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Modelling correlation as a stochastic process

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Abstract

Although market observations show that correlation between stocks, interest rates, e. g., is not deterministic, correlation is usually modelled as a fixed number. This article provides a new approach of modelling correlation as a stochastic process which has applications in many fields. We will show an example from financial markets where the stochasticity of correlation is a fundamental source of risk: the quanto.

Keywords: Correlation, stochastic processes, mean-reverting processes, finance, Fokker-Planck equation.

1 Introduction

Stochastic modelling is an essential tool in applications as different as finance, biology, medical science etc. As soon as there is more than one factor to consider, the question arises how to map the relationship between these factors. A widely used approach is to use correlated stochastic processes where the magnitude of correlation is measured by a single number $\rho \in [-1, 1]$, the correlation coefficient. In the case of two Brownian motions W^1 and W^2 correlated with ρ, one can express this concept by the symbolic notation

$$dW_t^1 dW_t^2 = \rho \, dt \ .$$

If $\rho = 0$ the Brownian motions are uncorrelated. The usual assumption about filtration etc. may hold. See for example Øksendal[5]. We will concentrate in the following on correlated Brownian motion. Further work must show how the idea can be extended to other distributions.
In finance correlated Brownian motions appear for example in the Heston model[2]

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t \ , \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dZ_t \ , \end{aligned}$$

where $\mu > 0, \kappa > 0, \theta > 0, \sigma > 0$ are constant. The first process is used to describe the movement of an underlying asset S and the second process describes the assumed behaviour of the volatility V.

Another example of coupled stochastic processes is the following Black-Scholes model for quantos (see also section 6):

$$\begin{aligned} dS_t^1 &= \mu_1 S_t^2 dt + \sigma_1 S_t^1 dW_t \ , \\ dS_t^2 &= \mu_2 S_t^2 dt + \sigma_2 S_t^2 dZ_t \ , \end{aligned}$$

with positive constants $\mu_1, \mu_2, \sigma_1, \sigma_2$. Here one of the stochastic differential equations is supposed to map the performance of a traded object (stock or index, for instance) in a currency A. The second stochastic differential equation describes the exchange rate between currency A and another currency B. Within both examples one usually assumes the Wiener processes to be correlated

$$dW_t \cdot dZ_t = \rho \, dt \ , \tag{1}$$

with a constant correlation factor $\rho \in [-1, 1]$. There are few approaches to extend this concept, see for example Burtschell et al. [1]. We will include randomness and time factor in the following. Moreover the process we propose possesses an intuitive interpretation which makes it a valuable modelling tool.

For motivating our model we look on the market behaviour of correlation. It indicates that 'CORRELATIONS ARE EVEN MORE UNSTABLE THAN VOLATILITIES' as mentioned by Wilmott[7]. Figure 1 shows the estimated correlations between Dow Jones and Euro/US-Dollar exchange rate on a daily basis. Hereby it is assumed that both processes follow a lognormal distribution. Figure 1 (A) clearly shows that correlation is not constant over time. Moreover, correlation seems

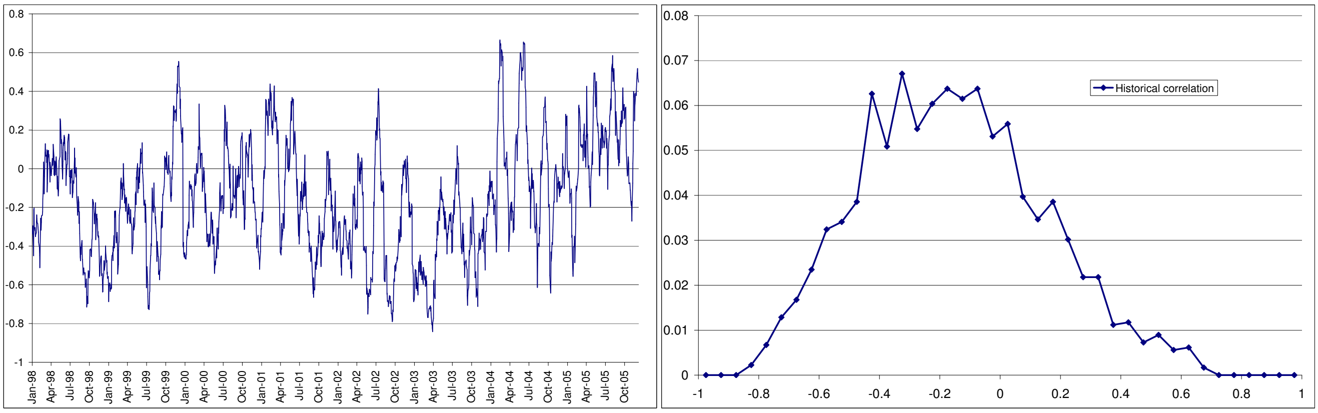

Figure 1: Correlation between Dow Jones and Euro/US-Dollar exchange rate, process and density. (A): Estimated historical correlation over time. (B): Empirical distribution.

even to be nondeterministic. In figure 1 (B) we receive a first impression how the density of a stochastic process modelling correlation could look like. That is the motivation for this report. We want to model correlation such that

- it is concentrated on $[-1, 1]$,
- it varies around a mean,
- the probability mass approaches zero in the boundary values,
- there is a suitable number of parameters to calibrate the model to market data.

