Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Lehrstuhl für Angewandte Mathematik
und Numerische Mathematik

Preprint BUW-AMNA 04/06

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September 2004

http://www.math.uni-wuppertal.de/org/Num/

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Summary. In radio frequency (RF) design, signals with widely separated time scales arise. To describe those circuits efficiently, a multidimensional signal model was developed. This approach transfers the circuit's differential-algebraic equations (DAE) to a multirate system of partial differential-algebraic equations (MPDAE). A structural analysis, based on the concept of underlying PDE systems and the index characterization of DAE systems, emphasises the entitlement of MPDAE-modelling.

1 Introduction - multidimensional signal model

In electronic circuit design the classical modified nodal analysis (MNA) leads to a system of differential-algebraic equations (DAE). Excluding controlled sources, the charge-flux oriented formulation of the network equations yields [ST98]

$$\begin{aligned} A_C\dot{q} + A_R r(A_R^\top u(t),t) + A_L \jmath_L(t) + A_V \jmath_V(t) + A_I \imath(t) &= 0, && \text{(1a)}\\ \dot{\Phi} - A_L^\top u(t) &= 0, && \text{(1b)}\\ A_V^\top u(t) - v(t) &= 0, && \text{(1c)}\\ q - q_C(A_C^\top u(t),t) &= 0, && \text{(1d)}\\ \Phi - \Phi_L(\jmath_L(t),t) &= 0. && \text{(1e)} \end{aligned}$$

In the following we will consider quasiperiodic input signals. To face widely separated time scales, that occur frequently in RF application, the quasiperiodic functions can be generalized to multivariate functions (MVF), where for each time scale a corresponding variable $t_1,\dots,t_m$ is introduced. A signal with m fundamental frequencies $\omega_i = 2\pi/T_i$, $i=1,\dots,m$ and $X(k_1,\dots,k_m) \in \mathbb{C}$ is described by

$$x(t) = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_m=-\infty}^{\infty} X(k_1,\dots,k_m)\exp((jk_1\omega_1 + \cdots + jk_m\omega_m)\,t).$$

Its MVF then reaches the form

$$\hat{x}(t_1,\dots,t_m) = \sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_m=-\infty}^{\infty} X(k_1,\dots,k_m)\exp(jk_1\omega_1 t_1+\cdots+jk_m\omega_m t_m).$$

Now, the time scales are decoupled and the MVF is periodic in each coordinate direction. The original signal is contained on the diagonal of the MVF and can be reconstructed by $x(t) = \hat{x}(t,\dots,t)$.

We apply the multidimensional signal model to the network equations and introduce MVFs of charges, fluxes, sources and of the state variables. Looking at the MVF of the charge function

$$\hat{q}_C(w,t_1,\dots,t_m) \quad \text{with} \quad \frac{\partial \hat{q}_C}{\partial w} =: \hat{C}(w,t_1,\dots,t_m),$$

we define $\tau_m := (t,\dots,t)^\top \in \mathbb{R}^m$ and get for the time derivative

$$\begin{aligned}
&\frac{d}{dt} q_C(A_C^\top u(t),t)\\
&\quad = \frac{d}{dt}\hat{q}_C(A_C^\top \hat{u}(\tau_m),\tau_m)\\
&\quad = \hat{C}(A_C^\top \hat{u}(\tau_m),\tau_m)A_C^\top \cdot \sum_{i=1}^{m}\frac{\partial \hat{u}(\tau_m)}{\partial t_i} + \sum_{i=1}^{m}\frac{\partial}{\partial t_i}\hat{q}_C(A_C^\top \hat{u}(\tau_m),\tau_m).
\end{aligned}$$

Therefore, we define $\hat{\tau}_m := (t_1,\dots,t_m)^\top$ and introduce the differential operator D_m with

$$D_m f(x(\hat{\tau}_m),\hat{\tau}_m) \;:=\; \frac{df}{d\hat{\tau}_m}\cdot \mathbb{1} \;=\; \sum_{i=1}^{m}\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t_i} + \frac{\partial f}{\partial t_i}\right).$$

