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Preprint BUW-AMNA 06/04

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September 2006

http://www.math.uni-wuppertal.de/org/Num/

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Summary. A multidimensional model yields an alternative strategy for the numerical simulation of frequency-modulated signals. Thus the differential algebraic equations (DAEs), which describe an electric circuit, change into warped multirate partial differential algebraic equations (MPDAEs). Houben [6] introduced an approach for solving efficiently initial-boundary value problems of such MPDAE systems. Thereby, envelope-modulated solutions of the DAEs are reproduced. In this paper, the technique is analysed for obtaining quasiperiodic solutions of the DAEs. The crucial question is if biperiodic solutions of the MPDAEs are generated automatically by Houben's approach provided that the initial values of a biperiodic solution are applied.

1 Introduction

In radio-frequency applications, electric circuits often produce oscillatory signals with widely-separated time scales. For example, the amplitude as well as the frequency of a high-frequency oscillation may change relatively slowly. A numerical simulation of the circuit demands to solve the corresponding time-dependent system of differential algebraic equations (DAEs), see [4]. Thus the simulation becomes inefficient, since fast oscillations limit the step size in time, whereas the slow time scale determines the total time interval.

A multivariate signal model yields an alternative strategy, where each separate time scale is given an own variable. Brachtendorf et al. [1] introduced the corresponding system of multirate partial differential algebraic equations (MPDAEs), which yields an efficient simulation of purely amplitude-modulated signals. Narayan and Roychowdhury [7] generalised the approach for signals, which are amplitude-modulated (AM) as well as frequency-modulated (FM). Accordingly, a system of warped MPDAEs arises, where the determination of an appropriate local frequency function is crucial for the efficiency of the multidimensional model. Rough choices produce unnecessary oscillations in the multivariate solutions of the system.

Houben [5, 6] introduced a minimisation criterion with respect to partial derivatives, which shall reduce oscillatory behaviour in the solutions of the MPDAE system. This strategy yields a formula for the unknown local frequency function depending on the multivariate solution. The approach can be used to solve initial-boundary value problems of MPDAEs, which reproduce envelope-modulated solutions of the DAEs.

The direct determination of quasiperiodic solutions of the DAEs demands to solve biperiodic boundary value problems of corresponding MPDAEs, see [10]. A method of characteristics can be used to compute biperiodic solutions of an MPDAE system efficiently, see [8]. This technique becomes inappropriate in case of initial-boundary value problems.

We investigate the performance of Houben's strategy when initial values from a biperiodic solution are given. If the resulting solution is biperiodic, too, then a method for biperiodic boundary value problems can be constructed based on the original strategy. Although the method of characteristics still seems to be superior for biperiodic problems, the results give more insight in the properties of Houben's method.

2 Multidimensional Model

The mathematical model of electric circuits yields a system of DAEs, see [4]. We consider a system of the form

$$\frac{\mathrm{d}\mathbf{q}(\mathbf{x})}{\mathrm{d}t} = \mathbf{f}(\mathbf{b}(t), \mathbf{x}(t)), \qquad \begin{array}{ll} \mathbf{x} : \mathbb{R} \to \mathbb{R}^k, & \mathbf{q} : \mathbb{R}^k \to \mathbb{R}^k, \\ \mathbf{b} : \mathbb{R} \to \mathbb{R}^l, & \mathbf{f} : \mathbb{R}^l \times \mathbb{R}^k \to \mathbb{R}^k, \end{array} \tag{1}$$

where $\mathbf{x}$ denotes unknown node voltages and branch currents. We assume that the predetermined input signals $\mathbf{b}$ vary relatively slowly. In contrast, the solution $\mathbf{x}$ shall include high-frequency oscillations, whose amplitude as well as frequency are changed slowly by the input signals. Thus the signals $\mathbf{x}$ include widely-separated time scales. Hence solving the DAEs (1) demands a huge number of time steps and a transient analysis becomes inefficient.

Brachtendorf et al. [1] introduced a multivariate signal model for purely AM signals, where each time scale is assigned an own variable. Narayan and Roychowdhury [7] generalised this model for signals including AM as well as FM. In case of two time scales, a multivariate function (MVF) $\hat{\mathbf{x}} : \mathbb{R}^2 \to \mathbb{R}^k$ and a local frequency amplification function $\nu : \mathbb{R} \to \mathbb{R}$ of the signal $\mathbf{x}$ are introduced. Thus an efficient model is achieved by decoupling the time scales.

