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On the weighted Bojanov–Chebyshev Problem and the sum of translates method of Fenton

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Abstract

Minimax and maximin problems are investigated for a special class of functions on the interval $[0,1]$. These functions are sums of translates of positive multiples of one kernel function and a very general external field function. Due to our very general setting the obtained minimax, equioscillation, and characterization results extend those of Bojanov, Fenton, Hardin, Kendall, Saff and Ambrus, Ball, Erdélyi. Moreover, we discover a surprising intertwining phenomenon of interval maxima, which provides new information even in the most classical extremal problem of Chebyshev.

Keywords: minimax problems, Chebyshev polynomials, weighted Bojanov problems, kernel function, sums of translates function

MSC2020 Classification code: 41A50

1 Introduction

Our starting point is the following theorem of Bojanov (see Theorem 1 in [3]).

Theorem 1.1. *Let $\nu_1, \nu_2, \ldots, \nu_n$ be positive integers. Given $[a,b]$ there exists a unique set of points $x_1^* \le x_2^* \le \ldots \le x_n^*$ such that*

$$\begin{aligned}\|(x-x_1^*)^{\nu_1}(x-x_2^*)^{\nu_2}\ldots(x-x_n^*)^{\nu_n}\| \\ = \inf_{a\le x_1\le x_2\le\ldots\le x_n\le b} \|(x-x_1)^{\nu_1}(x-x_2)^{\nu_2}\ldots(x-x_n)^{\nu_n}\|\end{aligned}$$

where $\|\cdot\|$ is the sup norm over $[a,b]$. Moreover, $a < x_1^* < x_2^* < \ldots < x_n^* < b$. The extremal polynomial $T(x) := (x-x_1^*)^{\nu_1}(x-x_2^*)^{\nu_2}\ldots(x-x_n^*)^{\nu_n}$ is uniquely determined by the following equioscillation property: there exists an array of $n+1$ points $a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$ such that

$$T(t_k) = (-1)^{\nu_{k+1}+\ldots+\nu_n}\|T\|$$

where $k = 0, 1, 2\ldots, n-1, n$.

This contains the classical and well-investigated Chebyshev problem—where all the $\nu_j = 1$. Generalizations of (the solution of) the original Chebyshev problem were extensively studied for systems $\Phi := \{\varphi_j\}_{j=1}^n$ with the so-called Chebyshev- or Haar property [13] that all the generated "polynomials" (linear combinations) $\sum_{j=1}^n \alpha_j \varphi_j$ have at most n zeroes (on the interval $[a,b]$), but direct adaptations of that approach abort very early as here for the Bojanov problem the occurring polynomials do not even form a vector space, not to speak of the fact that the number of zeroes can be as large as $\nu_1 + \cdots + \nu_n$.

Note also that different ordering of the same "multiplicity sequence" (ν_i) leads to different problems (as x_i is assumed to be ordered non-decreasingly). One can consider the "global version": $(x_1, \ldots, x_n) \in [0,1]^n$ only. However, we concentrate on the (harder) "local" version with given fixed order.

As in the Chebyshev problem, a natural extension is to investigate weighted maximum norms. Note that $\|wf\| = C \Leftrightarrow C/w(x) \le f(x) \le C/w(x)$ $(\forall a \le x \le b)$ with equality at some points. So it is logical to introduce $W(x) := 1/w(x)$ and consider $-CW(x) \le f(x) \le CW(x)$ $(\forall a \le x \le b)$; in this setup we are trying to *minimize the constant* C, needed for the validity of these bounds.

If $\nu_1 = \cdots = \nu_n = 1$, very general results are known, even for non-symmetric weights, i.e., when lower and upper bounds may be different:

$$CU(x) \le f(x) \le CV(x).$$

Extremal polynomials equioscillate between these bounds [15]. Such extremal polynomials are called "snake polynomials". For more on this direction of development we refer the reader to the book [16] or to, e.g., [17] or [5].

In our case we will consider only equal lower and upper bounds, or in other words symmetric weighted norms. This is natural from the point of view of the basic technicalities of our approach, which can deal with absolute values, but not with signs. However, in case of algebraic polynomials as above, it is easy to trace back the signs taken in various intervals between zeroes, and Bojanov's original results can easily be recovered. (We will leave these to the reader throughout, and concentrate on the arising other questions.) Let us only note here that our discussion will reveal that in the above Bojanov Theorem we can as well say that the unique system of optimizing nodes is characterized by the attainment, in all the intervals $[a, x_1^*], [x_1^*, x_2^*], \ldots, [x_{n-1}^*, x_n^*], [x_n^*, b]$, at some points $t_0, t_1, \ldots, t_n$, of the norm: $|T(t_i)| = \|T\|, (i = 0, 1, \ldots, n)$—so for characterization there is no need to assume *signed equioscillation*, but the only thing really necessary is the attainment of the norm (by whatever sign).

Our approach is equally well fit to non-integral, actually arbitrary positive real exponents ν_i, as soon as we do not insist on signs and deal with only absolute values. This means that we are addressing the Bojanov-type extremal problem for so-called *generalized algebraic polynomials*, c.f. [4], too. In fact soon we will see that much more general settings are dealt with in the paper.

