Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Lehrstuhl für Angewandte Mathematik
und Numerische Mathematik

Preprint BUW-AMNA 06/01

Roland Pulch

Transformation Qualities of Warped MPDAE Models

January 2006

http://www.math.uni-wuppertal.de/org/Num/

Transformation Qualities of Warped MPDAE Models

R. Pulch

Bergische Universität Wuppertal, Fachbereich C,
Arbeitsgruppe Angewandte Mathematik / Numerische Mathematik,
Gaußstr. 20, D-42119 Wuppertal, Germany.

Abstract
Radio frequency (RF) applications exhibit oscillatory signals, where the amplitude and the frequency changes slowly in time. Numerical simulations can be performed by a multidimensional model involving a warped multirate partial differential algebraic equation (MPDAE). Consequently, a frequency modulated signal demands a representation via a function in two variables as well as a univariate frequency function. The efficiency of this approach depends essentially on the determination of appropriate frequencies. However, the multidimensional representation is not specified uniquely by the corresponding RF signal. We prove that choices from a continuum of functions are feasible, which are interconnected by a specific transformation. Existence theorems for solutions of the MPDAE system demonstrate this degree of freedom. Furthermore, we perform numerical simulations to verify the transformation properties, where a voltage controlled oscillator is used.

1 Introduction

The modified nodal analysis represents a well established strategy for modelling electric circuits, see [4]. This network approach yields systems of differential algebraic equations (DAEs), which describe the transient behaviour of all node voltages and some branch currents. In particular, the determination of quasiperiodic solutions is a well known problem in radio frequency (RF) applications. These signals are characterised by a specific oscillating behaviour at several time scales, where the respective magnitudes differ significantly. Consequently, a transient analysis of the DAE system becomes inefficient, since the size of time steps is limited by the fastest oscillation, whereas the slowest time scale determines the time interval of the simulation. Time and frequency domain methods have been constructed for the direct computation of quasiperiodic solutions, see [2] and [11], for example. However, drawbacks may occur in these techniques in view of widely separated time scales or strong nonlinearities.

A multivariate model enables an alternative approach by decoupling the time scales of such signals, which generates an efficient representation of amplitude modulated signals. Based on this strategy, Brachtendorf et al. [1] remodelled the DAE into a multirate partial differential algebraic equation (MPDAE). Narayan and Roychowdhury [7] generalised the approach in view of frequency modulation and introduced a corresponding warped MPDAE. Multiperiodic solutions of this system reproduce quasiperiodic signals satisfying the DAE. The determination of a suitable local frequency function arising in the model is crucial for the efficiency of the technique.

In this paper, we analyse transformations of multivariate representations for frequency modulated signals. Thereby, the emphasis is on quasiperiodic functions. We perform investigations in the most frequent case of two time scales. Nevertheless, generalisations to several scales are straightforward. Firstly, a transformation of multivariate functions is examined, where all associated representations reproduce the same signal. Secondly, we retrieve the same degree of freedom for solutions of the warped MPDAE system, too. Moreover, a transformation to a representation with constant local frequency is feasible. Thus solutions can be carried over to the unwarped MPDAE system. Consequently, we apply theorems referring to the unwarped system, which are given by Roychowdhury [10], to obtain analogue results for the warped system.

The paper is organised as follows. In Sect. 2, we outline the multidimensional model for signals. Consequently, we define quasiperiodic signals via corresponding multivariate functions. A specific transformation of multivariate representations is constructed, which illustrates an intrinsic degree of freedom in the model. We introduce the warped MPDAE model briefly in Sect. 3. An underlying information transport along characteristic curves is described, which reflects the transformation properties. Accordingly, we formulate theorems concerning transformations of solutions, which represent the analogon of the results on the level of the signals. Based on these implications, the specification of additional boundary conditions is discussed. Finally, Sect. 4 includes numerical simulations using the warped MPDAE system, which demonstrate the degrees of freedom in the representations. An appendix contains the proofs of presented theorems.

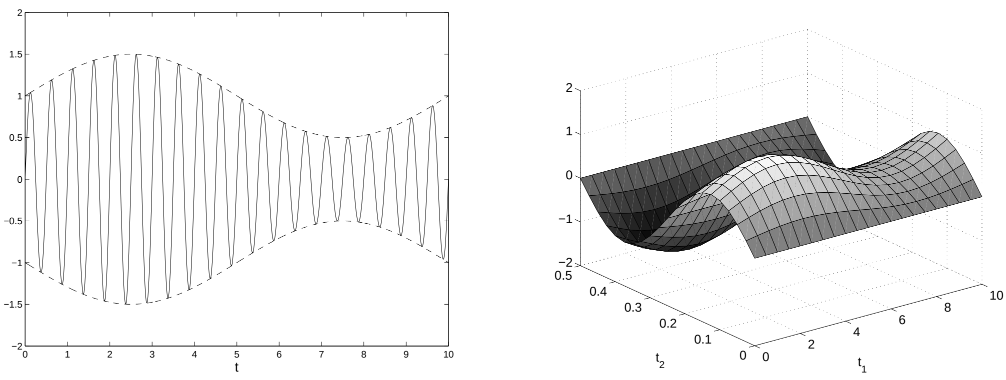

Figure 1: Amplitude modulated signal y (left) and corresponding MVF $\hat{y}$ (right).

2 Signal Model

2.1 Multivariate Signal Representation

To illustrate the multidimensional signal model, we consider the purely amplitude modulated signal

$$y(t) := \left[1 + \alpha \sin\left(\tfrac{2\pi}{T_1} t\right)\right] \sin\left(\tfrac{2\pi}{T_2} t\right). \tag{1}$$

The parameter $\alpha \in (0,1)$ determines the amount of amplitude modulation. Fig. 1 (left) depicts this function qualitatively. Hence many time steps are required to resolve all oscillations within the interval $[0, T_1]$ if $T_1 \gg T_2$ holds. Therefore we describe each separate time scale by an own variable and obtain

$$\hat{y}(t_1, t_2) := \left[1 + \alpha \sin\left(\tfrac{2\pi}{T_1} t_1\right)\right] \sin\left(\tfrac{2\pi}{T_2} t_2\right). \tag{2}$$

The new representation is called the *multivariate function* (*MVF*) of the signal (1). In the example, the MVF is biperiodic and thus already given by its values in the rectangle $[0, T_1[\times[0, T_2[$. Fig. 1 (right) shows this function. Since the time scales are decoupled, the MVF exhibits a simple behaviour. Accordingly, we need a relatively low number of grid points to represent the MVF sufficiently accurate. Yet the original signal (1) can be completely reconstructed by its MVF (2), because it is included on the diagonal, i.e.,

$$y(t) = \hat{y}(t, t). \tag{3}$$

In general, straightforward constructions of MVFs for purely amplitude modulated signals yield efficient representations.

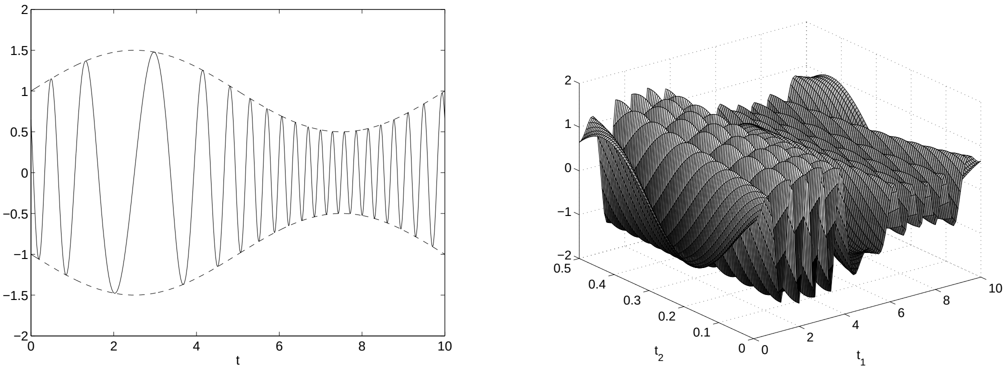

Figure 2: Frequency modulated signal x (left) and unsophisticated MVF $\hat{x}_1$ (right).