The outline of the remaining sections is as follows. First we present different approaches for modelling correlations as stochastic processes (section 2 and 3). In section 4 and 5 we examine these mathematical models analytically. Finally we analyse how theses processes can be used for pricing and apply the results exemplarily to a particular financial derivative, a quanto (section 6).

2 Construction of Fully Stochastically Correlated Brownian Motions

Before we introduce suitable processes to model correlation, we show how to construct Brownian motions correlated with a stochastic process. These will be called fully stochastically correlated

processes.
As this is an extension of the constant correlation case, it is a natural requirement that the new concept should cover the constant case. First we assume that

$$d\rho_t = a(t,\rho_t)dt + b(t,\rho_t)dK_t\ , \qquad \rho_0 \in [-1,1]\ ,$$

is an Ito process with a Brownian motion K and suitable functions a and b. Based on two independent Brownian motions V and W (also independent of K), we define

$$Z_t = \int_0^t \rho_s dW_s + \int_0^t \sqrt{1-\rho_s^2}dV_s\ , \qquad Z_0 = 0\ . \tag{2}$$

For proving that Z (2) is a Brownian motion, we remark that $Z_0 = 0$ and show that $E\left[Z_t^2\right] = t$. Because of the Ito-isometry, we can calculate:

$$E\left[Z_t^2\right] = E\left[\left(\int_0^t \rho_s dW_s\right)^2 + \left(\int_0^t \sqrt{1-\rho_s^2}dW_s\right)^2 + 2M_t\right] = t\ .$$

with $M_t := \int_0^t \rho_s dW_s \int_0^t \sqrt{1-\rho_s^2}dV_s$. We used that $E[M_t] = 0$ which follows from $M_0 = 0$ and

$$dM_t = \rho_t dW_t \int_0^t \sqrt{1-\rho_s^2}dV_s + \sqrt{1-\rho_t^2}dV_t \int_0^t \rho_s dW_s + \rho_t\sqrt{1-\rho_t^2}\underbrace{dW_t dV_t}_{=0}\ .$$

In a similar way, one verifies that $E\left[Z_t|\mathcal{F}_s\right] = Z_s, t \geq s$. Thus we have two Brownian motions W and Z correlated by the stochastic process ρ.
Especially it holds for W and Z that

$$E\left[Z_t \cdot W_t\right] = E\left[\int_0^t \rho_s ds\right]$$

which agrees for constant ρ with the symbolic expression

$$dZ_t \cdot dW_t = \rho\, dt\ .$$

This construction should be borne in mind throughout the remaining parts of this article.

3 Models for Fully Stochastic Correlation

We present two ways to construct stochastic processes which remain in the interval $[-1,1]$. For the first approach the new stochastic process is directly formulated as a function of Brownian motion. It stands out because of the ease of construction and the high degree of analytical tractability, but lacks intuitive interpretation. This is the reason why we focus on the second approach later on where the new stochastic process is described by a stochastic differential equation driven by Brownian motion.

The first and most important property of a process for modelling correlation is that the process must stay within the interval $[-1,1]$. Based on a Brownian motion W

$$X_t = \frac{2}{\pi}\arctan\left(\alpha(W_t+\gamma)\right)\ , \tag{3}$$

surely is in $[-1,1]$ since

$$f(x)=\frac{2}{\pi}\arctan\left(\alpha(x+\gamma)\right)\ ,$$

maps $(-\infty,\infty)$ to $(-1,1)$. Following this train of thought, we can also choose

$$f(x)=\frac{2}{1+x^2}-1\ , \tag{4}$$

or in general every transformation

$$f:(-\infty,\infty)\to[-1,1]\ ,\qquad f\in\mathcal{C}^2(\mathbb{R})\ .$$

Obviously not all of these possible processes are suitable. Exemplarily we analyse the expectation for (3). Note that the initial value is given by

$$X_0=\frac{2}{\pi}\arctan\left(\alpha\left(W_0+\gamma\right)\right)=\frac{2}{\pi}\arctan\left(\alpha\gamma\right)\ ,$$

since W denotes a Brownian motion. Using Ito's formula, we obtain the following representation for (3)

$$dX_t=-\frac{2\alpha^2}{\pi}\sin\left(\frac{\pi}{2}X_t\right)\cos^3\left(\frac{\pi}{2}X_t\right)dt+\frac{2\alpha}{\pi}\cos^2\left(\frac{\pi}{2}X_t\right)dW_t\ .$$

Thus the expectation is

$$E\left[X_t\right]\ =\ X_0-E\left[\frac{2\alpha^2}{\pi}\int\limits_0^t\sin(\frac{\pi}{2}X_s)\cos^3(\frac{\pi}{2}X_s)ds\right]\ .$$

This expectation depends on t and cannot be computed in general. Apart of this technical problem the approach lacks intuitive interpretation which is necessary as we want the process to map correlation. Because of its insufficient justification which also holds for similar transformations, we cannot consider these transformed Brownian motions as appropriate modelling tools. Therefore we present a second approach.

