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Summary. The numerical simulation of electric circuits including signals with largely differing time scales demands specific strategies. A multivariate model for signals, which exhibit amplitude as well as frequency modulation, yields a warped multirate partial differential algebraic equation (MPDAE). Corresponding initial boundary value problems lead to particular solution types. Two strategies for numerical simulation are discussed, which use contrary semidiscretisation techniques.

1 Introduction

Signals acting at widely separated time scales arise in radio frequency applications. The mathematical model of corresponding electric circuits consists in differential algebraic equations (DAEs). Integrating these systems demands a huge computational effort, since the fastest time scale restricts the step sizes. Consequently, numerical methods have to incorporate the specific structure of arising solutions in order to be efficient.

A multidimensional model yields a strategy for the simulation of amplitude and/or frequency modulated signals. Narayan and Roychowdhury [5] introduced an according warped multirate partial differential algebraic equation (MPDAE). The MPDAE solution of an initial boundary value problem reproduces a multitone DAE solution. Solving the MPDAE demands less effort than handling the DAE directly, since the model omits the computation of all fast oscillations. However, the warped MPDAE system includes a local frequency function, which is a priori unspecified. The determination of appropriate local frequencies is crucial for the efficiency of the model. Continuous phase conditions can be applied as additional boundary constraints to obtain suitable solutions.

We present two approaches for the numerical simulation of the initial boundary value problem, which both apply a semidiscretisation of the warped MPDAE system. On the one hand, we consider a method of Rothe type, which performs a discretisation in the slow time scale. On the other hand, we arrange a method of lines, which discretises the fast time scale. The properties of these two antipodal techniques are analysed. In particular, we discuss the inclusion of a continuous phase condition in view of an unknown local frequency. Finally, numerical results illustrate the performance of the two methods.

2 Warped MPDAE Model

To explicate the multidimensional model, we consider a multitone signal, which includes amplitude as well as frequency modulation, namely

$$x(t) = \left[1 + \alpha \sin\left(\tfrac{2\pi}{T_1} t\right)\right] \sin\left(\tfrac{2\pi}{T_2} t + \beta \sin\left(\tfrac{2\pi}{T_1} t\right)\right) \tag{1}$$

with parameters $0 < \alpha < 1$, $\beta > 0$. Fig. 1 illustrates this signal qualitatively. Assuming $T_1 \gg T_2$, many fast oscillations proceed during one slow oscillation of the modulation. Thus the number of time points to represent this signal increases drastically. Alternatively, we introduce an own variable for each separate time scale to model the amplitude modulation part

$$\hat{x}(t_1, t_2) = \left[1 + \alpha \sin\left(\tfrac{2\pi}{T_1} t_1\right)\right] \sin\left(2\pi t_2\right). \tag{2}$$

This representation is called the *multivariate function* (*MVF*) of the signal (1). Now the MVF is biperiodic and exhibits a simple structure in the rectangle $[0, T_1] \times [0, 1]$, which is also shown in Fig. 1. Hence we can resolve the MVF using relatively few grid points. The frequency modulation part is modelled by an additional time-dependent function

$$\Psi(t) = \frac{t}{T_2} + \frac{\beta}{2\pi} \sin\left(\tfrac{2\pi}{T_1} t\right). \tag{3}$$

The derivative $\nu := \Psi'$ plays the role of a *local frequency* belonging to the multitone signal (1). The function ν is T_1-periodic and features a simple behaviour, too. Nevertheless, we completely reconstruct the original signal via

$$x(t) = \hat{x}(t, \Psi(t)). \tag{4}$$

Thereby, Ψ is called a *warping function*, since it stretches the second time scale. Consequently, we obtain an efficient representation of the multitone signal by means of MVF and warping function/local frequency.

However, the multidimensional model is not unique. A family of MVFs and respective warping functions can describe the same signal. An inappropriate choice of the local frequency may yield a MVF, which exhibits many oscillations. Hence the identification of a local frequency with simple MVF determines the benefit of this representation.

Remark: The MVF concept is also convenient, if the first time scale is aperiodic and slowly varying. Consequently, the local frequency becomes aperiodic, too. In this case, we arrange the MVF in the domain $\mathbb{R}^+ \times [0, 1]$. Thus performing a step in t_1-direction already reproduces many fast oscillations.

