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Abstract

In financial mathematics, the fair price of options can be achieved by solutions of parabolic differential equations. The volatility usually enters the model as a constant parameter. However, since this constant has to be estimated with respect to the underlying market, it makes sense to replace the volatility by an according random variable. Consequently, a differential equation with stochastic input arises, whose solution determines the fair price in the refined model. Corresponding expected values and variances can be computed approximately via a Monte Carlo method. Alternatively, the generalised polynomial chaos yields an efficient approach for calculating the required data. Based on a parabolic equation modelling the fair price of Asian options, the technique is developed and corresponding numerical simulations are presented.

1 Introduction

Mathematics have been becoming an important tool in investment banking for the last decades. The major breakthrough can be connected to the work of Fisher Black and Myron Scholes [2] and independently Robert C. Merton [8] in 1973. They presented a framework for modelling share prices which allowed for a closed form solution of the fair pricing problem for European plain vanilla options. Hereby, a price is fair if it excludes arbitrage, i.e., it averts instantaneous risk-less profit. Although there have been various developments since that time, the so called Black-Scholes model is still a cornerstone in quantitative finance.

The Black-Scholes equation represents a parabolic partial differential equation. A corresponding end-boundary value problem yields the fair price of European options in the mathematical model. Likewise, parabolic operators arise for modelling the fair price of American options, Asian options and others. The volatility specifies the amount of changes in the underlying time-dependent share prices. In general, the volatility represents a constant parameter in the parabolic equation.

The exact volatility is not known and thus has to be estimated for achieving an approximation of the fair option price. Alternatively, we consider the volatility as a random variable, which exhibits an adequate distribution like uniform type, Gaussian type or others. Hence the deterministic parabolic equation changes into a parabolic equation with stochastic input, where the solution represents a random field. Again end-boundary value problems have to be solved. The expected value of the random field yields the fair price in the refined model.

On the one hand, the expected value as well as corresponding variances can be obtained approximately by a Monte Carlo simulation. However, a huge number of realisations is often necessary to achieve a sufficiently accurate result. On the other hand, the generalised polynomial chaos allows for an expansion of the random field using orthogonal polynomials. Wiener [10] introduced the homogeneous polynomial chaos for stochastic inputs with Gaussian distributions. This result was extended by Cameron and Martin [3] for arbitrary random fields of second order. Moreover, the generalised polynomial chaos yields an expansion in case of stochastic inputs with other distributions, see [1, 12]. Applying the generalised polynomial chaos, a system of parabolic equations has to be solved only once to obtain the desired numerical approximations.

Since the Black-Scholes equation can be solved explicitly, we consider a more complex model of an arithmetic-average-strike-call, which represents an Asian option, see [4]. Corresponding end-boundary value problems of a parabolic equation have to be solved numerically in a Monte Carlo simulation. We apply a method of lines and thus we obtain a stiff system of ordinary differential equations, cf. [5, 6]. Likewise, a method of lines also yields a numerical solution of the system resulting from the generalised polynomial chaos.

Well-posedness of the arising differential equations with random input or convergence analysis of corresponding numerical methods are not within the scope of this paper. According to the above descriptions, the paper is organised as follows. In Sect. 2, we introduce the idea to consider the volatility as a stochastic parameter. Consequently, a differential equation with random input is arranged. In Sect. 3, the Monte Carlo method, which allows for solving the problem numerically, is sketched. We demonstrate the application of the generalised polynomial chaos to the problem in detail. Finally, numerical results are presented in Sect. 4, where the two techniques are compared with respect to efficiency aspects.

2 Problem Definition

Black, Scholes and Merton [2, 8] proposed a geometric Brownian motion for modelling the dynamics of stock prices $S : [0, \infty) \times \Omega \to \mathbb{R}$ through time with respect to some probability space $(\Omega, \mathcal{A}, P)$. A stochastic differential equation arises in the sense of Itô, see for example Øksendal [9],

$$\mathrm{d}S_t = rS_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t, \tag{1}$$

where the initial value $S_0 = s_0$, the interest rate $r > 0$ and the volatility $\sigma > 0$ are predefined constants and $W : [0, \infty) \times \Omega \to \mathbb{R}$ is a Brownian motion. Note that S_t remains positive.