Let us point out that Bojanov in his papers used the classical, Chebyshev-Markov style approximation theoretic approach, with fine tracing of zeroes, monotonicity observations in case of moving of nodes, intertwining properties, zero counting etc. To the best of our knowledge, Bojanov's result was never extended to the weighted case, (not to speak of the domain of generalized algebraic polynomials) and neither to trigonometric polynomials.

In our previous paper [6] we used an approach similar to the one presented here to address general minimax problems on the torus, and obtained (generalized) trigonometric polynomial and also generalized algebraic polynomial versions of Bojanov's Theorem. However, for obtaining results on an interval, the approach involved a pull-back of the interval to the torus, with nodes on the interval corresponding to a pair of symmetric trigonometric nodes on the torus. This symmetry was crucial in transferring back the obtained extremal situation on the torus to a definite system in the interval: if the extremal situation was not symmetric, the backward transfer would not work. For this reason the method can at best be generalized to the weighted case if only we assume symmetry (that is, normalizing to the interval $[-1,1]$, evenness) of the weight also on the interval. We consider that this is possible, and would thus extend our unweighted Bojanov-type results in [6] even to the weighted case with even weights.

However, our aim here is to obtain results also for not necessarily even weights, our model case being Jacobi and even generalized Jacobi weights with arbitrary (though positive) and not necessarily symmetric powers. Therefore we dropped the transfer idea and worked out the method right for the interval case. To our surprise, new obstacles and phenomena occurred and the interval case was found to be different from the periodic (torus) case in a number of features. In particular, we had to assume new conditions, which was found to be not only technicalities, but essential conditions, need of which being shown by examples—see our Section 5. On the other hand we were striving to possibly widest generality, which again led us to new conditions, less restrictive than used before in similar works. We will comment on this aspect later in due course.

Let us now describe our approach and the related literature in brief. The original formulation of the Bojanov-type minimization problem can be immediately rephrased to a minimax problem by taking logarithm:

$$\log|(x-x_1)^{\nu_1}(x-x_2)^{\nu_2}\dots(x-x_n)^{\nu_n}| = \sum_{j=1}^{n}\nu_j\log|x-x_j|.$$

The main result of this change is that the original multiplicative problem is reformulated to an additive one. The extremal problem is then to minimize (in $x_1,\dots,x_n$) $\max_{[a,b]}\sum_{j=1}^n\nu_j\log|x-x_j|$. As said, we want to deal with the weighted norm case, too. That version is:

$$\textbf{minimize}\text{ (in } x_1,\dots,x_n) \qquad \|(x-x_1)^{\nu_1}(x-x_2)^{\nu_2}\dots(x-x_n)^{\nu_n}w(x)\|_{C([a,b]}\,.$$

After taking logarithm (and assuming that $w(x)\geq 0$) the problem turns to be

$$\textbf{minimize}\text{ (in } x_1,\dots,x_n) \qquad \max_{[a,b]}\ \log w(x)+\sum_{i=1}\nu_i\log|x-x_i|.$$

The extremal problem here can be interpreted (as is widespread in the approximation theory literature, see, e.g., [20]) in terms of (logarithmic) potential theory. Then $J(x):=\log w(x)$ is considered a fixed "outer field", while the "free charges" (of the amount ν_i) at the variable points x_i add to the total field a respective logarithmic term. So the "electrostatic law" is logarithmic, and the minimization problem asks for maximum energy minimization under the restrictions. We will not make use of potential theory in this paper, but from time to time we will refer to our functions in related terminology. In particular, the logarithm, (and the more general functions K to replace them in the sequel) will be termed "kernel functions", J the "outer field", the total sum above as the "potential" and the maximum of it as the "energy". Apart from names, these will bear no further potential theoretical importance here.

In this introduction let us formulate just one model result of our investigations, which shows the yields of the method in a concrete situation.

Theorem 1.2. *Let $r_1, r_2, \dots, r_n$ be positive numbers, $[a,b]$ a non-degenerate, compact interval, and w be an upper semicontinuous, non-negative weight function on $[a,b]$, assuming non-zero values at least at n interior points, plus one more point either in the interior or at some end of the interval $[a,b]$.*

Then there exists a unique extremizer set of points $(a \le) x_1^* \le x_2^* \le \dots \le x_n^* (\le b)$ such that

$$\|w(x)|x-x_1^*|^{r_1}|x-x_2^*|^{r_2}\dots|x-x_n^*|^{r_n}\| \\ = \inf_{a\le x_1\le x_2\le\dots\le x_n\le b} \|w(x)|x-x_1|^{r_1}|x-x_2|^{r_2}\dots|x-x_n|^{r_n}\|$$

where $\|\cdot\|$ is the sup norm over $[a,b]$, and in fact $a < x_1^* < x_2^* < \dots < x_n^* < b$. Moreover, the extremal generalized polynomial $T(x) := \prod_{i=1}^n |x-x_i^*|^{r_i}$ is uniquely determined by the following equioscillation property: there exists a system of $n+1$ points $a \le t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n \le b$ (actually interlacing with the x_i^* so that $a \le t_0 < x_1^* < t_1 < x_2^* < \dots < x_n < t_n \le b$) such that

$$T(t_k) = \|T\| \qquad (k = 0, 1, \dots, n).$$

As is pointed out above, one of the reasons to use a new method is to deal with situations where the system of "polynomials" do not form a vector space, and even the less a so-called Chebyshev or Haar system (defined by the restriction that there can be at most n zeroes of any linear combinations of the first n functions in the system). Surprisingly, however, we can derive conclusions, which seem to be unnoticed (and seemingly hard to prove by direct applications of the classical methods) even in the most classical case of the basic Chebyshev problem, see Corollary 4.1. So it seems that even if we lose some precision regarding signs (which on the other hand, we consider easy to recover at least in the extremal cases), but we on the other hand do gain some more insight into the structure of these classical problems, too.