The situation becomes more difficult in case of frequency modulation. For example, we examine the modified signal

$$x(t) := \left[1 + \alpha \sin\left(\tfrac{2\pi}{T_1} t\right)\right] \sin\left(\tfrac{2\pi}{T_2} t + \beta \cos\left(\tfrac{2\pi}{T_1} t\right)\right), \tag{4}$$

where the parameter $\beta > 0$ specifies the amount of frequency modulation. This signal is illustrated in Fig. 2 (left). A direct transition to a biperiodic MVF is also feasible in this situation and we obtain

$$\hat{x}_1(t_1, t_2) := \left[1 + \alpha \sin\left(\tfrac{2\pi}{T_1} t_1\right)\right] \sin\left(\tfrac{2\pi}{T_2} t_2 + \beta \cos\left(\tfrac{2\pi}{T_1} t_1\right)\right). \tag{5}$$

However, the MVF exhibits many oscillations in the rectangle $[0, T_1[\times[0, T_2[$, too, see Fig. 2 (right). The number of oscillations depends on the amount of frequency modulation β. Hence this multidimensional description is unsuited.

To achieve an efficient representation, Narayan and Roychowdhury [7] propose a readjusted model. Thereby, the MVF incorporates only the amplitude modulation part

$$\hat{x}_2(t_1, t_2) := \left[1 + \alpha \sin\left(\tfrac{2\pi}{T_1} t_1\right)\right] \sin\left(2\pi t_2\right), \tag{6}$$

where the second period is transformed to 1. This function has the same simple form as the MVF (2). The frequency modulation part is described by an additional time-dependent *warping function*

$$\Psi(t) := \frac{t}{T_2} + \frac{\beta}{2\pi} \cos\left(\tfrac{2\pi}{T_1} t\right). \tag{7}$$

We take the derivative of the warping function as a corresponding *local frequency function*, i.e.,

$$\nu(t) := \Psi'(t) = \frac{1}{T_2} - \frac{\beta}{T_1} \sin\left(\tfrac{2\pi}{T_1} t\right), \tag{8}$$

which represents an elementary T_1-periodic function in this example. Since we assume $T_1 \gg T_2$, it holds $\nu(t) > 0$ for a broad range of parameters β. The reconstruction of the original signal (4) reads

$$x(t) = \hat{x}_2(t, \Psi(t)), \tag{9}$$

where the warping function stretches the second time scale. Hence we obtain a powerful model for signals, which feature amplitude as well as frequency modulation with largely differing rates.

2.2 Definition of Quasiperiodic Signals

Commonly, a univariate function $x : \mathbb{R} \to \mathbb{C}$ is said to be two-tone quasiperiodic, if it can be represented by a two-dimensional Fourier series of the form

$$x(t) = \sum_{j_1,j_2=-\infty}^{+\infty} X_{j_1,j_2} \exp\left(i \left(\tfrac{2\pi}{T_1} j_1 + \tfrac{2\pi}{T_2} j_2\right) t\right) \tag{10}$$

with rates $T_1, T_2 > 0$ and coefficients $X_{j_1,j_2} \in \mathbb{C}$, where $i := \sqrt{-1}$ denotes the imaginary unit. However, we have to specify the kind of convergence in the arising series. The continuity of signals is guaranteed via absolute convergence, i.e.,

$$\sum_{j_1,j_2=-\infty}^{+\infty} |X_{j_1,j_2}| < \infty, \tag{11}$$

which implies also a uniform convergence. Moreover, the series becomes well defined with respect to permutations of the terms. In particular, an interchange of j_1 and j_2 is allowed. Assuming (11), the biperiodic MVF $\hat{x} : \mathbb{R}^2 \to \mathbb{C}$ of (10)

$$\hat{x}(t_1, t_2) := \sum_{j_1,j_2=-\infty}^{+\infty} X_{j_1,j_2} \exp\left(i \left(\tfrac{2\pi}{T_1} j_1 t_1 + \tfrac{2\pi}{T_2} j_2 t_2\right)\right). \tag{12}$$

is continuous, too. For locally integrable functions, weaker concepts of convergence are feasible. However, our aim is to compute solutions of differential equations and thus smooth functions are required. Using the representation (10), just sufficient conditions can be formulated. For example, the strong requirements

$$\sum_{j_1,j_2=-\infty}^{+\infty} |X_{j_1,j_2}| \left(|j_1| + |j_2|\right) < \infty \tag{13}$$

or

$$\sum_{j_1,j_2=-\infty}^{+\infty} |X_{j_1,j_2}| \left(|j_1|^2 + |j_1 j_2| + |j_2|^2\right) < \infty \tag{14}$$

guarantee $x \in C^1$ and $x \in C^2$, respectively. To omit the discussion of convergence properties, an alternative definition of quasiperiodic functions makes sense.

Definition 1 *A function* $x : \mathbb{R} \to \mathbb{C}$ *is* two-tone quasiperiodic *with rates* T_1 *and* T_2 *if a* (T_1, T_2)*-periodic function* $\hat{x} : \mathbb{R}^2 \to \mathbb{C}$ *exists satisfying* $x(t) = \hat{x}(t,t)$.

If $\hat{x}$ exhibits some smoothness, then the function x inherits the same smoothness. For signals of the form (10), the multivariate representation is given by (12), which implies quasiperiodicity in view of this definition, too.

Our aim is to analyse frequency modulated signals. Following the modelling in the previous subsection, we consider functions $x : \mathbb{R} \to \mathbb{C}$ of the form

$$x(t) = \sum_{j_1,j_2=-\infty}^{+\infty} X_{j_1,j_2} \exp\left(i\left(\tfrac{2\pi}{T_1} j_1 t + 2\pi j_2 \Psi(t)\right)\right), \tag{15}$$

where $\Psi : \mathbb{R} \to \mathbb{R}$ represents a warping function. The above discussion of convergence applies also to this case. Alternatively, we formulate a characterisation according to Definition 1.

Definition 2 *A function* $x : \mathbb{R} \to \mathbb{C}$ *is* frequency modulated two-tone quasiperiodic *with rate* T_1 *if there exists a* $(T_1, 1)$*-periodic function* $\hat{x} : \mathbb{R}^2 \to \mathbb{C}$ *and a function* $\Psi : \mathbb{R} \to \mathbb{R}$ *with* T_1*-periodic derivative* Ψ' *such that* $x(t) = \hat{x}(t, \Psi(t))$.

In this definition, it is not necessary that $\nu(t) := \Psi'(t) > 0$ holds for all t. However, frequency modulated signals, which arise in electric circuits, feature fast oscillations, whose frequencies vary slowly in time. Thus the positivity of the local frequency function ν and the demand $T_1 > \nu(t)^{-1}$ for all t is often required to obtain an efficient representation.

We recognise that all specifications of quasiperiodic functions imply the existence of corresponding MVFs. Hence quasiperiodic signals exhibit an inherent multidimensional structure and the transition to the multivariate model becomes natural.

Furthermore, the slow time scale may be aperiodic. In this case, we obtain a Fourier expansion of the type

$$x(t) = \sum_{j_2=-\infty}^{+\infty} X_{j_2}(t) \exp\left(i 2\pi j_2 \Psi(t)\right) \tag{16}$$

with time-dependent coefficients $X_{j_2} : \mathbb{R} \to \mathbb{C}$, which are called the envelopes. The envelopes introduce an amplitude modulation, whereas the warping function again describes a frequency modulation. A formal definition is given below.

Definition 3 *A function* $x : \mathbb{R} \to \mathbb{C}$ *is called* envelope modulated *if a function* $\hat{x} : \mathbb{R}^2 \to \mathbb{C}$*, which is periodic in the second variable with rate 1, and a function* $\Psi : \mathbb{R} \to \mathbb{R}$ *exist, where* $x(t) = \hat{x}(t, \Psi(t))$ *holds.*

We take $\nu := \Psi'$ as a local frequency of the signal again. The generalisation of the above definitions to vector-valued functions $\mathbf{x} : \mathbb{R} \to \mathbb{C}^k$ consists in demanding the conditions in each component separately.

2.3 Transformation of Signal Representations

For modelling a frequency modulated signal, the employed MVF and corresponding warping function is not unique. We obtain a fundamental result already for general signals, which do not necessarily feature periodicities in the time scales.

