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PARABOLIC DIFFERENTIAL-ALGEBRAIC MODELS IN ELECTRICAL NETWORK DESIGN*

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Abstract. In refined network analysis, a compact network model is combined with distributed models for semiconductor devices in a multidimensional approach. For linear RLC networks containing diodes as distributed devices, we construct a mathematical model that joins the differential-algebraic initial-value problem for the electric circuit with parabolic-elliptic boundary value problems modeling the diodes. For this mixed initial-boundary value problem of partial differential-algebraic equations a first existence and uniqueness result is given and its asymptotic behavior is discussed.

Key words. Semiconductors, electric network, parabolic partial differential-algebraic equations, multiscale systems, refined coupled systems.

AMS subject classifications. 93A30, 35M10, 34M15.

1. Introduction. One main task in today's modeling and simulation of complex, more refined technical systems is to adequately describe the joint usage of elements combining lumped and distributed parts. These parts are often defined by a model reduction process. In these combined parts, topology is sufficient to describe the spatial allocation of lumped elements in a network approach, whereas the spatial dimension is indispensable to correctly reflect the behavior of distributed elements, either in 1d, 2d or even 3d, depending on the application and the model accuracy needed. Examples for this approach can be found in mechanics (combining rigid and elastic or elastoplastic elements in flexible multibody systems) [17], in biology (describing the human blood circulation by models of varying spatial complexity) [14] or ecology (modeling river systems by a network of hydrodynamical elements for water level prediction) [15], to name but a few. From a mathematical point of view, this modeling approach leads to partial differential-algebraic equations, for short PDAE systems, that couple differential-algebraic models for the lumped part and partial differential equations for the distributed elements by appropriate boundary conditions and source terms. These systems may not only be characterized by their multiscale feature in space (lumped versus distributed elements), but also in time: the time scales of the subsystems can differ by several orders of magnitude.

In chip design, PDAE models have been successfully used in refined network modeling: since spatial dimensions of semiconductor technology shrink steadily, former secondary effects tend to influence or even conduct the general behavior of the electric networks. Therefore hyperbolic PDAE models [6] are introduced to incorporate transmission line effects to the description of highly integrated circuits of extremely fast clock rate, whereas the inclusion of thermal effects in SOI circuits constitutes a parabolic PDAE system [2]. A third example is given by taking into account the distributed nature of semiconductor devices. The stationary case, which is appropriate for circuits that are characterized by very different time scales related to the

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relaxation of diodes to equilibrium and to the electric currents in the network, can be modeled by elliptic PDAE systems based on stationary drift-diffusion equations [1]. For all three PDAE examples mentioned above, existence results have been derived that justify the PDAE modeling approach.

In this paper, we aim at generalizing the last example to a non-stationary description. We introduce a mathematical model for such a coupled system, where the electric network is given by a linear RLC-circuit, described by differential algebraic equations (DAEs), and the devices are modeled by the nonstationary drift-diffusion model, which is a set of nonlinear parabolic-elliptic partial differential equations (PDE). We will study the well-posedness of this mixed parabolic-elliptic PDAE system. For the pure system of nonstationary drift-diffusion equations Gajewski proved the first existence result [3]. The ideas from that paper are the basis of our investigations.

The work is organized as follows. In Section 2, we describe the modeling of the lumped part (DAE network equations for the circuit), distributed part (nonstationary drift-diffusion model for the diodes) and their interdependence based on suitable coupling conditions. In Section 3, we discuss topological conditions for the overall network. For the appropriate choice of these condition, an inspection of the system's total energy is needed. In fact, the diodes create voltage-defined currents that enter the current balance in the network equations. On the other hand, the circuit's node potentials define boundary conditions for the drift-diffusion system. The next section combines these results to shape the overall coupled system in a (spatially) weak formulation. Section 5 is devoted to the existence and uniqueness analysis of this system. Generalizing the total energy derived in Section 2 to a parametric Liapunov functional, uniform a-priori estimates can be derived for both lumped and distributed components. These enable us to generalize local existence and uniqueness results, obtained by a fixed-point argument, to global results. In Section 6, the asymptotic behavior of the joint system, especially its relaxation to equilibrium, is addressed; the result depends heavily on the network's topology. Finally, in the last section we draw some conclusions and give an outlook on a few open problems.

2. Refined modeling of networks with diodes. To start with, we outline the network equations to describe the shunt electric elements. Next, we specify the 1d model for a semiconductor device, the drift-diffusion system. Then we aim at putting both models together by specifying coupling conditions.

2.1. Network models for electric circuits. We consider a linear RLC network which contains semiconductor devices with two Ohmic contacts, like, for instance, diodes. Then our circuit is an electric network which connects diodes, linear capacitors, inductors and resistors, and independent voltage and current sources, $v(t) \in \mathbb{R}^{n_V}$ and $i(t) \in \mathbb{R}^{n_I}$. Applying Modified Nodal Analysis (MNA) [7, 13] to setup the network description, the vector of unknowns \boldsymbol{x} comprises all node potentials $\boldsymbol{u}(t) \in \mathbb{R}^n$, and the currents $\boldsymbol{j}_L(t) \in \mathbb{R}^{n_L}$ and $\boldsymbol{j}_V(t) \in \mathbb{R}^{n_V}$ through inductors and voltage sources, respectively: thus $\boldsymbol{x} = (\boldsymbol{u}, \boldsymbol{j}_L, \boldsymbol{j}_V)^{\mathsf{T}}$, for $t \in [0, T]$, say. This set of unknowns is supplemented by currents $\boldsymbol{\lambda} \in \mathbb{R}^{2d}$ at the boundaries of all d diodes. It is precisely through $\boldsymbol{\lambda}$ that the diodes are coupled as non-basic elements to the RLC network. Then the DAE network equation for the RLC part is given by

$$\begin{pmatrix} \boldsymbol{A}_{C}\boldsymbol{C}\boldsymbol{A}_{C}^{\top} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\boldsymbol{t}} + \begin{pmatrix} \boldsymbol{A}_{R}\boldsymbol{G}\boldsymbol{A}_{R}^{\dagger} & \boldsymbol{A}_{L} & \boldsymbol{A}_{V} \\ -\boldsymbol{A}_{L}^{\top} & \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{A}_{V}^{\top} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} \boldsymbol{A}_{\lambda}\boldsymbol{\lambda} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} + \begin{pmatrix} \boldsymbol{A}_{I}\boldsymbol{\imath} \\ \boldsymbol{0} \\ \boldsymbol{v} \end{pmatrix} = \boldsymbol{0}. \quad (2.1)$$

Here, the incidence matrices $A_C \in \mathbb{R}^{n \times n_C}$, $A_L \in \mathbb{R}^{n \times n_L}$ and $A_R \in \mathbb{R}^{n \times n_G}$ describe the branch-node relationships for capacitors, inductors and resistors; the incidence matrices $A_V \in \mathbb{R}^{n \times n_V}$ and $A_I \in \mathbb{R}^{n \times n_I}$ describe this relationship for voltage and current sources, respectively. In variance, $A_\lambda \in \mathbb{R}^{n \times 2d}$ matches the boundaries of the diodes with the corresponding network nodes. Furthermore, we have the capacitance, inductance and conductance matrices $C \in \mathbb{R}^{n_C \times n_C}$, $L \in \mathbb{R}^{n_L \times n_L}$ and $G \in \mathbb{R}^{n_G \times n_G}$, which are assumed to be positive-definite and symmetric. The network equations are supplemented with consistent initial data

$$\boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{2.2}$$

The consistency of the initial data (2.2) with the equation (2.1) will be discussed later.

2.2. The one-dimensional diode model. We model a diode as a line segment of length l, characterized by a doping profile C(x), $x \in (0, l)$. We neglect all thermal effects, and assume that two carriers are responsible for the diode's output current, that is, electrons with negative charge -q, and holes with positive charge q. The behavior of the diode is described in terms of number densities of electrons and holes, denoted by n = n(x,t), p = p(x,t), and of electrostatic potential, denoted by V = V(x,t), with $(x,t) \in (0,l) \times (0,T)$. These variables satisfy the following transient drift-diffusion system [12],

$$-q\partial_t n + \partial_x j_n = qR, \tag{2.3a}$$

$$q\partial_t p + \partial_x j_p = -qR, \tag{2.3b}$$

$$j_n = -q \left\{ -D_n \partial_x n + \mu_n n \partial_x V \right\}, \qquad (2.3c)$$

$$j_p = q \left\{ -D_p \partial_x p - \mu_p p \partial_x V \right\}, \qquad (2.3d)$$

$$-\epsilon \partial_x^2 V = q(C+p-n), \qquad (2.3e)$$

where j_n and j_p denote the current densities for electrons and holes.

In (2.3), D_n , D_p are the diffusivities and μ_n , μ_p are the mobilities for electrons and holes, respectively. Around thermodynamic equilibrium, they are linked by

$$D_n = U_T \mu_n, \qquad D_p = U_T \mu_p,$$

called Einstein's relations, where U_T is the (constant) thermal potential. Since we neglect thermal effects, we assume at once the validity of these relations. Moreover, for simplicity, we assume diffusivities and mobilities to be positive constants.

The recombination-generation term R = R(n, p) is assumed to have the structure:

$$R(n,p) = F(n,p) \cdot \left(\frac{np}{n_i^2} - 1\right).$$

$$(2.4)$$

Here F = F(n, p) is Lipschitz-continuous and satisfies the estimate

$$0 \le F(n,p) \cdot (1+|n|+|p|) \le \bar{F}, \tag{2.5}$$

where \bar{F} is a constant and n_i denotes the intrinsic carrier concentration.

System (2.3) is supplemented with the initial-boundary conditions

$$n(x,0) = n_0(x), \quad p(x,0) = p_0(x),$$
 (2.6a)

$$n(x,t) = n_D(x), \quad p(x,t) = p_D(x), \qquad x = 0, l,$$
 (2.6b)

$$V(0,t) = V_{\rm bi}(0) + u_1(t), \quad V(l,t) = V_{\rm bi}(l) + u_2(t), \tag{2.6c}$$

where $p_0, n_0 \in L^2([0, l])$ are positive functions, and

$$n_D = C/2 + \sqrt{(C/2)^2 + n_i^2}, \quad p_D = -C/2 + \sqrt{(C/2)^2 + n_i^2},$$

 $V_{\rm bi} = U_T \ln(n_D/n_i).$

The potentials u_1 , u_2 in (2.6c) represent the external electric potentials applied to the devices. They are not independent functions to be assigned, but they are to be determined by the equations for the electrical network.

REMARK 1. From a physical point of view, mobilities and diffusivities are bounded, strictly positive functions of x and $E = \partial_x V$ [11, 16]. In this paper we assume these functions to be constant. It would be also possible to treat a class of non-constant mobilities as in [3], omitting Einstein's relations, and assuming

$$\mu_n(x, E) = m_n + M_n(x, |E|), \quad D_n(x, E) = U_T m_n, \mu_p(x, E) = m_p + M_p(x, |E|), \quad D_p(x, E) = U_T m_p,$$

where m_n and m_p are positive constants and

$$|M_n(x,z)| z \le \overline{M}, |M_p(x,z)| z \le \overline{M}, \text{ for any } x \in [0,l], z > 0,$$

with $0 \leq \overline{M} < \min(m_n, m_p)$.

REMARK 2. Conditions (2.4), (2.5) are satisfied by recombination-generation terms of Shockley-Read-Hall (SRH) type,

$$R_{SRH}(n,p) = \frac{np - n_i^2}{\tau_p(n+n_i) + \tau_n(p+n_i)},$$

where n_i is the intrinsic carrier concentration, τ_p and τ_n are the lifetimes of electrons and holes, respectively [16].