The process

$$dX_t=\sqrt{(1-X_t)(1+X_t)}dW_t\ , \tag{5}$$

with initial value $X_0=x_0\in(-1,1)$ guarantees values in $[-1,1]$ (see also section 4). It is straightforward to generalise (5) to

$$dX_t=\alpha\sqrt{(1-X_t)(1+X_t)}dW_t\ ,\qquad \alpha\in\mathbb{R}^+\ ,\ X_0=x_0\in(-1,1)\ , \tag{6}$$

which also remains in $[-1,1]$, but allows more freedom.
The expectation of (6) can be computed as

$$E\left[X_t\right]\ =\ X_0+E\left[\alpha\int\limits_0^t\sqrt{(1-X)(1+X)}dW\right]=x_0.$$

which is constant over time. Thus there are two parameters to calibrate the model to market data: α and x_0.
We can extend (6) by adding a drift term. Motivated by the practical example in the introductory section we choose a mean-reverting process with - for simplicity - deterministic mean:

$$dX_t=\kappa(\theta-X_t)dt+\alpha\sqrt{1-X_t^2}dW_t\ ,\qquad X_0=x_0\in(-1,1)\ , \tag{7}$$

with constants $\kappa\geq 0,\alpha>0,\theta\in(-1,1)$. The assignment $\kappa=0$ yields model (6).

At this point we do not know much about (6) and (7). In the following section, we discuss analytical properties with focus on the boundary behaviour at -1 and 1. Afterwards we study the process with the aid of the Fokker-Planck equation in section 5.

4 Analytical Properties of (7) - Boundaries

First we classify the boundaries -1 and 1. Hereby we follow the notation of Karlin and Taylor[3]. We start with a general Ito diffusion

$$dx_t = a(x_t)dt + b(x_t)dW_t\ , \qquad x_0 \in \mathbb{R}\ . \tag{8}$$

Moreover we denote the left boundary by l and the right boundary by r in the sense $l \leq x \leq r$. Without loss of generality we concentrate on the left boundary in the following. The analysis of the right one works analogously. Now we introduce some notations, thereby we assume $x \in (l, r)$,

$$\begin{aligned}
s(v) &= \exp\left(-\int_{v_0}^{v} \frac{2a(w)}{b^2(w)}dw\right)\ , \qquad v_0 \in (l, x)\ , \\
S(x) &= \int_{x_0}^{x} s(v)dv\ , \qquad x_0 \in (l, x)\ , \\
S[c, d] &= S(d) - S(c) = \int_{c}^{d} s(v)dv\ , \qquad (c, d) \in (l, r)\ , \\
S(l, x] &= \lim_{a \to l} S[a, x]\ .
\end{aligned}$$

We already indicate that x_0 and v_0 will be of no relevance in the following. We use that $dS(x) = s(x)dx$. S is called the scale measure whereas M is the speed measure:

$$\begin{aligned}
m(x) &= \frac{1}{b^2(x)s(x)}\ , \\
M[c, d] &= \int_{c}^{d} m(x)dx\ .
\end{aligned}$$

Analogously we can use the relation $dM(x) = m(x)dx$. Lastly we need the expression

$$\Sigma(l) = \lim_{a \to l} \int_{a}^{x} M[v, x]dS(v)\ .$$

Using these notations we can classify the left boundary. The boundary l is called attractive if and only if there is an $x \in (l, r)$ such that

$$S(l, x] < \infty\ . \tag{9}$$

Furthermore l is classified as an attainable boundary if and only if

$$\Sigma(l) < \infty\ . \tag{10}$$

Otherwise it is unattainable.

We are going to determine $S(-1, x]$ for the mean-reverting process (7). We compute $s(x)$ as a first step:

$$-\log(s(v)) = \int_{v_0}^{v} \frac{2\kappa(\theta - w)}{\alpha^2(1 - w^2)}dw = \log\left((1 + v)^{\frac{\kappa}{\alpha^2}(\theta+1)} \cdot (1 - v)^{\frac{\kappa}{\alpha^2}(1-\theta)} \cdot c_0^{-1}\right)$$

with $c_0^{-1} = \left(\frac{1+v_0}{1-v_0}\right)^{-\frac{\theta\kappa}{\alpha^2}} \cdot (1-v_0^2)^{-\frac{\kappa}{\alpha^2}}$. Consequently

$$s(v) = c_0 \cdot (1+v)^{-\frac{\kappa}{\alpha^2}(\theta+1)} \cdot (1-v)^{-\frac{\kappa}{\alpha^2}(1-\theta)}.$$

For the behaviour near the left boundary $l=-1$ we have to investigate

$$S(l,x] = \lim_{a\to l} c_0 \cdot \int_a^x (1+v)^{-\frac{\kappa}{\alpha^2}(\theta+1)}(1-v)^{-\frac{\kappa}{\alpha^2}(1-\theta)} dv\ .$$

We note that $(1-v) > 0$ for $v \in [a,x] \in (-1,1)$ and $-\frac{\kappa}{\alpha^2}(1-\theta) \leq 0$. Therefore the term $(1-v)^{-\frac{\kappa}{\alpha^2}(1-\theta)}$ is bounded for $v \in [a,x]$:

$$\begin{aligned}(1-v)^{-\frac{\kappa}{\alpha^2}(1-\theta)} &\geq (1-a)^{-\frac{\kappa}{\alpha^2}(1-\theta)} > 0\ ,\\ (1-v)^{-\frac{\kappa}{\alpha^2}(1-\theta)} &\leq (1-x)^{-\frac{\kappa}{\alpha^2}(1-\theta)} < 2^{-\frac{\kappa}{\alpha^2}(1-\theta)}\ .\end{aligned}$$

Applying the first inequality we can estimate as follows

$$S(-1,x] \geq \lim_{a\to -1} (1-a)^{-\frac{\kappa}{\alpha^2}(1-\theta)} c_0 \int_a^x (1+v)^{-\frac{\kappa}{\alpha^2}(\theta+1)} dv\ .$$