Theorem 1 *If the signal* $x : \mathbb{R} \to \mathbb{C}$ *is represented by the MVF* $\hat{x} : \mathbb{R}^2 \to \mathbb{C}$ *and the warping function* $\Psi : \mathbb{R} \to \mathbb{R}$*, i.e.,* $x(t) = \hat{x}(t, \Psi(t))$*, then the MVF*

$$\hat{y} : \mathbb{R}^2 \to \mathbb{C}, \quad \hat{y}(t_1, t_2) := \hat{x}\left(t_1, t_2 + \Psi(t_1) - \Phi(t_1)\right) \tag{17}$$

satisfies $x(t) = \hat{y}(t, \Phi(t))$ *for an arbitrary function* $\Phi : \mathbb{R} \to \mathbb{R}$*.*

Thus if a representation of a signal exists using some MVF and warping function, then we can prescribe a new warping function and transform to another MVF, which yields the same information as before. Hence we apply the warping function or the associated local frequency function as free parameters to obtain an efficient multivariate representation.

In the following, we discuss the model on the level of local frequency functions. Considering $\nu := \Psi'$, $\mu := \Phi'$ and $\Psi(0) = \Phi(0) = 0$, the transformation (17) reads

$$\hat{y} : \mathbb{R}^2 \to \mathbb{C}, \quad \hat{y}(t_1, t_2) := \hat{x}\left(t_1, t_2 + \int_0^{t_1} \nu(s) - \mu(s) \, \mathrm{d}s\right). \tag{18}$$

In case of envelope modulated signals, see Definition 3, the fast time scale is periodic, whereas the slow time scale may be aperiodic. The transformation (18) preserves the periodicity in the second variable. Specifying an arbitrary local frequency function yields a corresponding MVF in view of Theorem 1. For quasiperiodic signals, the slow time scale is periodic, too. Due to Definition 2, the corresponding MVFs have to be biperiodic. Thus transformations are feasible only if they preserve the periodicities. To analyse the feasibility, we define an average frequency.

Definition 4 *If $\nu : \mathbb{R} \to \mathbb{R}$ represents a T-periodic locally integrable frequency function, then the* average frequency *is given by the integral mean*

$$\overline{\nu} := \frac{1}{T} \int_0^T \nu(s) \, \mathrm{d}s. \tag{19}$$

Theorem 1 implies the following result for transformations in the quasiperiodic case.

Theorem 2 *Let $\nu, \mu : \mathbb{R} \to \mathbb{R}$ be T_1-periodic locally integrable functions with the property $\overline{\nu} = \overline{\mu}$. If $\hat{x} : \mathbb{R}^2 \to \mathbb{C}$ is a $(T_1, 1)$-periodic function, then the function $\hat{y} : \mathbb{R}^2 \to \mathbb{C}$ defined by (18) is $(T_1, 1)$-periodic, too. Furthermore, if $x(t) = \hat{x}(t, \int_0^t \nu(s) \, \mathrm{d}s)$ holds for all $t \in \mathbb{R}$, then it follows $x(t) = \hat{y}(t, \int_0^t \mu(s) \, \mathrm{d}s)$ for all $t \in \mathbb{R}$.*

Hence a frequency modulated quasiperiodic signal implies a continuum of representations via MVFs and respective local frequency functions, which all exhibit the same average frequency. We can perform the transformation (18) on the level of the representation (15), too. The following theorem yields an according result. However, stronger conditions on the original representation are necessary to guarantee the absolute convergence in the transformed representation.

Theorem 3 *Let $x : \mathbb{R} \to \mathbb{C}$ be a frequency modulated two-tone quasiperiodic function defined by (15) with $\Psi(t) := \int_0^t \nu(s) \, \mathrm{d}s$, where $\nu : \mathbb{R} \to \mathbb{R}$ represents a T_1-periodic locally integrable function. The coefficients $X_{j_1,j_2} \in \mathbb{C}$ shall have the property*

$$\sum_{j_1,j_2=-\infty}^{+\infty} |X_{j_1,j_2}| \left(|j_1| + |j_1 j_2| + |j_2|^2 \right) < \infty. \tag{20}$$

If $\mu : \mathbb{R} \to \mathbb{R}$ is a locally integrable T_1-periodic function satisfying $\overline{\nu} = \overline{\mu}$, then x owns the representation

$$x(t) = \sum_{j_1,j_2=-\infty}^{+\infty} \tilde{X}_{j_1,j_2} \exp\left(i \left(\tfrac{2\pi}{T_1} j_1 t + 2\pi j_2 \Phi(t) \right) \right) \tag{21}$$

with $\Phi(t) := \int_0^t \mu(s) \, \mathrm{d}s$ and new coefficients $\tilde{X}_{j_1,j_2} \in \mathbb{C}$ satisfying

$$\sum_{j_1,j_2=-\infty}^{+\infty} \left| \tilde{X}_{j_1,j_2} \right| < \infty. \tag{22}$$

An important consequence of Theorem 2 is that transformations to representations with constant local frequency can be performed. Given a biperiodic MVF and periodic local frequency function ν, the constant frequency $\mu \equiv \overline{\nu}$ enables a corresponding transformation, which is feasible in view of $\overline{\mu} = \overline{\nu}$. Thus the following corollary unifies the definition of quasiperiodic signals with respect to different local frequencies.

Corollary 1 *A frequency modulated quasiperiodic function characterised by Definition 2 features a representation as quasiperiodic function with constant rates according to Definition 1 and vice versa.*

Hence no qualitative difference between quasiperiodic signals involving constant and stretched time scales exists. Therefore we will just speak of quasiperiodic functions in the following. Given a frequency modulated signal, it does not make sense to assign a constant fast rate. However, an according interpretation as an average rate is reasonable.

The choice of a local frequency function and according multivariate representation is important for the efficiency of the multidimensional signal model. Regarding quasiperiodic functions, we consider a MVF corresponding to a constant frequency $\overline{\nu}$. Now we may transform the model to any local frequency function of the type

$$\mu(s) := \overline{\nu} + \xi(s) \quad \text{with} \quad \overline{\xi} = 0. \tag{23}$$

The function ξ is the degree of freedom in our design of multidimensional models for quasiperiodic signals.

3 Warped MPDAE Model

3.1 Derivation of the Model

The numerical simulation of electric circuits employs a network approach, which yields systems of *differential algebraic equations* (*DAEs*), see [3]. Thereby, the system describes the transient behaviour of all node voltages and some branch currents. We write such a system in the general form

$$\frac{\mathrm{d}\mathbf{q}(\mathbf{x})}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}(t)) + \mathbf{b}(t), \tag{24}$$

where $\mathbf{x} : \mathbb{R} \to \mathbb{R}^k$ represents the unknown voltages and currents. The functions $\mathbf{q}, \mathbf{f} : \mathbb{R}^k \to \mathbb{R}^k$ correspond to a charge and a resistive term, respectively. Predetermined input signals are included in the time-dependent function $\mathbf{b} : \mathbb{R} \to \mathbb{R}^k$.

We demand $\mathbf{f}, \mathbf{b} \in C^0$ and $\mathbf{q} \in C^1$, since smooth solutions of (24) are desired. DAEs cause specific theoretical and numerical problems like the index concept or the need of consistent initial values, see [6].

Given some two-tone quasiperiodic input $\mathbf{b}$ with rates T_1 and T_2, a forced oscillation arises, which often leads to amplitude modulated signals. We assume that the solution inherits the time scales, i.e., $\mathbf{x}$ is also quasiperiodic with same rates. Consequently, we obtain the (T_1, T_2)-periodic MVFs $\hat{\mathbf{b}}$ and $\hat{\mathbf{x}}$, which represent the corresponding quasiperiodic signals. Brachtendorf et al. [1] introduced the according system of *multirate partial differential algebraic equations* (*MPDAEs*)

$$\frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_1} + \frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_2} = \mathbf{f}(\hat{\mathbf{x}}(t_1, t_2)) + \hat{\mathbf{b}}(t_1, t_2), \tag{25}$$

which results from a transformation of the DAEs (24) with respect to multivariate functions. An arbitrary solution of the system (25) yields a solution of the system (24) via the reconstruction

$$\mathbf{x}(t) = \hat{\mathbf{x}}(t, t). \tag{26}$$

The proof is straightforward and can be found in [10], for example.

