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Abstract

In the framework of the large homogenous portfolio model the prices of CDOs depend on the correlation between the single assets. In the present paper this correlation itself is modelled as a stochastic factor. The randomness follows a mean reverting process. Densities, prices and implied correlations are compared.

Keywords: Correlation, mean-reverting process, cdo, implied correlation, LHPM.

1 Model

We start with a short description of the large homogeneous portfolio model (LHPM). For more details see [3]. The LHPM can be used to model the default of a credit. It is based on N assets A_i which are constructed as

$$A_i = \rho X + \sqrt{1-\rho^2} Y_i. \tag{1}$$

$X, Y_1, ..., Y_N$ are independent standard normal random variables. The associated correlation matrix

$$\begin{pmatrix} 1 & \rho & \dots & \rho \\ \rho & 1 & \ddots & \rho \\ \vdots & \ddots & \ddots & \vdots \\ \rho & \dots & \rho & 1 \end{pmatrix}$$

is positive semi-definite if $\rho \in \left[-\frac{1}{N-1}, 1\right]$. Therefore we assume $\rho \in [0,1]$ throughout the following. By this construction the single assets are correlated with ρ. The model is time free, the focus is on one single point in time. The default is triggered when the asset's value falls below the barrier K. This barrier is identical for every asset. The default probability is

$$q = \Phi(K) \tag{2}$$

where Φ denotes the cumulative standard normal distribution.
We are interested in the loss fraction in the portfolio which is given as

$$X = \frac{1}{N} \sum_{i=1}^{N} 1_{\{A_i < K\}}. \tag{3}$$

This construction is the idea underlying the LHPM. In order to achieve further analytical tractability, in the LHPM an infinite number of assets is considered. Thus one takes the limit $N \longrightarrow \infty$ and therewith the loss fraction (3) becomes continuous. Then density and cumulative distribution function (cdf) are known in closed form. The density is given as

$$p(x) = \sqrt{\frac{1-\rho^2}{\rho^2}} \exp\left(\frac{1}{2}\left(\Phi^{-1}(x)\right)^2 - \frac{1}{2\rho^2}\left(\Phi^{-1}(p) - \sqrt{1-\rho^2}\Phi^{-1}(x)\right)^2\right). \tag{4}$$

The cdf is as follows

$$P(x) = \Phi\left(\frac{1}{\rho}\left(\sqrt{1-\rho^2}\Phi^{-1}(x) - \Phi^{-1}(p)\right)\right). \tag{5}$$

In the following the LHPM is equipped with stochastic correlation. Referring for more details to [1], we assume correlation to follow the stochastic process

$$d\rho_t = \kappa\left(\theta - \rho_t\right) dt + \sqrt{\rho_t\left(1-\rho_t\right)} dZ_t \tag{6}$$

with $\rho_0, \theta \in (0,1)$ and $\kappa > 0$. Hereby the drift term causes mean reverting behaviour. The coefficient function in the diffusion forces the process to stay in the interval $[0,1]$. Furthermore the boundaries are neither attainable nor attractive if

$$2\kappa \geq \max\left(\frac{1}{\theta}, \frac{1}{1-\theta}\right). \tag{7}$$

For definition of attainability and attractiveness see [2].
As the LHPM is time independent, we do not need to consider the whole process over time. Therefore we will only include the randomness from the transition density for $t \to \infty$ which becomes stationary and reads

$$f(x) = c \cdot \left(-2 + \frac{2}{x}\right)^{-\kappa\theta} \left(x\left(1-x\right)\right)^{-1+\kappa} \tag{8}$$

with $c \in \mathbb{R}^+$ such that $\int_0^1 f(x)dx = 1$. Thus in the following we will analyse the stochastic LHPM (SLHPM):

$$A_i = \rho X + \sqrt{1-\rho^2} Y_i. \tag{9}$$

where $\rho \sim P_f$ and P_f is the probability distribution implied by the stationary density f in (8). The probability that the loss fraction X is below a barrier $B \in [0,1]$ reads

$$P(X \leq B) = \int_0^1 \int_0^B p(x,\rho) f(\rho) dx d\rho.$$

2 Results

We want to calculate densities, CDO prices and implied correlations in SLHPM and compare them to LHPM. As there is no closed-form solution if the correlation is stochastic, ρ is simulated with the help of numerical algorithms.

