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Abstract

In radio frequency (RF) applications, electric circuits produce signals including widely separated time scales. A multidimensional representation yields an efficient model by decoupling the time scales. Consequently, a warped multirate partial differential algebraic equation (MPDAE) describes the circuit's behaviour. The appropriate determination of an arising local frequency function is crucial for the efficiency of this approach. Variational calculus implies a necessary condition to a specific solution, which exhibits a minimal amount of oscillations in the whole domain of dependence. We apply a similar strategy to minimise oscillatory performance in some boundary values only. Now variational calculus yields a boundary condition, which can easily be used in numerical methods. We compare the results of both minimisation criteria in a simulation of a warped MPDAE model.

1 Introduction

The mathematical modelling of electric circuits is based on differential algebraic equations (DAEs), see [2]. In communication electronics, radio frequency (RF) signals exhibit widely separated time scales. Thus transient analysis of the DAE system becomes inefficient, since the fastest time scale restricts the step sizes in time, whereas the slowest time scale determines the total time interval of the simulation. A specific multivariate model decouples the time scales of RF signals and thus generates an efficient representation. Accordingly, Brachtendorf et al. [1] introduced the multirate partial differential algebraic equation (MPDAE), which results from a transformation of the original DAE. This multidimensional approach can be applied successfully to simulate amplitude modulated signals.

Narayan and Roychowdhury [4] generalised this model for the simulation of RF signals, which include amplitude as well as frequency modulation. Now a local frequency function arises in the resulting warped MPDAE system. Inappropriate choices of these parameters cause many oscillations in the corresponding multivariate solution and thus increase the amount of computational work unnecessarily. Hence the efficiency of this approach depends on the determination of a suitable local frequency function. An additional condition is required to specify a complete solution. Houben [3] introduced a minimisation criterion, which reduces the amount of oscillations in solutions. This strategy can be applied to the MPDAE in combination with a mixture of initial and boundary conditions. For biperiodic boundary value problems, an alternative minimisation technique is feasible, where a corresponding variational calculus yields a necessary condition for an optimal solution, see [6]. Thereby, the amount of partial derivatives is minimised in the whole domain of dependence.

In this paper, we restrict the minimisation scheme to some boundary values. In general, the number of oscillations in the whole solution is low if and only if the number of oscillations in the used boundary layer is low. Thus the efficiency of the multidimensional approach is preserved. Accordingly, a variational calculus implies an additional condition for an optimal solution, where just boundary values are involved. Furthermore, a method of characteristics is qualified for computing biperiodic solutions of the MPDAE, see [5]. In this strategy, a discretisation based on characteristic curves yields a boundary value problem of DAE subsystems. Therefore restricting the minimisation to the boundary values allows to add easily the resulting condition to the boundary constraints in the method of characteristics. Consequently, we can use standard numerical techniques to solve the arising boundary value problem of DAEs.

The paper is organised as follows. In Sect. 2, we introduce briefly the warped MPDAE model. The minimisation criteria and the according constraints, which are obtained by variational calculus, are illustrated in Sect. 3. Finally, we present numerical simulations applying both criteria, where a forced Van der Pol oscillator is used as benchmark.

2 Warped MPDAE Model

In electric circuit simulation, a network approach yields systems of *differential algebraic equations* (*DAEs*), see [2], which we write in the form

$$\frac{\mathrm{d}\mathbf{q}(\mathbf{x})}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}(t)) + \mathbf{b}(t), \qquad \begin{array}{ll} \mathbf{x} : \mathbb{R} \to \mathbb{R}^k, & \mathbf{q} : \mathbb{R}^k \to \mathbb{R}^k, \\ \mathbf{b} : \mathbb{R} \to \mathbb{R}^k, & \mathbf{f} : \mathbb{R}^k \to \mathbb{R}^k. \end{array} \tag{1}$$