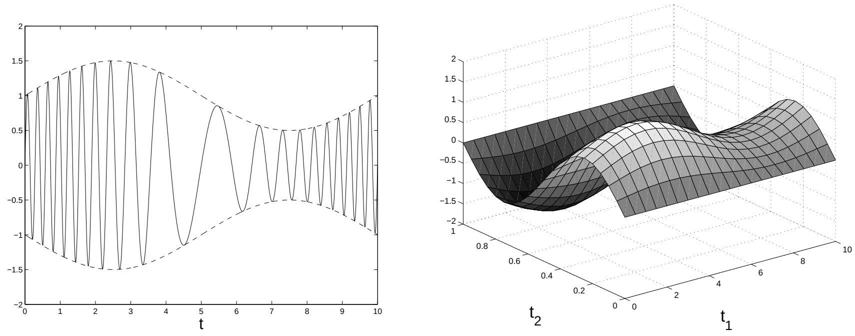

Fig. 1. Frequency modulated signal (left) and corresponding MVF (right).

Now we apply the multidimensional model in electric circuit simulation. A network approach yields *differential algebraic equations* (*DAEs*), which describe the transient behaviour of all node voltages and some branch currents, see [2]. In the following, we consider a semiexplicit DAE of index 1

$$\begin{aligned} \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} &= \mathbf{f}(\mathbf{y},\mathbf{z}) + \mathbf{b}(t) \\ \mathbf{0} &= \mathbf{g}(\mathbf{y},\mathbf{z}) + \mathbf{c}(t) \end{aligned} \tag{5}$$

with solutions $\mathbf{y}(t) \in \mathbb{R}^d$, $\mathbf{z}(t) \in \mathbb{R}^a$. The functions $\mathbf{b}(t) \in \mathbb{R}^d$, $\mathbf{c}(t) \in \mathbb{R}^a$ represent independent input signals. We assume that the input varies slowly the amplitude and frequency of fast oscillations in the solution. Thus the above multivariate representation becomes feasible. A transformation with respect to the reconstruction (4) changes the DAE model (5) into a *warped multirate partial differential algebraic equation* (*MPDAE*)

$$\begin{aligned} \frac{\partial\hat{\mathbf{y}}}{\partial t_1} + \nu(t_1)\frac{\partial\hat{\mathbf{y}}}{\partial t_2} &= \mathbf{f}(\hat{\mathbf{y}},\hat{\mathbf{z}}) + \mathbf{b}(t_1) \\ \mathbf{0} &= \mathbf{g}(\hat{\mathbf{y}},\hat{\mathbf{z}}) + \mathbf{c}(t_1). \end{aligned} \tag{6}$$

Now we solve the MPDAE system in a domain $[0, T_f] \times [0, 1]$ with arbitrary final time $T_f > 0$. Therefore we consider the initial boundary value problem (6) together with

$$\begin{array}{lll} \hat{\mathbf{y}}(0,t_2) = \mathbf{v}(t_2), & \hat{\mathbf{z}}(0,t_2) = \mathbf{w}(t_2) & \text{for all } t_2 \in \mathbb{R}, \\ \hat{\mathbf{y}}(t_1,t_2) = \hat{\mathbf{y}}(t_1,t_2+1), & \hat{\mathbf{z}}(t_1,t_2) = \hat{\mathbf{z}}(t_1,t_2+1) & \text{for all } t_1 \geq 0,\ t_2 \in \mathbb{R}. \end{array} \tag{7}$$

Thereby, the choice of the periodic initial values $\mathbf{v}$,$\mathbf{w}$ has to be consistent with respect to the DAE (5). An according MPDAE solution yields a complete DAE solution applying (4) with $\Psi(t) = \int_0^t \nu(\tau)\mathrm{d}\tau$. The reconstructed signal is uniquely defined by the initial values $\mathbf{y}(0) = \mathbf{v}(0)$, $\mathbf{z}(0) = \mathbf{w}(0)$. Hence the choice of the other values in $\mathbf{v}, \mathbf{w}$ just influence the efficiency of the model, since the resulting MVF depends on these initial functions. Using constant input $\mathbf{b} \equiv \mathbf{b}(0)$, $\mathbf{c} \equiv \mathbf{c}(0)$ in the DAE, a corresponding periodic solution represents a suitable initial state in general.

Assuming T_1-periodic input signals, biperiodic MPDAE solutions may exist. We can apply the problem (6),(7) to compute a biperiodic solution, too. We solve the MPDAE proceeding in t_1-direction until the solution enters a biperiodic steady state response. This strategy represents an advancement of transient analysis by using more information about the problem structure.