This expression converges to a constant $C \in \mathbb{R}$ if $\frac{\kappa}{\alpha^2}(\theta+1) < 1$, otherwise it diverges. Hence the boundary $l=-1$ is attractive if $\frac{\kappa}{\alpha^2}(\theta+1) < 1$. By applying the upper bound one shows that it is not attractice if $\frac{\kappa}{\alpha^2}(\theta+1) \geq 1$. Analogously we can deduce that the upper bound $r=1$ is not attractive if $\frac{\kappa}{\alpha^2}(1-\theta) \geq 1$. This boundary behaviour meets intuition. Increasing the parameter κ concentrates the process around the mean.
Furthermore the boundary behaviour is symmetric with respect to θ. For $|\theta|$ close to $|1|$ the parameter κ needs to be decisively larger to make the boundaries -1 respectively 1 not attractive.

Now we are going to figure out if the left boundary is attainable. We remark that only an attractive boundary can be attainable (see definition for Σ). For that reason we assume $\frac{\kappa}{\alpha^2}(\theta+1) < 1$ in the following. For notational simplicity, let c denote a suitable constant in the remaining part of this section. We consider

$$\begin{aligned}\Sigma(-1) &= \lim_{a\to -1} \int_a^x M[v,x] dS(v) = \lim_{a\to -1} \int_a^x S[a,v] dM(v)\\ &= \lim_{a\to -1} \int_a^x \int_a^v s(w)dw \frac{1}{b^2(v)s(v)} dv\ .\end{aligned}$$

First we calculate $\Sigma(-1)$ for $\kappa = 0$:

$$\Sigma(-1) = \frac{c}{2\alpha^2} \log\left(\frac{4}{(1-x)^2}\right) < \infty\ .$$

Hence the boundary $l=-1$ is attainable if $\kappa = 0$. Moreover we can deduce that

$$\begin{aligned}\Sigma(-1) &\leq c \lim_{a\to -1} \int_a^x \left(\int_a^v (1+w)^{-\frac{\kappa}{\alpha^2}(\theta+1)} dw\right) \frac{1}{\beta^2(v)s(v)} dv\\ &\leq c \lim_{a\to -1} \int_a^x \frac{(1+v)^{1-\frac{\kappa}{\alpha^2}(\theta+1)} - (1+a)^{1-\frac{\kappa}{\alpha^2}(\theta+1)}}{(1-v^2)(1+v)^{-\frac{\kappa}{\alpha^2}(\theta+1)}(1-v)^{-\frac{\kappa}{\alpha^2}(1-\theta)}} dv\\ &\leq c\left(\lim_{a\to -1} \int_a^x 1 dv - \lim_{a\to -1} (1+a)^{1-\frac{\kappa}{\alpha^2}(\theta+1)} \int_a^x (1+v)^{-\left(1-\frac{\kappa}{\alpha^2}(\theta+1)\right)} dv\right)\ .\end{aligned}$$

This last integral converges if $1-\frac{\kappa}{\alpha^2}(\theta+1)<1$ which clearly holds as $\theta\in(-1,1)$. The expression $(1+a)^{1-\frac{\kappa}{\alpha^2}(\theta+1)}$ converges because of the attractiveness condition $\frac{\kappa}{\alpha^2}(\theta+1)<1$. Thus the left boundary is attainable (and attractive as shown above) if $\frac{\kappa}{\alpha^2}(\theta+1)<1$. Otherwise the left boundary is not attractive and unattainable.

Following the same line of thought one derives that the right boundary $r=1$ is attractive and attainable if $\frac{\kappa}{\alpha^2}(1-\theta)<1$. Otherwise it is neither attractive nor attainable.
As we want the boundaries of our correlation process to be unattainable we will consider process (7) with

$$\kappa \geq \frac{\alpha^2}{1\pm\theta} \tag{11}$$

in the following.

The classification of the boundaries is a first step for a deeper understanding of the underlying processes. We investigate the transition densities for further insight.

5 Analytical Properties of (7) - Transition Density

We use the Fokker-Planck equation for determination of the transition density. For more information about the Fokker-Planck equation we refer to the book by Risken[6]. For simplicity we set $\alpha=1$. Assuming that the stochastic differential equation

$$dX_t = a(t,X_t)dt + b(t,X_t)dW_t\,,\ X_0 = x_0\,, \tag{12}$$

possesses a transition density $p(t,y|x_0)$, then p satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial t}p(t,x) + \frac{\partial}{\partial x}\left(a(t,x)p(t,x)\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(b(t,x)^2 p(t,x)\right) = 0\,. \tag{13}$$

We want to derive the transition density of (7) for $t\to\infty$. We demand the solution p to fulfill two structural conditions. Firstly p is required to be a density

$$\int_{-1}^{1} p(t,x)dx \;=\; 1\,. \tag{14}$$

Moreover we postulate that p preserves the expectation

$$\int_{-1}^{1} x\cdot p(t,x)dx \;\underset{t\to\infty}{\longrightarrow}\; \theta\,. \tag{15}$$