The method what we pursue was studied on the torus mainly for different reasons in our previous work [6]. For the history on the torus setup and the (different) antecedents in that settings we refer the reader to our previous paper [6] as well as to [1] and [14].

The real origins of the sum of translates approach are found in [11]. Fenton's original aim was to prove a conjecture of P.D. Barry from 1962 about the growth of entire functions, in which he succeeded in [8] by means of his sums of translates method and the found minimax theorem. Even if it turned out that Barry's original problem was already solved by Goldberg [12] by other methods a little earlier, later on Fenton showed further fruitful applications of his approach, see [9, 10]. Our main results extend and further Fenton's original work on the sum of translates function and related minimax problems. Here we do not discuss possible further applications of the sums of translates method in the theory of entire functions, but we consider it likely that our more general minimax results have the potential of being applied in a wide range of topics including entire functions.

What we do explore somewhat are a few applications to approximation theory. Apart from the above detailed Bojanov direction, we show that some seemingly unrelated questions, like, e.g., Chebyshev constants of sets E consisting of the union of k intervals, can also be dealt with. Here it bears a crucial relevance that we did not stop at logarithmically concave, or not even at continuous weight functions—like we did in the torus case in [6]—but generalized the approach to arbitrary upper semicontinuous kernels. Indeed, the k interval case corresponds to taking the weight $w := \sum_{j=1}^{k} \chi_{[a_j,b_j]}$, with the $[a_j, b_j]$ standing for the disjoint intervals and χ_A the characteristic or indicator function of the set A. To conquer the arising technical difficulties took us some time, but the achieved generality seems to be indeed better applicable.

However, the possibly most interesting (and surprising) findings will follow at Theorem 4.1 because the phenomena described here seem not having been observed so far, not even for the most classical case of the (unweighted, maximum norm) original Chebyshev problem. It is well-known and is of kind of folklore, even if it is more then difficult to reference it out, that Chebyshev nodes have the property that for any other node system some of the arising interval maxima stay below, while some of the interval maxima becomes larger than the maxima of the Chebyshev polynomial (which, by equioscillation, is attained as interval maxima for all the $n+1$ intervals between neighboring nodes and the endpoints). But we will prove, that this "intertwining property" of interval maxima is not at all a unique feature of the extremal Chebyshev nodes—in fact, *any two node systems* exhibit this intertwining property with each other.

This paper is not a stand-alone one, although we tried to be self-contained wherever possible. However, in the final, very detailed full picture a key role is played by a particular result, the so-called Homeomorphism Theorem (Theorem 2.1 below), without which we could have said much less (even if certain results, in particular minimax results, do not require its use). Interestingly, also this Homeomorphism Theorem has approximation theoretic applications in itself, too. To prove it requires essentially different techniques and considerations, so that we decided to dedicate a different paper to this result. This also allowed us to discuss that topic in an even broader generality than it was possible here. To the question of what one can possibly obtain in the full generality of the different kernels of [7] we want to return in a later work.

Let us record here that in spite of the wide generality of the paper, we opted for technical simplicity wherever it was possible. That is, there are decisions, which were consequently made throughout, on the restriction of full generality for the sake of gaining better transparency of the discussion. One immediate case is that we decided to avoid the fully general treatment of totally independent kernels and restricted to the case when all the kernels K_j are of the form $K_j = r_j K$, i.e., some constant multiple of each other. We can state a number of results even for the wider generality of arbitrary different kernels, but some of our methods do not go through and so here we have decided to prefer the full picture instead of getting even more general, yet less precise statements.

2 Basic settings

A function $K : (-1,0) \cup (0,1) \to \mathbb{R}$ will be called a *kernel function*[1] if it is concave on $(-1,0)$ and on $(0,1)$, and if it satisfies

$$\lim_{t \downarrow 0} K(t) = \lim_{t \uparrow 0} K(t). \tag{1}$$

By the concavity assumption these limits exist, and a kernel function has one-sided limits also at -1 and 1. We set

$$K(0) := \lim_{t \to 0} K(t), \quad K(-1) := \lim_{t \downarrow -1} K(t) \quad \text{and} \quad K(1) := \lim_{t \uparrow 1} K(t).$$

Note explicitly that we thus obtain the extended continuous function $K : [-1,1] \to \mathbb{R} \cup \{-\infty\} =: \underline{\mathbb{R}}$, and that we still have $\sup K < \infty$. Also note that a kernel function is almost everywhere differentiable.

We say that the kernel function K is *strictly concave* if it is strictly concave on both of the intervals $(-1,0)$ and $(0,1)$.