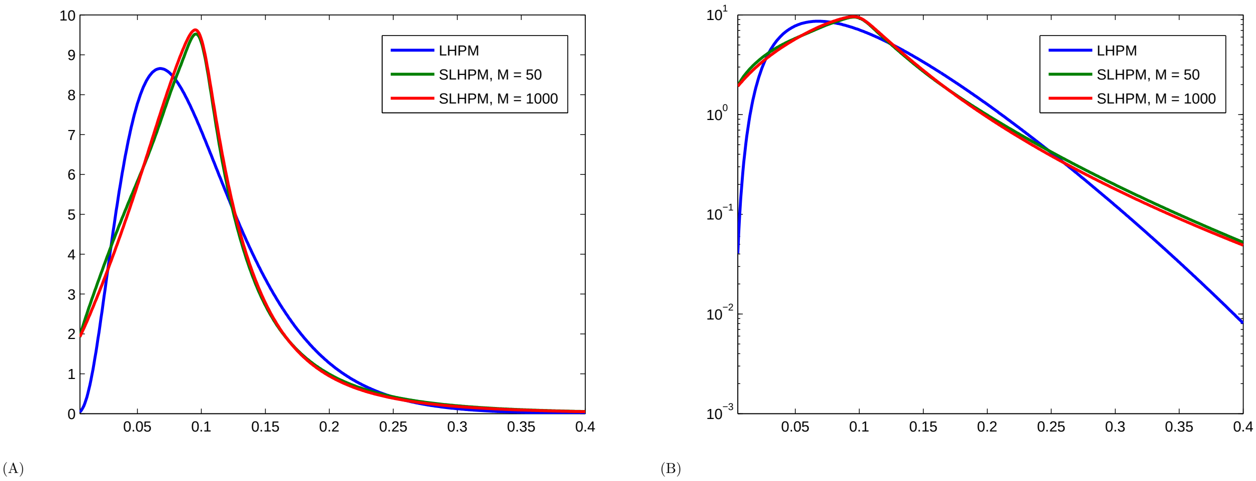


Figure 1: (A): Different number of simulations of (4) in SLHPM with $\theta = 0.3, \kappa = 5, q = 0.1$ compared to LHPM with $\rho = 0.3$. (B): Logarithmic coordinates on y-axes.

2.1 Density

We start by computing the densities. Hereby the SLHPM density is an estimated density. M random numbers for the correlation are drawn. For each of those the density is calculated. The mean over those M densities is considered as the SLHPM density. For making the two considered models comparable, in general the mean parameter θ in the SLHPM is identical with the constant correlation in the LHPM: $\rho = \theta$.

The first two pictures result from the parameter configuration $\theta = 0.3, \kappa = 5$ and $q = 0.1$. One observes that the maximum is shifted to the right in case of SLHPM. Further both tails are fatter for the stochastic model which becomes visible in Fig. 1 (B).

The dependence on κ is illustrated in the next graph. One observes that for the considered values of κ, the density becomes stable for high mean reversion factor, see Fig. 2.