For the mean reversion process (7) with (11) one can show that every two solutions of (13) are equal for $t\to\infty$ under certain conditions, see Risken[6], section 6.1. As a consequence it suffices to show that a stationary solution exists to know that it is the unique solution. This is how we proceed in the following: We will derive a stationary solution which fulfills (14) and (15). This solution suffices the conditions stated in Risken[6].
To obtain the stationary solution $p(x)=\lim_{t\to\infty} p(t,x)$, we consider the stationary Fokker-Planck equation. Firstly we examine the simplified case $\theta=0$:

$$(1-\kappa)p(x) + x(2-\kappa)p'(x) - \frac{1}{2}(1-x^2)p''(x) = 0. \tag{16}$$

We receive as a solution to (16)

$$p(x) = (1-x^2)^{(\kappa-1)} \left(c + b \int_x^1 \left(1-z^2\right)^{\kappa} dz \right) \tag{17}$$

with constants $b, c \in \mathbb{R}$. If $b = 0$, p is symmetric around $x = 0$. Hence

$$\int_{-1}^{1} x \cdot p(x) dx = 0 = \theta ,$$

and (15) holds. Thus we choose $b = 0$. The still free variable c must be chosen such that condition (14) is fulfilled. Since the antiderivative of p can be described via the hyper-geometric function

$$F(a,b,c,y) = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{2!c(c+1)}x^2 + ...$$

we obtain:

$$\int (1-x^2)^{(\kappa-1)} dx = x \cdot F(\frac{1}{2}, 1-\kappa, \frac{3}{2}, x^2).$$

Thus we must choose c as

$$c = \frac{1}{2 \cdot F(\frac{1}{2}, 1-\kappa, \frac{3}{2}, 1)}.$$

With this choice

$$p(x) = c(1-x^2)^{(\kappa-1)} \tag{18}$$

becomes a density.

As a next step we consider the full mean-reverting process (7) with $\theta \neq 0$. The density p is a solution of

$$(1-\kappa)p(x) + x(2-\kappa)h'(x) + \kappa\theta p'(x) - \frac{1}{2}(1-x^2)p''(x) = 0. \tag{19}$$

Lengthy calculations show that

$$p(x) \;=\; \left(\frac{1-x}{1+x}\right)^{-\kappa\cdot\theta} \left(1-x^2\right)^{\kappa-1} \cdot \left[c + b \int_x^1 e^{\left(-\kappa\left(2\theta \tanh^{-1}(z) + \log\left(z^2-1\right)\right)\right)} dz \right]$$

solves (19) with $b, c \in \mathbb{R}$. For $\theta = 0$ we want p to be equal to the above solution (18). This is only possible if $b = 0$. Thus the final solution is

$$p(x) = c\left(\frac{1-x}{1+x}\right)^{-\kappa\cdot\theta} \left(1-x^2\right)^{\kappa-1} , \tag{20}$$

with c such that $\int_{-1}^{1} p(x) = 1$ holds.

Figure 2 shows how the transition densities may look like. As we could not compute the constant c analytically, it is determined numerically. Thus $\int_{-1}^{1} p(x)dx \approx 1$ holds for the plots.

We can observe that

- p is concentrated on $[-1, 1]$.
- p is symmetric with respect to θ in the sense of $p^{\theta}(x) = p^{-\theta}(-x)$.
- If $\theta = 0$ the global maximum is attained at $x = 0$.

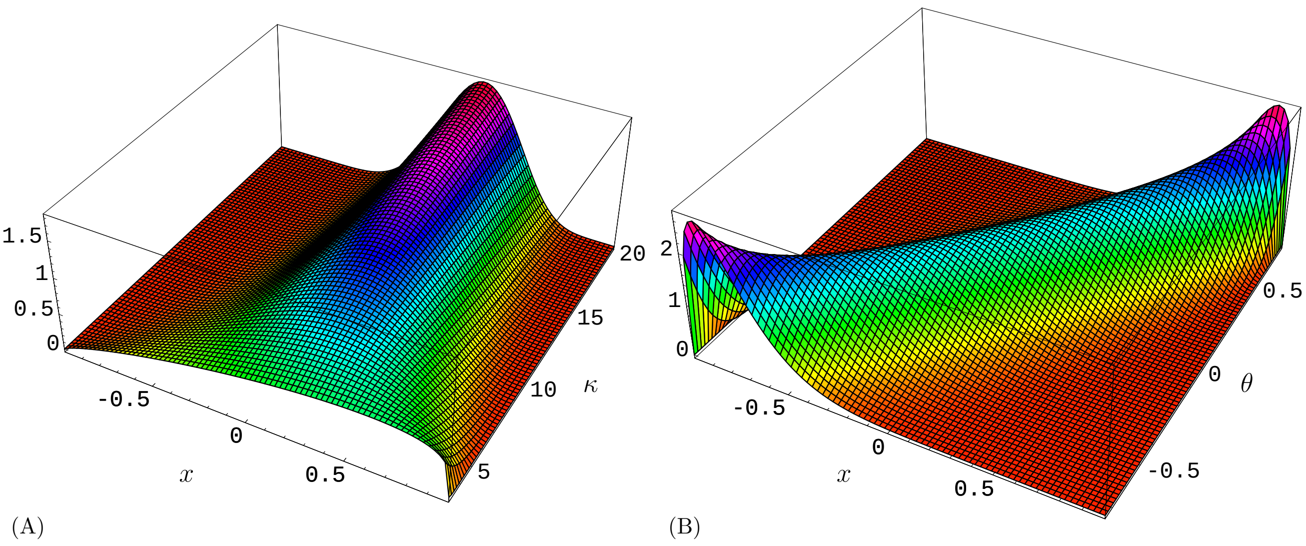

Figure 2: (A): Transition density (20) for varying $\kappa \in [3, 20]$, $\theta = 0.2$, $\int_{-1}^{1} p(x)dx \approx 1$, (B): Transition density (20) for varying $\theta \in [-0.75, 0.75]$, $\kappa = 12$, $\int_{-1}^{1} p(x)dx \approx 1$.