Thereby, $\mathbf{x} \in C^1$ denotes unknown node voltages and branch currents. We assume that the input signals $\mathbf{b}$ are periodic with a slow rate T_1. Furthermore, the system shall possess an inherent fast time scale. Hence the solution $\mathbf{x}$ becomes quasiperiodic. According to [4], the system of *warped multirate partial differential algebraic equations* (*MPDAEs*), which corresponds to the DAE system (1), reads

$$\frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_1} + \nu(t_1)\frac{\partial \mathbf{q}(\hat{\mathbf{x}})}{\partial t_2} = \mathbf{f}(\hat{\mathbf{x}}(t_1, t_2)) + \mathbf{b}(t_1), \qquad \hat{\mathbf{x}} : \mathbb{R}^2 \to \mathbb{R}^k, \quad \nu : \mathbb{R} \to \mathbb{R}. \tag{2}$$

Now $\hat{\mathbf{x}} \in C^1$ represents the *multivariate function* (*MVF*) describing the amplitude modulation in the signal $\mathbf{x}$. In addition, a *local frequency function* ν arises, which models the frequency modulation. The system (2) is underdetermined, since the local frequency is not specified a priori. We need an additional condition to fix a unique solution. An arbitrary solution of the MPDAE (2) yields a solution of the DAE (1) via the reconstruction

$$\mathbf{x}(t) := \hat{\mathbf{x}}(t, \Psi(t)) \quad \text{with} \quad \Psi(t) := \int_0^t \nu(\sigma)\,\mathrm{d}\sigma, \tag{3}$$

where the second time scale is stretched by the *warping function* Ψ. A biperiodic MVF $\hat{\mathbf{x}}$ with periods T_1 and 1 in connection with a T_1-periodic local frequency ν reproduces a quasiperiodic solution $\mathbf{x}$ of the original DAE (1). Thus we consider biperiodic boundary value problems of the MPDAE (2) in the following, where the rectangle $[0, T_1] \times [0, 1]$ is regarded in the domain of dependence.

In the *method of characteristics*, see [5], we discretise the t_1-direction first, e.g. using equidistant step size $h_1 := T_1/n$. Hence the points $t_1^j := (j-1)h_1$ arise for $j = 1, \ldots, n$. Let $\nu_1, \ldots, \nu_n$ be the corresponding discrete values of the local frequency function. A unique characteristic curve, which is determined by the local frequency function, runs through each point $(t_1^j, 0)$ at the boundary. Let $\tilde{\mathbf{x}}^j : \mathbb{R} \to \mathbb{R}^k$ be the solution on the jth curve, which depends on a parameter τ now. The corresponding characteristic systems are

$$\frac{\mathrm{d}\mathbf{q}(\tilde{\mathbf{x}}^j)}{\mathrm{d}\tau} = \mathbf{f}(\tilde{\mathbf{x}}^j(\tau)) + \mathbf{b}(\tau + t_1^j) \quad \text{for } \ j = 1, \ldots, n. \tag{4}$$

The periodicities yield boundary conditions, which interconnect these separate systems. The boundary conditions exhibit the form

$$\left(\tilde{\mathbf{x}}^1(0)^\top, \ldots, \tilde{\mathbf{x}}^n(0)^\top\right)^\top = \mathcal{B}\left(\tilde{\mathbf{x}}^1(\xi_1)^\top, \ldots, \tilde{\mathbf{x}}^n(\xi_n)^\top\right)^\top \qquad \text{with } \ \mathcal{B} \in \mathbb{R}^{nk \times nk}. \tag{5}$$

Thereby, the end points ξ_j as well as the matrix $\mathcal{B}$ depend on the local frequency function. Since the local frequency is unspecified a priori, we obtain a free boundary value problem of DAE subsystems given by (4) and (5). For fixed starting values $\nu_1, \ldots, \nu_n$, we can compute an approximation of $\xi_1, \ldots, \xi_n$ as well as $\mathcal{B}$ and thus evaluate the boundary conditions in a numerical method. However, we need a suitable additional condition to determine the unknown values $\nu_1, \ldots, \nu_n$.