Now, we are able to generalize the original DAE to the multirate system of partial differential-algebraic equations (MPDAE)

$$\begin{aligned}
A_C D_m \hat{q} + A_R \hat{r}(A_R^\top \hat{u}(\hat{\tau}_m),\hat{\tau}_m) + A_L \hat{\jmath}_L(\hat{\tau}_m) + A_V \hat{\jmath}_V(\hat{\tau}_m)&\\
+A_I \hat{\imath}(\hat{\tau}_m) &= 0, &\text{(2a)}\\
D_m \hat{\Phi} - A_L^\top \hat{u}(\hat{\tau}_m) &= 0, &\text{(2b)}\\
A_V^\top \hat{u}(\hat{\tau}_m) - \hat{v}(\hat{\tau}_m) &= 0, &\text{(2c)}\\
\hat{q} - \hat{q}_C(A_C^\top \hat{u}(\hat{\tau}_m),\hat{\tau}_m) &= 0, &\text{(2d)}\\
\hat{\Phi} - \hat{\Phi}_L(\hat{\jmath}_L(\hat{\tau}_m),\hat{\tau}_m) &= 0. &\text{(2e)}
\end{aligned}$$

As the MVF $\hat{x}$ contains the original signal on its diagonal, the DAE-solution x with $x = (u, \jmath_L, \jmath_V)^\top$ can be reconstructed by $x(t) = \hat{x}(t_m)$ via the MPDAE-solution $\hat{x} = (\hat{u}, \hat{\jmath}_L, \hat{\jmath}_V)^\top$. For more details we refer to [BWLB96].

In order to resolve structural properties for this transferred system , we apply the index concept to extract the algebraic and differential parts of the MPDAE as it was done for the original DAE in [ST98].

2 Index-1 networks

The differential-algebraic network equations (1) have differential index 1, if the following two topological conditions are fulfilled (see [Ti99]):

T1: There are no cutsets consisting of inductances and/or current sources only: $\ker(A_C, A_R, A_V)^\top = \{0\}$.
T2: There are no loops consisting of only capacitances and at least one voltage source: $\ker Q_C^\top A_V = \{0\}$.

To transfer this context to our partial differential-algebraic system, we rewrite (2) in a semi-explicit form. We assume passivity for the network elements: therefore the capacitance, inductance and conductance matrices

$$C(w, \hat{\tau}_m) := \frac{\partial \hat{q}_C(w, \hat{\tau}_m)}{\partial w}\,,\; L(w, \hat{\tau}_m) := \frac{\partial \hat{\Phi}_L(w, \hat{\tau}_m)}{\partial w}\,,\; G(w, \hat{\tau}_m) := \frac{\partial \hat{r}(w, \hat{\tau}_m)}{\partial w}$$

are positive definite (but not necessarily symmetric) with a globally bounded inverse.

Let Q_C be an orthogonal projector onto the kernel of $A_C^\top$ and its complement P_C such that $P_C = I - Q_C$, with the identity matrix I. Hence, equation (2a) only contains information about $P_C \hat{u}$ as

$$A_C^\top \hat{u} = A_C^\top (P_C + Q_C)\hat{u} = A_C^\top P_C \hat{u}.$$

Subsequently, we define two sets of network variables

$$\hat{y} = \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} = \begin{pmatrix} P_C \hat{u} \\ \hat{\jmath}_L \end{pmatrix} \quad \text{and} \quad \hat{z} = \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} = \begin{pmatrix} Q_C \hat{u} \\ \hat{\jmath}_V \end{pmatrix}$$

(to shorten notations, we skip the arguments of the multivariate functions).

We insert the charges (2d) in (2a) and multiply the equation by $P_C^\top$ from the left. With

$$A_C \hat{q}_C(A_C^\top \hat{u}, \hat{\tau}_m) = A_C \hat{q}_C(A_C^\top \hat{y}_1, \hat{\tau}_m) + Q_C^\top Q_C \hat{y}_1 =: H(\hat{y}_1, \hat{\tau}_m)$$

we obtain a PDE for $\hat{y}_1$:

$$D_m H(\hat{y}_1, \hat{\tau}_m) = -P_C^\top \left(A_R \hat{r}(A_R^\top [\hat{y}_1 + \hat{z}_1], \hat{\tau}_m) + A_L \hat{y}_2 + A_V \hat{z}_2 + A_I \hat{\imath} \right). \quad (3)$$

The Jacobian $H_1 := \frac{\partial H}{\partial \hat{y}_1} = A_C C(A_C^\top [\hat{y}_1 + \hat{z}_1], \hat{\tau}_m) A_C^\top + Q_C^\top Q_C$ is positive definite by construction.