2.3. Coupling conditions. Without loss of generality, we consider a network which contains exactly one diode. All the arguments below generalize in a straightforward way to networks containing a multitude of diodes. So, the drift-diffusion equations (2.3) for the diode are coupled to the electric network by

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \boldsymbol{A}_{\lambda}^{\top} \boldsymbol{u}(t), \qquad (2.7)$$

which relates the boundary data (u_1, u_2) to the node potentials $u(t) \in \mathbb{R}^n$ of the DAE model for the electric network, and furthermore by the term

$$oldsymbol{A}_{\lambda}oldsymbol{\lambda} = oldsymbol{A}_{\lambda} \begin{pmatrix} \lambda_1 \ \lambda_2 \end{pmatrix}$$

which appears in the circuit equations (2.1). The total electric current λ transferred from the device to the circuit depends on the solution of (2.3) and must satisfy

$$\lambda_1 + \lambda_2 = 0,$$

which asserts that the currents transmitted to the circuit at the two Ohmic contacts of the device are opposite. In other words, we require that the quantity

$$I := \lambda_1 = -\lambda_2$$

is a current and is conserved through the device, consistent with Kirchhoff's law. In the stationary case [1], the current I is given by the electric current due to charge transport, $J = j_n + j_p$. In the evolutionary case, J is not conserved anymore. Nevertheless, from (2.3a)–(2.3b) we can derive the conservation of total charge,

$$\partial_t(-qn+qp) + \partial_x J = 0.$$

Using Poisson's equation (2.3e), we find

$$\partial_x(-\epsilon\partial_t\partial_x V + J) = 0,$$

which implies

$$\begin{aligned} -\epsilon \partial_t \partial_x V(x,t) + J(x,t) &= -\epsilon \partial_t \partial_x V(0,t) + J(0,t) \\ &= -\epsilon \partial_t \partial_x V(l,t) + J(l,t). \end{aligned}$$

With these considerations we can identify the current conserved over the device

$$I(t) = -\epsilon \partial_t \partial_x V(x, t) + J(x, t), \qquad (2.8)$$

as the sum of the displacement current, due to the variation of charge, and the electric current, due to charge transport. Thus the current transmitted to the circuit reads

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} I(t).$$
(2.9)

We can render more explicit the current by integrating (2.8) over [0, l]. Using (2.6c) and (2.7), we obtain immediately

$$I(t) = -\frac{\epsilon}{l} \frac{\mathrm{d}}{\mathrm{d}t} (u_2(t) - u_1(t)) + \langle J \rangle (t)$$

= $\frac{\epsilon}{l} \mathbf{A}_D^{\top} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \langle J \rangle (t),$ (2.10)

where

$$\boldsymbol{A}_D = \boldsymbol{A}_\lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and we have used the average operator

$$\langle f \rangle (t) = \frac{1}{l} \int_0^l f(x, t) \, \mathrm{d}x$$

The same coupling current is obtained by the explicit solution of (2.3e),

$$V(x,t) = u_{1}(t) + V_{\rm bi}(0) + \frac{x}{l} \left(u_{2}(t) + V_{\rm bi}(l) - u_{1}(t) - V_{\rm bi}(0) \right) + \int_{0}^{x} \int_{0}^{x'} \frac{q}{\epsilon} \left(n(x'',t) - p(x'',t) - C(x'') \right) dx'' dx' - \frac{x}{l} \int_{0}^{l} \int_{0}^{x'} \frac{q}{\epsilon} \left(n(x'',t) - p(x'',t) - C(x'') \right) dx'' dx',$$
(2.11)

which gives an explicit formula for the electric field $E = \partial_x V$ (modulo sign),

$$E(x,t) = \frac{1}{l} \left(u_2(t) + V_{\rm bi}(l) - u_1(t) - V_{\rm bi}(0) \right) + \int_0^x \frac{q}{\epsilon} \left(n(x',t) - p(x',t) - C(x') \right) \mathrm{d}x' - \frac{1}{l} \int_0^l \int_0^{x'} \frac{q}{\epsilon} \left(n(x'',t) - p(x'',t) - C(x'') \right) \mathrm{d}x'' \,\mathrm{d}x', \tag{2.12}$$

These explicit formulas will be useful later.

In conclusion, using (2.9) and the definition (2.10) for the transmitted current, the coupling term $A_{\lambda}\lambda$ in the circuit equations is given by

$$\boldsymbol{A}_{\lambda}\boldsymbol{\lambda} = \frac{\epsilon}{l}\boldsymbol{A}_{D}\boldsymbol{A}_{D}^{\top}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{A}_{D}\left\langle J\right\rangle.$$
(2.13)

3. Topological conditions. In this section we discuss appropriate topological conditions for the electric network, which will be used in the final formulation of the problem. In the first part, we introduce a general decomposition into differential and algebraic circuit components, and require that the coupled system has topological index 1. Next, we discuss the energy of the coupled problem and show that additional topological conditions are needed which ensure that all differential components can be controlled by the total energy.

3.1. Decomposition of the circuit's unknowns. Using a well established procedure [19], we decompose the circuit's unknowns into a differential component and an algebraic component. Using (2.13), the first component of (2.1) becomes

$$\boldsymbol{M}_{C}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{A}_{R}\boldsymbol{G}\boldsymbol{A}_{R}^{\top}\boldsymbol{u} + \boldsymbol{A}_{L}\boldsymbol{\jmath}_{L} + \boldsymbol{A}_{V}\boldsymbol{\jmath}_{V} + \boldsymbol{A}_{D}\left\langle J\right\rangle + \boldsymbol{A}_{I}\boldsymbol{\imath} = 0$$

where we have introduced the symmetric matrix

$$\boldsymbol{M}_{C} := \boldsymbol{A}_{C}\boldsymbol{C}\boldsymbol{A}_{C}^{\top} + \frac{\epsilon}{l}\boldsymbol{A}_{D}\boldsymbol{A}_{D}^{\top} = (\boldsymbol{A}_{C}, \boldsymbol{A}_{D}) \begin{pmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{\epsilon}{l} \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_{C}^{\top} \\ \boldsymbol{A}_{D}^{\top} \end{pmatrix}$$

We denote by \boldsymbol{Q}_{CD} a projector onto the ker $\boldsymbol{M}_{C} = \ker(\boldsymbol{A}_{C}, \boldsymbol{A}_{D})^{\mathsf{T}}$, and set $\boldsymbol{P}_{CD} = \mathbf{Id} - \boldsymbol{Q}_{CD}$, such that $\boldsymbol{P}_{CD}\boldsymbol{Q}_{CD} = \boldsymbol{Q}_{CD}\boldsymbol{P}_{CD} = \mathbf{0}$. Then the network variables can be split into a differential component, $\boldsymbol{y} = (\boldsymbol{y}_{1}, \boldsymbol{y}_{2})^{\mathsf{T}} := (\boldsymbol{P}_{CD}\boldsymbol{u}, \boldsymbol{j}_{L})^{\mathsf{T}}$, and an algebraic component, $\boldsymbol{z} = (\boldsymbol{z}_{1}, \boldsymbol{z}_{2})^{\mathsf{T}} := (\boldsymbol{Q}_{CD}\boldsymbol{u}, \boldsymbol{j}_{V})^{\mathsf{T}}$. In terms of these components, the network equations read

$$\begin{pmatrix} \boldsymbol{H} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \boldsymbol{y}_{1} \\ \boldsymbol{y}_{2} \end{pmatrix} + \begin{pmatrix} \boldsymbol{P}_{CD}^{\top} \boldsymbol{A}_{R} \boldsymbol{G} \boldsymbol{A}_{R}^{\top} \boldsymbol{P}_{CD} & \boldsymbol{P}_{CD}^{\top} \boldsymbol{A}_{L} \\ -\boldsymbol{A}_{L}^{\top} \boldsymbol{P}_{CD} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{y}_{1} \\ \boldsymbol{y}_{2} \end{pmatrix}$$
(3.1a)
$$+ \begin{pmatrix} \boldsymbol{P}_{CD}^{\top} \boldsymbol{A}_{R} \boldsymbol{G} \boldsymbol{A}_{R}^{\top} \boldsymbol{Q}_{CD} & \boldsymbol{P}_{CD}^{\top} \boldsymbol{A}_{V} \\ -\boldsymbol{A}_{L}^{\top} \boldsymbol{Q}_{CD} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{z}_{1} \\ \boldsymbol{z}_{2} \end{pmatrix} + \begin{pmatrix} \boldsymbol{P}_{CD}^{\top} (\boldsymbol{A}_{I} \boldsymbol{\imath} + \boldsymbol{A}_{D} \langle J \rangle) \\ \boldsymbol{0} \end{pmatrix} = \boldsymbol{0},$$
(3.1b)
$$\begin{pmatrix} \boldsymbol{Q}_{CD}^{\top} \boldsymbol{A}_{R} \boldsymbol{G} \boldsymbol{A}_{R}^{\top} \boldsymbol{Q}_{CD} & \boldsymbol{Q}_{CD}^{\top} \boldsymbol{A}_{V} \\ \boldsymbol{A}_{V}^{\top} \boldsymbol{Q}_{CD} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{z}_{1} \\ \boldsymbol{z}_{2} \end{pmatrix}$$
(3.1b)
$$+ \begin{pmatrix} \boldsymbol{Q}_{CD}^{\top} \boldsymbol{A}_{R} \boldsymbol{G} \boldsymbol{A}_{R}^{\top} \boldsymbol{P}_{CD} & \boldsymbol{Q}_{CD}^{\top} \boldsymbol{A}_{L} \\ \boldsymbol{A}_{V}^{\top} \boldsymbol{P}_{CD} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{y}_{1} \\ \boldsymbol{y}_{2} \end{pmatrix} + \begin{pmatrix} \boldsymbol{Q}_{CD}^{\top} (\boldsymbol{A}_{I} \boldsymbol{\imath} + \boldsymbol{A}_{D} \langle J \rangle) \\ -\boldsymbol{v} \end{pmatrix} = \boldsymbol{0}.$$

Here $\boldsymbol{H} = \boldsymbol{M}_C + \boldsymbol{Q}_{CD}^{\top} \boldsymbol{Q}_{CD}$ is a positive-definite, symmetric matrix. By definition of \boldsymbol{Q}_{CD} , we have $\boldsymbol{Q}_{CD}^{\top} \boldsymbol{A}_D = 0$. Then system (3.1) can be written in the compact form

$$\tilde{\boldsymbol{A}}_{P}\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} + \boldsymbol{B}_{P}\boldsymbol{y} + \boldsymbol{C}_{P}\boldsymbol{z} + \boldsymbol{F}_{P}(\boldsymbol{\imath}) + \begin{pmatrix} \boldsymbol{A}_{D} \langle J \rangle \\ 0 \end{pmatrix} = 0, \qquad (3.2a)$$

$$\boldsymbol{B}_{Q}\boldsymbol{z} + \boldsymbol{C}_{Q}\boldsymbol{y} + \boldsymbol{F}_{Q}(\boldsymbol{\imath}, \boldsymbol{v}) = 0, \qquad (3.2b)$$

with obvious notation. The differential-algebraic system (3.2) has differential index-1 (for a given function $\langle J \rangle$) if (3.2b) can be solved for z as a function of y. Thus we need the invertibility of the linear map B_Q in the subspace (ker M_C) × \mathbb{R}^{n_V} . To this end, we assume the following topological conditions [6, 19]:

$$\ker(\boldsymbol{A}_D, \boldsymbol{A}_C, \boldsymbol{A}_R, \boldsymbol{A}_V)^{\top} = \{0\}, \qquad (3.3)$$

$$\ker \boldsymbol{Q}_{CD}^{\mathsf{T}} \boldsymbol{A}_{V} = \{0\}. \tag{3.4}$$

Then, due to the positive-definiteness of G, the matrix

$$\boldsymbol{B}_{Q} = \begin{pmatrix} \boldsymbol{Q}_{CD}^{\top} \boldsymbol{A}_{R} \boldsymbol{G} \boldsymbol{A}_{R}^{\top} \boldsymbol{Q}_{CD} & \boldsymbol{Q}_{CD}^{\top} \boldsymbol{A}_{V} \\ \boldsymbol{A}_{V}^{\top} \boldsymbol{Q}_{CD} & \boldsymbol{0} \end{pmatrix}$$

satisfies

$$\ker \boldsymbol{B}_Q = \ker \left((\boldsymbol{A}_R, \boldsymbol{A}_V)^\top \boldsymbol{Q}_{CD} \right) \times \ker \boldsymbol{Q}_{CD}^\top \boldsymbol{A}_V = \ker \boldsymbol{Q}_{CD} \times \{0\},$$

and hence \boldsymbol{z} is a linear function of $\boldsymbol{y},\,\boldsymbol{\imath}$ and \boldsymbol{v} . Explicitly, system (3.2) is equivalent to

$$\tilde{\boldsymbol{A}}_{P}\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} + \tilde{\boldsymbol{B}}_{P}\boldsymbol{y} + \tilde{\boldsymbol{F}}_{P}(\boldsymbol{\imath},\boldsymbol{v}) + \begin{pmatrix} \boldsymbol{A}_{D} \langle J \rangle \\ 0 \end{pmatrix} = 0, \qquad (3.5a)$$

$$\boldsymbol{z} = -\boldsymbol{B}^{-1} \left(\boldsymbol{C}_{Q} \boldsymbol{y} + \boldsymbol{F}_{Q} (\boldsymbol{\imath}, \boldsymbol{v}) \right), \qquad (3.5b)$$

with

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{Q}_{CD}^{\top} \boldsymbol{A}_{R} \boldsymbol{G} \boldsymbol{A}_{R}^{\top} \boldsymbol{Q}_{CD} + \boldsymbol{P}_{CD}^{\top} \boldsymbol{P}_{CD} & \boldsymbol{Q}_{CD}^{\top} \boldsymbol{A}_{V} \\ \boldsymbol{A}_{V}^{\top} \boldsymbol{Q}_{CD} & \boldsymbol{0} \end{pmatrix}, \qquad \tilde{\boldsymbol{B}}_{P} = \boldsymbol{B}_{P} - \boldsymbol{C}_{P} \boldsymbol{B}^{-1} \boldsymbol{C}_{Q}, \\ \tilde{\boldsymbol{F}}_{P} = \boldsymbol{F}_{P} - \boldsymbol{C}_{P} \boldsymbol{B}^{-1} \boldsymbol{F}_{Q}.$$

REMARK 3. The topological conditions (3.3) and (3.4) have simple physical interpretations. Condition (3.3) forbids cutsets composed of independent current sources and inductors. Condition (3.4) states that loops containing at least one voltage source and any number of capacitors and diodes are forbidden.