Consequently, the system of DAEs (1) is transformed into the system of warped MPDAEs

$$\frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_1} + \nu(t_1)\frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_2} = \mathbf{f}(\mathbf{b}(t_1), \hat{\mathbf{x}}(t_1, t_2)), \qquad \begin{array}{l} \hat{\mathbf{x}} : \mathbb{R}^2 \to \mathbb{R}^k, \\ \nu : \mathbb{R} \to \mathbb{R}. \end{array} \tag{2}$$

We assume $\mathbf{q}, \hat{\mathbf{x}} \in C^1$ and $\mathbf{b}, \mathbf{f}, \nu \in C^0$. The local frequency amplification ν is a priori unknown, too. The input signals vary just slowly and thus do not require a multivariate description. An arbitrary solution of the MPDAEs (2) yields a solution of the DAEs (1) using the reconstruction

$$\mathbf{x}(t) = \hat{\mathbf{x}}\left(t, \int_0^t \nu(\sigma)\mathrm{d}\sigma\right), \tag{3}$$

i.e., the MVF includes the original signal. In this general case, ν represents a local frequency amplification and thus ν is physically dimensionless.

This model is suitable only if the fast time scale is periodic, since we want to resolve many oscillations in a bounded and relatively small multidimensional domain. Hence two types of problems arise. Firstly, initial-boundary value problems of the system (2) read

$$\hat{\mathbf{x}}(0, t_2) = \mathbf{h}(t_2), \quad \hat{\mathbf{x}}(t_1, t_2) = \hat{\mathbf{x}}(t_1, t_2 + 1) \quad \text{for all } t_1 \geq 0,\ t_2 \in \mathbb{R} \tag{4}$$

with a predetermined periodic function $\mathbf{h} : \mathbb{R} \to \mathbb{R}^k$. The period is standardised to 1 and thus the second argument t_2 of the MVF becomes dimensionless. Hence ν in (3) includes the magnitude of the fast time scale and exhibits the physical dimension of a frequency now. The problems (4) are solved in a domain $[0, T] \times [0, 1]$ for some $T > 0$. Secondly, biperiodic boundary value problems exhibit the conditions

$$\hat{\mathbf{x}}(t_1, t_2) = \hat{\mathbf{x}}(t_1 + T_1, t_2) = \hat{\mathbf{x}}(t_1, t_2 + 1) \quad \text{for all } t_1, t_2 \in \mathbb{R}, \tag{5}$$

which correspond to the domain $[0, T_1] \times [0, 1]$. In this case, the input signals as well as the local frequency function have to be T_1-periodic. Note that we require a smooth solution of (2) to fulfil the biperiodicity condition (5).

Applying (3), solutions satisfying (4) reproduce envelope-modulated signals, whereas solutions fulfilling (5) yield quasiperiodic signals. Note that both envelope-modulated and quasiperiodic signals include FM here.

For appropriate choices of the local frequency functions, the corresponding MVF exhibits a simple structure in $[0, T] \times [0, 1]$. Thus we can compute the solution of the MPDAEs (2) using a relatively low number of grid points and achieve an efficient numerical simulation. The desired solution of the DAEs (1) is reconstructed via (3).

Solutions of the MPDAEs (2) corresponding to different local frequency functions are interconnected by a transformation, see [9]. If $\hat{\mathbf{x}}$ is a MVF satisfying the system for the local frequency ν, then the transformed MVF

$$\hat{\mathbf{y}}(t_1, t_2) := \hat{\mathbf{x}}\left(t_1, t_2 + \int_0^{t_1} \nu(\sigma) - \mu(\sigma)\ \mathrm{d}\sigma\right) \tag{6}$$

represents a solution of the system with local frequency μ. The initial values at $t_1 = 0$ are invariant in this transformation. Thus, for solving initial-boundary value problems (4), the local frequencies are completely free parameters, which can be used to achieve an efficient representation. In case of biperiodic problems (5), an additional requirement is necessary to preserve the periodicity in the slow time scale, namely

$$\int_0^{T_1} \mu(\sigma)\ \mathrm{d}\sigma = \int_0^{T_1} \nu(\sigma)\ \mathrm{d}\sigma, \tag{7}$$

which means that the average frequency coincides.