If the MVF is (T_1, T_2)-periodic, then the reconstructed signal is two-tone quasiperiodic with rates T_1, T_2. Thus the determination of quasiperiodic signals leads to the boundary value problem

$$\hat{\mathbf{x}}(t_1, t_2) = \hat{\mathbf{x}}(t_1 + T_1, t_2) = \hat{\mathbf{x}}(t_1, t_2 + T_2) \quad \text{for all } \ t_1 \in \mathbb{R}, \ t_2 \in \mathbb{R}. \tag{27}$$

If the slow time scale is aperiodic, then we obtain envelope modulated signals by solutions of the MPDAE, too. Therefore a mixture of initial and boundary conditions is considered, namely

$$\hat{\mathbf{x}}(0, t_2) = \mathbf{h}(t_2), \quad \hat{\mathbf{x}}(t_1, t_2 + T_2) = \hat{\mathbf{x}}(t_1, t_2) \quad \text{for all } \ t_1 \geq 0, \ t_2 \in \mathbb{R}, \tag{28}$$

where $\mathbf{h} : \mathbb{R} \to \mathbb{R}^k$ represents a prescribed T_2-periodic function, whose values have to be consistent with respect to the DAEs (24). The choice of appropriate initial values influences the efficiency of this approach. Note that the reconstructed signal (26) depends on the value $\mathbf{h}(0)$ only. For further details, we refer to [10].

Now we assume that the input signals exhibit a slow time scale only. Nevertheless, the DAE system (24) shall feature an inherent fast time scale. Consequently, multitone signals arise, which may be amplitude modulated as well as frequency modulated. Narayan and Roychowdhury [7] generalised the MPDAE model to this problem. The transition to MVFs implies a system of *warped multirate partial differential algebraic equations*, namely

$$\frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_1} + \nu(t_1) \frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_2} = \mathbf{f}(\hat{\mathbf{x}}(t_1, t_2)) + \mathbf{b}(t_1) \tag{29}$$

with the unknown solution $\hat{\mathbf{x}} : \mathbb{R}^2 \to \mathbb{R}^k$. Since we assume that the input $\mathbf{b}$ just acts on the slow time scale, a multivariate description is not necessary here. A local frequency function $\nu : \mathbb{R} \to \mathbb{R}$ arises, which depends on the same variable as $\mathbf{b}$ if the input causes the frequency modulation. In the following, we assume $\nu \in C^0$, since smooth solutions are discussed. An appropriate choice for the local frequencies is unknown a priori. Furthermore, the system (29) is autonomous in the second variable t_2, since the fast time scale is not forced by the input but inherent.

Solving the warped MPDAEs (29) for some given local frequency function, we obtain a solution of the DAEs (24) via

$$\mathbf{x}(t) = \hat{\mathbf{x}}(t, \Psi(t)) \quad \text{with} \quad \Psi(t) := \int_0^t \nu(s)\, \mathrm{d}s. \tag{30}$$

The proof operates similar to the case of constant time scales. Again periodicities are necessary to solve the system (29) in a bounded domain. We always consider a periodic fast time scale, where the period is standardised to $T_2 = 1$. The magnitude of the fast rate is included in the local frequency function. Initial-boundary value problems (28) of the system (29) determine envelope modulated signals, which exhibit frequency modulation.

If the input signals are T_1-periodic, then a $(T_1, 1)$-periodic MVF in addition to a T_1-periodic local frequency function yield a two-tone quasiperiodic signal in the reconstruction (30). Thus the boundary value problem (27) is considered. However, given a biperiodic solution $\hat{\mathbf{x}}$, the shifted function

$$\hat{\mathbf{y}}(t_1, t_2) := \hat{\mathbf{x}}(t_1, t_2 + c) \quad \text{for} \;\; c \in \mathbb{R} \tag{31}$$

also satisfies the system including the same local frequency function. Hence a solution of the MPDAE reproduces a continuum of signals solving the underlying DAE. Therefore the multidimensional approach reveals a degree of freedom by shifting, which is not directly transparent in the according DAE model (24). In a corresponding numerical method for the problem (27),(29), a specific solution has to be isolated from the continuum (31).

The system (29) is underdetermined, since an adequate local frequency function is unspecified a priori. In the multidimensional model, the local frequency shall generate a corresponding MVF of a simple form. Since the solution is unknown, we do not have the knowledge to prescribe a local frequency function appropriately. Thus alternative conditions have to be added to the system (29) in order to fix a suitable solution. Several strategies are feasible like criteria based on specific minimisations, cf. [5]. The use of multidimensional phase conditions for this purpose will be discussed in Subsect. 3.4.

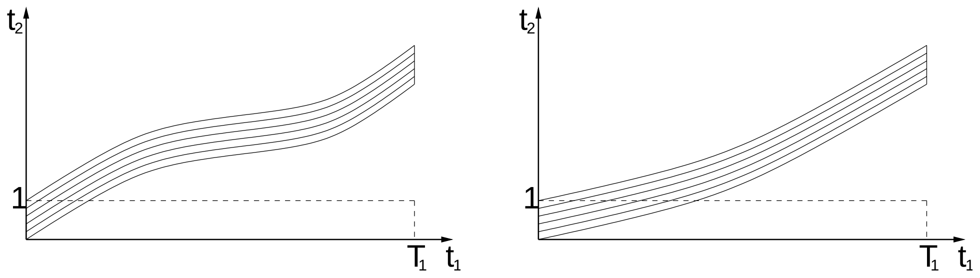

Figure 3: Characteristic projections for two different choices of local frequency functions.

3.2 Characteristic Curves

The warped MPDAE system (29) exhibits a specific transport of information, see [9]. The corresponding *characteristic system* reads

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}\tau} t_1(\tau) &= 1 \\ \tfrac{\mathrm{d}}{\mathrm{d}\tau} t_2(\tau) &= \nu(t_1(\tau)) \\ \tfrac{\mathrm{d}}{\mathrm{d}\tau} \mathbf{q}(\hat{\mathbf{x}}(\tau)) &= \mathbf{f}(\hat{\mathbf{x}}(\tau)) + \mathbf{b}(t_1(\tau)) \end{aligned} \tag{32}$$

with t_1, t_2 as well as $\hat{\mathbf{x}}$ depending on a parameter τ. Solutions of the system (32) are called *characteristic curves*. For fixed local frequency function ν, we solve the part with respect to the variables t_1, t_2 explicitly and obtain the *characteristic projections*

$$t_2 = \Psi(t_1) + c \quad \text{with} \quad \Psi(t_1) := \int_0^{t_1} \nu(s)\,\mathrm{d}s, \tag{33}$$

where $c \in \mathbb{R}$ represents an arbitrary constant. Hence the characteristic projections form a continuum of parallel curves in the domain of dependence. Fig. 3 illustrates this property.

In the initial-boundary value problem (28) belonging to (29), we consider the initial manifold

$$\mathcal{F}_0 := \{0\} \times [0, 1[. \tag{34}$$

The solution of this problem can be obtained via solving a collection of initial value problems corresponding to the characteristic system (32), see Fig. 3. However, this approach is not efficient, since each initial value problem demands the same amount of work as solving the original DAE system (24).

Likewise, the determination of biperiodic solutions can be discussed with respect to characteristic curves. Considering an arbitrary $(T_1, 1)$-periodic solution of (29), its initial values in (34) reproduce the complete solution via (32). Consequently, a biperiodic solution of (29) is already fixed by its initial values in the manifold (34) and the corresponding local frequency function ν. Solving the systems (32) yields final values in

$$\mathcal{F}_1 := \{T_1\} \times [\Psi(T_1), \Psi(T_1) + 1[. \tag{35}$$

The last equation in system (32) does not involve the local frequency. Using another T_1-periodic local frequency μ and identical initial state in (34) produces the same final state in

$$\mathcal{F}_2 := \{T_1\} \times [\Phi(T_1), \Phi(T_1) + 1[\quad \text{with} \quad \Phi(t_1) = \int_0^{t_1} \mu(s)\, \mathrm{d}s. \tag{36}$$

Hence the MVF corresponding to μ is biperiodic, too, if $\Phi(T_1) = \Psi(T_1)$ holds, i.e., $\overline{\nu} = \overline{\mu}$. Note that the reconstructed signal (30) is the same for all choices of the local frequency. This behaviour of the characteristic system motivates a transformation of solutions, which is investigated in the next subsection.