3.2. Total energy of the coupled problem. Next, we discuss the total physical energy of our coupled system. We will propose two expressions for this quantity, which will lead us to consider an additional topological condition for the network.

To start, we introduce the local physical energy $w_D(x,t)$ associated to the device

$$w_D = qU_T \left\{ n \left(\ln \frac{n}{n_i} - 1 \right) + p \left(\ln \frac{p}{n_i} - 1 \right) \right\} + \frac{\epsilon}{2} E^2,$$

where $E = \partial_x V$ [3]. For smooth solutions, w_D satisfies the following balance equation:

$$\partial_t w_D + \partial_x \left(-U_T \ln \frac{n}{n_i} j_n + U_T \ln \frac{p}{n_i} j_p \right) = -\frac{j_n^2}{q\mu_n n} - \frac{j_p^2}{q\mu_p p} - qU_T R \ln \frac{np}{n_i^2} - I \partial_x V, \quad (3.6)$$

where I = I(t) is defined by (2.8). The total energy of the device is given by

$$W_D(t) = \int_0^l w_D(x,t) \,\mathrm{d}x.$$

We can also consider the total energy W_C associated to the circuit, defined by

$$W_C = \frac{1}{2} \boldsymbol{u}^{\mathsf{T}} \boldsymbol{A}_C \boldsymbol{C} \boldsymbol{A}_C^{\mathsf{T}} \boldsymbol{u} + \frac{1}{2} \boldsymbol{j}_L^{\mathsf{T}} \boldsymbol{L} \boldsymbol{j}_L.$$

Multiplying the network equations (2.1) by \mathbf{x}^{T} , we obtain the following relation

$$\frac{\mathrm{d}W_C}{\mathrm{d}t} + \boldsymbol{u}^{\mathsf{T}}\boldsymbol{A}_R\boldsymbol{G}\boldsymbol{A}_R^{\mathsf{T}}\boldsymbol{u} + \boldsymbol{u}^{\mathsf{T}}\boldsymbol{A}_I\boldsymbol{\imath} + \boldsymbol{\jmath}_V^{\mathsf{T}}\boldsymbol{v} + \boldsymbol{u}^{\mathsf{T}}\boldsymbol{A}_\lambda\boldsymbol{\lambda} = 0.$$
(3.7)

An obvious choice for the total energy of the coupled problem is just the sum of the energy of the device and the energy of the circuit, $W = W_D + W_C$. Integrating (3.6) over the space domain [0, l], recalling the boundary conditions (2.6) and using (3.7), we find that W satisfies the equation

$$\frac{\mathrm{d}W}{\mathrm{d}t} + \boldsymbol{u}^{\mathsf{T}}\boldsymbol{A}_{R}\boldsymbol{G}\boldsymbol{A}_{R}^{\mathsf{T}}\boldsymbol{u} + \boldsymbol{u}^{\mathsf{T}}\boldsymbol{A}_{I}\boldsymbol{i} + \boldsymbol{j}_{V}^{\mathsf{T}}\boldsymbol{v}$$
$$= -\int_{0}^{l} \left(\frac{j_{n}^{2}}{q\mu_{n}n} + \frac{j_{p}^{2}}{q\mu_{p}p}\right) \mathrm{d}x - \int_{0}^{l} qU_{T}R\ln\frac{np}{n_{i}^{2}}\mathrm{d}x - \left[V_{\mathrm{bi}}(I-J)\right]_{0}^{l}.$$
 (3.8)

Here we have used the identity

$$\boldsymbol{u}^{\mathsf{T}}\boldsymbol{A}_{\lambda}\boldsymbol{\lambda} = -(u_2 - u_1)I. \tag{3.9}$$

It is possible to give an alternative definition for the total energy, where no boundary terms appear in the total energy balance equation. To this end, we introduce a special steady-state solution (n^{e}, p^{e}, V^{e}) of (2.3), which corresponds to zero external electric voltage source v and electric current source i, where the potentials applied to the diode vanish, and, furthermore, this solution shall satisfy the conditions:

$$n^{\mathrm{e}} = n_i \exp\left(V^{\mathrm{e}}/U_T\right), \qquad p^{\mathrm{e}} = n_i \exp\left(-V^{\mathrm{e}}/U_T\right).$$

A steady-state solution corresponding to these constraints represents a physical state in total thermodynamic equilibrium. The equilibrium voltage V^{e} is uniquely determined by the following nonlinear elliptic problem:

$$-\epsilon \partial_x^2 V = qC - qn_i \left(\exp\left(V/U_T\right) - \exp\left(-V/U_T\right)\right), \qquad (3.10)$$
$$V(x,t) = V_{\rm bi}(x), \qquad x = 0, l,$$

with

$$V_{\rm bi} = U_T \ln(n_D/n_i), \qquad n_D = C/2 + \sqrt{(C/2)^2 + n_i^2}.$$

Having introduced this steady state solution $(n^{\rm e}, p^{\rm e}, V^{\rm e})$, we can consider a shifted local energy w_D^* for the device:

$$w_D^* = w_D - w_D^{\mathbf{e}} - \left(\frac{\partial w_D}{\partial n}\right)^{\mathbf{e}} (n - n^{\mathbf{e}}) - \left(\frac{\partial w_D}{\partial p}\right)^{\mathbf{e}} (p - p^{\mathbf{e}}) - \left(\frac{\partial w_D}{\partial E}\right)^{\mathbf{e}} (E - E^{\mathbf{e}})$$
$$\equiv q U_T \left\{ n \left(\ln \frac{n}{n^{\mathbf{e}}} - 1 \right) + n^{\mathbf{e}} + p \left(\ln \frac{p}{p^{\mathbf{e}}} - 1 \right) + p^{\mathbf{e}} \right\} + \frac{\epsilon}{2} |E - E^{\mathbf{e}}|^2,$$

which is quadratic in the neighborhood of the equilibrium. Here the e-superscript signifies the evaluation at the equilibrium state, and $E^{\rm e} = \partial_x V^{\rm e}$. Let W_D^* denote the corresponding global energy, then the total energy $W^* = W_D^* + W_C$ satisfies

$$\frac{\mathrm{d}W^*}{\mathrm{d}t} + \boldsymbol{u}^{\mathsf{T}}\boldsymbol{A}_{R}\boldsymbol{G}\boldsymbol{A}_{R}^{\mathsf{T}}\boldsymbol{u} + \boldsymbol{u}^{\mathsf{T}}\boldsymbol{A}_{I}\boldsymbol{i} + \boldsymbol{j}_{V}^{\mathsf{T}}\boldsymbol{v}$$
$$= -\int_{0}^{l} \Big(\frac{j_{n}^{2}}{q\mu_{n}n} + \frac{j_{p}^{2}}{q\mu_{p}p}\Big)\mathrm{d}x - \int_{0}^{l} qU_{T}R\ln\frac{np}{n_{i}^{2}}\,\mathrm{d}x.$$
(3.11)

For both choices W and W^* , the components of the circuit variables which contribute to the total energy for the coupled problem are given by $(\boldsymbol{P}_C \boldsymbol{u}, \boldsymbol{j}_L)$, where $\boldsymbol{P}_C = \mathbf{Id} - \boldsymbol{Q}_C$, and \boldsymbol{Q}_C is a projector onto ker \boldsymbol{A}_C^{\top} . We require that these components contain the differential components $(\boldsymbol{P}_{CD}\boldsymbol{u},\boldsymbol{j}_L)$, that is, we demand $\boldsymbol{Q}_C \boldsymbol{P}_{CD}\boldsymbol{u} = 0$. By the definition of \boldsymbol{Q}_{CD} and \boldsymbol{P}_{CD} , this condition is equivalent to $\boldsymbol{Q}_C \boldsymbol{u} = \boldsymbol{Q}_C \boldsymbol{Q}_{CD} \boldsymbol{u} \equiv \boldsymbol{Q}_{CD} \boldsymbol{u}$, or $\boldsymbol{Q}_C = \boldsymbol{Q}_{CD}$, which reads as topological condition:

$$\boldsymbol{A}_{\boldsymbol{D}}^{\scriptscriptstyle \mathsf{I}}\boldsymbol{Q}_{\boldsymbol{C}}=\boldsymbol{0}.\tag{3.12}$$

If (3.12) is valid, then the index-1 topological conditions (3.3) and (3.4) reduce to

$$\ker(\boldsymbol{A}_C, \boldsymbol{A}_R, \boldsymbol{A}_V)^{\top} = \{0\}, \qquad (3.13)$$

$$\ker \boldsymbol{Q}_C^{\dagger} \boldsymbol{A}_V = \{0\}. \tag{3.14}$$

Also, the energy of the circuit assumes the simple form

$$W_C = \frac{1}{2} \boldsymbol{y}^{\mathsf{T}} \boldsymbol{A}_P \boldsymbol{y},$$

with

$$\boldsymbol{A}_{P} = \begin{pmatrix} \boldsymbol{A}_{C}\boldsymbol{C}\boldsymbol{A}_{C}^{\top} + \boldsymbol{Q}_{C}^{\top}\boldsymbol{Q}_{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L} \end{pmatrix}.$$
 (3.15)

In the following, we assume altogether the validity of (3.12)–(3.14).

REMARK 4. The topological conditions (3.12)–(3.14) have the following network interpretations: (i) the diode's terminals are connected by a path of capacitors, (ii) there is no cutset of inductors and current sources, and (iii) there is no loop of capacitors containing at least one voltage source, respectively.

4. Problem formulation. Assuming the topological condition (3.12)-(3.14), we can combine the evolutionary drift-diffusion model and the RLC-network equations from above and formulate the full coupled problem on a time interval [0, T]. Before doing so, the consistency of the initial data (2.2) needs to be briefly addressed. We identify $\mathbf{x}_0 = (\mathbf{u}_0, \mathbf{j}_{L0}, \mathbf{j}_{V0})^{\mathsf{T}}$ and define

$$m{y}_0 = (m{y}_{1,0},\,m{y}_{2,0}) = (m{P}_Cm{u}_0,\,m{\jmath}_{L0})^{\!\!\!\top}, \qquad m{z}_0 = (m{Q}_Cm{u}_0,\,m{\jmath}_{V0})^{\!\!\!\top}.$$

Then the differential part y_0 and the algebraic part z_0 of the initial data (2.2) must satisfy the consistency condition

$$\boldsymbol{B}_{Q}\boldsymbol{z}_{0} + \boldsymbol{C}_{Q}\boldsymbol{y}_{0} + \boldsymbol{F}_{Q}(\boldsymbol{\imath}(0), \boldsymbol{v}(0)) = 0.$$

This condition can be fulfilled by supplementing (3.2a) with arbitrary initial data $\boldsymbol{y}_{1,0}$ such that $\boldsymbol{Q}_C^{\top} \boldsymbol{y}_{1,0} = 0$. Then the algebraic part of the initial data is fixed by

$$oldsymbol{z}_0 = -oldsymbol{B}^{-1}\left(oldsymbol{C}_Qoldsymbol{y}_0 - oldsymbol{F}_Q(oldsymbol{\imath}(0), \,oldsymbol{v}(0))
ight) =: (oldsymbol{z}_{1,0}, \,oldsymbol{\jmath}_{V0})^{ op}.$$

In particular, the initial applied voltage will be $\boldsymbol{u}(0) = \boldsymbol{y}_{1,0} + \boldsymbol{z}_{1,0}$.

The final equations (4.1) for the full coupled problem are summarized in Box 4.1. The structure of the coupling of electric networks and devices has many resemblances

Box 4.1:	Mixed system of parabolic-elliptic IBVP and I	DAE-IVP.
Parabolic-Elliptic IBVP for the diode:		
	$\begin{split} -q\partial_t n + \partial_x j_n &= qR, j_n = -q \left(\mu_n n \partial_x V - U_T \mu_n \partial_x n\right), \\ q\partial_t p + \partial_x j_p &= -qR, j_p = -q \left(\mu_p p \partial_x V + U_T \mu_p \partial_x p\right), \\ -\epsilon \partial_x^2 V &= q(C - n + p), \end{split}$	(4.1a)
	with $R = F(n,p) \left(np/n_i^2 - 1 \right),$	
	$n(x,0) = n_0(x), p(x,0) = p_0(x)$	(4.1b)
	$n(x,t) = n_i^2/p(x,t) = n_D(x,t), \qquad (x=0,l)$ $V(0,t) = V_{\rm bi}(0) + u_1(t), \qquad V(l,t) = V_{\rm bi}(l) + u_2(t),$ with $V_{\rm bi} = U_{\rm bi} n_D$ $m_D = 0.000 {\rm eV}(2 + \sqrt{(C/2)^2 + m_D^2})$	(4.1c)
	with $V_{\rm bi} = U_T \text{in} \frac{n_i}{n_i}$, $n_D = C/2 + \sqrt{(C/2)^2 + n_i}$,	
Coupling inter	$\underbrace{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \boldsymbol{A}_{\lambda}^{\top}(\boldsymbol{y}_1 + \boldsymbol{z}_1), \qquad \langle J \rangle = \langle j_n + j_p \rangle; $	(4.1d)
DAE-IVP problem for the network:		
$ ilde{oldsymbol{A}}_{P} rac{\mathrm{d}oldsymbol{y}}{\mathrm{d}oldsymbol{y}} + ilde{oldsymbol{B}}_{P}oldsymbol{y} + ilde{oldsymbol{F}}_{P}(oldsymbol{x},oldsymbol{v}) + igg(oldsymbol{A}_{D}ig\langle J angle igg) = 0.$		

$$\frac{1}{dt} + \boldsymbol{B}_{P}\boldsymbol{y} + \boldsymbol{F}_{P}(\boldsymbol{i}, \boldsymbol{v}) + \begin{pmatrix} 0 \end{pmatrix} = 0, \qquad (4.1e)$$
$$\boldsymbol{z} = -\boldsymbol{B}^{-1} \left(\boldsymbol{C}_{O}\boldsymbol{y} + \boldsymbol{F}_{O}(\boldsymbol{i}, \boldsymbol{v}) \right),$$

$$\boldsymbol{y}(0) = \boldsymbol{y}_0. \tag{4.1f}$$

with the coupling arising from networks with transmission lines [6]. In both cases some subsystems describe refined electric network elements, which depend on the boundary (terminal) node potentials.

