Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Lehrstuhl für Angewandte Mathematik
und Numerische Mathematik

Preprint BUW-AMNA 04/09

M. Striebel, M.Günther

Towards One-Step Multirate Methods in Chip Design

September 2004

http://www.math.uni-wuppertal.de/org/Num/

Towards One-Step Multirate Methods in Chip Design*

Michael Striebel[1] and Michael Günther

Bergische Universität Wuppertal, Departement of Mathematics, Chair of Applied Mathematics/Numerical Analysis, D-42097 Wuppertal, Germany.
[striebel,guenther]@math.uni-wuppertal.de

1 Introduction

Large integrated electrical networks are usually build up by numerous coupled subcircuits of different functionality. These subcircuits are modelled independently and composed to one macro system by connecting them at the respective terminals, i. e. each pair of connected terminal nodes merge to one node (see Fig. 1, left).

From a modelling point of view, this procedure can be described by introducing virtual voltage sources at the boundary nodes (see Fig. 1, right). This approach preserves the macro circuits block structure and produces additional variables: *branch currents u through the coupling voltage sources*. These currents are determined by the property, that the node potentials of each pair of connected boundary nodes have to coincide.

Regarding r subcircuits, r differential-algebraic systems, coupled by algebraic equations arise:

$$\mathcal{F}_\lambda(x_\lambda, \frac{d}{dt} w_\lambda(x_\lambda), u, t) = 0, \quad (\lambda = 1, \ldots, r) \tag{1a}$$

$$\mathcal{G}(x_1, \ldots, x_r) = 0, \tag{1b}$$

where x_λ describes the node potentials and currents and w_λ the charges and fluxes of the $\lambda-$th subcircuit and u the coupling currents.

As the subcircuits constitute different functional units, the macro system often shows *multirate behaviour*, i. e. the subcircuits behave on different timescales. Thus *multirate methods* can be applied, that integrate subsystems

*This work is part of the project "Partielle Differential-Algebraische Multiskalensysteme für die Numerische Simulation von Hochfrequenz-Schaltungen" (No. 03GUNAVN), which is founded by the BMBF program "Multiskalensysteme in Mikro- und Optoelektronik ".

[1]The author is indebted to Infineon Technologies München, and especially to Drs. Feldmann and Schultz, for supporting his PhD project.

showing different transient behaviour with different stepsizes adjusted to each subcircuits activity level.

Fig. 1. Coupling: technical and modelling point of view

2 Partitioned Network

Coupled problems that can be described by the abstract model (1a, 1b) also occur in other applications (e. g. multi-body physics). To set up numerical methods that are adapted to simulating electrical networks, a closer look at their special properties is required.

2.1 Network Equations

For circuits that are designed in the described manner, charge oriented modified nodal analysis (MNA) yields network equations of the following form:

$$\begin{aligned} 0 &= A_{C_\lambda}\dot{q}_\lambda + A_{R_\lambda} r_\lambda(A_{R_\lambda}^t e_\lambda, t) + A_{L_\lambda} \jmath_{L_\lambda} + A_{V_\lambda} \jmath_{V_\lambda} + \\ &\quad + A_{I_\lambda} \imath_\lambda(t) + \boxed{A_{u_\lambda} u}, && \text{(2a)} \\ 0 &= \dot{\phi}_\lambda - A_{L_\lambda}^t e_\lambda, && \text{(2b)} \\ 0 &= A_{V_\lambda}^t e_\lambda - v_\lambda(t), && \text{(2c)} \\ 0 &= q_\lambda - q_{C_\lambda}(A_{C_\lambda}^t e_\lambda, t), && \text{(2d)} \\ 0 &= \phi_\lambda - \varphi_{L_\lambda}(\jmath_{L_\lambda}, t) && \text{(2e)} \end{aligned}$$

for the $\lambda-$th subcircuit ($\lambda = 1, \ldots, r$) and the overall coupling equation

$$0 = \sum_{\lambda=1}^{r} A_{u_\lambda}^t \cdot e_\lambda. \tag{3}$$

Here (2a) constitutes the current balance for each node with the element related currents through capacitances, resistances, inductances, voltage and current sources assembled by the incidence matrices $A_{C_\lambda}, A_{R_\lambda}, A_{L_\lambda}, A_{V_\lambda}, A_{I_\lambda}$. The additional (boxed) term $A_{u_\lambda} u$ reflects the coupling currents to adjoined subcircuits, i. e. through the virtual voltage sources. The appropriate incidence matrix A_{u_λ} filters out the adequate boundary nodes.