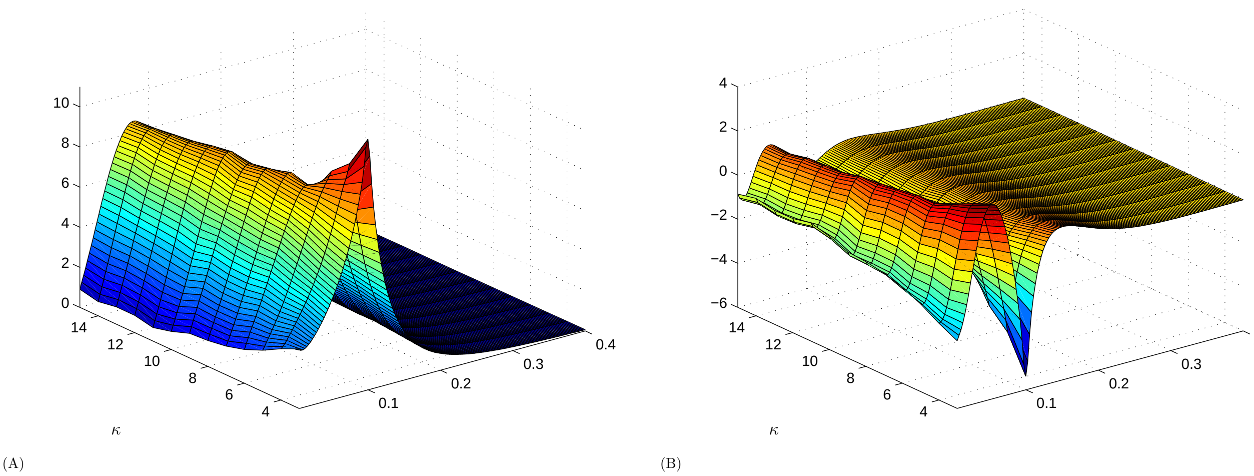
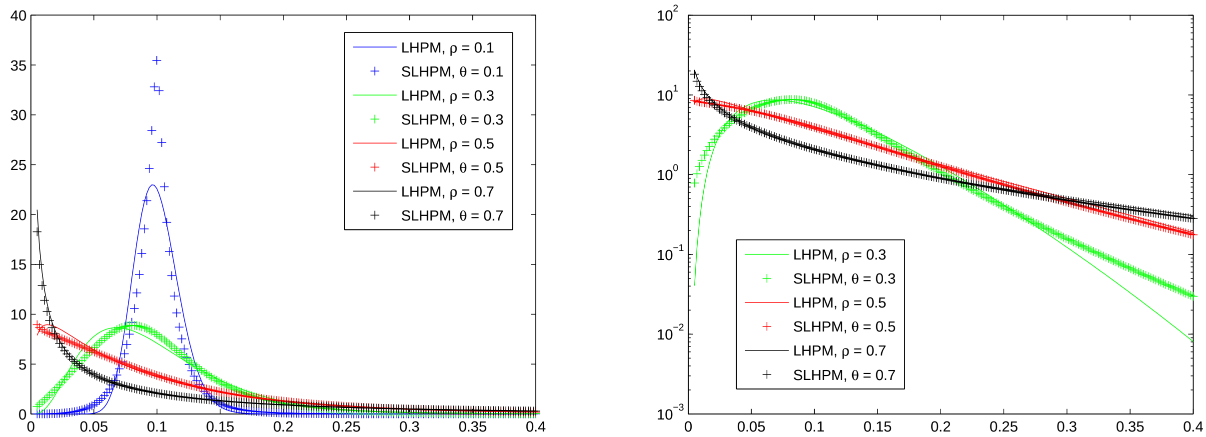


Figure 2: (A): Density (4) for fixed $\theta = 0.3, q = 0.1$ and varying $\kappa \in [3, 15]$ in SLHPM. $M = 500$. (B): Difference of densities to LHPM density with $\rho = 0.3$ measured as LHPM - SLHPM

In Fig. 2 (B) the difference between the densities for increasing κ in SLHPM and the LHPM

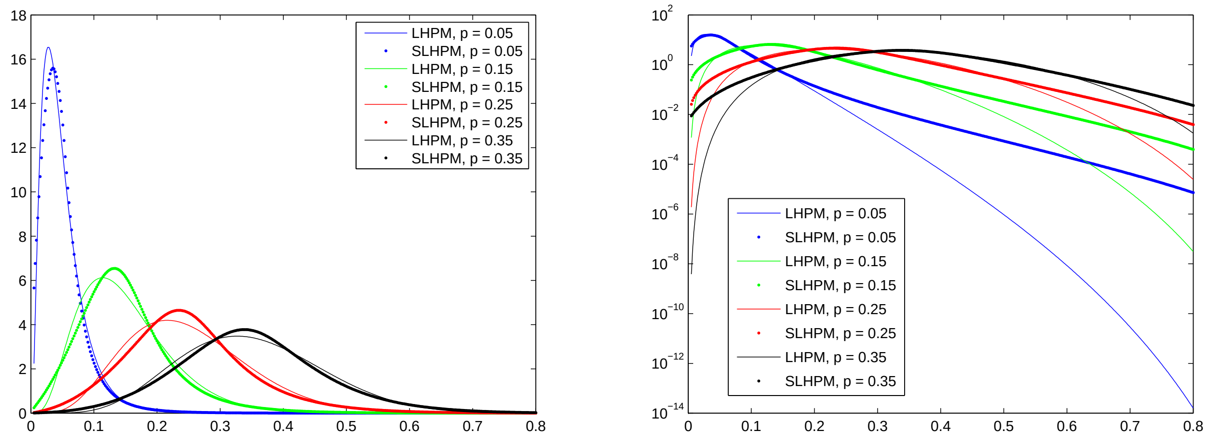
density is visualised. The difference does not vanish for high mean reversion. Neither does it vanish for large argument values because of the fat tails. The difference at $x = 0.4$ is between -0.055 and -0.015.
The next graph shows the LHPM densities and their stochastic counterparts for different mean θ and ρ respectively.