Based on this model, it is possible to determine the fair value of options $V(S, t)$ on the underlying S. By the virtue of Feynman-Kac, see [9] Thm. 8.2.1, the option price V can be expressed as the solution of a partial differential equation (PDE), namely the Black-Scholes equation

$$\frac{\partial V}{\partial t}(S,t) + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S,t) + rS\frac{\partial V}{\partial S}(S,t) - rV(S,t) = 0. \tag{2}$$

Certain assumptions are necessary to derive this result:

- There is no arbitrage.
- The interest rate r is fixed, positive and identical for borrowing and lending.
- The market is liquid, continuous and frictionless.
- There are no dividend payments on S.

For a discussion of these assumptions, see for example [11]. Nevertheless, the model can be generalised to other premises like the occurrence of dividend payments, which yields nonlinear PDEs.

In case of a European call, V depends on S, maturity $T > 0$ and strike $K > 0$. The European call guarantees a payment of

$$(S_T - K)^+ := \max(S_T - K, 0), \tag{3}$$

which yields an end condition at time T. The problem has to be solved backwards in time to obtain the essential information at time $t = 0$. The pay-off function (3) in combination with (1) determines the boundary conditions, which eventually allow for a closed form solution. As this is not topic of the paper at hand, we refer the interested reader to [4].

Although the Black-Scholes model is appealing from a theoretical point of view, there is one major drawback. The model does not lead to option prices matching market prices. The phenomenon of implied volatility skew makes that particularly apparent.

To improve the matching to market data, various refinements of the Black-Scholes model have been introduced, for example adding jumps or making σ and/or r stochastic processes themselves. Hereby, it is crucial that the modified model still allows for fast and accurate computation of options prices. In the following, we perform a refinement, which can easily be combined with others to improve the matching to market.

Looking at the Black-Scholes equation (2), one observes that it depends on the parameters r and σ. The initial value s_0 determines the interesting point in the corresponding solution. The market data enables to set r and s_0 easily. In contrast, σ is not directly observable but has to be estimated from market prices. That makes a model assumption necessary, which tells us how σ and market prices are linked. It is in the nature of assumptions that there is a particular degree of uncertainty. To reflect this uncertainty, we suggest to change σ into a random variable.

In contrast to (2), we consider a mathematical model, which does not exhibit solutions in closed form. Thus we choose an option type as example, which is more complex than the European option mentioned above: an Asian option. More specifically, we deal with an arithmetic-average-strike-call involving the pay-off function

$$(S_T - I_T/T)^+ \qquad \text{with} \quad I_t := \int_0^t S_\tau \, \mathrm{d}\tau. \tag{4}$$

Thus the value of the option depends on the stock price at the final time as well as on the average of the stock prices in time.

Again it is possible to identify the fair value of this option with a parabolic PDE. Under the usual assumptions in the Black-Scholes framework and the additional condition that I and S are stochastically independent, one can derive the following parabolic equation for the option value $V(S, I, t)$

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial I} - rV = 0, \tag{5}$$

where $S, I > 0$ and $0 < t < T$. The end condition with respect to time reads

$$V(S, I, T) = (S - I/T)^+ \quad \text{for} \;\; S, I \geq 0. \tag{6}$$

In addition, appropriate boundary conditions have to be specified. See again [4] for a thorough derivation. The PDE (5) is not analytically solvable and thus numerical methods are required. Furthermore, we transform (5) into a more tractable form from a numerical point of view. Setting $u : \mathbb{R}^+ \times [0,T] \to \mathbb{R}$ such that

$$V(S,I,t) = Su(I/S,t) \tag{7}$$

holds, a straightforward calculation shows that

$$\frac{\partial u}{\partial t} + (1-rx)\frac{\partial u}{\partial x} + \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{8}$$

with $x = I/S > 0$ represents an equivalent equation to (5). Accordingly, the type of this equation is still parabolic. In the limit case $x = 0$, the PDE (8) degenerates into a hyperbolic equation. A behaviour of hyperbolic type arises for relatively small volatilities, too.