Since the local frequency ν stands for an a priori unknown function, the system (6),(7) is underdetermined. Hence we require additional conditions to isolate special solutions. In [5], *continuous phase conditions* are proposed to achieve this purpose. Thereby, the idea is to control the phase in each cross section $t_1 = \text{const}$ of a MVF. In the following, we apply a specific phase condition to the (without loss of generality) first component of the solution $\hat{\mathbf{y}} = (\hat{y}^1, \ldots, \hat{y}^d)^T$, namely

$$\left. \frac{\partial \hat{y}^1}{\partial t_2} \right|_{t_2=0} = 0 \quad \text{for all } t_1. \tag{8}$$

If the involved functions are sufficiently smooth, then differentiating (8) with respect to t_1 and (6) with respect to t_2 implies

$$\left. \frac{\partial^2 \hat{y}^1}{\partial t_1 \partial t_2} \right|_{t_2=0} = 0 \quad \Rightarrow \quad \nu(t_1) \left. \frac{\partial^2 \hat{y}^1}{\partial {t_2}^2} \right|_{t_2=0} = \left. \frac{\partial f^1(\hat{\mathbf{y}}, \hat{\mathbf{z}})}{\partial t_2} \right|_{t_2=0} \quad \text{for all } t_1. \tag{9}$$

Thus to ensure that the phase condition determines the local frequency uniquely, we assume the existence of a solution satisfying (8) and

$$\left| \left. \frac{\partial^2 \hat{y}^1}{\partial {t_2}^2} \right|_{t_2=0} \right| \geq \delta \quad \text{for all } t_1 \tag{10}$$

with a constant $\delta > 0$ in the following.

Alternatively, Houben [3] introduces minimum demands, which shall reduce oscillations in MVFs. Using these criteria, the determination of a relatively simple MVF representation is guaranteed. However, minimum demands cause more computation work in comparison to the elementary condition (8), which we add directly to the boundary conditions in the underlying domain.

3 Semidiscretisation Techniques

Now we examine two numerical techniques for solving the MPDAE initial boundary value problem (6),(7), which both apply semidiscretisation.

Firstly, we perform a *Rothe method* (*RM*). For parabolic PDEs including a time and a space variable, this means that the time derivative is discretised and thus a sequence of ODE boundary value problems in space arises. Accordingly, a difference scheme replaces the derivative with respect to t_1 in (6). Assuming a positive local frequency, the implicit Euler scheme, for example, yields the subsequent DAE systems

$$\begin{aligned} \frac{\mathrm{d}\tilde{\mathbf{y}}_j}{\mathrm{d}t_2}(t_2) &= \frac{1}{\nu_j}\left\{\mathbf{f}(\tilde{\mathbf{y}}_j(t_2),\tilde{\mathbf{z}}_j(t_2)) + \mathbf{b}(jh_1) - \tfrac{1}{h_1}\left[\tilde{\mathbf{y}}_j(t_2) - \tilde{\mathbf{y}}_{j-1}(t_2)\right]\right\} \\ \mathbf{0} &= \mathbf{g}(\tilde{\mathbf{y}}_j(t_2),\tilde{\mathbf{z}}_j(t_2)) + \mathbf{c}(jh_1) \end{aligned} \tag{11}$$

for $j = 1, 2, \ldots$ with step size h_1, where the jth part is an approximation of the MPDAE solution in the layer $t_1 = jh_1$. The initial values correspond to $j = 0$. The periodicity and the phase condition (8) generate the boundary constraints

$$\tilde{\mathbf{y}}_j(0) = \tilde{\mathbf{y}}_j(1), \quad \tilde{\mathbf{z}}_j(0) = \tilde{\mathbf{z}}_j(1), \quad \tfrac{\mathrm{d}\tilde{y}_j^1}{\mathrm{d}t_2}(0) = 0. \tag{12}$$

The local frequency ν_j represents an unknown parameter in each system. Hence the RM consists in the successive handling of boundary value problems corresponding to parameter-dependent DAEs with $d + a$ unknown functions. The DAEs (11) inherit the index 1 from the DAE (5). Moreover, specific techniques can be used to determine the periodic solution $\tilde{\mathbf{y}}_j, \tilde{\mathbf{z}}_j$ and the parameter ν_j in view of phase conditions, see [4].