In case of widely separated time rates $T_1 \gg \overline{\nu}^{-1}$, solving initial value problems of the characteristic system (32) in the large time interval $[0, T_1]$ demands the computation of a huge number of oscillations. Consequently, methods of characteristics for solving the initial-boundary value problem (28) are inefficient. The analysis of MPDAE solutions via the characteristic system with initial values in $\mathcal{F}_0$ represents just a theoretical tool. On the other hand, initial values in

$$\mathcal{G}_0 := [0, T_1[\times \{0\} \tag{37}$$

determine a biperiodic solution completely, too. Characteristic projections starting in $\mathcal{G}_0$ can be used to construct an efficient numerical method to solve the boundary value problem (27), see [8, 9]. Thereby, each arising initial value problem of the characteristic system (32) is solved only in a short time interval.

3.3 Transformation of Solutions

In Subsect. 2.3, Theorem 1 demonstrates that the local frequency of an according signal is optional. Just the efficiency of a multivariate representation depends on an appropriate choice. We recover this arbitrariness of local frequency functions in the MPDAE model, too.

Theorem 4 *Let $\hat{\mathbf{x}} : \mathbb{R}^2 \to \mathbb{R}^k$ and $\nu : \mathbb{R} \to \mathbb{R}$ satisfy the warped MPDAE system (29). Given a function $\mu : \mathbb{R} \to \mathbb{R}$, the MVF $\hat{\mathbf{y}} : \mathbb{R}^2 \to \mathbb{R}^k$ specified by (18) represents a solution of the warped MPDAE system (29) corresponding to the local frequency function μ.*

Hence we can transform a solution of the MPDAE corresponding to a specific local frequency to another solution with arbitrary local frequency. Note that the deformation (18) does not change the initial manifold (34). In particular, we can transform a multivariate model to a corresponding constant local frequency $\mu \equiv 1$. Consequently, the left-hand sides of (25) and (29) coincide. Therefore solutions of both PDAE models can be tranformed in each other provided that the input signals involve just one time scale.

Corollary 2 *If $\hat{\mathbf{x}} : \mathbb{R}^2 \to \mathbb{R}^k$ and $\nu : \mathbb{R} \to \mathbb{R}$ satisfy the warped MPDAE system (29), then the MVF given by (18) with $\mu \equiv 1$ represents a solution of the MPDAE system (25) including the input $\hat{\mathbf{b}}(t_1, t_2) := \mathbf{b}(t_1)$. Vice versa, if the MPDAE system (25) with $\hat{\mathbf{b}}(t_1, t_2) := \mathbf{b}(t_1)$ exhibits a solution $\hat{\mathbf{x}} : \mathbb{R}^2 \to \mathbb{R}^k$, then using $\nu \equiv 1$ in (18) results in a solution of the warped MPDAE system (29) for arbitrary local frequency function $\mu : \mathbb{R} \to \mathbb{R}$.*

The transformation (18) changes neither the initial values at $t_1 = 0$ nor the periodicity in t_2. Thus transforming a solution of the initial boundary value problem (29),(28) yields another solution of the problem. In case of quasiperiodic signals, some restrictions with regard to the used transformation arise again, which are necessary to preserve the periodicities. Theorem 2 and Theorem 4 imply an according corollary.

Corollary 3 *The following assumptions shall hold:*

(i) $\nu : \mathbb{R} \to \mathbb{R}$ *is* T_1*-periodic,* *(iii)* $\overline{\nu} = \overline{\mu}$,

(ii) $\mu : \mathbb{R} \to \mathbb{R}$ *is* T_1*-periodic,* *(iv)* $\hat{\mathbf{x}} : \mathbb{R}^2 \to \mathbb{R}^k$ *is* $(T_1, 1)$*-periodic.*

If $\hat{\mathbf{x}}$ and ν fulfil the warped MPDAE system (29), then the MVF $\hat{\mathbf{y}} : \mathbb{R}^2 \to \mathbb{R}^k$ obtained by (18) represents a $(T_1, 1)$-periodic solution of the warped MPDAE system (29) including the local frequency function μ.

We note that a $(T_1, 1)$-periodic solution of (29) for constant frequency $\overline{\nu}$ can be transformed in a (T_1, T_2)-periodic solution of (25) with $T_2 = \overline{\nu}^{-1}$. This connection is caused by the standardisation of the second period in the warped system.

Moreover, these corollaries allows to apply results from the MPDAE model with constant rates in the context of the warped MPDAE model. Especially, it is proved that the existence of a quasiperiodic solution of the DAE system (24) implies the existence of a corresponding biperiodic solution of the MPDAE system (25), see [10]. Together with Corollary 2 and Corollary 3, we obtain the following important property.

Corollary 4 *Let the system of DAEs (24) have a two-tone quasiperiodic solution* $\mathbf{x} : \mathbb{R} \to \mathbb{R}^k$ *with rates* T_1, T_2*. Given an arbitrary* T_1*-periodic local frequency function* $\nu : \mathbb{R} \to \mathbb{R}$ *with* $\overline{\nu}^{-1} = T_2$ *in (29), the system of warped MPDAEs features a* $(T_1, 1)$*-periodic solution* $\hat{\mathbf{x}} : \mathbb{R}^2 \to \mathbb{R}^k$*, where* $\mathbf{x}(t) = \hat{\mathbf{x}}(t, \int_0^t \nu(s)\ \mathrm{d}s)$ *holds for all* t*.*

Based on Corollary 3, we can transform any multidimensional representation of a quasiperiodic signal to a reference solution including constant frequency $\overline{\nu}$. However, the corresponding MVF yields an inefficient model for frequency modulated signals as we have seen in Subsect. 2.1.

The degree of freedom in the transformation of biperiodic solutions is given by the function ξ in (23) again. Now a numerical technique for solving the biperiodic boundary value problem of the MPDAE system (29) is imaginable, where we prescribe a periodic function ξ with $\overline{\xi} = 0$. The average frequency $\overline{\nu}$ represents an unknown scalar. Since the MPDAE is autonomous in the second coordinate, a scalar phase condition is necessary to isolate a specific biperiodic solution. Examples are the conditions

$$\hat{x}_1(0, 0) = \eta \quad \text{or} \quad \left.\frac{\partial \hat{x}_1}{\partial t_2}\right|_{t_1 = t_2 = 0} = \eta \quad \text{with } \eta \in \mathbb{R} \tag{38}$$

in the (without loss of generality) first component of $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_k)^\top$. The parameter η has to be chosen appropriately. We can construct a method based on formula (23), where we specify the function ξ and determine the unknown $\overline{\nu}$ via such an additional equation. Unfortunately, this strategy does not work in case of widely separated time scales, since the problem becomes extremely sensitive. To understand this quality, we observe the deformation with respect to the second time scale caused by the transformation (18). For example, we prescribe ξ as a harmonic oscillation of relative amplitude ε. After half a period T_1, the deformation results in

$$\Delta t_2 = \int_0^{T_1/2} \overline{\nu} - \overline{\nu} \left[1 + \varepsilon \sin\left(\tfrac{2\pi}{T_1} s\right)\right] \mathrm{d}s = \frac{\varepsilon}{\pi} T_1 \overline{\nu}. \tag{39}$$

Hence $T_1 \overline{\nu} \gg 1$ implies an enormous amplification of the perturbance ε. Consequently, the respective MVF changes completely in the underlying domain $[0, T_1] \times [0, 1]$. Thus we do not expect an a priori specification of the local frequency to generate a suitable solution.