3 Houben's Method

A suitable local frequency function for representing the signals efficiently is unknown a priori. Inappropriate selections cause undesired oscillations in the MVFs, see [9]. Houben [5, 6] formulated the minimisation problem

$$s(t_1) := \int_0^1 \| \tfrac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_1}(t_1, u) \|^2 \, \mathrm{d}u \quad \longrightarrow \quad \text{min.} \qquad \text{for each } \; t_1 \geq 0 \tag{8}$$

using the Euclidean norm $\| \cdot \|$. Thus oscillatory behaviour is reduced via minimising the impact of the partial derivative with respect to the slow time scale. For example, a method of lines can be employed to solve the initial-boundary value problem (2),(4). Hence a corresponding optimal solution allows for using relatively large step sizes in the numerical simulation.

The demand (8) implies a necessary condition for an optimal solution:

$$\nu(t_1) = \frac{\int_0^1 \langle \mathbf{f}(\mathbf{b}(t_1), \hat{\mathbf{x}}(t_1, u)), \tfrac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_2}(t_1, u) \rangle \, \mathrm{d}u}{\int_0^1 \| \tfrac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_2}(t_1, u) \|^2 \, \mathrm{d}u} \quad \text{for all } \; t_1 \geq 0 \tag{9}$$

with the Euclidean inner product $\langle \cdot, \cdot \rangle$. This formula can be used to eliminate the unknown local frequency function. Thus initial-boundary value problems can be solved by proceeding in the slow time scale. Furthermore, the condition (9) is equivalent to the orthogonality relation

$$\int_0^1 \langle \tfrac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_1}(t_1, u), \tfrac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_2}(t_1, u) \rangle \, \mathrm{d}u = 0 \quad \text{for all } \; t_1 \geq 0. \tag{10}$$

In the following, we assume the existence of a smooth biperiodic solution $\hat{\mathbf{z}}$ corresponding to the periodic local frequency κ. Let $\mathbf{h} := \hat{\mathbf{z}}(0, \cdot)$ be its initial values. We investigate the results from the initial-boundary value problem (4) applying Houben's technique. In [3], the case of ordinary differential equations ($\mathbf{q}(\mathbf{x}) \equiv \mathbf{x}$) has already been considered.

An arbitrary solution $\hat{\mathbf{x}}, \nu$ of the MPDAEs (2) with the same initial values $\mathbf{h}$ can be obtained from the biperiodic solution $\hat{\mathbf{z}}$ via the transformation (6). We define the quantity

$$c := \int_0^{T_1} \nu(\sigma) - \kappa(\sigma) \, \mathrm{d}\sigma. \tag{11}$$

Using the transformation (6), it follows

$$\hat{\mathbf{x}}(T_1, t_2) = \hat{\mathbf{z}}(T_1, t_2 + c) = \hat{\mathbf{z}}(0, t_2 + c) = \hat{\mathbf{x}}(0, t_2 + c) \quad \text{for all } \; t_2 \in \mathbb{R}. \tag{12}$$

We have achieved the following result.

Theorem 1. *If $\hat{\mathbf{x}}, \nu$ is an arbitrary solution of the system (2) with initial values from a biperiodic solution, then it holds $\hat{\mathbf{x}}(T_1, t_2) = \hat{\mathbf{x}}(0, t_2 + c)$ for all t_2, i.e., the end values represent a time shift of the initial values.*

In Houben's approach, the question is if this time shift is equal to zero or not. In the formula (9) for the corresponding local frequency function, the arising integrals are invariant with respect to a shift in the fast time scale t_2. Thus the following theorem holds.

Theorem 2. *If $\hat{\mathbf{x}}, \nu$ is a solution of the system (2) with initial values from a biperiodic solution and satisfying (9), then it holds $\nu(0) = \nu(T_1)$ and thus the local frequency function is periodic.*

Note that this theorem does not imply that the corresponding MVF is biperiodic. Nevertheless, the local frequency becomes periodic and thus a biperiodic solution may result from Houben's approach. However, a proof is still missing.