Next, we are going to reformulate problem (4.1) in a functional setting which is more suitable for a mathematical treatment. We denote by $L^r = L^r([0, l])$, and $H^k = H^k([0, l])$ the usual spaces of functions, with norms $\|\cdot\|_{L^r}$, and $\|\cdot\|_{H^k}$, respectively. We also use the space L^2_+ of all functions in L^2 which are nonnegative almost everywhere. Let [0, T] be a bounded time interval. For any Banach space X, we denote by C([0, T]; X), $L^r([0, T]; X)$ and $H^k([0, T]; X)$ the usual spaces of functions defined on [0, T] with values in X. For our purposes, we introduce the following Banach spaces:

$$X = \{ u \in H^1 | u(0) = u(l) = 0 \},$$

$$Y = C([0,T]; L^2) \cap L^2([0,T]; X) \cap H^1([0,T]; X^*),$$

$$\mathcal{C}_L = C([0,T]; \mathbb{R}^{n+n_L}),$$

$$\mathcal{C}_V = C([0,T]; \mathbb{R}^{n+n_V}),$$

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where X^* is the dual space of X.

Before proceeding, note that the variables directly involved in problem (4.1) are the electron and hole density, n and p, and the network's differential part \boldsymbol{y} . In fact, only the gradient of the electric potential (the field $E = \partial_x V$) appears in the equations for n and p. Recalling the explicit representation of the field (2.12), we immediately see that E depends on the electric network only through the quantity $u_2 - u_1$. Then, although u_1 and u_2 depend both on \boldsymbol{y}_1 and \boldsymbol{z}_1 through (4.1d), we find that

$$u_2 - u_1 = -\boldsymbol{A}_D^{\top} \boldsymbol{y}_1, \tag{4.2}$$

since, by definition, $\mathbf{A}_D^{\top} \mathbf{z}_1 = 0$. We can now present the weak formulation of problem (4.1) which will be studied in the following sections.

DEFINITION 4.1 (Solutions to PDAE). We define the tuple $(n, p, V, \boldsymbol{y}, \boldsymbol{z})$ to be a solution of the PDAE problem (4.1) if the network algebraic component $\boldsymbol{z} \in C_V$ and the electric potential V are given by

$$\boldsymbol{z}(t) = -\boldsymbol{B}^{-1} \left(\boldsymbol{C}_{Q} \boldsymbol{y}(t) + \boldsymbol{F}_{Q}(\boldsymbol{\imath}(t), \boldsymbol{v}(t)) \right),$$

$$V(x, t) = V_{\text{bi}}(0) + u_{1}(t) + \int_{0}^{x} E(x', t) \, \mathrm{d}x, \quad \begin{pmatrix} u_{1} \\ u_{2} \end{pmatrix} = \boldsymbol{A}_{\lambda}^{\top}(\boldsymbol{y}_{1} + \boldsymbol{z}_{1}),$$

and the tuple (n, p, E, y) satisfies the following conditions:

- (i) $(n n^e), (p p^e) \in Y$, and $n, p \in C([0, T]; L^2_+)$ satisfy (4.1b);
- (ii) the unknown $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2)^{\top}$, belongs to \mathcal{C}_L and satisfies (4.1f);
- (iii) the electric field E is given by (2.11), with $u_2 u_1 = -\mathbf{A}_D^{\mathsf{T}} \mathbf{y}_1$;
- (iv) for all test-functions $\psi_n, \psi_p \in X$, n and p satisfy the weak formulation

$$(\partial_t n, \psi_n) + (U_T \mu_n \partial_x n - \mu_n n E, \partial_x \psi_n) + (R, \psi_n) = 0, \qquad (4.3a)$$

$$(\partial_t p, \psi_p) + (U_T \mu_p \partial_x p + \mu_p p E, \partial_x \psi_p) + (R, \psi_p) = 0, \qquad (4.3b)$$

where (\cdot, \cdot) is the usual L²-inner product;

(v) the network variable $oldsymbol{y}$ satisfies

$$\tilde{\boldsymbol{A}}_{P}\frac{d\boldsymbol{y}}{dt} + \tilde{\boldsymbol{B}}_{P}\boldsymbol{y} + \tilde{\boldsymbol{F}}_{P}(\boldsymbol{\imath},\boldsymbol{v}) + \begin{pmatrix} \boldsymbol{A}_{D} \langle J \rangle \\ 0 \end{pmatrix} = 0.$$
(4.4)

5. Existence of solutions. In this section, we study the existence and uniqueness of solutions to the PDAE problem (4.1). The existence proof resides heavily on some physically based a priori estimates which will allow the prolongation of a local solution. Local existence and uniqueness can be established by the Banach-Caccioppoli fixed-point theorem, for sufficiently small times. These results are presented in two separate parts: the first one deals with a priori estimates; in the second one, local existence and uniqueness are addressed, and a global existence result is stated and proved.

5.1. A priori estimates. In this first part, we assume the existence of a solution $(n, p, V, \boldsymbol{y}, \boldsymbol{z})$ to (4.1), in an arbitrary time interval [0, T]. For any function $g(n, p, E, \boldsymbol{y})$, we introduce the notation $\delta g = g - g^{\mathrm{e}}$, with $g^{\mathrm{e}} = g(n^{\mathrm{e}}, p^{\mathrm{e}}, E^{\mathrm{e}}, \boldsymbol{y}^{\mathrm{e}})$, and $E^{\mathrm{e}} = \partial_x V^{\mathrm{e}}, \boldsymbol{y}^{\mathrm{e}} = 0$. We also write $\delta V = V - V^{\mathrm{e}}$.

Following Gajewski [3], for any $\alpha > 0$, we introduce the Liapunov functional:

$$\begin{aligned} \mathcal{H}_{\alpha}(\delta n, \delta p, \delta E, \delta \boldsymbol{y}) &= \int_{0}^{l} q U_{T} \left\{ (n+\alpha) \left(\ln \frac{n+\alpha}{n^{\mathrm{e}}+\alpha} - 1 \right) + n^{\mathrm{e}} \right. \\ &+ (p+\alpha) \left(\ln \frac{p+\alpha}{p^{\mathrm{e}}+\alpha} - 1 \right) + p^{\mathrm{e}} \right\} \mathrm{d}x + \int_{0}^{l} \frac{\epsilon}{2} |\delta E|^{2} \mathrm{d}x + \frac{1}{2} \boldsymbol{y}^{\mathsf{T}} \boldsymbol{A}_{P} \boldsymbol{y} \\ &=: \mathcal{H}_{\alpha}^{*}(t) + \frac{1}{2} \boldsymbol{y}^{\mathsf{T}} \boldsymbol{A}_{P} \boldsymbol{y}, \end{aligned}$$

where A_P is defined in (3.15). The parameter α is needed to ensure that \mathcal{H}_{α} is well defined, since n and p may vanish locally.

REMARK 5. Formally, as α tends to zero, the Liapunov functional \mathcal{H}_{α} tends to the total physical energy W^* of the coupled system, defined in Section 3.2. This assertion can be stated in a precise way by observing that the function

$$g(z) = \begin{cases} z \ln z, & \text{if } z > 0, \\ 0, & \text{if } z = 0, \end{cases}$$

is continuous for $z \in [0, \infty)$. Then the functional $\mathcal{H} = \lim_{\alpha \to 0} \mathcal{H}_{\alpha}$ is well defined and we have $\mathcal{H} = W^*$.

For any solution of the full problem (4.1), the quantity $\mathcal{H}_{\alpha}(\delta n, \delta p, \delta E, \delta y)$ is a function of time only. Its time derivative is

$$\frac{\mathrm{d}\mathcal{H}_{\alpha}}{\mathrm{d}t} = q\left(\partial_t n, U_T \delta \ln(n+\alpha)\right) + q\left(\partial_t p, U_T \delta \ln(p+\alpha)\right) \\ + \epsilon\left(\partial_t \delta E, \delta E\right) + \boldsymbol{y}^{\mathsf{T}} \boldsymbol{A}_P \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t},$$
(5.1)

where the L^2 -scalar product (\cdot, \cdot) is employed. Writing (4.3) with $\psi_n = U_T \delta \ln(n+\alpha)$ and $\psi_p = U_T \delta \ln(p+\alpha)$, the first two terms at the right-hand side of (5.1) become

$$q(\partial_t n, U_T \delta \ln(n+\alpha)) + q(\partial_t p, U_T \delta \ln(p+\alpha)) = (j_n, -U_T \partial_x \delta \ln(n+\alpha)) + (j_p, U_T \partial_x \delta \ln(p+\alpha)) - qU_T (R, \delta \ln((n+\alpha)(p+\alpha))) = (j_n, \partial_x \delta \Phi_n - \partial_x \delta V) + (j_p, \partial_x \delta \Phi_p - \partial_x \delta V) - qU_T (R, \delta \ln((n+\alpha)(p+\alpha))) = (j_n, \partial_x \delta \Phi_n) + (j_p, \partial_x \delta \Phi_p) - qU_T (R, \delta \ln((n+\alpha)(p+\alpha))) - (J, \delta E),$$
(5.2)

where $\Phi_n, \Phi_p, \Phi_n^{e}, \Phi_p^{e}$ are defined by the relations

$$n + \alpha = n_i \exp\left(\frac{V - \Phi_n}{U_T}\right), \qquad p + \alpha = n_i \exp\left(-\frac{V - \Phi_p}{U_T}\right),$$
$$n^{e} + \alpha = n_i \exp\left(\frac{V^{e} - \Phi_n^{e}}{U_T}\right), \qquad p^{e} + \alpha = n_i \exp\left(-\frac{V^{e} - \Phi_p^{e}}{U_T}\right).$$

These new quantities can be interpreted as α -shifted 'quasi-Fermi levels', and coincide with the physical quasi-Fermi levels as α tends to zero. The third term at the righthand side of (5.1) can be rewritten as

$$\epsilon \left(\partial_t \delta E, \delta E\right) = \left[\epsilon \delta V \partial_t \partial_x \delta V\right]_0^l - q \left(\partial_t (\delta n - \delta p), \delta V\right)$$

$$= \left[\epsilon \delta V \partial_t \partial_x \delta V\right]_0^l - \left(\partial_x J, \delta V\right)$$

$$= -\left[\left(-\epsilon \partial_t \partial_x \delta V + J\right) \delta V\right]_0^l + \left(J, \partial_x \delta V\right)$$

$$= -I(u_2 - u_1) + \left(J, \delta E\right).$$
(5.3)

Finally, the definition of \tilde{A}_P , the circuit equations (3.5) and the identity (2.13) yields

$$\boldsymbol{y}^{\mathsf{T}}\boldsymbol{A}_{P}\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \boldsymbol{y}^{\mathsf{T}}\tilde{\boldsymbol{A}}_{P}\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} - \frac{\epsilon}{l}\boldsymbol{y}_{1}^{\mathsf{T}}\boldsymbol{A}_{D}\boldsymbol{A}_{D}^{\mathsf{T}}\frac{\mathrm{d}\boldsymbol{y}_{1}}{\mathrm{d}t}$$
$$= -\boldsymbol{y}^{\mathsf{T}}\tilde{\boldsymbol{B}}_{P}\boldsymbol{y} - \boldsymbol{y}^{\mathsf{T}}\tilde{\boldsymbol{F}}_{P}(\boldsymbol{\imath},\boldsymbol{v}) - \boldsymbol{y}_{1}^{\mathsf{T}}\boldsymbol{A}_{\lambda}\boldsymbol{\lambda}.$$
(5.4)

In conclusion, we obtain the following result:

LEMMA 5.1. For the functional \mathcal{H}_{α} , we have

$$\mathcal{H}_{\alpha}(t) - \int_{0}^{t} \int_{0}^{l} (j_{n}\partial_{x}\delta\Phi_{n} + j_{p}\partial_{x}\delta\Phi_{p}) \,\mathrm{d}x \,\mathrm{d}s + \int_{0}^{t} \int_{0}^{l} qU_{T}R\delta\ln\left((n+\alpha)(p+\alpha)\right) \,\mathrm{d}x \,\mathrm{d}s$$
$$= \mathcal{H}_{\alpha}(0) - \int_{0}^{t} (\boldsymbol{y}^{\top}\tilde{\boldsymbol{B}}_{P}\boldsymbol{y} + \boldsymbol{y}^{\top}\tilde{\boldsymbol{F}}_{P}) \,\mathrm{d}s.$$
(5.5)