3 Minimisation Techniques

Unqualified choices of the local frequency function generate many oscillations in the corresponding MVF. Thus the idea is to reduce oscillatory behaviour by minimising the amount of respective partial derivatives. Let $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_k)^\top$ be the components of the MVF and $\hat{\mathbf{x}} \in C^2$. In [6], the *functional*

$$\gamma\left(\hat{\mathbf{x}}\right) := T_1 \int_0^{T_1} \int_0^1 \sum_{l=1}^{k} w_l \Big(\frac{\partial \hat{x}_l}{\partial t_1} \Big)^2 \, \mathrm{d}t_2 \, \mathrm{d}t_1 \tag{6}$$

is investigated, where $w_1, \ldots, w_k \geq 0$ represent constant weights. Regarding all biperiodic solutions of (2), we assume the existence of a global minimum $\hat{\mathbf{x}}_{\mathrm{opt}}$ with respect to this functional. Biperiodic solutions of the warped MPDAE system feature specific transformation properties. Based on a transformation formula, we obtain certain competitive solutions, see [6]. Using this relation, a *variational calculus* implies a necessary condition for the optimal MVF, namely

$$r(t_1) := \int_0^1 \sum_{l=1}^{k} w_l \cdot \frac{\partial^2 \hat{x}_l}{\partial {t_1}^2} \cdot \frac{\partial \hat{x}_l}{\partial t_2} \, \mathrm{d}t_2 = 0 \quad \text{for all} \;\; t_1 \in \mathbb{R}. \tag{7}$$

We recognise that this constraint involves values related to the MVF in all points of the domain $[0, T_1] \times [0, 1]$, which is required to perform the minimisation everywhere.

On the contrary, we restrict the minimisation to the boundary values in the layer $t_2 = 0$ now. Accordingly, the new functional reads

$$\delta\left(\hat{\mathbf{x}}\right) := T_1 \int_0^{T_1} \sum_{l=1}^{k} w_l \Big(\left. \frac{\partial \hat{x}_l}{\partial t_1} \right|_{t_2=0} \Big)^2 \, \mathrm{d}t_1. \tag{8}$$

Again we assume the existence of an optimal biperiodic solution, i.e. a global minimum of (8). In this case, an analogue variational calculus yields the necessary condition

$$s(t_1) := \sum_{l=1}^{k} w_l \cdot \left. \frac{\partial^2 \hat{x}_l}{\partial {t_1}^2} \right|_{t_2=0} \cdot \left. \frac{\partial \hat{x}_l}{\partial t_2} \right|_{t_2=0} = 0 \quad \text{for all} \;\; t_1 \in \mathbb{R}. \tag{9}$$

We can evaluate the derivative with respect to t_1 within the layer $t_2 = 0$. In contrast, the other derivative is perpendicular to this layer and requires points from outside for an approximative evaluation. However, if the system (1) represents an *ordinary differential equation* (*ODE*), i.e. $\mathbf{q}(\mathbf{x}) \equiv \mathbf{x}$, we use the MPDAE (2) to replace this derivative. Assuming $\nu(t_1) > 0$, we obtain

$$s(t_1) = \sum_{l=1}^{k} w_l \cdot \left.\frac{\partial^2 \hat{x}_l}{\partial {t_1}^2}\right|_{t_2=0} \cdot \frac{1}{\nu(t_1)} \cdot \left[f_l(\hat{\mathbf{x}}(t_1, 0)) + b_l(t_1) - \left.\frac{\partial \hat{x}_l}{\partial t_1}\right|_{t_2=0} \right] = 0 \qquad (10)$$

for all $t_1 \in \mathbb{R}$. Now we are able to approximate the derivatives with respect to t_1 by difference formulae within the layer $t_2 = 0$. Thereby, we employ just boundary values arising in (5). For example, an approximation of second order is given by

$$\begin{aligned} s_j \quad := \quad & \sum_{l=1}^{k} w_l \cdot \tfrac{1}{{h_1}^2} \left[\tilde{x}_l^{j-1}(0) - 2\tilde{x}_l^{j}(0) + \tilde{x}_l^{j+1}(0) \right] \\ & \qquad \cdot \left\{ f_l(\tilde{\mathbf{x}}^j(0)) + b_l(t_1^j) - \tfrac{1}{2h_1} \left[\tilde{x}_l^{j+1}(0) - \tilde{x}_l^{j-1}(0) \right] \right\} = 0 \end{aligned} \qquad (11)$$

for $j = 1, \ldots, n$. In this form, we can add the constraints (11) to the conditions (5) to solve the boundary value problem resulting in the method of characteristics. Thus we obtain n additional equations to fix the unknown values $\nu_1, \ldots, \nu_n$.