Secondly, we apply a *method of lines* (*ML*). Now the derivative with respect to t_2 is substituted by a difference formula in the MPDAE. We employ symmetric differences and obtain a large DAE system including the subunits

$$\begin{aligned} \frac{\mathrm{d}\bar{\mathbf{y}}_i}{\mathrm{d}t_1}(t_1) &= \mathbf{f}(\bar{\mathbf{y}}_i(t_1),\bar{\mathbf{z}}_i(t_1)) + \mathbf{b}(t_1) - \nu(t_1)\tfrac{1}{2h_2}\left[\bar{\mathbf{y}}_{i+1}(t_1) - \bar{\mathbf{y}}_{i-1}(t_1)\right] \\ \mathbf{0} &= \mathbf{g}(\bar{\mathbf{y}}_i(t_1),\bar{\mathbf{z}}_i(t_1)) + \mathbf{c}(t_1) \end{aligned} \tag{13}$$

for $i = 1, \ldots, n_2$ with step size $h_2 = 1/n_2$. The ith component represents an approximation of the MPDAE solution in the layer $t_2 = (i-1)h_2$. The periodicity allows to identify $\bar{\mathbf{y}}_{n_2+1} = \bar{\mathbf{y}}_1$, $\bar{\mathbf{y}}_0 = \bar{\mathbf{y}}_{n_2}$ and thus to eliminate these unknown. Since the local frequency ν is unidentified, too, we have to incorporate the phase condition (8) via a difference formula. For example,

$$0 = \tfrac{\partial \hat{y}^1}{\partial t_2}(t_1, 0) \doteq \tfrac{1}{2h_2}\left[\bar{y}_2^1(t_1) - \bar{y}_{n_2}^1(t_1)\right] \tag{14}$$

gives an additional algebraic relation. Consequently, the ML yields an initial value problem of DAEs with dimension $n_2(d + a) + 1$. However, if we see ν as a part of the solution, then the index of the system (13),(14) is at least 2 even for an original DAE (5) of index 1. Furthermore, a suitable consistent choice of a starting value $\nu(0)$ is necessary.

As mentioned in the previous section, the initial boundary value problem can be used to determine a biperiodic solution by transient analysis. If we want to compute this steady state response directly, then a method of characteristics becomes favourable, see [6]. Moreover, the employed information transfer generates an inherent potential for parallelism. In contrast, the solution of the initial boundary value problem implies a sequential structure.

4 Numerical Results

We apply both semidiscretisation methods for the numerical simulation of a voltage controlled Van der Pol oscillator. The corresponding system reads

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -10(u^2-1)v - (2\pi w)^2 u \\ 0 &= w - b(t), \end{aligned} \tag{15}$$

which represents a semiexplicit DAE of index 1. If the input signal b is constant, a periodic steady state response arises. Otherwise, a time-dependent input signal produces frequency modulation. We choose the function

$$b(t) = 1 + \tfrac{1}{2}\sin\left(\tfrac{2\pi}{T_1}t\right) \quad \text{with} \quad T_1 = 1000. \tag{16}$$

Since the involved time scales are widely separated, we use the corresponding MPDAE model and treat problem (6),(7). As initial values, the periodic response of (15) corresponding to $b \equiv 1$ is employed. In the RM (11), we solve the periodic boundary value problems via a finite difference method including trapezoidal rule. In the ML (13), the initial value problems are integrated by the implicit Euler scheme. The used step sizes are equidistant, namely $h_1 = 20$ and $h_2 = 0.01$ in both techniques.

Fig. 2 illustrates the computed local frequencies. Since both functions respond to the input signal, the local frequencies are physically reasonable. Fig. 3 and Fig. 4 show the MPDAE solutions for u and v, respectively. The MVF of u features a constant amplitude, whereas the MVF of v includes amplitude modulation. The component w just reproduces the input signal. Investigating these MVFs, we recognise that assumption (10) is satisfied with $\delta \approx 80$.

Finally, we observe the corresponding DAE solutions. The results of the RM and the ML are used in the reconstruction (4). The outcome for u is shown in Fig. 5. Thereby, a reference solution was computed via an initial value problem of (15) using trapezoidal rule. In the first few cycles, both semidiscretisation methods exhibit a frequency, which is too high in comparison to the reference signal. In the RM, the local frequency even increases incorrectly for smaller step sizes h_1, whereas the frequency remains the same in the ML. In later cycles, all signals exhibit a significant phase shift to each other, which reflects a certain sensitivity, see [6]. Nevertheless, amplitude and shape agree in all three signals.