Proof. Using (5.2)–(5.4), and $\boldsymbol{y}_1^{\mathsf{T}} \boldsymbol{A}_{\lambda} \boldsymbol{\lambda} = -(u_2 - u_1)I$, the identity (5.1) becomes

$$\frac{\mathrm{d}\mathcal{H}_{\alpha}}{\mathrm{d}t} = \int_{0}^{l} (j_{n}\partial_{x}\delta\Phi_{n} + j_{p}\partial_{x}\delta\Phi_{p}) \,\mathrm{d}x + \int_{0}^{l} qU_{T}R\,\delta\ln\big((n+\alpha)(p+\alpha)\big)\mathrm{d}x - \boldsymbol{y}^{\mathsf{T}}\tilde{\boldsymbol{B}}_{P}\boldsymbol{y} - \boldsymbol{y}^{\mathsf{T}}\tilde{\boldsymbol{F}}_{P}.$$

An integration over any time-interval [0, t] (t < T) yields the thesis. \Box

The identity shown in the previous lemma is the key result to prove some a priory bounds for the solution, stated in the following lemmas:

LEMMA 5.2. Assume that the recombination term is of type (2.4)-(2.5), and the mobilities are bounded. Then there exist constants C_1 , C_2 independent of t such that for any solution n, p of (4.3) holds

$$\|n(\cdot,t)\|_{L^1} + \|p(\cdot,t)\|_{L^1} + \|E(\cdot,t)\|_{L^2}^2 + |\boldsymbol{y}(t)|^2 \le C_1 e^{C_2 t}.$$
(5.6)

Proof. Recalling the identity (5.5), we estimate its left-hand side term by term. The current can be written as

$$j_n = q\mu_n \left(U_T \partial_x n - n \partial_x V \right) = q\mu_n \left(\alpha \partial_x V - (n+\alpha) \partial_x \Phi_n^{\rm e} - (n+\alpha) \partial_x \delta \Phi_n \right).$$

Therefore, using the non-negativity of n and the boundedness of the mobility, we get

$$-\int_0^l j_n \partial_x \delta \Phi_n \mathrm{d}x \ge -q \int_0^l \mu_n \left(\frac{n+\alpha}{2} |\partial_x \Phi_n^{\mathrm{e}}|^2 + \frac{\alpha}{2} |\partial_x V|^2\right) \mathrm{d}x$$
$$\ge -c_1 (1 + \|n\|_{L^1} + \|E\|_{L^2}^2),$$

for some constant $c_1 > 0$. The estimate for j_p is analog. For the recombinationgeneration integral, using $n^e p^e = n_i^2$, and $(a-1)\ln(a) \ge 0$, we can estimate

$$\int_{0}^{l} \left(q U_T R \delta \ln((n+\alpha)(p+\alpha)) \right) \mathrm{d}x$$

$$\geq -\frac{\alpha q U_T}{n_i^2} \int_{0}^{l} F\left(\ln\left(\frac{n+\alpha}{n^{\mathrm{e}}+\alpha}\right) + \ln\left(\frac{p+\alpha}{p^{\mathrm{e}}+\alpha}\right) \right) (n+p-n^{\mathrm{e}}-p^{\mathrm{e}}) \mathrm{d}x.$$

Then, using the estimate (2.5) for F, and the inequality

$$\left|\frac{\ln(z+a)}{z+a}\right| \le \max\left(\frac{1}{e}, \left|\frac{\ln a}{a}\right|\right), \quad \text{for all } z \ge 0, \, a > 0,$$

we get

$$\int_0^t q U_T R\delta \ln((n+\alpha)(p+\alpha)) \, \mathrm{d}x \ge -\alpha c_2 \big(1 + \|n\|_{L^1} + \|p\|_{L^1}\big).$$

Combining the above estimates with (5.5), we obtain

$$\mathcal{H}_{\alpha}(t) \le c_3 + c_4 \int_0^t \left(1 + \|n\|_{L^1} + \|p\|_{L^1} + \|E\|_{L^2}^2 + |\mathbf{y}|^2 \right) \mathrm{d}x, \tag{5.7}$$

using positive constants c_3, c_4 . On the other end, applying the inequality

$$z-a \leq c \left[z \left(\ln \frac{z}{a} - 1 \right) + a \right] + a \epsilon(c), \quad \text{for all } z \geq 0, \, a > 0, \, c > 0, \tag{5.8}$$

with $\epsilon(c) = c(e^{1/c} - 1) - 1 > 0$, we find

$$qU_T\left(\|n\|_{L^1} + \|p\|_{L^1}\right) + \frac{\epsilon}{2} \|\delta E\|_{L^2}^2 \le \mathcal{H}^*_{\alpha} + qU_T(e-1) \int_0^l (n^e + p^e + 2\alpha) \mathrm{d}x.$$

Then we can conclude that

,

$$1 + ||n||_{L^1} + ||p||_{L^1} + ||E||_{L^2}^2 + |\mathbf{y}|^2 \le c_5 \mathcal{H}_{\alpha} + c_6,$$
(5.9)

for some positive constants $c_5,\,c_6.$ Combining (5.7)–(5.9) and using Gronwall's lemma, we deduce the thesis. \Box

LEMMA 5.3. Under the hypothesis of Lemma 5.2, there exists a constant c = c(T) such that for all $t \leq T$ we have

$$\|n(\cdot,t)\|_{L^{2}}^{2} + \|p(\cdot,t)\|_{L^{2}}^{2} + \int_{0}^{t} \left(\|\partial_{x}n(\cdot,s)\|_{L^{2}}^{2} + \|\partial_{x}p(\cdot,s)\|_{L^{2}}^{2}\right) \mathrm{d}s \leq c.$$
 (5.10)

Proof. We consider the weak formulation (4.3a) with $\psi_n = \delta n$. Integrating with respect to time, we find

$$\frac{1}{2} \left(\left\| \delta n(\cdot,t) \right\|_{L^2}^2 - \left\| \delta n(\cdot,0) \right\|_{L^2}^2 \right) + U_T \mu_n \int_0^t \int_0^l \left| \partial_x \delta n \right|^2 \mathrm{d}x \, \mathrm{d}s$$
$$= -U_T \mu_n \int_0^t \int_0^l \partial_x n^e \partial_x \delta n \, \mathrm{d}x \, \mathrm{d}s + \int_0^t (\mu_n n \partial_x V, \partial_x \delta n) \, \mathrm{d}s - \int_0^t (R,\delta n) \, \mathrm{d}s.$$

We can estimate

$$-\int_0^t \int_0^l \partial_x n^e \partial_x \delta n \, \mathrm{d}x \, \mathrm{d}s \leq \frac{1}{2} \int_0^t \left(\|\partial_x n(\cdot, \tau)\|_{L^2}^2 + \|\partial_x n^e\|_{L^2}^2 \right) \mathrm{d}s.$$

Furthermore, integrating by parts and using the Poisson equation, we obtain

$$\begin{split} \int_{0}^{t} &(n\partial_{x}V,\partial_{x}\delta n)\,\mathrm{d}s = \int_{0}^{t} &(\delta n\partial_{x}V,\partial_{x}\delta n)\,\mathrm{d}s + \int_{0}^{t} &(n^{\mathrm{e}}E,\partial_{x}\delta n)\,\mathrm{d}s \\ &\leq \frac{q}{2\epsilon}\int_{0}^{t} \int_{0}^{l} &(C-n+p)\,(\delta n)^{2}\,\mathrm{d}x\,\mathrm{d}s + \int_{0}^{t} &\left(c\,\|E\|_{L^{2}}^{2} + \frac{U_{T}}{4}\,\|\partial_{x}\delta n\|_{L^{2}}^{2}\right)\,\mathrm{d}s \\ &\leq c(T)\,\left\{1+\|n\|_{L^{2}}^{2}+\|p\|_{L^{2}}^{2}\right\} + \frac{q}{2\epsilon}\int_{0}^{t} \int_{0}^{l} &n^{2}(p-n)\,\mathrm{d}x\,\mathrm{d}s + \frac{U_{T}}{4}\int_{0}^{t} \|\partial_{x}\delta n\|_{L^{2}}^{2}\,\mathrm{d}s. \end{split}$$

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Here, we have used Lemma 5.2 to estimate all linear terms in n and p and the L^2 -norm of E. For the recombination-generation (2.4), we have

$$|(R,\delta n)| \leq \int_0^l \left| F\Big(\frac{np}{n_i^2} - 1\Big) \delta n \right| dx \leq c \left(1 + \|n\|_{L^2}^2 + \|p\|_{L^2}^2\right),$$

employing $Fn \leq \overline{F}$. Putting everything together, we get the estimate

$$\begin{aligned} \|\delta n(t)\|_{L^{2}}^{2} &+ \frac{U_{T}\mu_{n}}{2} \int_{0}^{t} \int_{0}^{l} |\partial_{x}\delta n|^{2} \mathrm{d}x \,\mathrm{d}s \\ &\leq c(T) \left\{ 1 + \int_{0}^{t} (\|n\|_{L^{2}}^{2} + \|p\|_{L^{2}}^{2}) \,\mathrm{d}s \right\} + \frac{q}{\epsilon} \int_{0}^{t} \int_{0}^{l} n^{2}(p-n) \,\mathrm{d}x \,\mathrm{d}s, \end{aligned}$$

which leads to

$$\|n(t)\|_{L^{2}}^{2} + \int_{0}^{t} \|\partial_{x}n\|_{L^{2}}^{2} \,\mathrm{d}s \leq c(T) \Big\{ 1 + \int_{0}^{t} (\|n\|_{L^{2}}^{2} + \|p\|_{L^{2}}^{2}) \,\mathrm{d}s \Big\} + c_{n} \int_{0}^{t} \int_{0}^{t} n^{2}(p-n) \,\mathrm{d}x \,\mathrm{d}s,$$

for some positive constant c_n . Proceeding in the same way for p, we find

$$\|p(t)\|_{L^{2}}^{2} + \int_{0}^{t} \|\partial_{x}p\|_{L^{2}}^{2} \,\mathrm{d}s \leq c(T) \Big\{ 1 + \int_{0}^{t} (\|n\|_{L^{2}}^{2} + \|p\|_{L^{2}}^{2}) \,\mathrm{d}s \Big\} - c_{p} \int_{0}^{t} \int_{0}^{l} p^{2}(p-n) \,\mathrm{d}x \,\mathrm{d}s,$$

for some positive constant c_p . Then the last two estimates are multiplied by c_p and c_n , respectively, such that in the sum of these estimates the whole cubic term is negative: $-(p^2 - n^2)(p - n) \leq 0$. Finally, invoking Gronwall's lemma, we can conclude the proof. \Box

Combining the previous Lemmas, we arrive directly at the following a priori estimate:

LEMMA 5.4. There exists a constant c = c(T) such that for all $t \leq T$ we have

$$\|n(\cdot,t)\|_{L^{2}}^{2} + \|p(\cdot,t)\|_{L^{2}}^{2} + |\boldsymbol{y}|^{2} + \int_{0}^{t} \left(\|\partial_{x}n(\cdot,\tau)\|_{L^{2}}^{2} + \|\partial_{x}p(\cdot,\tau)\|_{L^{2}}^{2}\right) \mathrm{d}s \leq c.$$

5.2. Global existence and uniqueness. Next, we prove the main result of this paper, asserting global existence and uniqueness of solutions to problem (4.1).

THEOREM 5.5 (Global existence & uniqueness). Let the source functions i, v be continuous, the network matrices be symmetric, positive definite with conditions (3.12)-(3.14), and assume constant diffusivities and mobilities. Then problem (4.1) admits a unique solution on the time interval [0,T] for any $T \in (0, \infty)$.

The proof is based on the prolongation of the unique local solution ensured by the following Theorem.

THEOREM 5.6 (Local existence & uniqueness). Let the source functions \mathbf{i} and \mathbf{v} be continuous, the network matrices be symmetric, positive definite and the topological conditions (3.12)-(3.14) be fulfilled, and diffusivities and mobilities be constant. Then problem (4.1) admits a unique solution provided T > 0 is sufficiently small.