Further, we call it *monotone*[2] if

$$K \text{ is monotone decreasing on } (-1,0) \text{ and increasing on } (0,1). \tag{M}$$

By concavity, under the monotonicity condition (M) the values $K(-1)$ and $K(1)$ are also finite. If K is strictly concave, then (M) implies *strict monotonicity*:

$$K \text{ is strictly decreasing on } [-1,0) \text{ and strictly increasing on } (0,1], \tag{SM}$$

where we have extended the assertion to the finite endpoint values, too.

A kernel function K will be called *singular* if

$$K(0) = -\infty. \tag{∞}$$

This condition is fundamental in the intended applications, so generally in this paper we will confine ourselves to singular kernels; but it is of some interest how

[1] The terminology used by Fenton in [11] is that K is a *cusp*, perhaps better fitting to his settings where K is not assumed to satisfy the singularity condition (∞) below, but rather the "derivative singularity" condition $\lim_{t\to 0\pm 0} K'(t) = \pm\infty$.

[2] These conditions—and more, like C^2 smoothness and strictly negative second derivatives—were assumed on the kernel functions in the ground-breaking paper of Fenton [11].

much of the theory goes through for other, possibly non-singular kernels. To these—more involved—cases we plan to dedicate a separate study, while here we take full advantage of the assumption of singularity. See also Remark 2.1 below.

Let $n \in \mathbb{N} = \{1, 2, \dots, \}$ be fixed. We will call a function $J : [0, 1] \to \underline{\mathbb{R}}$ an *external n-field function*[3], or—if the value of n is unambiguous from the context—simply a *field* or *field function*, if it is bounded above on $[0, 1]$, and it assumes finite values at more than n different points, where we count the points 0 and 1 with weight[4] 1/2 only, while the points in $(0, 1)$ are accounted for with weight 1. Therefore, for a field function[5] J the set $(0, 1) \setminus J^{-1}(\{-\infty\})$ has at least n elements, and if it has precisely n elements, then either $J(0)$ or $J(1)$ is finite.

Further, we consider the *open simplex*

$$S := S_n := \{\mathbf{y} : \mathbf{y} = (y_1, \dots, y_n) \in (0, 1)^n,\ 0 < y_1 < \dots < y_n < 1\},$$

and its closure the *closed simplex*

$$\overline{S} := \{\mathbf{y} : \mathbf{y} \in [0, 1]^n,\ 0 \le y_1 \le \dots \le y_n \le 1\}.$$

For any given $n \in \mathbb{N}$, kernel function K, constants $r_j > 0$ $j = 1, \dots, n$, and a given field function J we will consider the *pure sum of translates function*

$$f(\mathbf{y}, t) := \sum_{j=1}^{n} r_j K(t - y_j) \quad (\mathbf{y} \in \overline{S},\ t \in [0, 1]), \tag{2}$$

and also the *(weighted) sum of translates function*

$$F(\mathbf{y}, t) := J(t) + \sum_{j=1}^{n} r_j K(t - y_j) \quad (\mathbf{y} \in \overline{S},\ t \in [0, 1]). \tag{3}$$

Note that the functions J, K can take the value $-\infty$, but not $+\infty$, therefore the sum of translates functions can be defined meaningfully. Furthermore, if $g, h : A \to \underline{\mathbb{R}}$ are extended continuous functions on some topological space A,

[3] Again, the terminology of kernels and fields came to our mind by analogy, which in case of the logarithmic kernel $K(t) := \log|t|$ and an external field $J(t)$ arising from a weight $w(t) := \exp(J(t))$ are indeed discussed in logarithmic potential theory. However, in our analysis no further potential theoretic notions and tools will be applied. This is so in particular because our analysis is far more general, allowing different and almost arbitrary kernels and fields; yet the resemblance to the classical settings of logarithmic potential theory should not be denied.

[4] The weighted counting makes a difference only for the case when $J^{-1}(\{-\infty\})$ contains the two endpoints; with only $n-1$ further interior points in $(0, 1)$ the weights in this configuration add up to n only, whence the node system is considered inadmissible.

[5] To keep the coherence with our companion paper [7], we do not assume a priori that a field function must be upper semicontinuous. However, in this paper this light extra assumption will be needed throughout–whence it will be signaled in the formulation of all the assertions relying also on this additional assumption.

then their sum is extended continuous, too; therefore, $f : \overline{S} \times [0,1] \to \underline{\mathbb{R}}$ is extended continuous. Note that for any $\mathbf{y} \in \overline{S}$ the function $f(\mathbf{y}, \cdot)$ is finite valued on $(0,1) \setminus \{y_1, \ldots, y_n\}$. Moreover, $f(\mathbf{y}, 0) = -\infty$ can happen only if some $y_j = 0$ (whence also $y_1 = 0$ for sure) and $K(0) = -\infty$ or if some $y_j = 1$ (and whence $y_n = 1$ for sure) and $K_j(-1) = -\infty$. Analogous statement can be made about the equality $f(\mathbf{y}, 1) = -\infty$. Recall that J is finite at more than n, i.e., at least at $n + 1/2$ points in the above weighted sense, so, in particular, J is finite on at least n points of $(0,1)$. Thus either $F(\mathbf{y}, \cdot)$ is finite valued at least on one point of $(0,1)$, or if not, then $y_1, \ldots, y_n$ are pairwise distinct, belong to $(0,1)$, all $J(y_j) \in \mathbb{R}$, $j = 1, \ldots, n$ and still at least one of $J(0), J(1)$, so also one of $F(\mathbf{y}, 0), F(\mathbf{y}, 1)$, must be finite. Therefore, $F(\mathbf{y}, \cdot)$ is not constant $-\infty$ and $\sup_{t \in [0,1]} F(\mathbf{y}, t) > -\infty$, always [6].