The pay-off function (6) implies the corresponding end condition

$$u(x,T) = (1 - x/T)^+ \quad \text{for } \; x > 0. \tag{9}$$

Considering the meaning of the involved variables, the boundary conditions result in

$$\frac{\partial u}{\partial t}(0,t) + \frac{\partial u}{\partial x}(0,t) = 0 \quad \text{for } \; t \in [0,T] \tag{10}$$

and

$$\lim_{x\to\infty} u(x,t) = 0 \quad \text{for } \; t \in [0,T]. \tag{11}$$

The boundary condition (11) has to be approximated on a finite domain using

$$u(L,t) = 0 \quad \text{for } \; t \in [0,T] \tag{12}$$

with some constant $L > 0$. Often the choice $L := 1$ already yields sufficiently accurate approximations of the original formulation. An according asymptotical analysis can verify the quality of this approximation.

Thus we obtain an end-boundary value problem of the parabolic PDE (8). Since the PDE is linear, a corresponding solution $u \in C^2([0,L]\times[0,T])$ exists. However, the dependence of the solution u on the parameter σ as well as σ^2 is nonlinear.

Now we assume that the volatility represents a random variable $\sigma : \Omega \to \mathbb{R}$ corresponding to some probability space $(\Omega, \mathcal{A}, P)$. More precisely, we demand

$$\sigma(\omega) = g(\xi(\omega)) \quad \text{with } \; g : \mathbb{R} \to \mathbb{R}, \tag{13}$$

where the random variable $\xi : \Omega \to \mathbb{R}$ exhibits a standard distribution of Gaussian type, beta-type, etc. For example, a reasonable choice is

$$\sigma(\omega) = a + b\xi(\omega) \tag{14}$$

with ξ uniformly distributed in $[-1, 1]$ and constants $a > b > 0$. Hence it holds $\mathbb{E}(\sigma) = a$ and $\mathrm{Var}(\sigma) = b^2/3$.

Consequently, the solution of the PDEs (5) and (8) become random fields

$$V : \mathbb{R}^+ \times \mathbb{R}^+ \times [0, T] \times \Omega \to \mathbb{R}$$

and for the infinite or bounded case, respectively,

$$u : \mathbb{R}^+ \times [0, T] \times \Omega \to \mathbb{R} \quad \text{or} \quad u : [0, L] \times [0, T] \times \Omega \to \mathbb{R}.$$

Each realisation of σ implies a corresponding fair price of the option. Thus we obtain a refined fair price by the expected value of the random fields at time $t = 0$. Furthermore, the variance of the random field yields essential information about the stochastic variables. Due to the structure of the transformation (7), it holds

$$\begin{aligned} \mathbb{E}(V(S, I, t, \cdot)) &= S\, \mathbb{E}(u(I/S, t, \cdot)) \\ \mathrm{Var}(V(S, I, t, \cdot)) &= S^2\, \mathrm{Var}(u(I/S, t, \cdot)). \end{aligned} \tag{15}$$

Hence we can apply the simplified equation (8) to determine the crucial data of the problem. Now efficient methods are necessary to answer the problem with stochastic input.

3 Numerical Techniques

We consider the end-boundary value problem (8),(9),(10),(12). Let the involved volatility σ be a random variable of the form (13) depending on a variable ξ with some standard distribution. The solution u becomes a random field due to the stochastic parameter. Consequently, we are interested in the expected value and the variance of this random field.

A well-known approach to obtain an approximation of corresponding expected values and variances consists in a Monte Carlo method. In our case, a finite number of realisations $\sigma_1, \ldots, \sigma_n$ for the volatility are produced according to the underlying distribution. Each realisation σ_k yields a corresponding deterministic solution u_k of the end-boundary value problem. Given the set of solutions $u_1, \ldots, u_n$, the expected value is approximated by the mean value

$$\mathbb{E}(u(x, t, \cdot)) \doteq \frac{1}{n} \sum_{k=1}^{n} u_k(x, t) \tag{16}$$

and the variance is substituted by the sample variance

$$\mathrm{Var}(u(x, t, \cdot)) \doteq \frac{1}{n-1} \sum_{l=1}^{n} \left(u_l(x, t) - \frac{1}{n} \sum_{k=1}^{n} u_k(x, t) \right)^2 . \tag{17}$$