(A) (B)

Figure 3: (A): Density (4) for fixed $\kappa = 12, q = 0.1$ and varying θ in SLHPM and varying ρ in LHPM, respectively. $M = 500$. (B): Logarithmic scale.

One observes that the difference between LHPM and SLHPM induced densities are relatively large for low correlation. But the higher correlation ρ in LHPM and θ in SLHPM, the smaller this difference seems to become.
The comparison for varying default probabilities q is presented in the next graph. The stochastic densities still feature fat tails, see Fig 4:



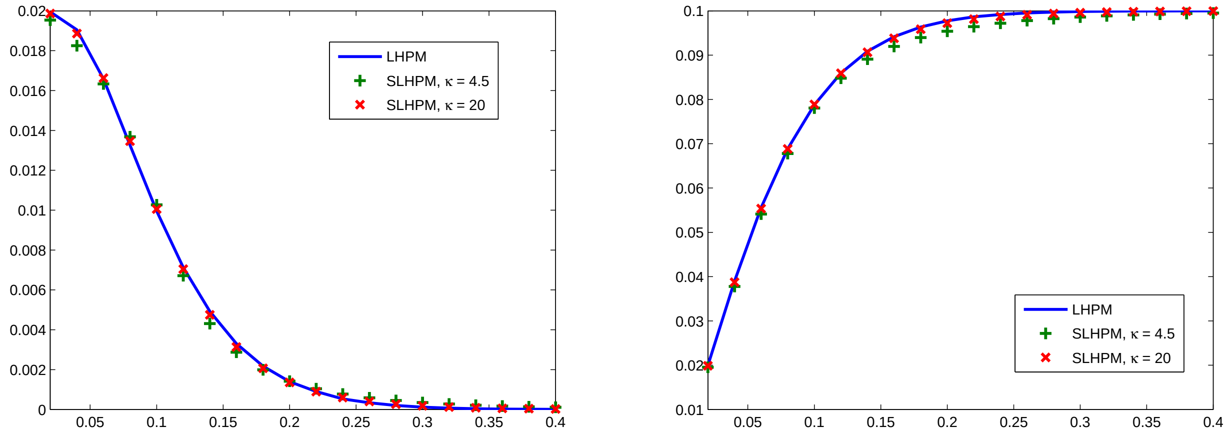
(A) (B)

Figure 4: (A): Density (4) for constant $\rho = 0.3$ and for $\theta = 0.3, \kappa = 12$. Default probability q varies. $M = 500$. (B): Logarithmic scale

Furthermore the maximum is always shifted towards higher loss fraction in case of the SLHPM.