On the other hand, a minimisation demand for biperiodic solutions has been introduced in [9]. Similar to this approach, we consider the formulation

$$\gamma := \int_0^{T_1} \int_0^1 \| \tfrac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_1}(v,u) \|^2 \, \mathrm{d}u \, \mathrm{d}v \quad \longrightarrow \quad \text{min.} \tag{13}$$

here. A variational calculus yields the necessary condition

$$\int_0^1 \langle \tfrac{\partial^2 \mathbf{q}(\hat{\mathbf{x}})}{\partial {t_1}^2}(t_1, u), \tfrac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_2}(t_1, u) \rangle \, \mathrm{d}u = 0 \quad \text{for all } \; t_1 \geq 0, \tag{14}$$

which an optimal solution has to satisfy. Thereby, the periodicity of the solution in t_1 is crucial to obtain this requirement. A biperiodic solution, which minimises (8), also represents a minimum of (13). This fact yields the following statement.

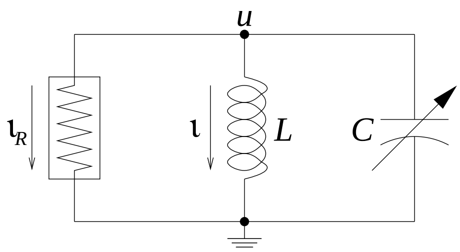
Theorem 3. *Given a biperiodic solution $\hat{\mathbf{x}}, \nu$ of the system (2), which is minimal with respect to Houben's criterion (8), then the MVF $\hat{\mathbf{x}}$ satisfies the orthogonality property (10) as well as (14).*

This result indicates that a solution obtained by Houben's approach is not biperiodic in general. If it is biperiodic, then two orthogonality properties are satisfied, which do not seem to be equivalent. Likewise, we consider an optimal biperiodic solution with respect to (13). This solution may become better by a transformation (6) to the optimal local frequency (9). Consequently, we may loose the periodicity as the price to be paid for the further reduction of the impact of partial derivatives.

4 Illustrative Example

As test example, we consider a voltage controlled oscillator, which is illustrated in Fig. 1 (left). The mathematical model of this circuit can be written as a system of ordinary differential equations (ODEs). We apply this formulation, since the examinations on the periodicity do not differ significantly if ODEs instead of DAEs are considered. The system reads

$$\dot{u} = (-\imath_R(u) - \imath)\,/(Cb(t)), \qquad \dot{\imath} = u/L \tag{15}$$



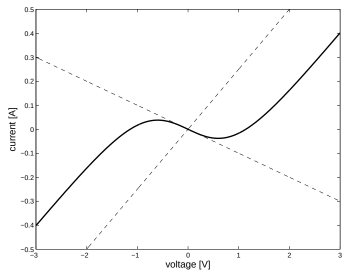


Fig. 1. Circuit diagram of voltage controlled oscillator (left) and current-voltage relation $\imath = \imath_R(u)$ of nonlinear resistor (right).

with the node voltage u and the branch current $\imath$. For the input signal, we choose the slowly varying oscillation

$$b(t) = 1 + 0.8 \cos\left(\tfrac{2\pi}{T_1} t\right) \quad \text{with } T_1 = 1 \text{ ms} \quad (f := {T_1}^{-1} = 1 \text{ kHz}). \tag{16}$$

The current-voltage relation of the nonlinear resistor is given by

$$\imath_R(u) = (G_0 - G_\infty) U_k \tanh(u/U_k) + G_\infty u. \tag{17}$$

The used parameters are $C = 1$ nF, $L = 1$ μH, $U_k = 1$ V, $G_0 = -0.1$ A/V, $G_\infty = 0.25$ A/V. Fig. 1 (right) shows the corresponding relation (17).

For constant input $b \equiv 1$, the system (15) exhibits a periodic limit cycle with a frequency of about 4 MHz. The input signal (16) changes the capacitance and thus introduces a frequency modulation. Since the input is periodic, a quasiperiodic signal arises. Consequently, we transform the ODEs (15) into a system (2) of partial differential equations (PDEs). We compute a biperiodic solution of the system via the method presented in [9]. Its initial values are used to apply Houben's strategy now.

To solve the initial-boundary value problem (4), we use a method of lines. The integrals in (9) are replaced by finite sums evaluated on the lines. The derivatives with respect to t_2 are substituted by BDF2-formulae, see [2], which are applied in the PDEs (2) as well as in the local frequencies (9). The arising system of ODEs is solved by trapezoidal rule in the interval $[0, T_1]$, where a relatively high accuracy is demanded in the step size control.