- The probability mass vanishes when getting away from the mean. In particular, it approaches zero in the boundary values.

These properties agree with intuition and what we demanded in the introductory section. Especially they correspond to figure 1. Therefore we consider the mean-reverting process (7) as suitable for modelling correlation.

Remark I: Numerical calculations indicate that (20) also fulfills

$$\int_{-1}^{1} xp(x)dx = \theta \,.$$

which is still to be proven analytically.

Remark II: The transition density can be used for calibrating the parameters, see for example Wilmott [8].

6 Practical Example - Quantos

A quanto is a cross-currency option. It is a combination of a regular option (European, American, Asian etc.) which pays off in currency A and a currency option which hedges against the exchange risk between A and another currency B. The quanto pays off in B. Thus the payoff is defined with respect to an underlying noted in currency A. Then this payoff is converted to currency B.
As an example we can think of a call on the Dow Jones whose payoff is paid in Euro

$$(Dow_T - Strike)^+ \cdot exchange - rate_0 \,.$$

In the following we assume a Black-Scholes world with lognormal distribution for the underlying asset

$$dS_t = \mu_S S_t dt + \sigma_S S_t dW_t^S$$

and lognormally distributed exchange rate

$$dX_t = \mu_X X_t dt + \sigma_X X_t dW_t^X \,.$$

If the correlation between W^S and W^X is constant,

$$dW_t^S dW_t^X = \rho dt ,$$

an analytical solution for pricing a quanto call exists. It turns out that pricing a quanto reduces to pricing a call in the Black-Scholes setting with the adjusted constant dividend yield

$$r_f - r_h + \rho\sigma_S\sigma_X .$$

See for example Wilmot[7]. Hereby r_f denotes the risk-free interest rate in the currency in which S is traded (foreign currency). The risk-free interest rate of the other currency is r_h (home).

The remaining question is how to incorporate stochastic correlation in this framework. We assume the model to be arbitrage free. Thus the expected return of one unit of home currency, exchanged, risk-free invested in the foreign country and re-exchanged must equal the risk-free return on one unit:

$$\frac{1}{X_0}\exp(r_f T)E\left[X_T\right] = \exp(r_h T) . \tag{21}$$

The exchange rate X_t follows a geometric Brownian motion and thus $E\left[X_T\right] = X_0\exp(\mu_X T)$. Insertion in (21) yields the restriction

$$\mu_X = r_h - r_f . \tag{22}$$

In addition the arbitrage argument leads to a second restriction. Analogous argumentation as above makes clear that

$$\frac{1}{X_0}\frac{1}{S_0}E\left[S_T X_T\right] = \exp(r_h T) . \tag{23}$$

must hold. Hereby the left side describes the re-exchanged expectation of an investment of one home currency unit into the underlying S.
For calculating $E\left[S(T)X(T)\right]$ we need to compute

$$\begin{aligned} d(S_t X_t) &= S_t dX_t + X_t dS_t + dS_t dX_t \\ &= S_t X_t \left[(\mu_S + \mu_X)dt + \sigma_S dW_t^S + \sigma_X dW_t^X + \rho_t\sigma_S\sigma_X dt\right] . \end{aligned}$$

Ito's formula implies

$$d(\ln(x_t)) = \frac{1}{x_t}dx_t - \frac{1}{2}\frac{1}{x_t^2}(dx_t)^2 .$$

Application to $S_t X_t$ leads to

$$\int_0^T \frac{d(S_t X_t)}{S_t X_t} = (\mu_S + \mu_X)T + \sigma_S W_T^S + \sigma_X W_T^X - \frac{1}{2}\left(\left(\sigma_S^2 + \sigma_X^2\right)T\right) .$$

A further application of Ito's formula leads to

$$E\left[S_T X_T\right] = S_0 X_0 E\left[\exp\left((\mu_S + \mu_X)T + \sigma_S\sigma_X\int_0^T \rho dt\right)\right] . \tag{24}$$