Inserting the fluxes (2e) in (2b), we directly obtain a PDE for $\hat{y}_2$:

$$D_m \hat{\Phi}_L(\hat{y}_2, \hat{\tau}_m) = A_L^\top [\hat{y}_1 + \hat{z}_1]. \quad (4)$$

Besides the differential equations (3) for $\hat{y}_1$ and (4) for $\hat{y}_2$ we are left with equation (2a) multiplied by $Q_C^\top$ from the left and (2c). Using $Q_C \hat{z}_1 = \hat{z}_1$ and $P_C^\top P_C \hat{z}_1 = 0$, we have

$$\begin{pmatrix} Q_C^\top \left(A_R \hat{r}(A_R^\top[\hat{y}_1 + Q_C\hat{z}_1], \hat{\tau}_m) + A_L\hat{y}_2 + A_V\hat{z}_2 + A_I\hat{\imath}\right) + P_C^\top P_C\hat{z}_1 \\ A_V^\top[\hat{y}_1 + Q_C\hat{z}_1] - \hat{v} \end{pmatrix} = 0. \tag{5}$$

The Jacobian with respect to $\hat{z}$

$$B := \begin{pmatrix} Q_C^\top A_R G(A_R^\top[\hat{y}_1 + \hat{z}_1])A_R^\top Q_C + P_C^\top P_C & Q_C^\top A_V \\ A_V^\top Q_C & 0 \end{pmatrix}$$

is regular, iff T1 and T2 hold, see [Ti99]. Thus, demanding the topological conditions, we are able to rewrite (2) in a semi-explicit form:

$$\begin{aligned} D_m\hat{y} &= F(\hat{y}, \hat{z}, \hat{\tau}_m), \\ 0 &= h(\hat{y}, \hat{z}, \hat{\tau}_m), \end{aligned}$$

where the algebraic equation is resolvable for $\hat{z} = \varphi(\hat{y}, \hat{\tau}_m)$. Hence, we are able to derive the underlying PDE

$$D_m\hat{y} = F(\hat{y}, \varphi(\hat{z}, \hat{\tau}_m), \hat{\tau}_m).$$

3 Index-2 networks

To investigate the differences in the index-2 case, we split the network equations until it is possible to resolve them for all the different sets of network variables.

After the first splitting $\hat{u} = P_C\hat{u} + Q_C\hat{u}$ in the index-1 case, we determined the algebraic equations (5)

$$Q_C^\top \left(A_R r(A_R^\top[P_C\hat{u} + Q_C\hat{u}], \hat{\tau}_m) + A_L\hat{\jmath}_L + A_V\hat{\jmath}_V + A_I\hat{\imath}\right) = 0, \tag{5a}$$

$$A_V^\top[P_C\hat{u} + Q_C\hat{u}] - \hat{v} = 0. \tag{5b}$$

In the index-2 case T1 and/or T2 are violated and the Jacobian relating to $Q_C\hat{u}$ and $\hat{\jmath}_V$ is not regular anymore. Therefore, (5a) and (5b) contain hidden constraints and further splittings of the network variables are necessary.

Lemma 1. *If T1 and/or T2 are violated, the MPDAE (2) is equivalent to the semi-explicit system*

$$\begin{aligned} A_C D_m \hat{q}_C(A_C^\top\hat{u}, \hat{\tau}_m) + P_C^\top \left(A_R\hat{r}(A_R^\top\hat{u}, \hat{\tau}_m) + A_L\hat{\jmath}_L + A_V\hat{\jmath}_V + A_I\hat{\imath}\right) &= 0, \\ D_m\hat{\Phi}_L(\hat{\jmath}_L, \hat{\tau}_m) - A_L^\top\hat{u} &= 0. \end{aligned}$$

$$\text{Index 1} \begin{cases} \bar{P}_{V-C}^\top \left(A_V^\top\hat{u} - \hat{v}\right) = 0, \\ P_{R-CV}^\top Q_{V-C}^\top Q_C^\top \left(A_R\hat{r}(A_R^\top\hat{u}, \hat{\tau}_m) + A_L\hat{\jmath}_L + A_I\hat{\imath}\right) = 0, \\ P_{V-C}^\top Q_C^\top \left(A_R\hat{r}(A_R^\top\hat{u}, \hat{\tau}_m) + A_L\hat{\jmath}_L + A_V\hat{\jmath}_V + A_I\hat{\imath}\right) = 0. \end{cases}$$

$$\text{Index 2} \begin{cases} Q_{CRV}^\top \left(A_L\hat{\jmath}_L + A_I\hat{\imath}\right) = 0, \\ \bar{Q}_{V-C}^\top \left(A_V^\top P_C\hat{u} - \hat{v}\right) = 0. \end{cases}$$

Proof. The orthogonal projectors used to obtain this semi-explicit description are defined as follows (see [ST98]):

projector	Q_{V-C}	$\bar{Q}_{V-C}$	Q_{R-CV}	Q_{CRV}
onto	$\ker A_V^\top Q_C$	$\ker Q_C^\top A_V$	$\ker A_R^\top Q_C Q_{V-C}$	$\ker(A_C, A_R, A_V)^\top$

with complements denoted by P and the corresponding subindex.