3.4 Continuous Phase Conditions

If the local frequency is an unidentified function in the MPDAE (29), then we need an additional postulation to determine the complete solution. Houben [5] specifies

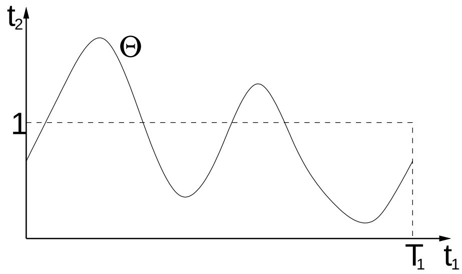

Figure 4: General curve Θ satisfying the continuous phase condition in domain of dependence.

an optimisation condition, which minimises oscillatory behaviour in MVFs and thus yields simple representations. Another possibility consists in demanding a continuous phase condition, cf. [7]. Thereby, the phase in each cross section with $t_1 = \text{const.}$ is controlled, which often produces elementary MVFs, too. Without loss of generality, we choose the first component of $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_k)^\top$. Examples for continuous phase conditions are

$$\hat{x}_1(t_1, 0) = \eta(t_1) \quad \text{for all } t_1 \in \mathbb{R} \tag{40}$$

or

$$\left.\frac{\partial \hat{x}_1}{\partial t_2}\right|_{t_2=0} = \eta(t_1) \quad \text{for all } t_1 \in \mathbb{R} \tag{41}$$

including a predetermined slowly varying function $\eta : \mathbb{R} \to \mathbb{R}$. We add such a phase condition as additional boundary condition in time domain. In general, we do not have a priori knowledge about the specification of the function η. Nevertheless, applying constant choices, i.e., $\eta \equiv \text{const.}$, is often successful. Note that (40),(41) represent conditions for scalar functions depending on t_1, which agrees to the structure of the unknown parameters $\nu(t_1)$.

The degree of freedom in transformations of MPDAE solutions can be used to justify the existence of solutions satisfying some phase condition. For example, we discuss the constraint (41) setting $\eta \equiv 0$. We assume the existence of a biperiodic solution $\hat{\mathbf{x}} \in C^1$ of (29). The smoothness and periodicity implies that, for each $t_1 \in \mathbb{R}$, a corresponding $\Theta(t_1) \in \mathbb{R}$ exists such that

$$\left.\frac{\partial \hat{x}_1}{\partial t_2}\right|_{t_2=\Theta(t_1)} = 0 \quad \text{for all } t_1 \in \mathbb{R}. \tag{42}$$

For $t_1 = 0$, we select some $\Theta(0) \in [0, 1[$. Motivated by the implicit function theorem, we postulate that a certain choice $t_2 = \Theta(t_1)$ with $\Theta \in C^1$ exists, which forms an isolated curve in the domain of dependence. The periodicity of $\hat{\mathbf{x}}$ produces a T_1-periodic function Θ. Fig. 4 illustrates this quality. Now we transform the curve $t_2 = \Theta(t_1)$ onto the line $t_2 = 0$, which yields the new MVF

$$\hat{\mathbf{y}}(t_1, t_2) := \hat{\mathbf{x}}(t_1, t_2 + \Theta(t_1)) = \hat{\mathbf{x}}\left(t_1, t_2 + \int_0^{t_1} \Theta'(s)\ \mathrm{d}s + \Theta(0)\right). \tag{43}$$

The involved translation by $\Theta(0)$ is of the type (31) and thus results in a solution again. Furthermore, the periodicity yields

$$\overline{\Theta'} = \frac{1}{T_1}\int_0^{T_1} \Theta'(s)\ \mathrm{d}s = \frac{\Theta(T_1) - \Theta(0)}{T_1} = 0. \tag{44}$$

Hence Theorem 4 implies that the function (43) represents a biperiodic solution of the MPDAE satisfying the phase condition (41) with $\eta \equiv 0$.

4 Illustrative Example

To demonstrate the application of the multidimensional strategy, we consider the tanh-based LC oscillator in Fig. 5. This circuit is similar to an example investigated in [7]. The node voltage u and the branch current $\imath := I_L$ through the inductance are unknown time-dependent functions. The current-voltage relation of the nonlinear resistor reads

$$I_R = g(u) := (G_0 - G_\infty)U_0 \tanh\left(\frac{u}{U_0}\right) + G_\infty u, \tag{45}$$

where the used parameters are $G_0 = -0.1$ A/V, $G_\infty = 0.25$ A/V and $U_0 = 1$ V. An independent input signal b specifies the capacitance C, which produces a voltage controlled oscillator. We write the arising mathematical model in the form

$$\begin{aligned} \dot{u} &= (-\imath - g(u))/(C_0 w) \\ \dot{\imath} &= u/L \\ 0 &= w - b(t) \end{aligned} \tag{46}$$

with the unknown functions $u, \imath, w$ and parameters $C_0 = 100$ nF and $L = 1$ μH. Hence system (46) represents a semi-explicit DAE of index 1. For constant input, the system exhibits a periodic oscillation of a high frequency. We choose the input signal

$$b(t) = 1 + 0.9\sin\left(\tfrac{2\pi}{T_1}t\right) \tag{47}$$

with slow rate $T_1 = 30$ μs, which produces frequency modulated signals. Consequently, we apply the respective warped MPDAE model (29) and determine a

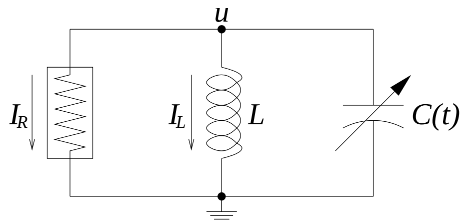

Figure 5: Circuit of voltage controlled LC oscillator.

biperiodic solution satisfying the phase condition (41). A finite difference method employing centred differences on a uniform grid yields a numerical approximation.

Firstly, we select the function $\eta \equiv 0$ in phase condition (41). Fig. 6 illustrates the results of the multidimensional model. According to the input signal, the local frequency is high in areas, where the capacitance is low. Since this behaviour is typical for LC oscillators, the used phase condition identifies a physically reasonable frequency function. The MVFs $\hat{u}$ and $\hat{\imath}$ exhibit a weak and a strong amplitude modulation, respectively. The MVF $\hat{w}$ reproduces the input signal.

Now we employ the solution of the MPDAE to reconstruct a corresponding quasiperiodic response of the DAE (46) via (30). For comparison, an initial value problem of the DAE is solved by trapezoidal rule. Fig. 7 shows the resulting signals. We observe a good agreement of both functions, which confirms the relations between univariate and multivariate model.

Secondly, we perform the simulation using the same techniques but considering $\eta(t_1) = 0.8 \sin(2\pi t_1/T_1)$ in phase condition (41). The arising local frequency μ is depicted in Fig. 8 (left). We recognise that the difference $\Delta\nu := \mu - \nu$ with respect to the previous frequency, see Fig. 8 (right), satisfies $\overline{\Delta\nu} = 0$, i.e., $\overline{\nu} = \overline{\mu}$. However, the function $\Delta\nu$ is nontrivial in comparison to the difference between the two choices of the function η. Fig. 9 (left) illustrates the corresponding MVF $\hat{u}$ in the latter simulation. For comparison, a transformation of the previous MVF $\hat{u}$ is shown in Fig. 9 (right), where we apply formula (18) with the updated frequency μ. Both functions demonstrate a good agreement. These results clarify that the solutions belonging to the two different phase conditions are located in the same continuum of transformed solutions. Moreover, the simple choice $\eta \equiv 0$ in the phase condition (41) already yields a very efficient representation.