2.2 Price

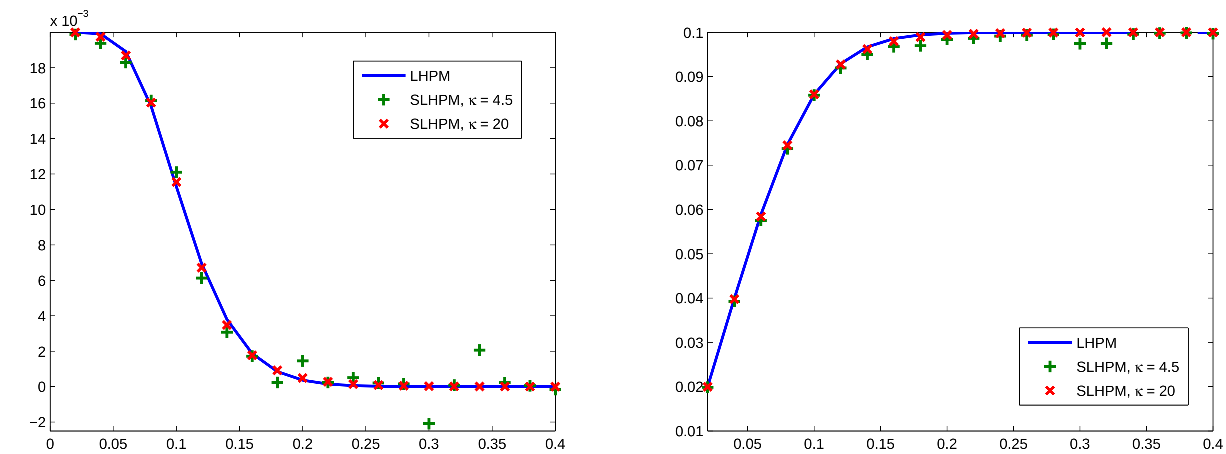
As a next step the influence of randomness on CDO prices is analysed. We consider an artificial CDO with detachment points $u_1 = 0.02, u_2 = 0.04$ and so on.
Fig. 5 shows the prices of tranches of a CDO with notional one. Thus we actually look at the loss fraction itself. The tranches $[u_{i+1}, u_i]$ and the pure equity tranches $[0, u_i]$ are considered. The interest rate is set to zero.



(A) (B)

Figure 5: (A): Tranche prices for LHPM with $\rho = 0.3$ and for SLHPM with $\theta = 0.3, \kappa = 4.5, 20$. $q = 0.1$. $M = 1000$. (B): Equity tranches.

One observes that the prices from LHPM and SLHPM are very close to each other. The higher the mean reversion, the closer the prices are.
The next graph shows the same comparison with θ and ρ being lowered (Fig. 6).



(A) (B)

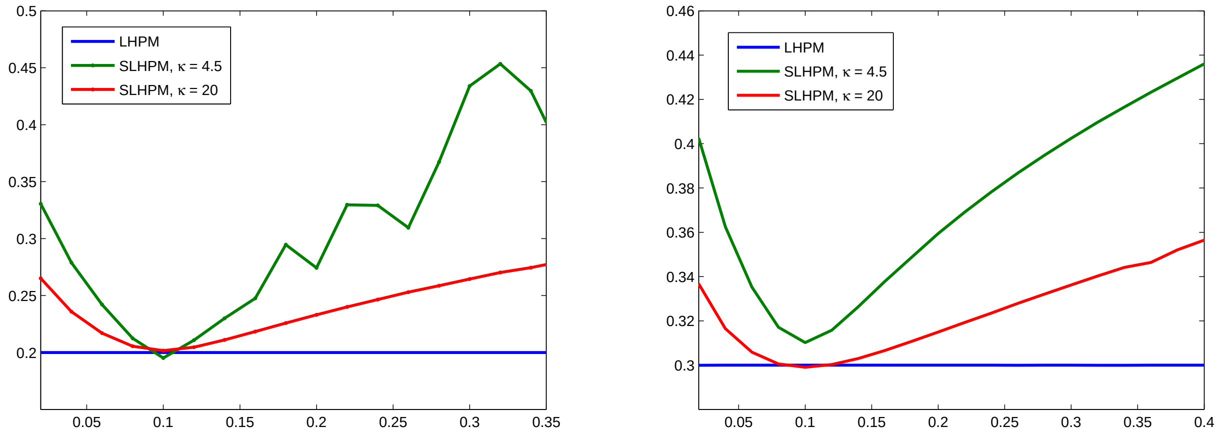
Figure 6: (A): Tranche prices for LHPM with $\rho = 0.2$ and for SLHPM with $\theta = 0.2, \kappa = 4.5, 20$. $q = 0.1$. $M = 1000$. (B): Equity tranches.

Again one notices that generally the prices of LHPM and SLHPM are rather close.