Other simulations, for example using a smaller value T_1, indicate an even more problematical behaviour of the semidiscretisation methods, where also too high amplitudes may arise. Moreover, the use of a BDF2 scheme, see [1], to proceed in t_1-direction leads to less accurate results in both RM and ML. Applying trapezoidal rule in the ML causes significant inaccuracies, which reflect the higher index of the semidiscretised system. Thus the application of semidiscretisation techniques seems to be critical, at least if the boundary constraint (8) is involved.

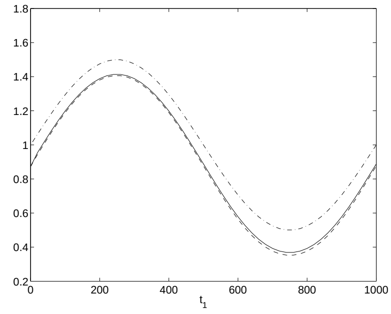

Fig. 2. Local frequency computed by RM (—) and ML (- -), respectively, together with input signal (- · -).

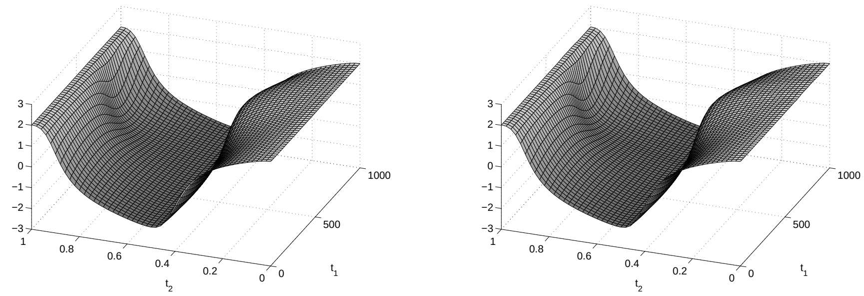

Fig. 3. MPDAE solution for u computed by RM (left) and ML (right).

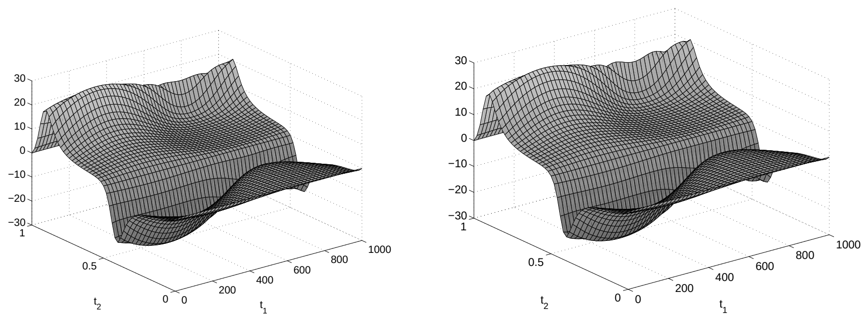

Fig. 4. MPDAE solution for v computed by RM (left) and ML (right).

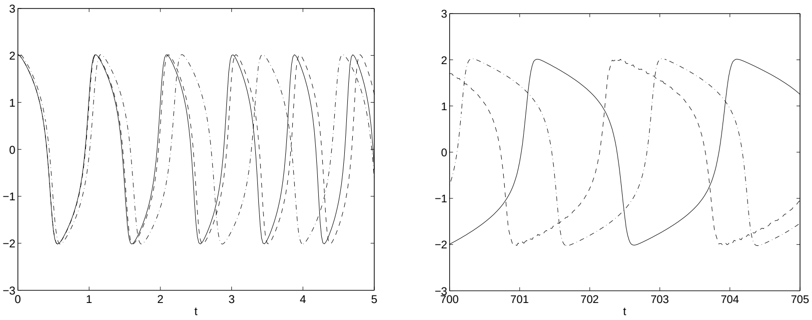

Fig. 5. DAE solution for u in time intervals $[0, 5]$ (left) and $[700, 705]$ (right) from RM (—), ML (- -) and transient integration (- · -).

5 Conclusions

The MPDAE model provides an alternative approach for the numerical simulation of multitone signals. Two techniques based on semidiscretisation for solving initial boundary value problems of MPDAEs have been presented, namely a Rothe method and a method of lines. Thereby, a specific boundary constraint is applied to identify the local frequency function. Numerical results demonstrate that both techniques exhibit problems in computing an accurate solution. Hence further theoretical examinations with respect to feasibility and stability of semidiscretisation methods are necessary in this context.

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