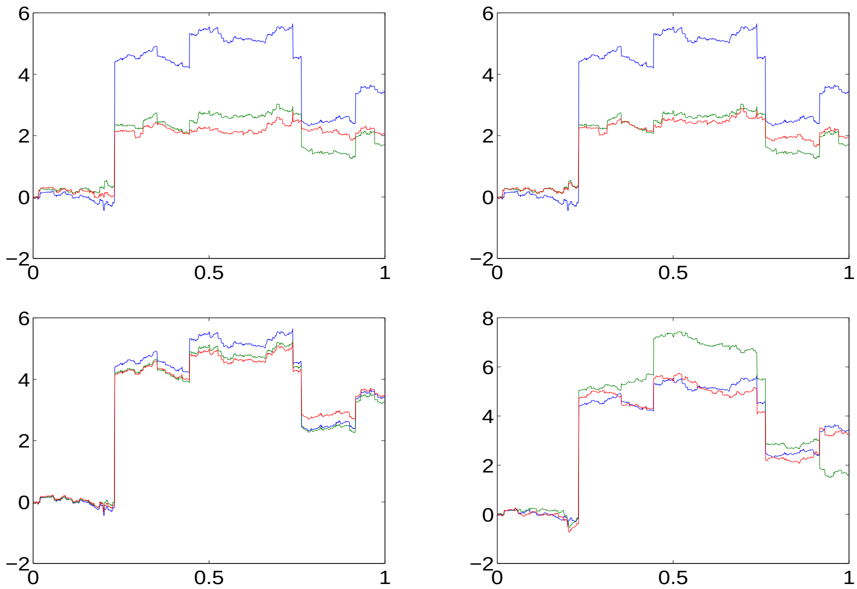

Figure 8: Trajectories of 3-dimensional Lévy paths with dependence specified by canonical Clayton Lévy copula using different parametrizations. Blue line: canon process; red line: 2nd component process; green line: 3rd component process

dependence is kept at a high $\kappa = 5, \theta = 5$. One can observe that opposing jump directions together with nearly equal jump sizes produce mirror images in some intervals. This parallels our former findings on the Lévy density of the bivariate Clayton copula.

Conclusion

This study established canonical Lévy copulas for ample dependence modelling in dynamical settings. It introduced the notion of a driving force component into the Lévy world by means of canonization. As with known Lévy copulas, structure modelling of multivariate dependence was achieved to remain decoupled from marginal aspects.

Simulation by series representation was found to be tremendously simplified by the inverse modelling approach via the canon. Both the ease of path generation and the decisive impact on multidimensional process evolution was recognized with the aid of graphical illustrations.

Future research includes application to financial derivative pricing and extension to other parsimonious copula ingredients. Performance in high dimension is of true interest because this is expected advantageous with canonical Lévy copulas.

References

AAS, K., H. BAKKEN, C. CZADO AND A. FRIGESSI (2006), Pair-copula constructions of multiple dependence, *Norsk Regnesentral.*

CONT, R. AND P. TANKOV (2004), *Financial modelling with jump processes*, Chapman & Hall/CRC.

EMBRECHT, P., A. MCNEIL AND D. STRAUMANN (2002), Correlation and dependence in risk management: Properties and Pitfalls, *Risk Management: Value at Risk and Beyond*, 176–223.

KALLSEN, J. AND P. TANKOV (2004), Characterization of dependence of multidimensional Lévy Processes using Lévy copulas, *Journal of Multivariate Analysis*, 97, 1551–1573.

NELSON, R. (2006), *An introduction to Copulas*, Springer.

TANKOV, P. (2004), *Lévy Processes in Finance: Inverse Problems and Dependence Modelling*, Ph.d., Ecole Polytechnique.

TANKOV, P. (2006), Simulation and option pricing in Lévy copula models. Preprint, avaiable from www.institut.math.jussieu.fr/∼tankov/.

WHELAN, N. (2004), Sampling from Archimedean Copulas, *Quantitative Finance*, 4(3), 339–352.