Proof. We introduce the Banach space

$$Z = (n^{\rm e}, p^{\rm e}) + C([0,T]; L^2 \times L^2) \cap L^2([0,T]; X^2),$$

and fix the pair of functions $(\hat{n}, \hat{p}) \in Z$, and $\hat{\boldsymbol{y}} = (\hat{\boldsymbol{y}}_1, \hat{\boldsymbol{y}}_2)^\top \in \mathcal{C}_L$, with $\hat{n}(., 0) = n_0(.)$, $\hat{p}(., 0) = p_0(.)$, and $\hat{\boldsymbol{y}}(0) = \boldsymbol{y}_0$. We consider the following linearized problem for (n, p):

$$(\partial_t n, \psi_n) + \left(U_T \mu_n \partial_x n - \mu_n \hat{n}^+ \hat{E}, \partial_x \psi_n \right) + \left(\hat{R}, \psi_n \right) = 0, \tag{5.11}$$

$$(\partial_t p, \psi_p) + \left(U_T \mu_p \partial_x p + \mu_p \hat{p}^+ \hat{E}, \partial_x \psi_p \right) + \left(\hat{R}, \psi_p \right) = 0, \tag{5.12}$$

for all test-functions $(\psi_n, \psi_p) \in X^2$. Here, $\hat{R} = R(\hat{n}^+, \hat{p}^+), g^+ := \max(g, 0)$, and

$$\hat{E}(x,t) = \frac{1}{l} \left(V_{\rm bi}(l) - V_{\rm bi}(0) - \boldsymbol{A}_D^{\top} \hat{\boldsymbol{y}}_1 \right) + \int_0^x \frac{q}{\epsilon} \left(\hat{n}(x',t) - \hat{p}(x',t) - C(x') \right) \mathrm{d}x' \\ - \frac{1}{l} \int_0^l \int_0^{x'} \frac{q}{\epsilon} \left(\hat{n}(x'',t) - \hat{p}(x'',t) - C(x'') \right) \mathrm{d}x'' \mathrm{d}x'.$$
(5.13)

Also, we consider the following problem for $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2)^{\top}$:

$$\tilde{\boldsymbol{A}}_{P}\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} + \tilde{\boldsymbol{B}}_{P}\hat{\boldsymbol{y}} + \tilde{\boldsymbol{F}}_{P}(\boldsymbol{\imath},\boldsymbol{v}) + \begin{pmatrix} \boldsymbol{A}_{D}\langle\hat{J}\rangle\\ 0 \end{pmatrix} = 0, \qquad (5.14)$$

with

$$\hat{J}(t) = q \left(U_T \mu_n \partial_x \hat{n} - \mu_n \hat{n} \hat{E} - U_T \mu_p \partial_x \hat{p} - \mu_p \hat{p} \hat{E} \right).$$
(5.15)

The decoupled equations (5.11)-(5.12), (5.15) admit a unique solution, (n, p, y), which satisfies $(n, p) \in (n^{e}, p^{e}) + Y^{2}$, $y \in C_{L}$, and have the initial data $n(., 0) = n_{0}(.)$, $p(., 0) = p_{0}(.)$, $y(0) = y_{0}$ of our problem. In fact, the existence of a unique solutions y to (5.13) follows immediately by time integration of the equation over [0, t]. The existence of a unique solution (n, p) to (5.11)-(5.12), follows by standard results for linear parabolic equations with discontinuous coefficients (see [10], Th. III.4.2). It is sufficient to note that (5.13) yields the estimate

$$\|\hat{E}\|_{L^{\infty}} \leq c \left(1 + |\hat{\boldsymbol{y}}| + \|\hat{n}\|_{L^{1}} + \|\hat{p}\|_{L^{1}}\right), \qquad (5.16)$$

which implies immediately

$$\|\mu_n \hat{n}^+ \hat{E}\|_{L^2} \le \mu_n \|\hat{n}\|_{L^2} \|\hat{E}\|_{L^{\infty}} \le c \|\hat{n}\|_{L^2} \left(1 + |\hat{\boldsymbol{y}}| + \|\hat{n}\|_{L^1} + \|\hat{p}\|_{L^1}\right).$$

It follows that $\mu_n \hat{n}^+ \hat{E}$, $\mu_p \hat{p}^+ \hat{E} \in L^2$, so that equations (5.11)–(5.12) are well defined for any choice of (\hat{n}, \hat{p}) , and we can apply the above mentioned existence results.

Thus, assigning $(\hat{n}, \hat{p}, \hat{y})$ and solving the resulting problems (5.11), (5.12) and (5.14) for (n, p, y), we obtain an operator Q, defined by

$$(\hat{n}, \hat{p}, \hat{y}) \rightarrow (n, p, y) =: Q(\hat{n}, \hat{p}, \hat{y}),$$

which maps $Z \times C_L$ into $Z \times C_L$. We introduce the norm

$$\begin{aligned} |||(n, p, \boldsymbol{y})|||^{2} &= \max_{0 \le t \le T} \left(||n(\cdot, t)||_{L^{2}}^{2} + ||p(\cdot, t)||_{L^{2}}^{2} + |\boldsymbol{y}(t)|^{2} \right) \\ &+ \int_{0}^{T} \left(||\partial_{x} n(\cdot, t)||_{L^{2}}^{2} + ||\partial_{x} p(\cdot, t)||_{L^{2}}^{2} \right) \mathrm{d}s, \qquad (n, p, \boldsymbol{y}) \in Z \times \mathcal{C}_{L}. \end{aligned}$$

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We are going to prove that Q is strictly contractive with respect to this norm, for T small enough, in the set

$$S_a = \{(n, p, \boldsymbol{y}) \in Z \times \mathcal{C} : n(., 0) = n_0(.), p(., 0) = p_0(.), \boldsymbol{y}(0) = \boldsymbol{y}_0, |||(n, p, \boldsymbol{y})|||^2 \le a\},$$

where the constant satisfies: $a > ||n_0||_{L^2}^2 + ||p_0||_{L^2}^2 + ||\boldsymbol{y}_0|^2$. To this end, we consider the triplets $(\hat{n}_1, \hat{p}_1, \hat{\boldsymbol{y}}_1), (\hat{n}_2, \hat{p}_2, \hat{\boldsymbol{y}}_2) \in S_a$ and set

$$(n_1, p_1, \boldsymbol{y}_1) = Q(\hat{n}_1, \hat{p}_1, \hat{\boldsymbol{y}}_1), \quad \hat{E}_1 = \hat{E}(\hat{n}_1, \hat{p}_1, \hat{\boldsymbol{y}}_1), (n_2, p_2, \boldsymbol{y}_2) = Q(\hat{n}_2, \hat{p}_2, \hat{\boldsymbol{y}}_2), \quad \hat{E}_2 = \hat{E}(\hat{n}_2, \hat{p}_2, \hat{\boldsymbol{y}}_2),$$

where $\hat{E} = \hat{E}(\hat{n}, \hat{p}, \hat{y})$ is defined by (5.13). We write (5.11) with $n = n_1$ and $n = n_2$, both with test function $\psi_n = n_1 - n_2$. Subtracting the two equations and integrating over the time interval [0, t], we obtain

$$\frac{1}{2} \|n_1 - n_2\|_{L^2}^2 + U_T \mu_n \int_0^t \|\partial_x (n_1 - n_2)\|_{L^2}^2 \,\mathrm{d}s$$
$$= \mu_n \int_0^t \left(\hat{n}_1^+ \hat{E}_1 - \hat{n}_2^+ \hat{E}_2, \,\partial_x (n_1 - n_2)\right) \,\mathrm{d}s - \int_0^t \left(\hat{R}_1 - \hat{R}_2, \,n_1 - n_2\right) \,\mathrm{d}s,$$

where $\hat{R}_i = R(\hat{n}_i^+, \hat{p}_i^+), i = 1, 2$. Using a weighted Cauchy-Schwarz inequality, we get

$$\frac{1}{2} \|n_1 - n_2\|_{L^2}^2 + U_T \mu_n \int_0^t \|\partial_x (n_1 - n_2)\|_{L^2}^2 \,\mathrm{d}s$$

$$\leq c(\eta) \int_0^t \{\|\hat{n}_1^+ \hat{E}_1 - \hat{n}_2^+ \hat{E}_2\|_{L^2}^2 + \|\hat{R}_1 - \hat{R}_2\|_{L^2}^2\} \,\mathrm{d}s$$

$$+ \eta \int_0^t \{\|n_1 - n_2\|_{L^2}^2 + \|\partial_x (n_1 - n_2)\|_{L^2}^2\} \,\mathrm{d}s,$$
(5.17)

for some arbitrary positive real number η . To estimate the first term on the right-hand side of (5.17), we first observe that (5.13) and the definition of \hat{E}_1 and \hat{E}_2 , yield

$$\|\hat{E}_1 - \hat{E}_2\|_{L^{\infty}} \leq c \left(|\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2| + \|\hat{n}_1 - \hat{n}_2\|_{L^1} + \|\hat{p}_1 - \hat{p}_2\|_{L^1} \right).$$
(5.18)

Then, using the embedding of L^2 in L^1 , we can estimate

$$\begin{aligned} \|\hat{n}_{1}^{+}\hat{E}_{1}-\hat{n}_{2}^{+}\hat{E}_{2}\|_{L^{2}}^{2} &\leq \|\hat{n}_{1}\|_{L^{2}}^{2}\|\hat{E}_{1}-\hat{E}_{2}\|_{L^{\infty}}^{2} + \|\hat{n}_{1}-\hat{n}_{2}\|_{L^{2}}^{2}\|\hat{E}_{2}\|_{L^{\infty}}^{2} \\ &\leq c(a)\left(|\hat{y}_{1}-\hat{y}_{2}|^{2} + \|\hat{n}_{1}-\hat{n}_{2}\|_{L^{2}}^{2} + \|\hat{p}_{1}-\hat{p}_{2}\|_{L^{2}}^{2}\right). \end{aligned}$$

For the recombination term in (5.17), using its definition (2.4), we get

$$\left\|\hat{R}_{1}-\hat{R}_{2}\right\|_{L^{2}}^{2} \leq c(a)\left(\left\|\hat{n}_{1}-\hat{n}_{2}\right\|_{L^{2}}^{2}+\left\|\hat{p}_{1}-\hat{p}_{2}\right\|_{L^{2}}^{2}\right)$$

Using the above estimates in (5.17), we can conclude that

$$\left(\frac{1}{2} - \eta T\right) \max_{[0,T]} \|n_1 - n_2\|_{L^2}^2 + (U_T \mu_n - \eta) \int_0^t \|\partial_x (n_1 - n_2)\|_{L^2}^2 \,\mathrm{d}s \leq c(a,\eta) T \max_{[0,T]} \left(|\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2|^2 + \|\hat{n}_1 - \hat{n}_2\|_{L^2}^2 + \|\hat{p}_1 - \hat{p}_2\|_{L^2}^2\right).$$
(5.19)

An analogous estimate holds for $p_1 - p_2$.

Next, we write (5.14) for $\boldsymbol{y} = \boldsymbol{y}_1$, $\boldsymbol{y} = \boldsymbol{y}_2$, subtract the resulting equations and multiply the result by $\boldsymbol{y}_1 - \boldsymbol{y}_2$. After integrating with respect to time, we find

$$\frac{1}{2}(\boldsymbol{y}_1 - \boldsymbol{y}_2)^{\mathsf{T}} \tilde{\boldsymbol{A}}_P(\boldsymbol{y}_1 - \boldsymbol{y}_2) = -\int_0^t (\boldsymbol{y}_1 - \boldsymbol{y}_2)^{\mathsf{T}} \left\{ \tilde{\boldsymbol{B}}_P(\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2) + \begin{pmatrix} \boldsymbol{A}_D \langle \hat{J}_1 - \hat{J}_2 \rangle \\ 0 \end{pmatrix} \right\} \mathrm{d}s.$$

Employing positive definiteness, $k_A |\mathbf{y}|^2 \leq \mathbf{y}^{\top} \tilde{\mathbf{A}}_P \mathbf{y}$, the above equality implies

$$(k_A - \eta) |\boldsymbol{y}_1 - \boldsymbol{y}_2|^2 \le c(\eta) \int_0^t \left\{ |\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2|^2 + |\langle \hat{J}_1 - \hat{J}_2 \rangle|^2 \right\} \mathrm{d}s, \qquad (5.20)$$

investing again $\eta > 0$. For the coupling term, we find

$$\langle \hat{J}_1 - \hat{J}_2 \rangle = \frac{q}{l} \int_0^l \left\{ -\mu_n (\hat{n}_1 \hat{E}_1 - \hat{n}_2 \hat{E}_2) - \mu_p (\hat{p}_1 \hat{E}_1 - \hat{p}_2 \hat{E}_2) \right\} \mathrm{d}x,$$

since we have $\int_0^l \partial_x (\hat{n}_1 - \hat{n}_2) dx = \int_0^l \partial_x (\hat{p}_1 - \hat{p}_2) dx = 0$. Then we can estimate

$$\begin{aligned} |\langle \hat{J}_1 - \hat{J}_2 \rangle|^2 &\leq c \left(\|\hat{n}_1 \hat{E}_1 - \hat{n}_2 \hat{E}_2\|_{L^2}^2 + \|\hat{p}_1 \hat{E}_1 - \hat{p}_2 \hat{E}_2\|_{L^2}^2 \right) \\ &\leq c(a) \left(|\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2|^2 + \|\hat{n}_1 - \hat{n}_2\|_{L^2}^2 + \|\hat{p}_1 - \hat{p}_2\|_{L^2}^2 \right). \end{aligned}$$

Using this result in (5.20), we find

$$(k_A - \eta) \max_{[0,T]} |\boldsymbol{y}_1 - \boldsymbol{y}_2|^2 \le c(a, \eta) T \max_{[0,T]} (|\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2|^2 + \|\hat{n}_1 - \hat{n}_2\|_{L^2}^2 + \|\hat{p}_1 - \hat{p}_2\|_{L^2}^2).$$

Eventually, we sum this result with the result (5.19) for $n_1 - n_2$, and with the analog estimate for $p_1 - p_2$. Choosing η small enough, we deduce

$$\begin{aligned} |||(n_1 - n_2, p_1 - p_2, \boldsymbol{y}_1 - \boldsymbol{y}_2)|||^2 &\leq c(a) T \max_{[0,T]} \left(|\hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2|^2 + \|\hat{n}_1 - \hat{n}_2\|_{L^2}^2 + \|\hat{p}_1 - \hat{p}_2\|_{L^2}^2 \right) \\ &\leq c(a) T |||(\hat{n}_1 - \hat{n}_2, \, \hat{p}_1 - \hat{p}_2, \, \hat{\boldsymbol{y}}_1 - \hat{\boldsymbol{y}}_2)|||^2. \end{aligned}$$

Then there exists a time T > 0, such that Q becomes a contraction on S_a . Thus, by Banach's fixed-point theorem, we obtain the existence of a unique fixed-point $(n_*, p_*, \boldsymbol{y}_*) \in S_a$.