Thus with the choice

$$\mu_S = r_h - \mu_X - \sigma_S\sigma_X\frac{1}{T}\int_0^T \rho dt , \tag{25}$$

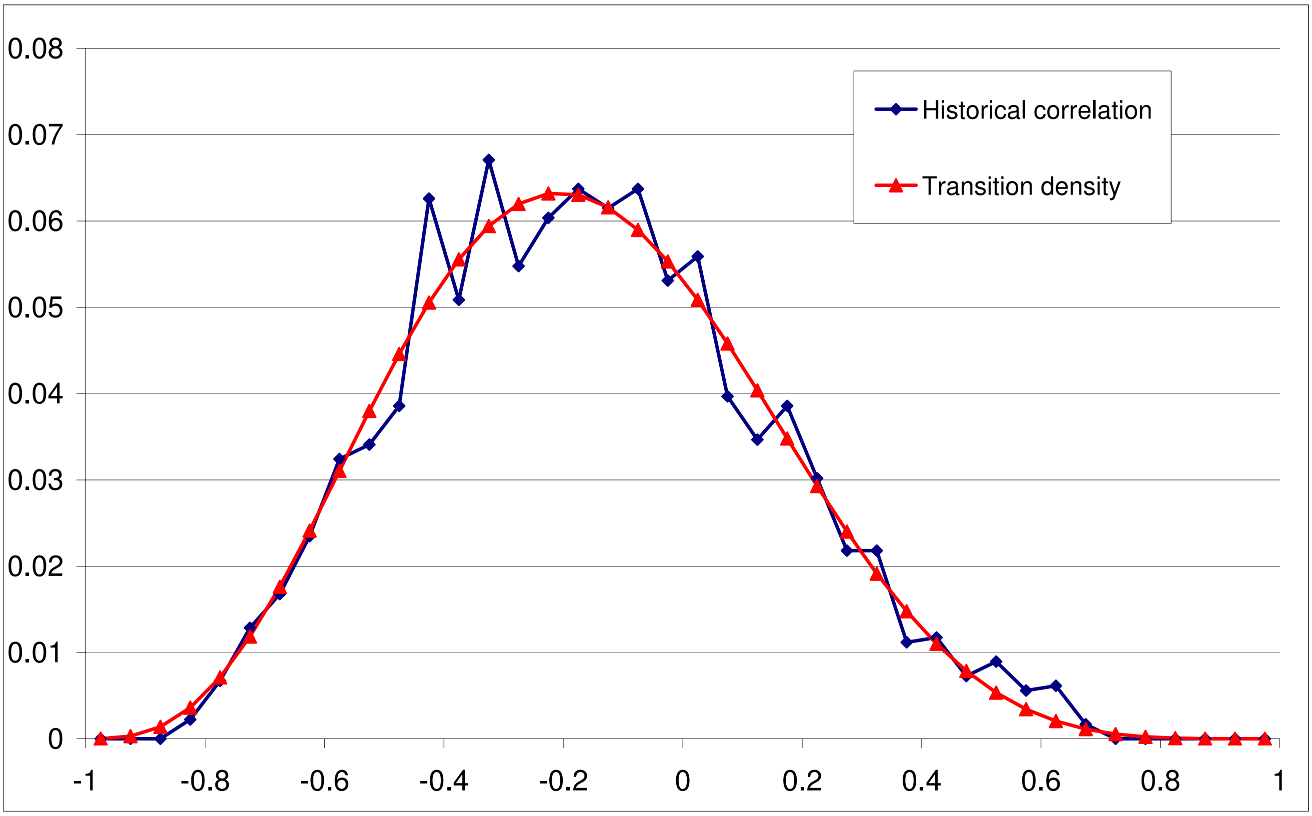

Figure 3: Correlation between Dow Jones and Euro/US-Dollar exchange rate, density; empirical distribution vs. density (20) computed with $\theta = -0.1, \kappa = 10.6$.

the no-arbitrage condition (23) is fulfilled. Note that the choice

$$\mu_S = r_h - \mu_X - \ln E\left[\sigma_S \sigma_X \frac{1}{T}\int_0^T \rho dt\right] ,$$

would also be consistent with (23), but in that case the stochasticity would have been lost. Now we can use a conditional Monte Carlo approach. We simulate paths for ρ. Then we obtain a constant expression for μ_S for every path. Interpreting (25) as return minus continuous dividend

$$r_h - (r_h - r_f + \sigma_S \sigma_X \int_0^T \rho_t dt) ,$$

we can simply use the Black-Scholes price for stocks with continuous dividend payments and get a solution for every path. Afterwards we compute the fair price of the quanto as the mean over all Black-Scholes prices.

We test if stochastic correlation leads to different prices for quantos compared to the constant correlation case. We consider the following process for stochastic correlation

$$d\rho_t = \kappa \cdot (\theta - \rho_t)dt + 1 \cdot \sqrt{1-\rho_t^2}dW_t . \tag{26}$$

As above W denotes a Brownian motion. If not mentioned otherwise $\theta = -0.1, \kappa = 10.6$. The parameter choice results from a least-square fitting of (7) to the historical data from figure 1. This fitting is surprisingly good, see figure 3.

Underlying and exchange rate are supposed to follow a lognormal distributions with $\sigma_S = 0.2$ and $\sigma_X = 0.4$, respectively. The risk free interest rates amounts to $r_h = 0.05$ and $r_f = 0.03$. The underlying starts in 100, the starting value of the exchange rate is 1. The strike amounts to 120. For numerical integration the Milstein scheme is used, see for example Kloeden and Platen[4]. The number of paths amounts to 10000, and the step size is set to 0.001. In none of the cases the boundaries $l = -1$ and $r = 1$ have been exceeded thus we consider the Milstein scheme as an appropriate choice. This effect perfectly agrees with the result that the boundaries are not attractive for this parameter configuration.

Figures 4 and 5 show prices computed for constant $\rho = \theta$ using the analytic formula (continuous line). The crosses show the prices determined by simulation using our correlation process (7).