2.3 Implied Correlation

For getting more insight we calculate the implied correlations. Therefore we first simulate the prices with the stationary density (8). Afterwards we compute the correlation which has to be inserted into the LHPM to get this price. This input correlation is considered as implied correlation throughout this section. The calculation bases on the equity tranche prices.

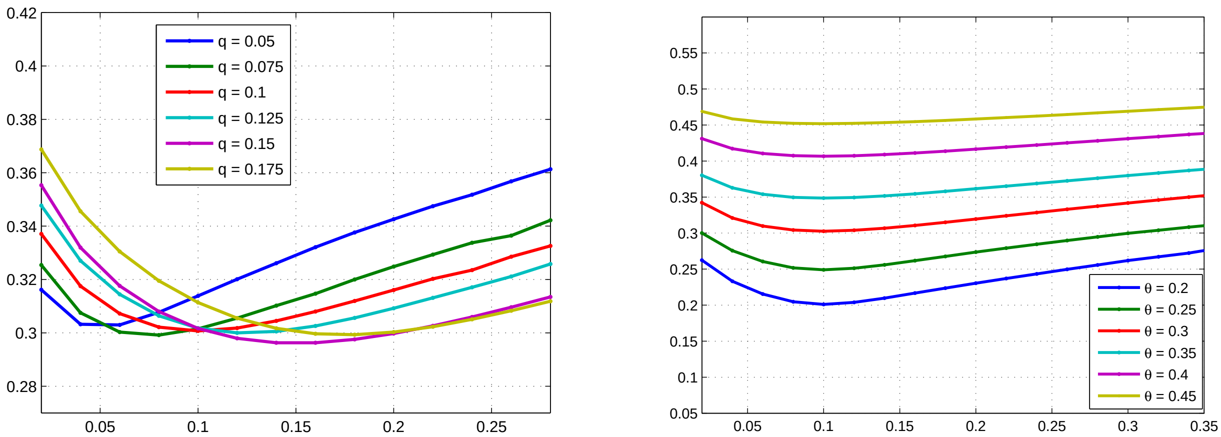
First Fig.7 depicts the implied correlation corresponding to figure 5 and 6.



(A) (B)

Figure 7: (A): Implied correlation for LHPM with $\rho = 0.2$ and for SLHPM with $\theta = 0.2, \kappa = 4.5, 20$. $q = 0.1$. $M = 1000$. (B): Implied correlation for LHPM with $\rho = 0.3$ and for SLHPM with $\theta = 0.3, \kappa = 4.5, 20$. $q = 0.1$. $M = 1000$.

Although there is no obvious difference in the tranche price, the implied correlations do differ. There is a clear smile effect in the SLHPM. The lowest implied correlation is measured hereby for tranche with detachment point 0.1 which also is the default probability.



(A) (B)

Figure 8: (A) Implied correlation in SLHPM for varying default probability q with $\theta = 0.3, \kappa = 20$. $M = 500$. (B) Implied correlation in SLHPM for varying mean θ with $\kappa = 20, q = 0.1$. $M = 500$.

Fig. 8 (A) shows the implied correlation smile for different default probabilities $q = 0.05, 0.075, \ldots, 0.175$. The minimum of implied correlation - the smile - moves along with default probability.

Further one can observes that lower default probabilities lead to steeper smiles.
Fig. 8 (B) visualises the changes in the smile according to the level of input correlation changes. Although the smile is clearly visible for all levels of correlation, it flattens with increasing θ. This goes in line with the result that LHPM and SLHPM become closer for increasing correlation, see also fig. 3.

3 Conclusion

Including stochastic correlation in the LHPM leads to a structurally different model. Particularly with regards to the tails of the corresponding distribution, the probabililty for rare events is decisively larger in the SLHPM. Mean reverting stochastic correlation causes fat tails. The difference between the computed fair prices is small if a relatively simple payoff function is considered as the CDO above. But calculating the implied correlation makes the difference obvious. This structural difference will most probably become important if more complex products are valued.

References

[1] C. van Emmerich, Modelling correlation as a stochastic process, *working paper* (2006).

[2] S. Karlin and H. M. Taylor, *A second course in stochastic processes* (Academic Press, 1981).

[3] P. Schönbucher, *Credit derivatives pricing models* (John Wiley & Sons, 2003).