Following [4], each deterministic end-boundary value problem can be solved numerically via a method of lines, for example. Consequently, the solution is discretised with respect to the lines $x_j = jh$ using $h = L/N$ for $j = 0, 1, \ldots, N$, which yields the functions $u_j(t) = u(x_j, t)$. The corresponding partial derivatives in (8) are replaced by difference formulae

$$\frac{\partial u}{\partial x}(x_j, t) = \frac{1}{2h}\left[u_{j+1}(t) - u_{j-1}(t)\right] + \mathcal{O}(h^2) \tag{18}$$

and

$$\frac{\partial^2 u}{\partial x^2}(x_j, t) = \frac{1}{h^2}\left[u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)\right] + \mathcal{O}(h^2) \tag{19}$$

for $j = 1, \ldots, N-1$. The derivative appearing in boundary condition (10) is substituted via a backward difference formula

$$\frac{\partial u}{\partial x}(x_0, t) = \frac{1}{2h}\left[-3u_0(t) + 4u_1(t) - u_2(t)\right] + \mathcal{O}(h^2). \tag{20}$$

Thus the discretisation schemes are consistent of order 2. The boundary condition (12) implies directly $u_N \equiv 0$. Furthermore, condition (9) yields final values

$$u_j(T) = (1 - x_j/T)^+ \quad \text{for all } \ j. \tag{21}$$

Using the abbreviation $y := (u_0, u_1, \ldots, u_{N-1})$, we obtain an end value problem of ordinary differential equations (ODEs)

$$\dot{y}(t) = Ay(t), \quad t \in [0, T]. \tag{22}$$

The system is linear, since a matrix $A \in \mathbb{R}^{N \times N}$ arises. However, the matrix depends nonlinearly on σ. We consider the decomposition

$$A = A_1 + A_2 + A_3,$$

where A_1, A_2 correspond to the discretisations (18) and (19), respectively, and A_3 follows from the minor part (20). Furthermore, the matrices A_1, A_2 exhibit a band structure, see [4]. Hence an LU-decomposition of the matrix A causes a computational work of $\mathcal{O}(N^2)$. We can apply ODE solvers for stiff systems to achieve a numerical solution of the discretised problem as described in [6]. Thus the method of lines represents the crucial step in the Monte Carlo simulation.

Alternatively, a polynomial expansion of random fields yields another approach to obtain approximations of expected values and variances. Wiener [10] introduced the homogenous polynomial chaos based on expansions using Hermite polynomials in case of Gaussian random variables. Cameron and Martin [3] proved that the corresponding expansion is convergent for an arbitrary functional, which is quadratically integrable with respect to the underlying probability space. In case of other distributions, expansions using respective orthogonal polynomials yield according convergence results, which implies the theory of generalised polynomial chaos, see [1, 12].

To apply the generalised polynomial chaos, we assume that the random process u corresponding to (8) exhibits finite second moments, i.e.,

$$\int_{\omega\in\Omega} (u(x,t,\omega))^2 \; \mathrm{d}P(\omega) < \infty$$

for each $x \in [0, L]$ and $t \in [0, T]$. It follows that the random process exhibits an expansion of the form

$$u(x,t,\omega) = \sum_{i=0}^{\infty} v_i(x,t)\Phi_i(\xi(\omega)) \tag{23}$$

with (a priori unknown) coefficient functions $v_i : [0, L] \times [0, T] \to \mathbb{R}$ and orthogonal polynomials $\Phi_i : \mathbb{R} \to \mathbb{R}$. Let the degree of Φ_i be equal to i. If the random variable ξ exhibits a uniform distribution, for example, then the Legendre polynomials represent the optimal choice with respect to the speed of convergence in general. Nevertheless, other classical distributions imply corresponding basis polynomials, see [12].

The series (23) converges pointwise. The coefficient functions satisfy

$$v_i(x,t) = \int_{\omega\in\Omega} u(x,t,\omega)\Phi_i(\xi(\omega)) \; \mathrm{d}P(\omega) \quad \text{for } \; i = 0, 1, 2, \ldots \tag{24}$$

given (without loss of generality) orthonormal polynomials. The integrals (24) depend on the parameters x and t. Thus the smoothness of the coefficient functions v_i follows from the smoothness of the solutions of (8) for each $\sigma(\omega)$ in (13) applying elementary theorems for parameter-dependent integrals.