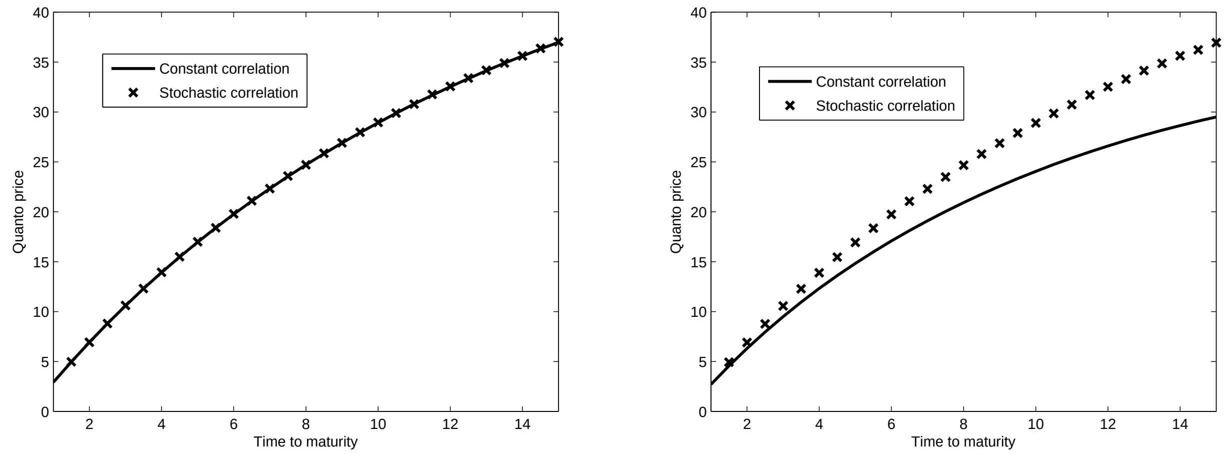

Figure 4: Comparison of prices for quanto with fixed and stochastic ρ, (A): $\rho_0 = -0.1$, (B): $\rho_0 = 0$.

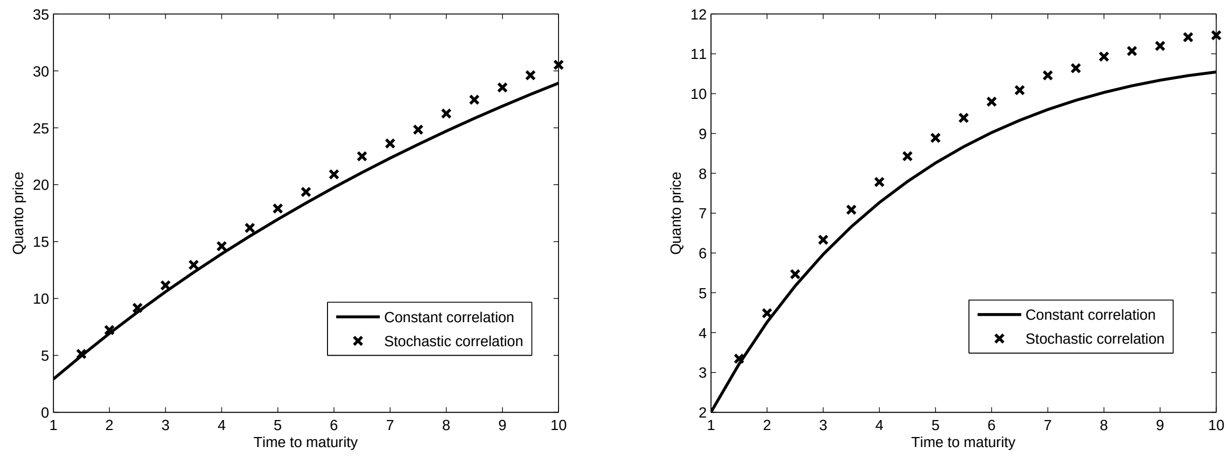

Figure 5: Comparison of prices for quanto with fixed and stochastic ρ; (A): $\rho_0 = -0.1, \kappa = 1.06$, (B): $\rho_0 = 0.4, \theta = 0.4, \kappa = 1.06$.

In figure 4 (A) there is no decisive difference between the prices calculated with constant correlation and the prices calculated with (25) where ρ_t follows (26). The reason is that we do not use the whole path of correlation but eventually only the distribution at T. One can easily verify that with $\rho_0 = \theta$ the Milstein scheme generates random variables at T with mean θ. As the mean-reverting factor κ is relatively high this distribution is strongly concentrated around the mean. In contrast to that, in figure 4 (B) we can observe a difference: The prices for stochastic correlation are higher as for constant correlation. This is due to the fact that the expectation of

correlation changes over time. Consequently the additional freedom we win by using stochastic processes for modelling correlation does have an influence on pricing.
In figure 5 (A) initial value and mean are equal but here the mean reverting factor κ is smaller compared to figure 4. We observe that stochastic correlation leads to higher prices than constant correlation although the expectation is equal. Thus pricing with constant correlation means neglecting correlation risk. The effect is even stronger if θ is higher as seen in figure 5 (B).

7 Conclusion

Based on independent Brownian motions V, W, K and a given stochastic process

$$d\rho_t = a(t,\rho_t)dt + b(t,\rho_t)dK_t \ , \qquad \rho_0 \in [-1,1] \ ,$$

we constructed a further Brownian motion

$$Z_t = \int_0^t \rho_t dW_s + \int_0^t \sqrt{1-\rho_t^2} dV_s \, .$$

Interpreting ρ_t as a correlation process Z and W are stochastically correlated. We presented suitable functions a and b for modelling the correlation process. We focussed on (7)

$$d\rho_t = \kappa(\theta-\rho_t)dt + \alpha\sqrt{1-\rho_t^2}dW_t \ , \qquad \rho_0 = r_0 \ .$$

Analysis of boundary behaviour and transition density showed that it fulfills the natural features we expect correlation to possess.
Finally we showed how to incorporate the concept of fully stochastic correlation into the modelling of a financial product, the quanto. We observed that correlation risk is neglected if we choose constant correlation instead of the here presented fully stochastic correlation.

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