Further, for any fixed $\mathbf{y} \in \overline{S}$ and $t \neq y_1, \ldots, y_n$ there exists a relative (with respect to $\overline{S}$) open neighborhood of $\mathbf{y} \in \overline{S}$ where $f(\cdot, t)$ is concave (hence continuous). Indeed, such a neighborhood is $B(\mathbf{y}, \delta) := \{\mathbf{x} \in \overline{S} : \|\mathbf{x} - \mathbf{y}\| < \delta\}$ with

$$\delta := \min_{j=1,\ldots,n} |t - y_j|,$$

where $\|\mathbf{v}\| := \max_{j=1,\ldots,n} |v_j|$.

We introduce the *singularity set* of the field function J as

$$X := X_J := \{t \in [0,1] : \ J(t) = -\infty\}, \tag{4}$$

and note that $X^c := [0,1] \setminus X$ has cardinality exceeding n (in the above described, weighted sense), in particular $X \neq [0,1]$. Similarly, the singularity set of $F(\mathbf{y}, \cdot)$ is

$$\widehat{X} := \widehat{X}(\mathbf{y}) := \{t \in [0,1] : \ F(\mathbf{y}, t) = -\infty\} \subsetneq [0,1].$$

Of course, $X \subseteq \widehat{X}(\mathbf{y})$, and if the kernel K is singular, then we have

$$\widehat{X}(\mathbf{y}) \cap (0,1) = (X \cup \{y_1, \ldots, y_n\}) \cap (0,1),$$

and if additionally K is finite valued on $\{-1, 1\}$–whence in particular when it is monotone–then

$$\widehat{X}(\mathbf{y}) = X \cup \{y_1, \ldots, y_n\}.$$

Accordingly, an interval $I \subseteq [0,1]$ with $I \subseteq \widehat{X}(\mathbf{y})$ will be called *singular*.

Writing $y_0 := 0$ and $y_{n+1} := 1$ we also set for each $\mathbf{y} \in \overline{S}$ and $j \in \{0, 1, \ldots, n\}$

$$I_j(\mathbf{y}) := [y_j, y_{j+1}],$$
$$m_j(\mathbf{y}) := \sup_{t \in I_j(\mathbf{y})} F(\mathbf{y}, t),$$

[6] Note that our somewhat complicated-looking assumptions on the weighted count of points of finiteness of J is *the exact condition* to ensure this irrespective of the concrete choice of the kernels in general.

and

$$\overline{m}(\mathbf{y}) := \max_{j=0,\dots,n} m_j(\mathbf{y}) = \sup_{t\in[0,1]} F(\mathbf{y},t),$$
$$\underline{m}(\mathbf{y}) := \min_{j=0,\dots,n} m_j(\mathbf{y}).$$

As has been said above, for each $\mathbf{y} \in \overline{S}$ we have that $\overline{m}(\mathbf{y}) = \sup_{t\in[0,1]} F(\mathbf{y},t) \in \mathbb{R}$ is finite. Observe that an interval $I \subseteq [0,1]$ is contained in $\widehat{X}(\mathbf{y})$, i.e., I is singular, if and only if $F(\mathbf{y},\cdot)|_I \equiv -\infty$. In particular $m_j(\mathbf{y}) = -\infty$ exactly when $I_j(\mathbf{y}) \subseteq \widehat{X}(\mathbf{y})$. A node system $\mathbf{y}$ is called *singular* if there is $j \in \{0,1,\dots,n\}$ with $I_j(\mathbf{y})$ singular, i.e., $m_j(\mathbf{y}) = -\infty$; and a node system $\mathbf{y} \in \partial S = \overline{S} \setminus S$ is called *degenerate*. If the kernels are singular, then each degenerate node system is singular. Furthermore, for a non-degenerate node system $\mathbf{y}$ we have $m_j(\mathbf{y}) = -\infty$ if and only if $\operatorname{rint} I_j(\mathbf{y}) \subseteq X$. Here rint denotes the relative interior of a set with respect to $[0,1]$.

If K is singular, then the functions m_j are extended continuous, this is not immediately obvious due to the arbitrariness of J, but proven in [7] as Lemma 3. We record this fact here explicitly for later reference.

Proposition 2.1. *Let K be a singular kernel function and J be an arbitrary n-field function. Then for each $j \in \{0,1,\dots,n\}$ the function*

$$m_j : \overline{S} \to \underline{\mathbb{R}}$$

is continuous (in the extended sense). Moreover, $\overline{m}, \underline{m} : \overline{S} \to \mathbb{R}$ are extended continuous and $\overline{m}$ is finite valued and continuous in the usual sense.