To obtain a numerical approximation, the series (23) has to be truncated. Thus we perform a stochastic discretisation using the finite sum

$$\tilde{u}(x,t,\omega) = \sum_{i=0}^{M} v_i(x,t)\Phi_i(\xi(\omega)). \tag{25}$$

To determine the involved coefficient functions, we apply a Galerkin approach. Inserting the approximation (25) in the linear PDE (8) yields the residual

$$\begin{aligned} r(x,t,\omega) \;\; = \;\; & \sum_{i=0}^{M} \frac{\partial v_i}{\partial t}(x,t)\Phi_i(\xi(\omega)) + (1 - rx)\sum_{i=0}^{M} \frac{\partial v_i}{\partial x}(x,t)\Phi_i(\xi(\omega)) \\ & + \frac{x^2}{2}\sum_{i=0}^{M} \frac{\partial^2 v_i}{\partial x^2}(x,t) g(\xi(\omega))^2 \Phi_i(\xi(\omega)), \end{aligned}$$

which represents a random field. We demand that the residual is orthogonal with respect to the space spanned by the first $M+1$ polynomials. Let $\langle f(\cdot)\rangle$ be the expected value of a function $f:\Omega\to\mathbb{R}$ with respect to the probability space $(\Omega,\mathcal{A},P)$. Consequently, the orthogonality relation reads

$$\langle r(x,t,\cdot)\Phi_l(\xi(\cdot))\rangle = 0 \quad \text{for} \quad l=0,1,\dots,M.$$

Thus we obtain a linear system of parabolic equations, namely

$$\frac{\partial v_l}{\partial t} + (1-rx)\frac{\partial v_l}{\partial x} + \frac{x^2}{2}\sum_{i=0}^{M}\frac{\langle g^2\Phi_i\Phi_l\rangle}{\langle\Phi_l^2\rangle}\cdot\frac{\partial^2 v_i}{\partial x^2} = 0 \tag{26}$$

for $l=0,1,\dots,M$. The coefficient functions satisfying this system are not the same as in (23) but an approximation, which is optimal in some sense. The corresponding matrix, which couples the equations, is given by

$$B\in\mathbb{R}^{(M+1)\times(M+1)}, \quad B_{li} = \frac{\langle g^2\Phi_i\Phi_l\rangle}{\langle\Phi_l^2\rangle}. \tag{27}$$

Hence the matrix is constant and can be computed numerically for a given distribution of (13). Thereby, we have to calculate the matrix only once. Furthermore, if g is a polynomial and ξ exhibits a classical distribution like uniform distribution, Gaussian distribution, etc., then an analytical computation of this matrix is feasible. The coupling of the parabolic equations arises just in the third term of (26), since this part contains the random volatility.

Repeating the Galerkin approach for the first boundary condition (10) results in the corresponding system

$$\frac{\partial v_l}{\partial t}(0,t) + \frac{\partial v_l}{\partial x}(0,t) = 0 \quad \text{for} \quad t\in[0,T] \quad \text{and} \quad l=0,1,\dots,M, \tag{28}$$

which consists of decoupled equations. Similarly, the second boundary condition (12) yields

$$v_l(L,t) = 0 \quad \text{for} \quad t\in[0,T] \quad \text{and} \quad l=0,1,\dots,M. \tag{29}$$

The corresponding final values from (9) are deterministic and thus we obtain

$$v_0(x,T) = (1-x/T)^+ \quad \text{and} \quad v_l(x,T) = 0 \quad \text{for} \quad l=1,2,\dots,M. \tag{30}$$

Hence the generalised polynomial chaos yields an end-boundary value problem of a linear system with parabolic equations. A solution of the system generates an approximation to the expected value by

$$\mathbb{E}(u(x,t,\cdot)) \doteq \mathbb{E}(\tilde{u}(x,t,\cdot)) = v_0(x,t) \tag{31}$$

and to the variance via

$$\mathrm{Var}(u(x,t,\cdot)) \doteq \mathrm{Var}(\tilde{u}(x,t,\cdot)) = \sum_{l=1}^{M} v_l(x,t)^2 \langle \Phi_l^2 \rangle, \tag{32}$$

where $\Phi_0 \equiv 1$ is assumed without loss of generality.