Finally, we simulate a case of widely separated time scales by setting $T_1 = 100$ ms in (47). The used phase condition is (41) with $\eta \equiv 0$. To validate the applicability of the inherent MPDAE structure outlined in Subsect. 3.2, a method of characteristics produces the numerical solution now. Involved DAE systems are discretised by trapezoidal rule. Nevertheless, the finite difference method, which

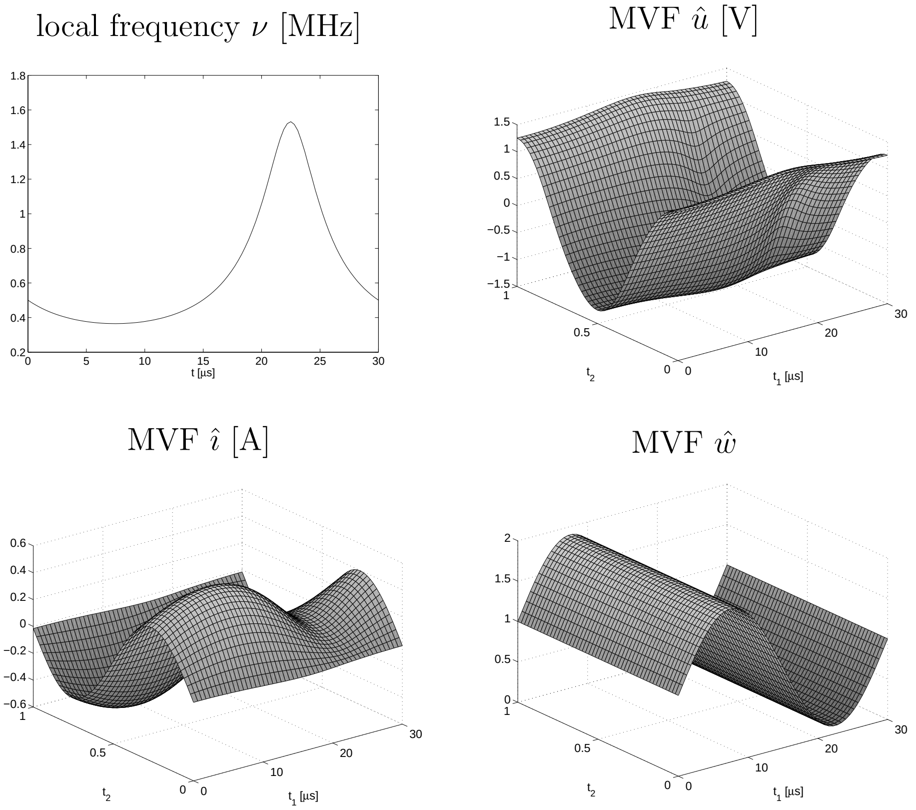

Figure 6: MPDAE solution for rate $T_1 = 30\ \mu$s.

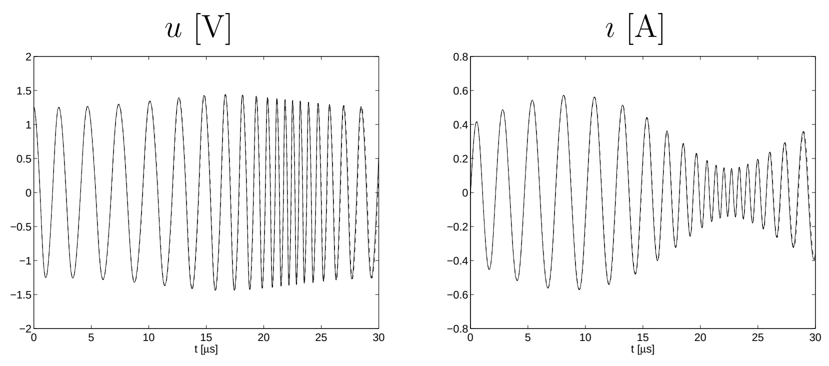

Figure 7: DAE solution for rate $T_1 = 30\ \mu$s reconstructed by MPDAE solution (solid line) and computed by transient integration (dashed line).

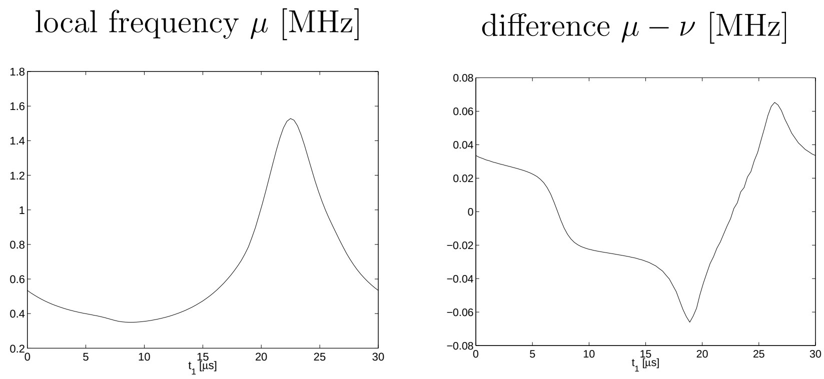

Figure 8: Local frequency for modified phase condition (left) and difference to previous local frequency (right) in case of $T_1 = 30$ μs.

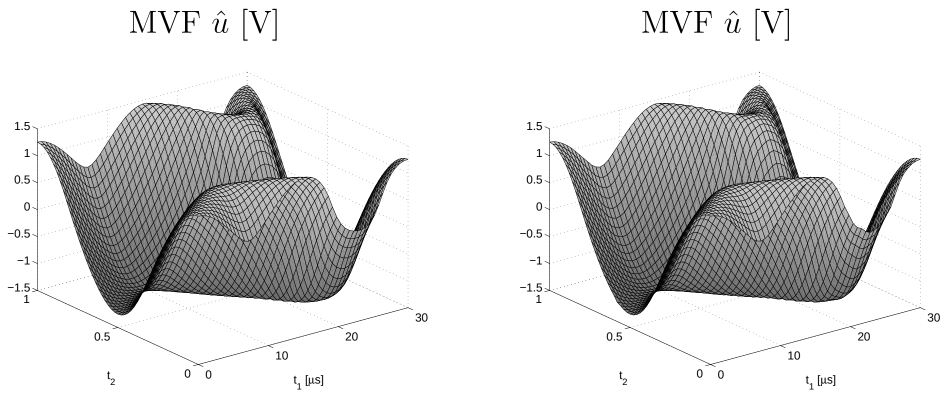

Figure 9: MVF $\hat{u}$ for second phase condition (left) and transformation of MVF $\hat{u}$ for first phase condition (right) in case of $T_1 = 30$ μs.

has been applied in the previous simulations, yields the same solution. Fig. 10 depicts the results of the MPDAE model. Local frequency as well as MVFs exhibit the same behaviour as in the case $T_1 = 30$ μs. Just the amplitude modulation in $\hat{u}$ disappears. The corresponding DAE solution features about 66 000 oscillations during the period T_1 here, which can be reconstructed by these results.

5 Conclusions

Multidimensional models of a quasiperiodic signal are not unique but interconnected by a transformation formula. We obtain quasiperiodic solutions of DAEs via determining multiperiodic solutions of MPDAEs. Accordingly, the warped MPDAE system exhibits a continuum of solutions, which reproduce the same

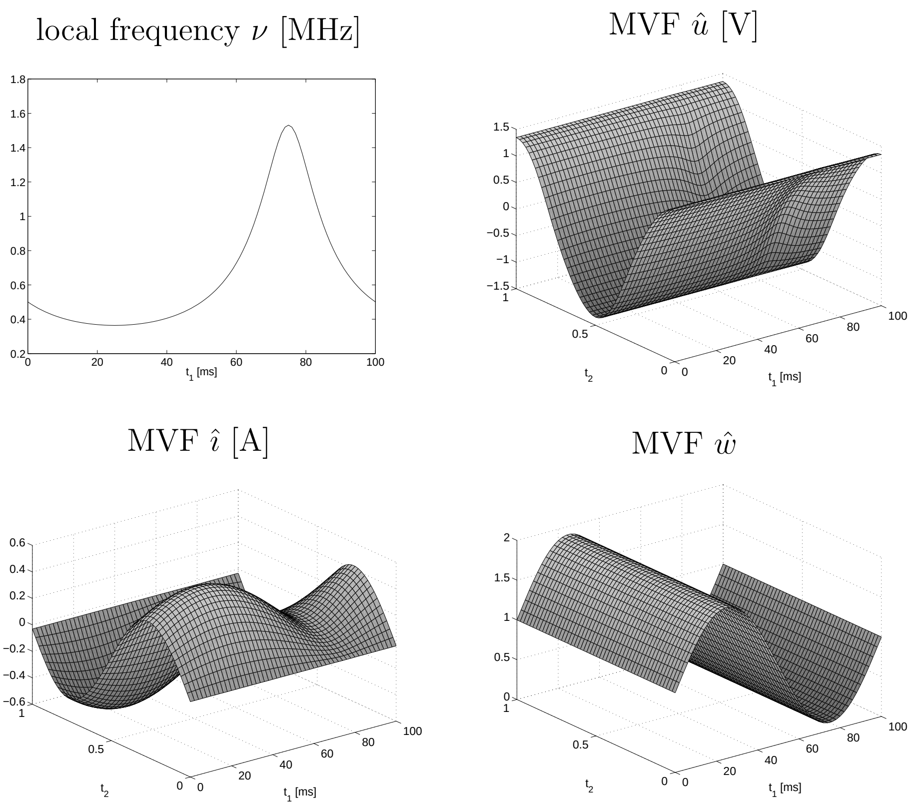

Figure 10: MPDAE solution for rate $T_1 = 100$ ms.

quasiperiodic response of the underlying DAE system. However, the average frequency of all representations coincides. Thus solutions of the warped system can be transformed to solutions of the unwarped system, which enables the use of respective theoretical results. In particular, the existence of quasiperiodic solutions of DAEs implies the existence of corresponding multiperiodic solutions of warped MPDAEs for a broad class of local frequency functions. Moreover, the arising transformation formula allows for the analysis of additional conditions for the determination of a priori unknown local frequency functions. Numerical simulations based on the warped MPDAE system confirm the transformation qualities of the multidimensional model.