The flux–node potential correlation (2b),the node – source voltage dependency (2c) and the charge and flux defining equations (2d,2e) are not affected by coupling to other subcircuits, as the information exchange is done solely via coupling currents.

The linear coupling equation (3) states, that the potentials at the boundary nodes of connected subcircuits have to coincide.

2.2 Index properties

The overall system (1a,1b) is made up of r subsystems – each with inner variables x_λ ($\lambda = 1, \ldots, r$) – that are coupled by one equation and one variable u respectively. Hence, several index-1 conditions are assumed to be fulfilled, according to the subsystems and the overall system:

(C1) The overall system (1a,1b) has index 1 (with respect to $x_1, \ldots, x_r, u$).
(C2) All systems (1a) define index-1 systems with respect to x_λ (and u given as input).
(C3) For all $\lambda \in \{1, \ldots, r\}$, the overall system (1a,1b) has index-1 with respect to x_λ and u (and $x_i, \forall i \neq \lambda$ given as input).

Remark 1. The condition (C3) concerns the index-1 property of the system $\mathcal{F}_i = 0, \mathcal{G} = 0$ and is not implied by (C1) and (C2) in general.

Topological Conditions

In analogy to the procedure described in [6], topological conditions to guarantee the index conditions (C1)-(C3) can be derived. Therefor (1a,1b) is transformed into the semi-explicit systems for $\lambda = 1, \ldots, r$

$$\begin{aligned} \dot{y}_\lambda(t) &= f_\lambda(z_\lambda, u, t), \\ 0 &= h_\lambda(y_\lambda, z_\lambda, u, t) \end{aligned} \tag{4a}$$

coupled by the algebraic equation

$$0 = g(z_1, \ldots, z_r) \tag{4b}$$

where z_λ identifies the node potentials and inner currents and y_λ defines the charges and fluxes (see also [2]).

3 Multirate Methods

The basic idea of *multirate methods* is to prevent parts to be integrated more often than necessary to guarantee given error tolerances. This is done by using different stepsizes that are suitable for the different levels of activity at each

time. In the case of problems that are already given in the form of coupled subsystems like (1a,1b) it is convenient to assume, that these subsystems have no intrinsic *multirate potential*.

Remark 2. Multirate methods have to interweave approximations working on different time grids. This causes an overhead that has to be outbalanced by the reduction of computational costs for the discretisation of the less active (latent) parts. Hence systems showing multirate behaviour are said to have *multirate potential* if the different timescales are widley seperated, the latent parts are larger than the active ones and the coupling amongst subsystems representing different activity levels is weak.

3.1 Multirate schemes for ODE systems

The concept of onestep multirate methods can be described with a system of two coupled ODEs:

$$\dot{y}_L = f_L(y_L, y_A), \quad y_L(t_0) = y_{L,0}, \tag{5a}$$
$$\dot{y}_A = f_A(y_L, y_A), \quad y_A(t_0) = y_{A,0}. \tag{5b}$$

The idea is to compute one *macrostep* of the latent part (subscript L) with the stepsize $\mathcal{H}_L$, i. e. get an approximation $y_{L,1} \approx y_L(t_0 + \mathcal{H}_l)$ and to perform q *microsteps* with stepsizes $\mathcal{H}_{A,\mu}(\mu = 1, \ldots, q)$ for the active part (see Fig. 2). In its most general way this procedure can be defined as follows:

$$\begin{aligned}
y_{L,1} &= y_{L,0} + \sum_{i=1}^{s_L} b_i^L \cdot k_i^L, \\
y_{A,\mu} &= y_{A,\mu-1} + \sum_{i=1}^{s_A} b_i^A \cdot k_i^{A,\mu} \quad (\mu = 1, \ldots, q), \\
k_i^L &= \Phi_L(h_l; y_{L,0}, Y_i^A, k_1^L, \ldots, k_{s_L}^L) \quad (i = 1, \ldots, s_L), \\
k_i^{A,\mu} &= \Phi_A(h_{a,\mu}; y_{A,\mu-1}, Y_i^{L,\mu}, k_1^{A,\mu}, \ldots, k_{s_A}^{A,\mu}) \quad (i = 1, \ldots, s_A),
\end{aligned}$$

where Φ_* denotes an s_* stage IRK or ROW scheme with coefficients $\alpha^*, \beta^*, \gamma^*, \nu^*$ ($* \in \{\mathrm{L,A}\}$).