We achieve a corresponding numerical solution of the end-boundary value problem (26),(28),(29),(30) by a method of lines following the previous deterministic case. We discretise

$$z_{i,j}(t) = v_i(x_j,t) \quad \text{for} \;\; j = 0,1,\ldots,N-1 \quad \text{and} \quad i = 0,1,\ldots,M$$

and arrange for $z : [0,T] \to \mathbb{R}^{N(M+1)}$ the ordering

$$z := \left(z_{0,0}, z_{1,0}, \ldots, z_{M,0}, z_{0,1}, \ldots, z_{M,N-2}, z_{0,N-1}, z_{1,N-1}, \ldots, z_{M,N-1}\right)^{\top}.$$

Thus the method of lines leads to a linear system of ODEs

$$\dot{z} = (A_1 \otimes I + A_2 \otimes B + A_3 \otimes I)z \tag{33}$$

using Kronecker products and the unit matrix $I \in \mathbb{R}^{(M+1)\times(M+1)}$. The required final values follow directly from (30). Hence the constant matrix in (33) exhibits a block structure, where submatrices correspond to the stochastic discretisation. This matrix inherits the band shape from A_1 and A_2. We may use common ODE integrators for stiff systems to solve the according end value problem numerically.

4 Simulation Results

We solve the problem from Sect. 2 by numerical simulations in MATLAB [7] (version 7.1.0). Thereby, we assume that the volatility is a stochastic parameter with uniform distribution $\sigma \in [0.3, 0.5]$. Thus the stochastic input exhibits the form (14) with constants $a = 0.4$ and $b = 0.1$. Furthermore, we choose the interest rate $r = 0.1$ and the final time $T = 0.5$.

Firstly, we perform a Monte Carlo method (M.C.) as described in Sect. 3. Thus end-boundary value problems (8),(9),(10),(12) are solved using different numbers of realisations $n = 10^2, 10^3, 10^4$ for comparison. Marsaglia's subtract-with-borrow algorithm yields the pseudo random numbers for σ. Setting $L = 1$, we apply the step size $h = 0.005$ in the method of lines. Each system of ODEs (22) exhibits $N = 200$ equations. A Rosenbrock-Wanner scheme of second order, see [6], yields a numerical solution of the end value problems (21),(22). In the ODE solver, we perform a local error control with relative tolerance $\varepsilon_r = 10^{-3}$ and absolute tolerance $\varepsilon_a = 10^{-6}$. The approximations (16) and (17) for expected values and variances, respectively, are evaluated at time $t = 0$.

We note that more sophisticated techniques for computing the expected values and variances of the problem exist like quasi-Monte Carlo methods or techniques achieving variance reduction, for example. However, an ordinary Monte Carlo simulation is able to solve the considered test problem sufficiently accurate using a moderate number of realisations here.

Secondly, we solve the end-boundary value problem (26),(28),(29),(30) corresponding to the polynomial chaos (P.C.) only once. We use just $M + 1 = 4$ basis functions. The arising matrix (27) is computed according to (14) with the Legendre polynomials. We arrange $L = 1$ and $h = 0.005$ in the method of lines again. The system of ODEs (33) includes $N(M + 1) = 800$ equations now. The corresponding end value problem is solved numerically by the Rosenbrock-Wanner method with the above tolerances $\varepsilon_r, \varepsilon_a$. Figure 1 illustrates the resulting coefficient functions. On the one hand, the component $i = 0$ determines the expected values due to (31). On the other hand, the subsequent components yield the variances following (32). The magnitude of the coefficient functions and their contribution to the variance is shown in the following table.

Table 1: Magnitude of coefficient functions.

	$i = 2$	$i = 3$	$i = 4$
$\max \lvert v_i \rvert$	$1.6 \cdot 10^{-2}$	$1.2 \cdot 10^{-3}$	$9.8 \cdot 10^{-5}$
$\max \lvert v_i \rvert^2 \langle \Phi_i^2 \rangle$	$9.1 \cdot 10^{-5}$	$3.0 \cdot 10^{-7}$	$1.4 \cdot 10^{-9}$

Thus the stochastic discretisation (25) features a fast rate of convergence and a relatively low number of coefficient functions already yields sufficiently accurate results.

Now we compare the approximations for expected values and variances, which are obtained by M.C. and P.C., respectively, at the crucial time point $t = 0$. Table 2 shows the maximum differences between the results in an absolute sense. As the number of realisations increases, the differences become smaller. To increase the accuracy in each method, the step size in the method of lines as well as the tolerances in the ODE integrator have to be refined. Furthermore, the relative computation times of the simulations are demonstrated in relation to P.C. by Table 2, too. The computational effort in M.C. increases linearly with the number of realisations. On the contrary, the P.C. exhibits a constant lower computation time.