Acknowledgements

This work is part of the BMBF programme “Multiskalensysteme in Mikro- und Optoelektronik” within the project “Partielle Differential-Algebraische Multiskalensysteme für die Numerische Simulation von Hochfrequenz-Schaltungen” (No. 03GUNAVN). The author is indebted to Dr. A. Bartel (University of Wuppertal) for helpful discussions. Furthermore, the author thanks T. Mei (University of Minnesota) for informations about the used example of the voltage controlled oscillator.

References

[1] Brachtendorf, H. G.; Welsch, G.; Laur, R.; Bunse-Gerstner, A.: Numerical steady state analysis of electronic circuits driven by multi-tone signals. Electrical Engineering 79 (1996), pp. 103-112.

[2] Chua, L. O.; Ushida, A.: Algorithms for computing almost periodic steady-state response of nonlinear systems to multiple input frequencies. IEEE CAS 28 (1981) 10, pp. 953-971.

[3] Günther, M.; Feldmann, U.: CAD based electric circuit modeling in industry I: mathematical structure and index of network equations. Surv. Math. Ind. 8 (1999), pp. 97-129.

[4] Ho, C.W.; Ruehli, A.E.; Brennan, P.A.: The modified nodal approach to network analysis. IEEE Trans. CAS 22 (1975) pp. 505-509.

[5] Houben, S.H.M.J.: Simulating multi-tone free-running oscillators with optimal sweep following. In: Schilders, W.H.A., ter Maten, E.J.W., Houben,

S.H.M.J. (eds.): Scientific Computing in Electrical Engineering, Mathematics in Industry, Springer, 2004, pp. 240-247.

[6] Kampowsky, W.; Rentrop, P.; Schmitt, W.: Classification and numerical simulation of electric circuits. Surv. Math. Ind. 2 (1992), pp. 23-65.

[7] Narayan, O.; Roychowdhury, J.: Analyzing oscillators using multitime PDEs. IEEE Trans. CAS I 50 (2003) 7, pp. 894-903.

[8] Pulch, R.: Finite difference methods for multi time scale differential algebraic equations. Z. Angew. Math. Mech. 83 (2003) 9, pp. 571-583.

[9] Pulch, R.: Multi time scale differential equations for simulating frequency modulated signals. Appl. Numer. Math. 53 (2005) 2-4, pp. 421-436.

[10] Roychowdhury, J.: Analysing circuits with widely-separated time scales using numerical PDE methods. IEEE Trans. CAS I 48 (2001) 5, pp. 578-594.

[11] Ushida, A.; Chua, L. O.: Frequency-domain analysis of nonlinear circuits driven by multi-tone signals. IEEE CAS 31 (1984) 9, pp. 766-779.

Appendix: Proofs of Theorems

Proof of Theorem 1 :

We obtain directly by the transformation formula

$$\begin{aligned} x(t) &= \hat{x}(t, \Psi(t)) \\ &= \hat{x}(t, \Phi(t) + \Psi(t) - \Phi(t)) \\ &= \hat{y}(t, \Phi(t)) \end{aligned}$$

and thus the proof is complete. $\square$

Proof of Theorem 2 :

The periodicity of $\hat{y}$ in the second variable t_2 is clear. The periodicity in the first variable t_1 follows from

$$\begin{aligned} \int_0^{t_1+T_1} \nu(s) - \mu(s)\ \mathrm{d}s &= \int_0^{t_1} \nu(s) - \mu(s)\ \mathrm{d}s + \int_{t_1}^{t_1+T_1} \nu(s) - \mu(s)\ \mathrm{d}s \\ &= \int_0^{t_1} \nu(s) - \mu(s)\ \mathrm{d}s \\ &\quad + \int_{t_1}^{t_1+T_1} \nu(s)\ \mathrm{d}s - \int_{t_1}^{t_1+T_1} \mu(s)\ \mathrm{d}s \\ &= \int_0^{t_1} \nu(s) - \mu(s)\ \mathrm{d}s + T_1\overline{\nu} - T_1\overline{\mu} \\ &= \int_0^{t_1} \nu(s) - \mu(s)\ \mathrm{d}s. \end{aligned}$$

The property

$$x(t) = \hat{x}\left(t, \int_0^t \nu(s)\ \mathrm{d}s\right) = \hat{y}\left(t, \int_0^t \mu(s)\ \mathrm{d}s\right)$$

is implied by Theorem 1. $\square$

Proof of Theorem 3 :

Let x be of the form (15) with property (20). Thus it holds $x \in C^1$. We define the biperiodic function $\hat{x} \in C^1$ via

$$\hat{x}(t_1, t_2) = \sum_{j_1,j_2=-\infty}^{+\infty} X_{j_1,j_2} \exp\left(i\left(\tfrac{2\pi}{T_1} j_1 t + 2\pi j_2 t\right)\right).$$

Due to (20), the series converges absolutely. The new representation $\hat{y}$ is defined by (18). Now the question is if $\hat{y}$ owns a representation by means of an absolutely convergent Fourier series, too.

Sufficient for this property is $\hat{y} \in C^1$ and $\frac{\partial^2 \hat{y}}{\partial t_1 \partial t_2} \in C^0$. The partial derivatives of $\hat{y}$ read

$$
\begin{aligned}
\frac{\partial \hat{y}}{\partial t_1} &= \frac{\partial \hat{x}}{\partial t_1} + (\nu(t_1) - \mu(t_1)) \frac{\partial \hat{x}}{\partial t_2} \\
\frac{\partial \hat{y}}{\partial t_2} &= \frac{\partial \hat{x}}{\partial t_2} \\
\frac{\partial^2 \hat{y}}{\partial t_1 \partial t_2} &= \frac{\partial^2 \hat{x}}{\partial t_1 \partial t_2} + (\nu(t_1) - \mu(t_1)) \frac{\partial^2 \hat{x}}{\partial {t_2}^2}.
\end{aligned}
$$

Hence the properties are fulfilled if $\hat{x} \in C^1$, $\frac{\partial^2 \hat{x}}{\partial t_1 \partial t_2} \in C^0$, $\frac{\partial^2 \hat{x}}{\partial {t_2}^2} \in C^0$ holds. Now it is exactly condition (20), which guarantees these requirements. $\square$

Proof of Theorem 4 :

In the following, to simplify the notation, we omit the location

$$(\tilde{t_1}, \tilde{t_2}) := (t_1, t_2 + \int_0^{t_1} \nu(s) - \mu(s) \; \mathrm{d}s)$$

of evaluations corresponding to $\hat{\mathbf{x}}$ given in (18). Now we calculate directly

$$
\begin{aligned}
\frac{\partial \mathbf{q}(\hat{\mathbf{y}})}{\partial t_1} + \mu(t_1) \frac{\partial \mathbf{q}(\hat{\mathbf{y}})}{\partial t_2} &= \frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_1} + (\nu(t_1) - \mu(t_1)) \frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_2} + \mu(t_1) \frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_2} \\
&= \frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_1} + \nu(t_1) \frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_2} \\
&= \mathbf{f}\,(\hat{\mathbf{x}}) + \mathbf{b}(t_1) \\
&= \mathbf{f}(\hat{\mathbf{y}}) + \mathbf{b}(t_1),
\end{aligned}
$$

which verifies that the MVF $\hat{\mathbf{y}}$ satisfies the MPDAE (29) including the local frequency function μ. $\square$