To show that this fixed-point is the unique solution to (4.1), it is sufficient to prove the nonnegativity of n_* and p_* . To this end, we consider $n_*^- = \min(n_*, 0) \in Y$. Writing (5.11) for the test-function $\psi_n = n_*^-$, after time integration, we obtain

$$\frac{1}{2} \|n_*^-(t)\|_{L^2}^2 + U_T \mu_n \int_0^t \|\partial_x n_*^-\|_{L^2}^2 \,\mathrm{d}s$$
$$= \int_0^t \int_0^t (\mu_n n_*^+ E_* \partial_x n_*^- - R(n_*^+, p_*^+)n_*^-) \,\mathrm{d}x \,\mathrm{d}s = \int_0^t \int_0^t F(n_*^+, p_*^+)n_*^- \,\mathrm{d}x \,\mathrm{d}s \le 0$$

Hence $n_*^-(t)$ vanishes almost everywhere, that is, n_* is nonnegative almost everywhere. In the same way we can prove the nonnegativity of p_* . \Box

This section is concluded with the proof of Theorem 5.5. It suffices to notice that the local solution ensured by Theorem 5.6 can be prolonged to arbitrary time intervals, due the a priori a bound provided by Lemma 5.4.

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6. Asymptotic behavior. In this section, we study the asymptotic behavior of the solution to (4.1) as t tends to infinity. For a single device without coupling with an electric circuit, it is know that all solutions relax to equilibrium [3], if the applied potential vanishes. Therefore, in the coupled system, we need to demand appropriate topological conditions on the electric network to enable such a result. We derive sufficient conditions on the network's topology during the development of this section. The resulting asymptotic result is stated at the end of the section.

Recalling Lemma 5.1 and the definition of j_n , j_p , Φ_n and Φ_p , for arbitrary times $t_1 \ge t_0 \ge 0$ we find

$$\begin{aligned} \mathcal{H}_{\alpha}(t_{1}) + \frac{q}{2} \int_{t_{0}}^{t_{1}} \int_{0}^{l} \left(\mu_{n}(n+\alpha) \left| \partial_{x} \delta \Phi_{n} \right|^{2} + \mu_{p}(p+\alpha) \left| \partial_{x} \delta \Phi_{p} \right|^{2} \right) \mathrm{d}x \, \mathrm{d}s \\ + q \int_{t_{0}}^{t_{1}} \int_{0}^{l} \frac{U_{T}F}{n_{i}^{2}} \delta\left((n+\alpha)(p+\alpha) \right) \delta \ln\left((n+\alpha)(p+\alpha) \right) \mathrm{d}x \, \mathrm{d}s \\ &\leq \mathcal{H}_{\alpha}(t_{0}) - \int_{t_{0}}^{t_{1}} \left(\mathbf{y}^{\top} \tilde{\mathbf{B}}_{P} \mathbf{y} + \mathbf{y}^{\top} \tilde{\mathbf{F}}_{P} \right) \mathrm{d}s \\ &+ \alpha q \int_{t_{0}}^{t_{1}} \int_{0}^{l} \left\{ \frac{\mu_{n}}{2} \left| \delta E - \frac{E^{\mathrm{e}} \delta n}{n^{\mathrm{e}} + \alpha} \right|^{2} + \frac{\mu_{p}}{2} \left| \delta E - \frac{E^{\mathrm{e}} \delta p}{p^{\mathrm{e}} + \alpha} \right|^{2} \\ &+ \frac{U_{T}F}{n_{i}^{2}} \delta(n+p) \delta \ln\left((n+\alpha)(p+\alpha) \right) \right\} \mathrm{d}x \, \mathrm{d}s. \end{aligned}$$

Notice, $\partial_x \Phi_n^{\rm e} = \alpha (\partial_x V^{\rm e})/(n^{\rm e} + \alpha)$. Since the last integral term on the right-hand side is bounded, we can pass to the limit as α tends to zero. The result is

$$\mathcal{H}(t_1) + \frac{q}{2} \int_{t_0}^{t_1} \int_0^l \left(\mu_n n \left| \partial_x \delta \phi_n \right|^2 + \mu_p p \left| \partial_x \delta \phi_p \right|^2 \right) \mathrm{d}x \, \mathrm{d}s$$
$$\leq \mathcal{H}(t_0) - \int_{t_0}^{t_1} \left(\boldsymbol{y}^\top \tilde{\boldsymbol{B}}_P \boldsymbol{y} + \boldsymbol{y}^\top \tilde{\boldsymbol{F}}_P \right) \mathrm{d}s, \qquad (6.1)$$

where ϕ_n and ϕ_p are the physical quasi-Fermi levels and

$$\begin{aligned} \mathcal{H}(t) &= \int_0^l q U_T \left\{ n \left(\ln \frac{n}{n^{\mathrm{e}}} - 1 \right) + n^{\mathrm{e}} + p \left(\ln \frac{p}{p^{\mathrm{e}}} - 1 \right) + p^{\mathrm{e}} \right\} \mathrm{d}x \\ &+ \int_0^l \frac{\epsilon}{2} |\delta E|^2 \mathrm{d}x + \frac{1}{2} \boldsymbol{y}^\top \boldsymbol{A}_P \boldsymbol{y} \\ &= \mathcal{H}^*(t) + \frac{1}{2} \boldsymbol{y}_1(t)^\top \boldsymbol{A}_C \boldsymbol{C} \boldsymbol{A}_C^\top \boldsymbol{y}_1(t) + \frac{1}{2} \boldsymbol{y}_2(t)^\top \boldsymbol{L} \boldsymbol{y}_2(t). \end{aligned}$$

We would like to use (6.1) for studying the asymptotic behavior of the solution to problem (4.1) established above. This can be done only after a careful consideration of the last term on the right-hand side. Explicitly, recalling the definition (3.15) of A_P , a comparison of the circuit's energy balance equation (3.7) with identity (5.4) yields

$$\boldsymbol{y}^{\mathsf{T}} \tilde{\boldsymbol{B}}_{P} \boldsymbol{y} + \boldsymbol{y}^{\mathsf{T}} \tilde{\boldsymbol{F}}_{P} = \boldsymbol{u}^{\mathsf{T}} \boldsymbol{A}_{R} \boldsymbol{G} \boldsymbol{A}_{R}^{\mathsf{T}} \boldsymbol{u} + \boldsymbol{u}^{\mathsf{T}} \boldsymbol{A}_{I} \boldsymbol{\imath} + \boldsymbol{j}_{V}^{\mathsf{T}} \boldsymbol{v}, \qquad (6.2)$$

where $(\boldsymbol{z}_1, \boldsymbol{z}_2) \equiv (\boldsymbol{Q}_C \boldsymbol{u}, \boldsymbol{j}_V)$ is given in terms of $(\boldsymbol{y}_1, \boldsymbol{y}_2) \equiv (\boldsymbol{P}_C \boldsymbol{u}, \boldsymbol{j}_L)$ by the equations (3.1b) for the algebraic network part. Identity (6.2) shows that the quadratic part of

 $\boldsymbol{y}^{\top} \tilde{\boldsymbol{B}}_{P} \boldsymbol{y}$ is controlled by the voltage drops over resistors, that is, by $|\boldsymbol{P}_{R}\boldsymbol{u}|^{2}$, where $\boldsymbol{P}_{R} = \mathbf{Id} - \boldsymbol{Q}_{R}$, and \boldsymbol{Q}_{R} denotes the orthogonal projector onto the kernel of $\boldsymbol{A}_{R}^{\top}$. On its turn, $|\boldsymbol{P}_{R}\boldsymbol{u}|^{2}$ is controlled by the differential node potentials \boldsymbol{y}_{1} if we demand ker $\boldsymbol{A}_{R}^{\top} \subset \ker \boldsymbol{A}_{C}^{\top}$. From the viewpoint of circuits, there have to be enough resistors – these are going to drain the capacitors' energy. We can express this demand by the topological condition

$$\boldsymbol{A}_{C}^{\dagger}\boldsymbol{Q}_{R}=0. \tag{6.3}$$

In circuit's language, the terminals of any capacitor are connected by a path of resistors and for the orthogonal projector P_C from above, this condition reads: $P_C = P_C P_R$. Thus we can estimate (with constant c' > 0)

$$|\boldsymbol{P}_{C}\boldsymbol{u}|^{2} = |\boldsymbol{P}_{C}\boldsymbol{P}_{R}\boldsymbol{u}|^{2} \leq |\boldsymbol{P}_{R}\boldsymbol{u}|^{2} \leq c' |\boldsymbol{u}^{\top}\boldsymbol{A}_{R}\boldsymbol{G}\boldsymbol{A}_{R}^{\top}\boldsymbol{u}|.$$

Condition (6.3) is not sufficient for our purposes. In fact, in (6.2), we still need to evaluate the linear term $\boldsymbol{y}^{\top} \tilde{\boldsymbol{F}}_{P} = (\boldsymbol{y}_{1} + \boldsymbol{z}_{1})^{\top} \boldsymbol{A}_{I} \boldsymbol{\imath} + \boldsymbol{z}_{2}^{\top} \boldsymbol{v}$. This term can be controlled by the quadratic term depending on $\boldsymbol{y}_{1}, \boldsymbol{\imath}, \boldsymbol{v}$, if \boldsymbol{z} does not depend on \boldsymbol{y}_{2} . To see how this translates in topological conditions, we write explicitly the equations (3.1b) for the algebraic network part:

$$\begin{aligned} \boldsymbol{Q}_{C}^{\top}\boldsymbol{A}_{R}\boldsymbol{G}\boldsymbol{A}_{R}^{\top}(\boldsymbol{z}_{1}+\boldsymbol{y}_{1}) + \boldsymbol{Q}_{C}^{\top}\boldsymbol{A}_{V}\boldsymbol{z}_{2} + \boldsymbol{Q}_{C}^{\top}\boldsymbol{A}_{L}\boldsymbol{y}_{2} + \boldsymbol{Q}_{C}^{\top}\boldsymbol{A}_{I}\boldsymbol{\imath} &= 0, \\ \boldsymbol{A}_{V}^{\top}\boldsymbol{z}_{1} + \boldsymbol{A}_{V}^{\top}\boldsymbol{y}_{1} - \boldsymbol{v} &= 0. \end{aligned}$$

Due to the topological conditions (3.12)–(3.14), this system is solvable for $\boldsymbol{z} = (\boldsymbol{z}_1, \boldsymbol{z}_2)^{\mathsf{T}}$. If we assume the further topological condition

$$\boldsymbol{A}_{\boldsymbol{L}}^{\dagger}\boldsymbol{Q}_{\boldsymbol{C}}=\boldsymbol{0},\tag{6.4}$$

then y_2 is excluded from these algebraic equations, and thus we have the estimate

$$|\boldsymbol{z}_{1}|^{2}, |\boldsymbol{j}_{V}|^{2} \leq |\boldsymbol{z}|^{2} \leq c_{1} \{ |\boldsymbol{y}_{1}|^{2} + |\boldsymbol{i}|^{2} + |\boldsymbol{v}|^{2} \}.$$
 (6.5)

The new topological condition (6.4) is analogous to (3.12), and states that any inductor is in a loop of capacitors.