Remark 2.1. Amazingly, under the singularity condition on the kernel K, virtually no assumption on J is needed for this continuity result (save that $\sup J < \infty$), see [7, Proposition 6]. Without the singularity condition, however, much of the theory fails to go through, or goes through only under other assumptions, or holds only in a different form. Just to mention two crucial handicaps: When K is not singular, the maximum functions m_j may be discontinuous, and the below cited Homeomorphism Theorem 2.1 fails to hold.

Remark 2.2. For a singular kernel K, for an arbitrary n-field function J, for any node system $\mathbf{w} \in \overline{S}$ and $k \in \{0,\dots,n\}$, if $I_k(\mathbf{w})$ is degenerate or singular, then $m_k(\mathbf{w}) = -\infty$, hence $m_k(\mathbf{w}) < \overline{m}(\mathbf{w})$.

We will be primarily interested in the minimax and maximin expressions

$$M(S) := \inf_{\mathbf{x}\in S} \overline{m}(\mathbf{x}) = \inf_{\mathbf{x}\in S} \max_{j=0,1,\dots,n} m_j(\mathbf{x}), \tag{5}$$

and

$$m(S) := \sup_{\mathbf{x}\in S} \underline{m}(\mathbf{x}) = \sup_{\mathbf{x}\in S} \min_{j=0,1,\dots,n} m_j(\mathbf{x}). \tag{6}$$

In this respect an essential role is played by the *regularity set*

$$\begin{aligned} Y := Y_n := Y_n(X) &:= \{\mathbf{y} \in S : \mathbf{y} \text{ is non-singular}\} \\ &= \{\mathbf{y} \in S : I_j(\mathbf{y}) \not\subseteq \widehat{X}(\mathbf{y}) \text{ for } j = 0, 1, \ldots, n\} \\ &= \{\mathbf{y} \in S : m_j(\mathbf{y}) \neq -\infty \text{ for } j = 0, 1, \ldots, n\}. \end{aligned} \tag{7}$$

Since J is an n-field function necessarily it holds $Y \neq \emptyset$, and we trivially have

$$m(Y) := \sup_{\mathbf{x} \in Y} \underline{m}(\mathbf{x}) = \sup_{\mathbf{x} \in S} \underline{m}(\mathbf{x}) = m(S) = \sup_{\mathbf{x} \in \overline{S}} \underline{m}(\mathbf{x}) =: m(\overline{S}).$$

It will turn out, as a minuscule byproduct of our results (see Theorem 3.1), that also

$$M(Y) := \inf_{\mathbf{x} \in Y} \overline{m}(\mathbf{x}) = \inf_{\mathbf{x} \in S} \overline{m}(\mathbf{x}) = M(S)$$

holds. As a matter of fact, for a singular, strictly concave kernel function K and an upper semicontinuous n-field function there is a unique $\mathbf{w} \in \overline{S}$ such that $\overline{m}(\mathbf{w}) = M(S)$, in fact $\mathbf{w} \in Y$ and it is the unique node system in $\overline{S}$ with $\underline{m}(\mathbf{w}) = m(S)$, see Corollary 3.2. Before we can prove these facts, some further preparation is needed.

If the kernel K is singular (a main assumption in this paper), then we also have

$$\begin{aligned} Y &= \{\mathbf{y} \in S : I_j(\mathbf{y}) \not\subseteq X \cup \Big(\textstyle\bigcup_{1 \le i \le n} \{y_i\}\Big) \text{ for } j = 0, 1, \ldots, n\} \\ &= \{\mathbf{y} \in S : \operatorname{rint} I_j(\mathbf{y}) \not\subseteq X \text{ for } j = 0, 1, \ldots, n\}. \end{aligned} \tag{8}$$

An important fact is that the regularity set does not depend on the kernel function K, except for the fact whether it is singular or not. Moreover, it only depends on the singularity set X of J, but not on the actual function J itself. Note also that we have $S = Y$ if and only if X is nowhere dense. In case the kernel K is singular, it follows from (8) that the regularity set is an open subset of S.

We also introduce the *interval maxima vector function*

$$\mathbf{m}(\mathbf{w}) := (m_0(\mathbf{w}), m_1(\mathbf{w}), \ldots, m_n(\mathbf{w})) \in \underline{\mathbb{R}}^{n+1} \quad (\mathbf{w} \in \overline{S})$$

and the both ways extended *interval maxima difference function* or simply *difference function*

$$\begin{aligned} \Phi(\mathbf{w}) &:= (m_1(\mathbf{w}) - m_0(\mathbf{w}), m_2(\mathbf{w}) - m_1(\mathbf{w}), \ldots, m_n(\mathbf{w}) - m_{n-1}(\mathbf{w})) \\ &=: (\Phi_1(\mathbf{w}), \ldots, \Phi_n(\mathbf{w})) \in [-\infty, \infty]^n, \end{aligned} \tag{9}$$

whose maximal domain of definition is

$$\begin{aligned} \mathcal{D} := \mathcal{D}_\Phi := \{\mathbf{w} \in \overline{S} : &\text{ for each } i = 1, \ldots, n \\ &\text{either } m_i(\mathbf{w}) \neq -\infty \text{ or } m_{i-1}(\mathbf{w}) \neq -\infty\}. \end{aligned}$$

Note that $\mathbf{m} : \overline{S} \to \underline{\mathbb{R}}^{n+1}$, hence also $\Phi : \mathcal{D} \to [-\infty, +\infty]^n$ are extended continuous functions, by Proposition 2.1.