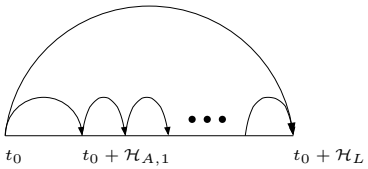

Fig. 2. Macro- and microsteps

As the subsystems are coupled, the computation of the weights for each part depends on information on the other one at some supporting timepoints:

$$Y_i^A \approx y_A(t_0 + \alpha_i^L \mathcal{H}_L) \quad (i = 1, \ldots, s_L), \tag{6a}$$

$$Y_i^{L,\mu} \approx y_L(t_0 + \sum_{\nu=1}^{\mu-1} \mathcal{H}_{A,\nu} + \alpha_i^A \mathcal{H}_{A,\mu}) \quad (i = 1, \ldots, s_A; \mu = 1, \ldots, q). \tag{6b}$$

There are different strategies to compute these values. Explicitly done *extra-/interpolation* [4] destroyes the onestep character of the method. *Generalised multirate* [5] a RK-based method calculates $Y_i^{L,\mu}$ and Y_i^A in RK-like manner using the stage increments $k_i^L, k_i^{A,\mu}$. *Mixed multirate* [1], ROW-based, builds up on generalised multirate. It decomposes the computation of one macrostep with its inner microstep to a so-called "compound step" and "later microsteps". For the former the incremental formulation of generalise multirate is used. In the latter dense-output is used for the coupling.

3.2 Mixed multirate scheme for coupled index-1 DAE systems

To get a mixed multirate scheme for coupled index-1 DAE systems of semi-explicit form (see (4a,4b) with $r = 2$),

$$\begin{array}{ll} \dot{y}_L = f_L(z_L, u) & \dot{y}_A = f_A(z_A, u) \\ 0 = h_L(y_L, z_L, u) & 0 = h_A(y_A, z_A, u) \\ \multicolumn{2}{c}{0 = g(z_L, z_A),} \end{array} \tag{7}$$

it is natural to assume the coupling variable u (defined by $0 = g$) to behave latent like y_L and z_L. With the index assumption (C3) $[\dot{y}_L = f_L, 0 = h_L, 0 = g]$ is an index-1 system with respect to z_L, u and $[\dot{y}_A = f_A, 0 = h_A]$ is of index 1 with respect to z_A if (C2) holds.

The mixed multirate ansatz for ODEs can be brought forward to the coupled semi-explicit problem (7). The *compound steps regulations* read :

$$\begin{pmatrix} y_{L,1} \\ z_{L,1} \\ u_1 \end{pmatrix} = \begin{pmatrix} y_{L,0} \\ z_{L,0} \\ u_0 \end{pmatrix} + b_L^t \begin{pmatrix} l_L \\ k_L \\ p \end{pmatrix}, \quad \begin{pmatrix} y_{A,1} \\ z_{A,1} \end{pmatrix} = \begin{pmatrix} y_{A,0} \\ z_{A,0} \end{pmatrix} + b_A^t \begin{pmatrix} l_A \\ k_A \end{pmatrix} \tag{8a}$$

with weights $b^\lambda := (b_1^\lambda, \ldots, b_s^\lambda)^t$ and increments $l_i^\lambda, k_i^\lambda, p_i$ defined by

$$\left(\begin{array}{c|c|c} \multicolumn{2}{c|}{\texttt{blkdiag}\left(\begin{pmatrix} \mathrm{I}_{y_\lambda} & -\mathcal{H}_\lambda \gamma^{(\lambda)} \frac{\partial f_\lambda}{\partial z_\lambda} \\ -\gamma^{(\lambda)} \frac{\partial h_\lambda}{\partial y_\lambda} & -\gamma^{(\lambda)} \frac{\partial h_\lambda}{\partial z_\lambda} \end{pmatrix}_{\lambda = L, A}\right)} & \begin{array}{c} -\mathcal{H}_L \gamma^{(L)} \frac{\partial f_L}{\partial u} \\ -\gamma^{(L)} \frac{\partial h_A}{\partial u} \\ \hline -\frac{1}{m} \cdot \mathcal{H}_A \nu^{(A,L)} \frac{\partial f_A}{\partial u} \\ -\frac{1}{m} \cdot \nu^{(A,L)} \frac{\partial h_A}{\partial u} \end{array} \\ \hline -\gamma^{(L)} \frac{\partial g}{\partial z_1} & -m \cdot \nu^{(L,A)} \frac{\partial g}{\partial z_2} \quad \Big| \quad \end{array}\right)$$