Firstly, we apply $m = 100$ lines in the semidiscretisation to demonstrate the optimal solution. Fig. 2 shows the resulting optimal local frequency, which is periodic in view of our discussions. The local frequency is physically reasonable, since it becomes low for high capacitances and vice versa. The corresponding optimal MVFs are illustrated in Fig. 3. On the one hand, we recognise that $\hat{u}$ is nearly constant in the slow time scale, which is caused by the minimisation. On the other hand, $\hat{\imath}$ exhibits a slight change in the slow time scale, which describes an AM signal and thus can not be reduced further.

Secondly, we compare the initial values at $t_1 = 0$ with the end values at $t_1 = T_1$ for several numerical simulations using different numbers of lines, namely $m = 25, 50, 100$. Table 1 demonstrates the maximum of the differences obtained from the discrete values on the lines. We recognise that the differences become smaller for an increasing accuracy in the method. This behaviour indicates that the exact solution is biperiodic or nearly (except for small differences) biperiodic. In [3], other numerical simulations, where a Van-der-Pol-oscillator is used, indicate that it can not be excluded that the resulting solution is biperiodic, too.

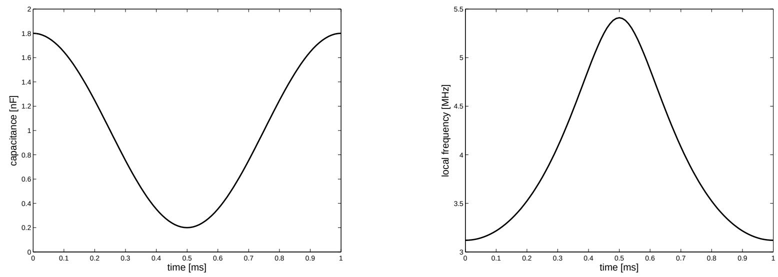


Fig. 2. Capacitance Cb [nF] (left) and optimal local frequency ν [MHz] (right).

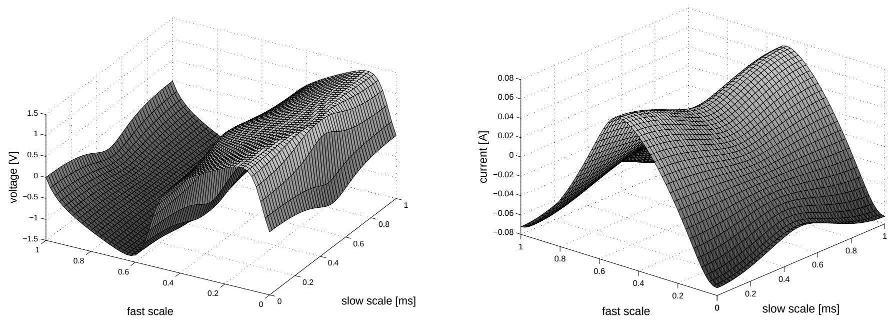


Fig. 3. Optimal MVFs $\hat{u}$ [V] (left) and $\hat{\imath}$ [A] (right).

Table 1. Maximum differences between initial and end values.

number of lines	$m = 25$	$m = 50$	$m = 100$
max $\lvert\hat{u}(0,\cdot) - \hat{u}(T_1,\cdot)\rvert$	$8 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	$4 \cdot 10^{-5}$
max $\lvert\hat{\imath}(0,\cdot) - \hat{\imath}(T_1,\cdot)\rvert$	$3 \cdot 10^{-3}$	$8 \cdot 10^{-4}$	$1 \cdot 10^{-6}$

5 Conclusions

The approach of Houben permits to solve initial-boundary value problems of warped MPDAEs efficiently, which yields envelope-modulated signals. We consider initial values of a biperiodic solution to investigate the determination of quasiperiodic signals. It follows that the resulting local frequency function becomes periodic in this case. However, it is still an open question if the corresponding MVF is always biperiodic. We performed numerical simulations with Houben's strategy via a method of lines. The results illustrate that it can not be excluded that the arising solution is automatically biperiodic. In practice, the resulting MVFs seem to be biperiodic or at least nearly (except for a small difference) biperiodic. If the solution is exactly biperiodic, then a method for computing biperiodic solutions of the warped MPDAEs can be constructed based on Houben's technique. For example, the initial conditions in the method of lines are just replaced by periodic boundary conditions.

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