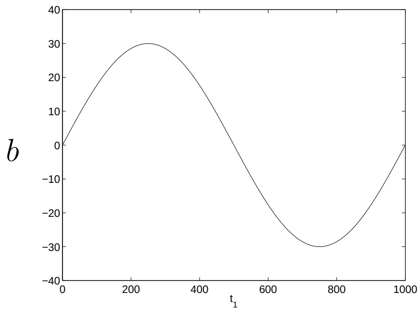
If the system (1) represents a DAE, then the derivatives of $\hat{\mathbf{x}}$ in (8) may be replaced by corresponding derivatives of the function $\mathbf{q}(\hat{\mathbf{x}})$. In an analogue calculus, information from the MPDAE (2) can be applied to replace the resulting term now. Furthermore, if the system (1) features only some explicit components, i.e. $q_p(\mathbf{x}) \equiv x_p$ for some $p \in P \subset \{1, \ldots, k\}$, then setting $w_l = 0$ for $l \notin P$ allows to employ the above conditions, too.

4 Test Results

As test example, we consider a forced Van der Pol oscillator. The corresponding system consists of two ODEs, namely

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -10 \left({x_1}^2 - 1 \right) x_2 - 4\pi^2 x_1 + 30 \sin \left(\tfrac{2\pi}{T_1} t \right). \qquad (12)$$

We choose the slow rate $T_1 = 1000$. In addition, the system exhibits a fast time scale with a magnitude about 1. Hence quasiperiodic solutions emerge, which include amplitude as well as frequency modulation. Consequently, we apply the warped MPDAE model in connection with the additional conditions from the previous section. We set the weights to $w_1 = w_2 = 1$. The optimal solution corresponding to the functional (6) is computed via a finite difference method



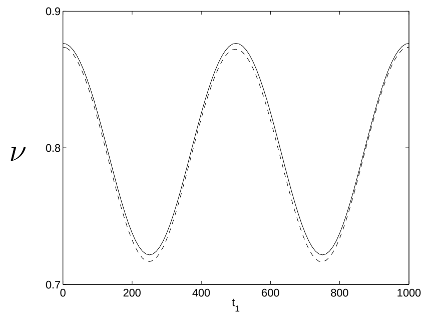


Figure 1: Input signal (left) and optimal local frequency functions (right) computed by criterion in whole domain (solid line) and by criterion in boundary values (dashed line).

using symmetric differences on a uniform grid. Considering the functional (8), we apply the method of characteristics, where the arising boundary value problem of ODEs is solved by a shooting method including trapezoidal rule.

Fig. 1 illustrates the resulting optimal local frequency functions. Let $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ be the optimal MVFs corresponding to the functionals (6) and (8), respectively. Fig. 2 and Fig. 3 show these MVFs. The discrepancy between both MVFs exhibits the typical behaviour of transformations involving biperiodic solutions. Although the minimisation is done with respect to boundary values only, the MVF $\hat{\mathbf{z}}$ features a simple structure, too. The values of the functional considering the whole domain of dependence result to $\gamma(\hat{\mathbf{y}}) = 3913$ and $\gamma(\hat{\mathbf{z}}) = 10405$. Vice versa, it holds $\delta(\hat{\mathbf{y}}) = 5630$ and $\delta(\hat{\mathbf{z}}) = 90$.