In the following, we will use the just defined projectors to filter out non-trivial information from the algebraic equations, as the variables of interest lie in the kernel of the antecedent matrices. To make the successive steps more comprehensible, equations extracted from (5a) and (5b) are denoted using a subindex: (5a$_i$) and (5b$_i$). If the differential operator is applied to an equation (x), it is denoted by (x$'$).

Regarding equation (5b), we only get information about $Q_C P_{V-C}\hat{u}$ as $A_V^\top Q_C Q_{V-C} = 0$. Furthermore, multiplying (5b) by $\bar{Q}_{V-C}^\top$ from the left reveals the linear combination

$$\bar{Q}_{V-C}^\top \left(A_V^\top P_C \hat{u} - \hat{v} \right) = 0, \tag{5b$_1$}$$

which does not appear in the index-1 case, as T2 implies $\bar{Q}_{V-C} = 0$. We will refer to this equation later.

To determine $Q_C P_{V-C}\hat{u}$ from the part $\bar{P}_{V-C}^\top \cdot (5b)$ of the equation, we have to mutiply by $Q_C^\top A_V$ from the left and add $Q_{V-C}^\top Q_{V-C} P_{V-C}\hat{u} = 0$:

$$\left(Q_C^\top A_V A_V^\top Q_C + Q_{V-C}^\top Q_{V-C} \right) P_{V-C}\hat{u} = Q_C^\top A_V \bar{P}_{V-C}^\top \left(\hat{v} - A_V^\top P_C \hat{u} \right). \tag{5b$_2$}$$

With $H_2 := Q_C^\top A_V A_V^\top Q_C + Q_{V-C}^\top Q_{V-C}$ positive definite, we can resolve for $P_{V-C}\hat{u}$, which leads to

$$Q_C P_{V-C}\hat{u} = Q_C H_2^{-1} Q_C^\top A_V \bar{P}_{V-C}^\top \left(\hat{v} - A_V^\top P_C \hat{u} \right).$$

At the moment we have the splitting

$$\hat{u} = \left[P_C + Q_C (P_{V-C} + Q_{V-C}) \right] \hat{u}$$

and still need equations for $Q_C Q_{V-C}\hat{u}$ and $\hat{\jmath}_V$.

To split equation (5a) in the right manner, we have a look at its derivative, as $\hat{u}$ is the argument of the nonlinear function $\hat{r}(\cdot)$. In our case, we apply the differential operator D_m, which yields

$$D_m \hat{u}(\hat{\tau}_m) = \frac{\partial \hat{u}}{\partial t_1} + \cdots + \frac{\partial \hat{u}}{\partial t_m}\ .$$

With the abbreviation $G := G(A_R^\top \hat{u}, \hat{\tau}_m)$, we obtain

$$Q_C^\top A_R G A_R^\top [P_C + Q_C P_{V-C} + Q_C Q_{V-C}] D_m \hat{u} + Q_C^\top (A_L D_m \hat{\jmath}_L + A_I D_m \hat{\imath}) + Q_C^\top A_V D_m \hat{\jmath}_V = 0. \tag{5a$'$}$$

Multiplying by $Q_{V-C}^\top$ from the left strikes off $D_m\hat{\jmath}_V$ and we obtain an equation for $P_{R-CV}D_m\hat{u}$ as $Q_C Q_{V-C} Q_{R-CV} =: Q_{CRV}$ and $A_R^\top Q_{CRV} = 0$. Thus, we also multiply by $P_{R-CV}^\top$ from the left and get

$$\begin{aligned} &Q_{V-C}^\top Q_C^\top A_R G A_R^\top Q_C Q_{V-C} P_{R-CV} D_m\hat{u} \qquad (5\mathrm{a}'_1)\\ &= -P_{R-CV}^\top Q_{V-C}^\top Q_C^\top \left(A_R G A_R^\top [P_C + Q_C P_{V-C}] D_m\hat{u} + A_L D_m\hat{\jmath}_L + A_I D_m\hat{\imath}\right). \end{aligned}$$

To resolve for $P_{R-CV}D_m\hat{u}$, we add $Q_{R-CV}^\top Q_{R-CV} P_{R-CV} D_m\hat{u} = 0$, which leads to $H_4 := H_4(A_R^\top\hat{u}, \hat{\tau}_m) := Q_{V-C}^\top Q_C^\top A_R G A_R^\top Q_C Q_{V-C} + Q_{R-CV}^\top Q_{R-CV}$ positive definite.