$$\cdot \begin{pmatrix} l_{L,i} \\ \dfrac{k_{L,i}}{l_{A,i}} \\ \dfrac{k_{A,i}}{p_i} \end{pmatrix} = \texttt{Rhs}(m, \mathcal{H}_L, \mathcal{H}_A; l_L, k_L, l_A, k_A, p) \quad (i = 1, \ldots, s) \tag{8b}$$

with stepsizeratio $m := \frac{\mathcal{H}_L}{\mathcal{H}_A}$

Remark 3. In the *later microsteps* it remains, to solve the system $[\dot{y}_A = f_a, 0 = h_a]$ with respect to y_A, z_A and $u(t)$ entering the right-hand-side via dense output: $u(t_0 + \xi \cdot \mathcal{H}_L) \approx u_0 + \sum_{i=1}^{s} b_{L,i}(\xi) \cdot p_i$ with $\xi \in (0, 1)$

Mixed multirate for coupled network (2a-e,3)

The coupled network equations (2a-e,3) are transfered to the semi-explicit formulation (4a,4b). With fixed projectors $Q_{C_\lambda}, Q_{V-C_\lambda}$, the variables of both formulations identify each other via linear operators $\hat{\vartheta}_\lambda, \vartheta_\lambda$ where :

$$\begin{array}{l|l} z_\lambda = \hat{\vartheta}_\lambda((e_\lambda^t, \jmath_{L_\lambda}^t, \jmath_{V_\lambda}^t)^t) & (e_\lambda^t, \jmath_{L_\lambda}^t, \jmath_{V_\lambda}^t)^t = \vartheta(z_\lambda) \\ \quad = \hat{\vartheta} \circ \vartheta(z_\lambda) & \quad = \vartheta \circ \hat{\vartheta}((e_\lambda^t, \jmath_{L_\lambda}^t, \jmath_{V_\lambda}^t)^t). \end{array} \tag{9}$$

The semi-explicit problem (4a,4b) and its associated method (8a,8b) is suitable to derive order conditions to get adequate coefficients, such that $\|\mathrm{err}_\lambda^{(\mathrm{se})}\| = \mathcal{O}(\mathcal{H}_\lambda^{p+1})^\dagger$.

As the transformation between the two formulations is not invertible, it is not possible to carry forward the attained method to a method that draws directly on the coupled network. To obtain such "network-regulations" with demanded accuracy ($\|\mathrm{err}_\lambda^{(\mathrm{mna})}\| = \mathcal{O}(\mathcal{H}_\lambda^{p+1})$)† in terms of node voltages and currents $(e, \jmath_L, \jmath_V)$ however, there is another way:

- Based upon the idea of (8a,8b) regulations with an (undefined) coefficient set can be deduced from the network formulation (2a-e,3).
- The same transformation that carries over the network formulation to the semi-explicit one, applied to the above regulations yields instructions that coincide with (8a,8b).
- Regarding (9), it holds that $\|\mathrm{err}_\lambda^{(\mathrm{mna})}\| \leq \|\vartheta\| \cdot \|\mathrm{err}_\lambda^{(\mathrm{se})}\|$ for the same coefficientset. Hence, if a coefficientset is chosen properly for the semi-explicit formulation, it is also suitable for the network formulation.

Finally a Block-Gaussian elimination and some linear transformation allow to eliminate the charges and fluxes q_λ, ϕ_λ. This guarantees charge- and flux-conservation and enables error-check and stepsize control based directly on the node potentials and currents $(e_\lambda, \jmath_{L_\lambda}, \jmath_{V_\lambda})$ (see also [3]).