From the above it follows that for $\mathbf{w} \in S$ we have $\mathbf{m}(\mathbf{w}) \neq (-\infty, \dots, -\infty)$, and in case $\mathbf{m}(\mathbf{w}) \notin \mathbb{R}^{n+1}$—that is, if some of the maxima $m_i(\mathbf{w}) = -\infty$—then we must also have either $\mathbf{w} \notin \mathcal{D}_\Phi$, or $\mathbf{w} \in \mathcal{D}_\Phi$ but $\Phi(\mathbf{w}) \notin \mathbb{R}^n$, some coordinate becoming (positive or negative) infinite.

A key foothold for our below investigation is the next (very special case of the main) result of [7], see Corollary 2.2 therein.

Theorem 2.1. *For $n \in \mathbb{N}$ let $r_1, \dots, r_n > 0$, suppose that the singular kernel function K is strictly monotone* (SM) *and take an arbitrary n-field function J. Consider the sum of translates function F as in* (3).

Then the corresponding difference function (9)*, restricted to Y, is a locally bi-Lipschitz homeomorphism between Y and $\mathbb{R}^n$.*

Note that the theorem contains, among other things, the already non-trivial fact that $Y \subseteq S$ must be a (simply) connected domain.

The reader finds our main results in Theorems 3.1, 3.2, Corollary 3.1, and Theorem 4.1. In the course of proof we prove several perturbation lemmas in Sections 3 and 4. The necessity of the conditions is discussed in Section 5, while applications are to be found in the final sections.

3 Perturbation lemmas

A variant of the next lemma is contained in [6] (see Lemma 11.5; but also [19], Lemma 10 on p. 1069). A similar but slightly simpler form was given by Fenton in [11] (though not formulated there explicitly, see around formula (15) in [11]).

Lemma 3.1 (Interval perturbation lemma). *Let K be any kernel function. Let $0 < \alpha < a < b < \beta < 1$ and $p, q > 0$. Set*

$$\mu := \frac{p(a-\alpha)}{q(\beta - b)}. \tag{10}$$

(a) If K satisfies (M) *and $\mu \geq 1$, then for every $t \in [0, \alpha]$ we have*

$$pK(t-\alpha) + qK(t-\beta) \leq pK(t-a) + qK(t-b). \tag{11}$$

(b) If K satisfies (M) *and $\mu \leq 1$, then* (11) *holds for every $t \in [\beta, 1]$.*

(c) If $\mu = 1$ then (11) *holds for every $t \in [0, \alpha] \cup [\beta, 1]$.*

(d) In case of a strictly concave kernel function (a), (b), and (c) hold with strict inequality in (11).

(e) If K is a monotone kernel function, then for every $t \in [a, b]$

$$pK(t-\alpha) + qK(t-\beta) \geq pK(t-a) + qK(t-b), \tag{12}$$

with strict inequality if K is strictly monotone.

Proof. Rearranging (11) and dividing by $q(\beta-b)$ yields the equivalent assertion

$$\frac{p(a-\alpha)}{q(\beta-b)}\frac{K(t-\alpha)-K(t-a)}{a-\alpha} \le \frac{K(t-b)-K(t-\beta)}{\beta-b}.$$

This expresses the inequality $\mu c \le C$ with μ defined in (10) and

$$c := \frac{K(t-\alpha)-K(t-a)}{a-\alpha}, \qquad C := \frac{K(t-b)-K(t-\beta)}{\beta-b}$$

being the slopes of the chords of the graph of the kernel function K raised above the points $t-a, t-\alpha$ and $t-\beta, t-b$, respectively. Note that $t-\beta < t-b < t-a < t-\alpha$, and all of them are in $[0,1]$ if $t \in [\beta,1]$, while all of them are in $[-1,0]$ if $t \in [0,\alpha]$. It follows that these points lie in the same interval of concavity of K, and the slope of the previous chord exceeds that of the chord to the right from it: that is, we have $c \le C$. In particular, for $\mu = 1$ (c) follows immediately, even with strict inequality provided that K is strictly concave, as then even $c < C$ holds.

It remains to see when we may have $\mu c \le C$, to which it suffices to see $\mu c \le c$.

Now if $c \le 0$ then this holds with $\mu \ge 1$, and if $c \ge 0$ then it holds with $\mu \le 1$. In view of monotonicity, however, we surely have $c \le 0$ whenever $t-a, t-\alpha \le 0$, i.e., when $t \in [0,\alpha]$, so that in this case $\mu \ge 1$ suffices; and the same way for $t \in [\beta,1]$ we have $t-a, t-\alpha > 0$ and $c \ge 0$, so that $\mu \le 1$ suffices. Altogether we obtain both assertions (a) and (b).

Similar arguments yield the strict inequalities in (a) and (b), whenever K is strictly concave. Altogether (d) is proved.