Now, we have to regard the splitting

$$\hat{u} = [P_C + Q_C(P_{V-C} + Q_{V-C}(P_{R-CV} + Q_{R-CV}))]\,\hat{u}$$

and have left the two equations $Q_{R-CV}^\top Q_{V-C}^\top\cdot(5\mathrm{a}')$ as well as $P_{V-C}^\top\cdot(5\mathrm{a}')$.

The first one is a hidden constraint, which the index-1 equations are lacking as T1 implies $Q_{CRV} = 0$. Using the PDE (4) for $\hat{\jmath}_L$ we obtain with the abbreviation $L := L(\hat{\jmath}_L, \hat{\tau}_m)$

$$\begin{aligned} &Q_{CRV}^\top \left(A_L L^{-1} A_L^\top [P_C + Q_C P_{V-C} + Q_C Q_{V-C} P_{R-CV} + Q_{CRV}]\hat{u} + A_I D_m\hat{\imath}\right)\\ &\quad = 0. \qquad (5\mathrm{a}'_2) \end{aligned}$$

Replacing $Q_{CRV}\hat{u}$ by $Q_{CRV}Q_{CRV}\hat{u}$ and adding $P_{CRV}^\top P_{CRV} Q_{CRV}\hat{u} = 0$ yields

$$\begin{aligned} &Q_{CRV}\hat{u} \;=\\ &\quad - H_5^{-1} Q_{CRV}^\top \left(A_L L^{-1} A_L^\top [P_C + Q_C P_{V-C} + Q_C Q_{V-C} P_{R-CV}]\hat{u} + A_I D_m\hat{\imath}\right) \end{aligned}$$

with $H_5 := H_5(\hat{\jmath}_L, \hat{\tau}_m) := Q_{CRV}^\top A_L L^{-1} A_L^\top Q_{CRV} + P_{CRV}^\top P_{CRV}$ positive definite. Here, we have to apply the differential operator D_m one more time to obtain an equation for $Q_{CRV}D_m\hat{u}$.

As we now have determined all parts of $D_m\hat{u}$, the second equation $P_{V-C}^\top\cdot(5\mathrm{a}')$ yields $\bar{P}_{V-C}D_m\hat{\jmath}_V$:

$$\begin{aligned} &P_{V-C}^\top Q_C^\top \left(A_R G A_R^\top D_m\hat{u} + A_L D_m\hat{\jmath}_L + A_I D_m\hat{\imath}\right)\\ &\qquad + Q_C^\top A_V[\bar{P}_{V-C} + \bar{Q}_{V-C}] D_m\hat{\jmath}_V \;=\; 0. \qquad (5\mathrm{a}'_3) \end{aligned}$$

We multiply by $A_V^\top Q_C$ from the left and add $\bar{Q}_{V-C}^\top \bar{Q}_{V-C} \bar{P}_{V-C} D_m\hat{\jmath}_V = 0$ to obtain the positive definite matrix $H_3 := A_V^\top Q_C Q_C^\top A_V + \bar{Q}_{V-C}^\top \bar{Q}_{V-C}$ and

$$\bar{P}_{V-C}D_m\hat{\jmath}_V = -H_3^{-1} A_V^\top Q_C \bar{P}_{V-C}^\top Q_C^\top \left(A_R G A_R^\top D_m\hat{u} + A_L D_m\hat{\jmath}_L + A_I D_m\hat{\imath}\right).$$

Finally, we have a look at the derivative of equation (5b$_1$):

$$\bar{Q}_{V-C}^\top A_V^\top P_C D_m\hat{u} - \bar{Q}_{V-C}^\top D_m\hat{v} = 0. \qquad (5\mathrm{b}'_1)$$

With $P_C D_m\hat{u} = -H_1^{-1} P_C^\top \left(A_R r(A_R^\top\hat{u}) + A_L\hat{\jmath}_L + A_V\hat{\jmath}_V + A_I\hat{\imath}\right)$ from (3), we get

$$\bar{Q}_{V-C}^\top A_V^\top H_1^{-1} P_C^\top A_V [\bar{P}_{V-C} + \bar{Q}_{V-C}] \hat{\jmath}_V$$
$$= -\bar{Q}_{V-C}^\top \left(D_m \hat{v} + A_V^\top H_1^{-1} P_C^\top \left(A_R r(A_R^\top \hat{u}) + A_L \hat{\jmath}_L + A_I \hat{\imath} \right) \right).$$