$^\dagger \mathrm{err}_\lambda^\star$ denotes the local error after one step for the λ-th subsystem

4 Hierarchical mixed multirate

The already known multirate schemes deal with two different levels of activity. However coupled problems like (1a,1b) need *n-level-multirate schemes*[‡] with stepsizes $\mathcal{H}_1 > \ldots > \mathcal{H}_n$. Transfering the 2-level-mixed multirate to n-level-schemes in a straight forward way produces a bunch of coupling coefficients. *Hierarchical mixed multirate* is a new approach in dealing with an arbitrary amount of activity levels and still limits the amount of coupling-factors. The main idea is to nest compound steps and later micro-steps in a way, that at each time merely a two-level multirate scheme is engaged: (see Fig. 3)

(i) Group remaining subsystems in terms of activity level. This yields k_i virtual blocks consisting of subsystems showing similar behaviour. If $k_i = 1$ employ later micro steps – the coupling to other subsystems/blocks is given by dense output – until endpoint is reached, skip to (iv).
(ii) Build up a sorted stack (top down, decreasing stepsizes). Apply a compound step with the stacks top as latent block and its associated stepsize and all the other blocks combined to one as active block with the stepsize associated to the stacks bottom. The coupling to already integrated subsystems is given by dense output.
(iii) Remove the stacks top. The new endpoint is the one reached by the macrostep. Skip to (i).
(iv) Enlarge the set of remainig subsystems by the ones that produced the last endpoint. If the endpoint is the endpoint of integration as demanded it is finished. Else forget the endpoint and skip to (i).

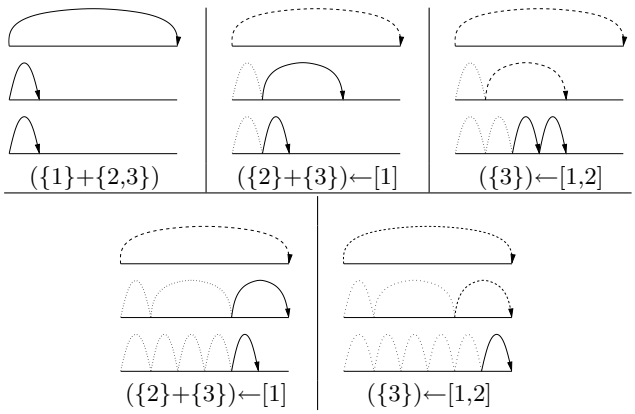

Fig. 3. Hierarchical mixed multirate for three blocks

[‡] $n \leq r$ as some subsystems may show the same activity level.

Numerical Tests

A first hierarchical multirate-method of order 2 has recently been implemented in `MATLAB`. It can deal with an arbitrary amount of subcircuits with grouping them in terms of activity levels.

First testruns were done with a three-block circuit (with 3/5/3 nodes) "behaving like" $\sin(\omega t)$ with $\omega = 1, 10, 100$ respectively. This yields promising results (see Table 1) as the mid-latent and latent block are calculated ten and hundred times less than in a corresponding singlerate.

Table 1. absolute distance multirate $(0.1, 0.01, 0.001)$ to singlerate (0.001)

$4 \cdot 10^{-13}$	$1 \cdot 10^{-2}$	$4 \cdot 10^{-11}$
$1 \cdot 10^{-3}$	$1 \cdot 10^{-6}$	$1 \cdot 10^{-8}$
$2 \cdot 10^{-3}$	$1 \cdot 10^{-2}$	$1 \cdot 10^{-3}$
	$3 \cdot 10^{-3}$	
	$5 \cdot 10^{-3}$	

5 Conclusions

A multirate scheme for circuit simulation that can deal with an arbitrary amount of subsystems has been derived. Now numerical tests have to be done with industry related examples. Furthermore the order of the method has to be enlarged to order three introducing adapted B-series and a stepsizecontrol adapted to coupled problems has to be derived.

References

1. A. Bartel, M. Günther and A. Kværnø: Multirate Methods in Electrical Circuit Simulation. In: A.M. Anile et al. (Ed.): Progress in Industrial Mathematics at ECMI 2000, Springer 2002, 258-265.
2. Günther, M., Arnold, M.: Coupled simulation of partitioned differential-algebraic network model. In preparation.
3. M. Günther:Simulating digital circuits numerically – a charge-oriented ROW approach Numer. Math. (1998) **79**: 203-212
4. C. W. Gear and R. R. Wells: Multirate linear multistep methods, BIT **24**, 484-502 (1984).
5. A. Kværnø and P. Rentrop: Low Order Multirate Runge-Kutta Methods in Electric Circuit Simulation. Preprint Nr. 99/1, IWRMM Universität Karlsruhe
6. Tischendorf, C.: Topological index calculation of differential-algebraic equations in circuit simulation. Surv. Math. Ind., **8**, 187–199 (1999)