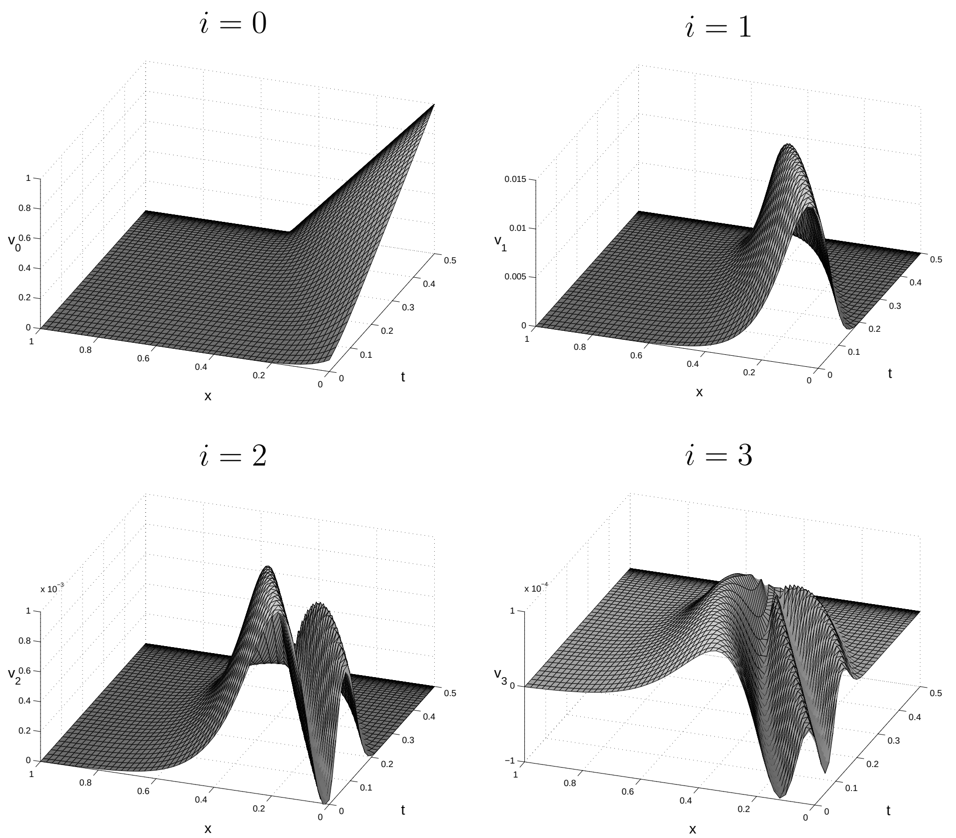

Figure 1: Coefficient functions from polynomial expansion.

Table 2: Comparison between results obtained by P.C. and by M.C. with different numbers of realisations at initial time $t = 0$.

method	M.C. $n = 10^2$	M.C. $n = 10^3$	M.C. $n = 10^4$	P.C.
diff. in exp. value	$6.1 \cdot 10^{-4}$	$4.7 \cdot 10^{-4}$	$1.9 \cdot 10^{-4}$	—
diff. in variance	$8.6 \cdot 10^{-6}$	$1.6 \cdot 10^{-6}$	$1.0 \cdot 10^{-6}$	—
rel. comp. times	2.0	20.9	185.9	1.0