To verify the optimality of the computed solution corresponding to the functional (8), we determine some competitive solutions employing the underlying transformation formulae. Thereby, the original local frequency ν_{opt} is transformed to a new function, which represents just one example of a feasible transformation, i.e.

$$\nu(t_1) := \nu_{\text{opt}}(t_1) + \alpha_0 \left(\frac{1}{T_1} \int_0^{T_1} \nu_{\text{opt}}(\sigma) \, \mathrm{d}\sigma \right) \sin\left(\frac{2\pi}{T_1} t_1 \right) \tag{13}$$

with parameter $\alpha_0 \in \mathbb{R}$. Let $\hat{\mathbf{x}}_{\alpha_0}$ be the corresponding MVF. Table 1 illustrates the resulting values of the functional (8). In fact, the minimum is reached for $\alpha_0 = 0$, which reproduces the computed optimal local frequency.

Table 1: Values of functional δ for competitive solutions.

α_0	-10^{-4}	-10^{-5}	-10^{-6}	-10^{-7}	0	10^{-7}	10^{-6}	10^{-5}	10^{-4}
$\delta(\hat{\mathbf{x}}_{\alpha_0})$	296.0	94.44	90.53	90.30	90.28	90.30	90.47	92.84	182.7

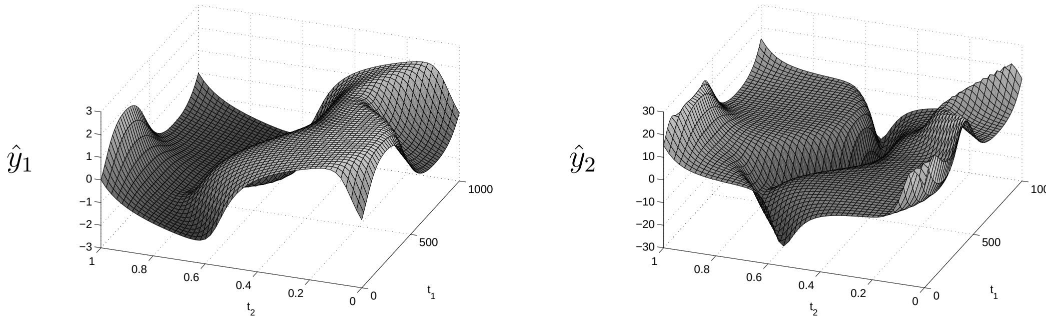


Figure 2: Optimal MVFs $\hat{y}_1$ (left) and $\hat{y}_2$ (right) for minimisation in whole domain.

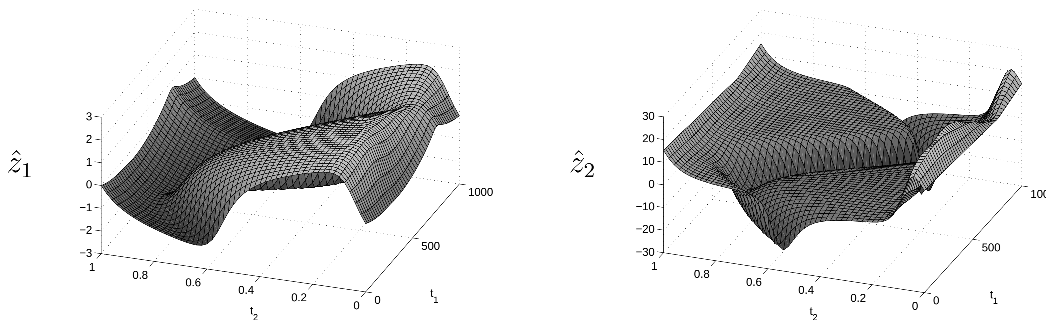


Figure 3: Optimal MVFs $\hat{z}_1$ (left) and $\hat{z}_2$ (right) for minimisation in boundary values.

5 Conclusions

The warped MPDAE model provides an efficient alternative for simulating quasi-periodic signals, which exhibit amplitude as well as frequency modulation. Adequate solutions can be obtained via a minimisation criterion, which ensures a simple structure of MVFs in the whole domain of dependence. An alternative criterion has been presented, where the minimisation is restricted to some boundary values. Hence a variational calculus yields an according condition, which can be added to boundary conditions in a numerical scheme, especially in a method of characteristics. Although just boundary values are considered in the minimisation, the resulting solutions feature a simple form.

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