Thus, provided conditions (6.3) and (6.4) hold, identity (6.2) gives the estimate

$$oldsymbol{y}^{ op} ilde{oldsymbol{B}}_Poldsymbol{y}+oldsymbol{y}^{ op} ilde{oldsymbol{F}}_P\,\geq\,c_u\left|oldsymbol{y}_1
ight|^2-c_s\left\{|oldsymbol{\imath}|^2+|oldsymbol{v}|^2
ight\}$$

for some constants c_u and c_s . Using this in (6.1), we find

$$\mathcal{H}(t_{1}) + \frac{q}{2} \int_{t_{0}}^{t_{1}} \int_{0}^{t} \left(\mu_{n} n \left| \partial_{x} \delta \phi_{n} \right|^{2} + \mu_{p} p \left| \partial_{x} \delta \phi_{p} \right|^{2} \right) \mathrm{d}x \,\mathrm{d}s + c_{u} \int_{t_{0}}^{t_{1}} \left| \boldsymbol{y}_{1} \right|^{2} \,\mathrm{d}s \leq \mathcal{H}(t_{0}) + c_{s} \int_{t_{0}}^{t_{1}} \left(\left| \boldsymbol{\imath} \right|^{2} + \left| \boldsymbol{v} \right|^{2} \right) \mathrm{d}s,$$
(6.6)

which implies for all $t_1 \ge t_0 \ge 0$

$$\mathcal{F}(t_1) \leq \mathcal{F}(t_0), \quad \text{with} \quad \mathcal{F}(t) := \mathcal{H}(t) - c_s \int_0^t \left(\left| \boldsymbol{\imath} \right|^2 + \left| \boldsymbol{\upsilon} \right|^2 \right) \mathrm{d}s. \quad (6.7)$$

Next, we assume that

$$\lim_{t \to \infty} \left(|\boldsymbol{\imath}(t)|^2 + |\boldsymbol{v}(t)|^2 \right) = 0, \qquad \int_0^t \left(|\boldsymbol{\imath}|^2 + |\boldsymbol{v}|^2 \right) \mathrm{d}s \le M, \quad \text{for all } t \ge 0, \tag{6.8}$$

for some real constant M (the electric sources decay fast in time or they are turned off after a finite time). Thus inequality (6.6) reads

$$\frac{q}{2} \int_0^t \int_0^l (\mu_n n \left| \partial_x \delta \phi_n \right|^2 + \mu_p p \left| \partial_x \delta \phi_p \right|^2) \, \mathrm{d}x \, \mathrm{d}s + c_u \int_0^t |\boldsymbol{y}_1|^2 \, \mathrm{d}s \le \mathcal{H}(0) + c_s M.$$

and in turn, there exists an unbounded, increasing sequence of times (t_j) , such that

$$\lim_{j \to \infty} \left[\int_0^l \left(\mu_n n \left| \partial_x \phi_n \right|^2 + \mu_p p \left| \partial_x \phi_p \right|^2 \right) \mathrm{d}x + c_u |\boldsymbol{y}_1|^2 \right]_{t=t_j} = 0.$$
(6.9)

Furthermore, to deduce relaxations for n, p and E, we consider the identity

$$\int_0^l \left(n \left| \partial_x \phi_n \right|^2 + p \left| \partial_x \phi_p \right|^2 \right) \mathrm{d}x$$

=
$$\int_0^l \left\{ n \left(E - U_T \frac{\partial_x n}{n} \right)^2 + p \left(E + U_T \frac{\partial_x p}{p} \right)^2 \right\} \mathrm{d}x$$

=
$$\int_0^l \left\{ (n+p) \left| E \right|^2 - 2U_T E \,\partial_x (n-p) + 4U_T^2 \left(\left| \partial_x \sqrt{n} \right|^2 + \left| \partial_x \sqrt{p} \right|^2 \right) \right\} \mathrm{d}x.$$

For the middle term, we will apply integration by parts, the Poisson equation (2.3e), the explicit formula (2.12) for the field E, and the identity $n_D - p_D = C$ at the boundary. In this way, we can estimate

$$\begin{split} \int_{0}^{l} E \,\partial_{x}(n-p) \,\mathrm{d}x &= \left[E\left(n-p\right)\right]_{0}^{l} - \int_{0}^{l} (\partial_{x}E) \left(n-p\right) \,\mathrm{d}x \\ &\leq c \left(1 + |u_{2}-u_{1}| + ||n-p-C||_{L^{1}}\right) - \frac{q}{\epsilon} \,||n-p-C||_{L^{2}}^{2} - \frac{q}{\epsilon} \int_{0}^{l} C(n-p-C) \,\mathrm{d}x \\ &\leq c^{*} + \frac{c_{u}}{\mu U_{T}} |\boldsymbol{y}_{1}|^{2} - \frac{q}{2\epsilon} \,||n-p-C||_{L^{2}}^{2} = c^{*} + \frac{c_{u}}{\mu U_{T}} |\boldsymbol{y}_{1}|^{2} - \frac{1}{2} \,||\partial_{x}E||_{L^{2}}^{2}, \end{split}$$

where $\mu = \min\{\mu_n, \mu_p\}$ and c^* are constants. Combining the above relations, we get

$$\int_{0}^{l} \{ \mu_{n} n \left| \partial_{x} \phi_{n} \right|^{2} + \mu_{p} p \left| \partial_{x} \phi_{p} \right|^{2} \} dx + c_{u} \left| \boldsymbol{y}_{1} \right|^{2} + c^{*} \mu U_{T} \\
\geq c \left(\left\| \partial_{x} \sqrt{n} \right\|_{L^{2}}^{2} + \left\| \partial_{x} \sqrt{p} \right\|_{L^{2}}^{2} + \left\| \partial_{x} E \right\|_{L^{2}}^{2} \right),$$
(6.10)

for some constant c. Then, evaluating the previous inequality at $t = t_j$, we find that the following sequence is bounded:

$$\left\{ \left\| \sqrt{n} \right\|_{H^1}^2 + \left\| \sqrt{p} \right\|_{H^1}^2 + \left\| E \right\|_{H^1}^2 \right\}_{t=t_j}.$$

Notice that $|E|_{H^1} \leq c' \{ 1 + |A_D u| + |\partial_x E| \}$. Using the compact embedding of H^1 into L^4 , we deduce the existence of functions $\bar{n}, \bar{p} \in L^2$ such that $\sqrt{\bar{n}}, \sqrt{\bar{p}} \in H^1$ and

$$n(t_j) \to \bar{n}, \quad p(t_j) \to \bar{p}, \qquad \text{in } L^2 \quad (j \to \infty).$$

Recalling (6.9), we have $\lim y_1(t_j) = 0$ (as $j \to \infty$) and thus from (6.5) and (6.8) we deduce the limits

$$\lim_{j \to \infty} \boldsymbol{u}(t_j) = 0, \quad \lim_{j \to \infty} \boldsymbol{j}_V(t_j) = 0.$$
(6.11)

Using the first limit, it follows for the potential

$$V(x,t_j) \rightarrow \bar{V}(x) = V_{\rm bi}(0) + \frac{x}{l} [V_{\rm bi}]_0^l + \int_0^x \int_0^{x'} \frac{q}{\epsilon} (\bar{n} - \bar{p} - C) \, \mathrm{d}x'' \, \mathrm{d}x'$$
$$- \frac{x}{l} \int_0^l \int_0^{x'} \frac{q}{\epsilon} (\bar{n} - \bar{p} - C) \, \mathrm{d}x'' \, \mathrm{d}x'$$

in $H^1 \hookrightarrow C$. Then, proceeding as in [3], we can prove that

$$\bar{n} = n^{\mathrm{e}}, \quad \bar{p} = p^{\mathrm{e}}, \quad \bar{V} = V^{\mathrm{e}}.$$

So far, all convergence results holds for a specific sequence of times (t_j) . To pass from the convergence along the specific sequence (t_j) to the convergence in time, we use the function $\mathcal{F}(t)$. Since it is decreasing and bounded, it converges to some real number as t tends to infinity. It follows that also \mathcal{H} converges and, recalling (6.11), we have

$$\lim_{t \to \infty} \mathcal{H}(t) = \lim_{j \to \infty} \mathcal{H}^*(t_j) + \lim_{j \to \infty} \frac{1}{2} \boldsymbol{\mathcal{I}}_L(t_j)^\top \boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{I}}_L}(t_j).$$
(6.12)

For the first limit on the right-hand side of (6.12), using the above convergence results and the estimate

$$\begin{aligned} \mathcal{H}^* &= \int_0^l q U_T \bigg\{ (n-n^{\mathrm{e}})^2 \int_0^1 \frac{(1-\theta) \mathrm{d}\theta}{n^{\mathrm{e}} + \theta(n-n^{\mathrm{e}})} + (p-p^{\mathrm{e}})^2 \int_0^1 \frac{(1-\theta) \mathrm{d}\theta}{p^{\mathrm{e}} + \theta(p-p^{\mathrm{e}})} \bigg\} \mathrm{d}x \\ &+ \int_0^l \frac{\epsilon}{2} \, |\delta E|^2 \mathrm{d}x \le c \left(\|n-n^{\mathrm{e}}\|_{L^2}^2 + \|p-p^{\mathrm{e}}\|_{L^2}^2 + \|E-E^{\mathrm{e}}\|_{L^2}^2 \right), \end{aligned}$$

we find

$$\lim_{j \to \infty} \mathcal{H}^*(t_j) = 0.$$
(6.13)

To prove that $\boldsymbol{\jmath}_L$ converges for large times, we recall (2.1) and observe that

$$\int_0^t \left| \frac{\mathrm{d} \boldsymbol{\jmath}_L}{\mathrm{d} t} \right|^2 \mathrm{d} s = \int_0^t \left| \boldsymbol{L}^{-1} \boldsymbol{A}_L^\top \boldsymbol{u} \right|^2 \mathrm{d} s \le \left| \boldsymbol{L}^{-1} \boldsymbol{A}_L^\top \right|^2 \int_0^t |\boldsymbol{u}|^2 \, \mathrm{d} s \le c.$$

It follows that there exists some $\bar{\pmb{\jmath}}_L \in \mathbb{R}^{n_L}$ such that

$$\lim_{t \to \infty} \boldsymbol{j}_L(t) = \bar{\boldsymbol{j}}_L. \tag{6.14}$$

Then, (6.12), (6.13) and (6.14) give immediately

$$\lim_{t \to \infty} \mathcal{H}^*(t) = \lim_{t \to \infty} \frac{1}{2} \boldsymbol{y}_1^{\mathsf{T}}(t) \boldsymbol{A}_C \boldsymbol{C} \boldsymbol{A}_C^{\mathsf{T}} \boldsymbol{y}_1(t) = 0.$$
(6.15)

Finally, using the estimate (5.8), we can prove that

$$||n - n^{e}||_{L^{1}} + ||p - p^{e}||_{L^{1}} + ||E - E^{e}||_{L^{2}}^{2} \le c(\epsilon)\mathcal{H}^{*} + \epsilon \text{ for all } \epsilon > 0.$$

Passing to the limit, from the arbitrariness of ϵ we deduce that

$$\lim_{t \to \infty} \{ \|n - n^{\mathbf{e}}\|_{L^{1}} + \|p - p^{\mathbf{e}}\|_{L^{1}} + \|V - V^{\mathbf{e}}\|_{H^{1}} \} = 0$$

We can summarize the results of this section in the following theorem:

THEOREM 6.1 (Asymptotic behavior). Under the assumptions of Theorem 5.5, let the network satisfy the further topological conditions:

$$\boldsymbol{A}_{L}^{\top}\boldsymbol{Q}_{C}=\boldsymbol{0}, \qquad \qquad \boldsymbol{A}_{C}^{\top}\boldsymbol{Q}_{R}=\boldsymbol{0}$$

and let the sources satisfy the following decaying condition:

$$\lim_{t \to \infty} \left(|\boldsymbol{\imath}(t)|^2 + |\boldsymbol{v}(t)|^2 \right) = 0, \qquad \int_0^t \left(|\boldsymbol{\imath}|^2 + |\boldsymbol{v}|^2 \right) \mathrm{d}s \le M, \quad \text{for all } t \ge 0.$$

Then, as time tends to infinity, the solution of the PDAE problem (4.1) approaches the equilibrium state in the following sense: there exists $\bar{\boldsymbol{j}}_L \in \mathbb{R}^{n_L}$ such that

$$\begin{split} & n \to n^e, \ p \to p^e, \quad in \ L^1, \\ & V \to V^e \quad in \ H^1, \\ & \boldsymbol{u} \to 0, \ \boldsymbol{\jmath}_V \to 0, \ \boldsymbol{\jmath}_L \to \bar{\boldsymbol{\jmath}}_L \quad in \ Euclidean \ norm. \end{split}$$

7. Conclusions. In refined network analysis, network equations describing lumped elements are augmented by PDE models for distributed effects. This leads to PDAE models. In this paper, we have investigated one example of a parabolic-elliptic PDAE system, which results from a semiconductor description by the nonstationary drift-diffusion model within electric networks. To prove the well-posedness of the overall system in an index-1 setting, we had to combine uniform a-priori estimates for all lumped and distributed differential components with generalized energy functionals in a Banach fixed-point framework. This result heavily depends on the validity of topological conditions that link the semiconductor part to the electrical network. If additional topological conditions are assumed that ensure for the power loss of the network, we are able to show that the joint system approximates asymptotically the equilibrium state.

In our opinion, future research has to be devoted to two tasks. The first one is the generalization of the analytical results to more sophisticated geometries (including 2d and 3d models), semiconductor models and topologies (higher-index systems). The latter calls for combining the concepts of weakly ill-posed PDE systems with the perturbation index for differential-algebraic equations. The second task is the derivation of dynamic iteration schemes tailored to the mixed and multiscale nature of the system. Here, adaptivity in time and space, based on appropriate error estimates, is a prerequisite. Also, in order to analyse the convergence of the overall procedure, the error propagation during the iteration has to be linked to the numerical discretization schemes for the PDE and DAE parts.

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