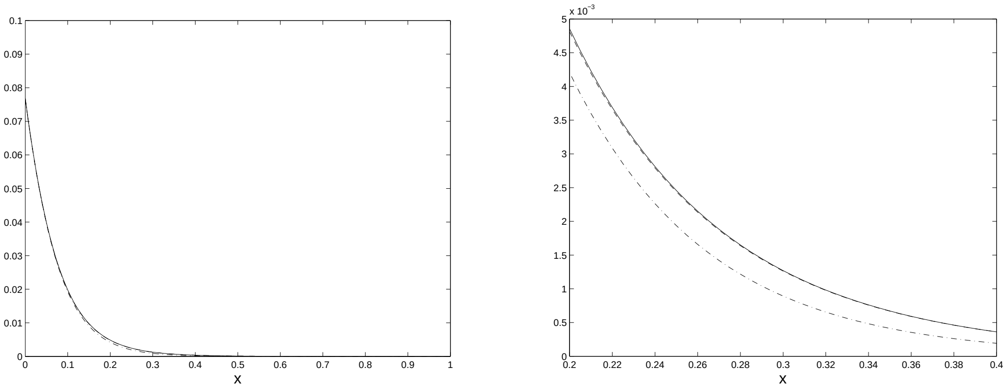

Figure 2: Expected values at $t = 0$ for P.C. (—) and M.C. with $n = 10^4$ (- -) together with deterministic solution using $\sigma = 0.4$ (- $\cdot$ -). Left: total interval $x \in [0, 1]$, Right: zoom in $x \in [0.2, 0.4]$.

For a further discussion, Figure 2 illustrates the achieved expected values at time $t = 0$. We recognise that the approximations of M.C. and P.C. nearly coincide for a large number of realisations n. Moreover, the deterministic solution for $\sigma = 0.4$ is displayed, since this volatility corresponds to the expected value of the stochastic input parameter (14). The difference

$$\mathbb{E}(u(x, t; \sigma(\xi(\cdot)))) - u(x, t; \mathbb{E}(\sigma(\xi(\cdot))))$$

can be seen as an estimate of the nonlinearity in the problem. We observe a significant difference in Figure 2 (right), which indicates that the refined model introduced in Sect. 2 yields new information in comparison to the ordinary modelling. Furthermore, Figure 3 shows the computed variances at time $t = 0$. For a lower number of realisations n, the approximation from M.C. exhibits a large error in a relative sense. Thus a huge number of realisations is necessary for a sufficiently accurate approximation of the variance. In this case, P.C. represents a more efficient technique to solve the problem, see Table 2.

The above simulations refer to the modified PDE (8), since a lower number of independent variables is achieved in comparison to the underlying PDE (5). Nevertheless, we obtain results corresponding to (5) directly by applying the transformations (15).

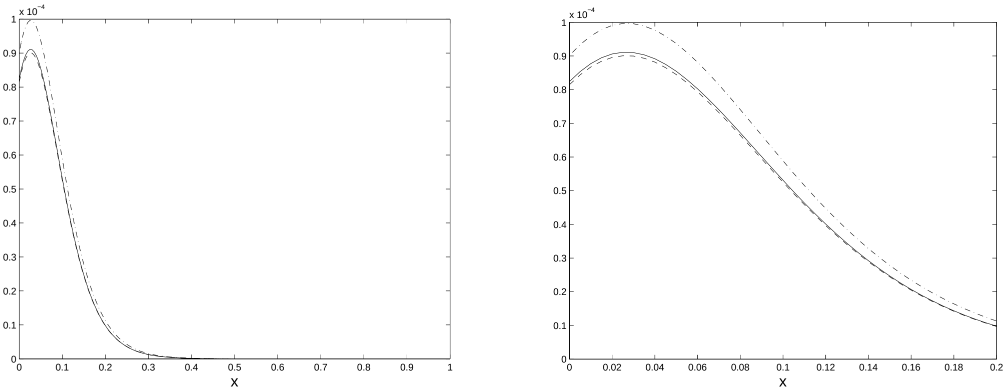

Figure 3: Variances at $t = 0$ for P.C. (—) and M.C. with $n = 10^2$ (- · -) as well as $n = 10^4$ (- -). Left: total interval $x \in [0, 1]$, Right: zoom in $x \in [0, 0.2]$.

5 Conclusions

A refined model for fair prices of options has been presented, where the volatility represents a stochastic input parameter to a parabolic partial differential equation. A Monte Carlo method produces approximations of desired expected values and variances. However, a stochastic discretisation based on the generalised polynomial chaos yields a more efficient technique. The strategy can be used for a large class of distributions corresponding to the random volatility. Moreover, the problem definition as well as the constructed technique applies directly to other models involving parabolic equations. Further refined models introducing several stochastic input parameters may be useful, too, where the multidimensional case of the generalised polynomial chaos yields an according technique. If a financial derivative does not allow for marked-to-market calibration, then the concept of random volatility reflects the uncertainties in historical parameter estimation.

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