Replacing $\bar{Q}_{V-C}\hat{\jmath}_V$ by $\bar{Q}_{V-C}\bar{Q}_{V-C}\hat{\jmath}_V$ and adding $\bar{P}_{V-C}^\top \bar{P}_{V-C}\bar{Q}_{V-C}\hat{\jmath}_V = 0$ yields

$$\bar{Q}_{V-C}\hat{\jmath}_V =$$
$$-H_6^{-1}\bar{Q}_{V-C}^\top \left[D_m \hat{v} + A_V^\top H_1^{-1} P_C^\top \left(A_R r(A_R^\top \hat{u}) + A_L \hat{\jmath}_L + A_V \bar{P}_{V-C}\hat{\jmath}_V + A_I \hat{\imath} \right) \right]$$

with $H_6 := H_6(A_C^\top \hat{u}, \hat{\tau}_m) := \bar{Q}_{V-C}^\top A_V^\top H_1^{-1} A_V \bar{Q}_{V-C} + \bar{P}_{V-C}^\top \bar{P}_{V-C}$ positive definite. Again, another differentiation is needed to obtain an expression for $\bar{Q}_{V-C} D_m \hat{\jmath}_V$. □

Corollary 1. *The system defined in Lemma 1 is equivalent to the index-2 semi-explicit (but not Hessenberg) system*

$$\begin{aligned} D_m \hat{y} &= f(\hat{y}, \hat{v}, \hat{w}, \hat{\tau}_m), && \text{(6a)}\\ 0 &= g_1(\hat{y}, \hat{v}, \hat{w}, \hat{\tau}_m), && \text{(6b)}\\ 0 &= g_2(\hat{y}, \hat{\tau}_m), && \text{(6c)} \end{aligned}$$

with three sets of network variables

$$\hat{y} = \begin{pmatrix} P_C \hat{u} \\ \hat{\jmath}_L \end{pmatrix}, \ \hat{v} = \begin{pmatrix} Q_C P_{V-C} \hat{u} \\ Q_C Q_{V-C} P_{R-CV} \hat{u} \\ \bar{P}_{V-C} \hat{\jmath}_V \end{pmatrix} \quad \textit{and} \quad \hat{w} = \begin{pmatrix} Q_{CRV} \hat{u} \\ \bar{Q}_{V-C} \hat{\jmath}_V \end{pmatrix}.$$

Now applying the differential operator D_m to (6c), we are able to resolve $g := (g_1, g_2)^\top$ for $\hat{z} := (\hat{v}, \hat{w})^\top = \Psi(\hat{y}, \hat{\tau}_m)$ and to derive the underlying PDE

$$D_m \hat{y} = f(\hat{y}, \Psi(\hat{y}, \hat{\tau}_m), \hat{\tau}_m).$$

Of course, when thinking of solving the MPDAE, we do not use the concept of the underlying PDE, but it is a helpful tool to use the analogy of the MPDAE network equations to DAE-systems when transferring the index concept.

A special characteristics index was proposed in [Wa00] for linear hyperbolic PDAEs. As our network MPDAE is of hyperbolic type, we can proceed similarly. Defining a characteristic system leads to a continuous set of DAEs. In our special case, the characteristic curves are straight lines in the direction of the diagonal and the DAEs have the same structure as the original system (1). Thus, it is natural to use the index for this DAE system to characterize the MPDAE. Perturbation estimates and other suitable PDAE index concepts proposed in [GW00] are reserved to future work.

4 Conclusions

In this paper we have analysed a system of multirate partial differential-algebraic equations, which arises when a multidimensional signal model is applied to the MNA network equations. We showed, that the MPDAE inherits all the characteristics of the original network DAE. In both index-1 and index-2 cases, an underlying PDE can be found, i.e. the MPDAE can be reduced to a PDE on a manifold. Index concepts can be transferred and therefore no additional stability problems have to be expected when solving the network equations via the multidimensional approach. And, exploiting its special structure, the MPDAE can be solved very efficiently, e.g. with a method of characteristics proposed in [PG02].

Acknowledgement. This work has been supported within the federal BMBF project with the grant number 03GUNAVN.

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