Assertion (e) is immediate, since under the condition (M) for $t \in [a,b]$ we have $K(t-\alpha) \ge K(t-a)$ and also $K(t-\beta) \ge K(t-b)$, with strict inequality whenever K is strictly monotone. $\square$

Let us record the following, trivial but extremely useful fact as a separate lemma.

Lemma 3.2 (Trivial lemma). *Let $f, g, h : D \to \underline{\mathbb{R}}$ be extended upper semicontinuous functions on some Hausdorff topological space D and let $\emptyset \ne A \subseteq B \subseteq D$ be arbitrary. Assume*

$$f(t) < g(t) \qquad \textit{for all } t \in A. \tag{13}$$

If $A \subseteq B$ is a compact set, then

$$\max_A (f+h) < \sup_B (g+h) \qquad \text{unless} \qquad h \equiv -\infty \quad \textit{on} \quad A. \tag{14}$$

Proof. It is obvious that $\sup_A(f+h) \le \sup_B(g+h)$. If A is compact, $f+h$ attains its supremum at some point $a \in A$. If $h(a) = -\infty$, then also $\max_A(f+h) = f(a)+h(a) = -\infty$, and the strict inequality in (14) follows, unless $h+g \equiv -\infty$ all over B, hence in particular all over A. In this case, however, we must have

$h \equiv -\infty$ all over A, since the strict inequality in the condition (13) entails that $g > -\infty$ on A. Therefore the statement (14) is proved whenever $h(a) = -\infty$. Now, if $h(a) > -\infty$ is finite, then we necessarily have $\max_A(f+h) = f(a) + h(a) < g(a) + h(a) \le \sup_B(g+h)$, and we are done also in this case. The proof is complete. $\square$

Remark 3.1. The upper semicontinuity of the field function J is needed for the validity of the previous lemma, on which our arguments rely heavily. This is why we need to assume this upper semicontinuity in the main results.

Theorem 3.1 (**Minimax equioscillation**). *Let $n \in \mathbb{N}$, let K be a singular, strictly concave and strictly monotone* (SM) *kernel function and let $J : [0,1] \to \underline{\mathbb{R}}$ be an upper semicontinuous field function.*

For $j = 1, \ldots, n$ consider $r_j > 0$ and the sum of translates function as in (3)*.*

Then there is a minimum point $\mathbf{w} \in \overline{S}$ of $\overline{m}$ in $\overline{S}$ (a minimax point), i.e.,

$$M(S) = \inf_{\mathbf{x} \in \overline{S}} \overline{m}(\mathbf{x}) = \inf_{\mathbf{x} \in \overline{S}} \max_{j=0,1,\ldots,n} m_j(\mathbf{x}) = \overline{m}(\mathbf{w}) := \max_{j=0,1,\ldots,n} m_j(\mathbf{w}). \tag{15}$$

Any such minimum point $\mathbf{w} \in \overline{S}$ is an equioscillation point*, i.e., it satisfies*

$$m_0(\mathbf{w}) = m_1(\mathbf{w}) = \ldots = m_n(\mathbf{w}). \tag{16}$$

Furthermore, the point $\mathbf{w}$ is non-singular, i.e., belongs to the regularity set Y, so in particular to the open simplex S.

Proof. By continuity of $\overline{m}$, see Proposition 2.1, some minimum point on the compact set $\overline{S}$ must exist. Let $\mathbf{w} \in \overline{S}$ be any such minimum point. In the following we set first to prove that $\mathbf{w}$ is an equioscillation point, i.e., $m_j(\mathbf{w}) = \overline{m}(\mathbf{w})$ for $j = 0, 1, \ldots, n$. Assume for a contradiction that $m_j(\mathbf{w}) < \overline{m}(\mathbf{w})$ for some $j \in \{0, 1, \ldots, n\}$.

Case 1. First we consider the case when $I_j = [w_j, w_{j+1}] \subseteq (0,1)$, and note that then $0 < j < n$, since $w_0 = 0 < w_j \le w_{j+1} < 1 = w_{n+1}$. Consider the following sets of indices with positions of the w_i at the left resp. right endpoint of I_j:

$$\mathcal{L} := \{i \le j : \ w_i = w_j\}, \qquad \mathcal{R} := \{i \ge j+1 \ : \ w_i = w_{j+1}\}.$$

Note that in principle I_j can be degenerate, i.e., $w_j = w_{j+1}$ can hold, but we have defined the index sets $\mathcal{L}, \mathcal{R}$ as disjoint. Further, we set

$$L := \sum_{i \in \mathcal{L}} r_i, \qquad R := \sum_{i \in \mathcal{R}} r_i, \qquad \text{and} \qquad \mathcal{I} := \{0, \ldots, n+1\} \setminus (\mathcal{L} \cup \mathcal{R}).$$

We apply Lemma 3.1 with $\alpha := w_j - Rh, a := w_j, b := w_{j+1}, \beta := w_{j+1} + Lh, p := L$ and $q := R$, with a small, but positive $h > 0$, to be specified suitably later on. As now the value of μ in (10) is exactly 1, the mentioned lemma yields the strict inequality

$$LK(t-(w_j-Rh))+RK(t-(w_{j+1}+Lh)) < LK(t-w_j)+RK(t-w_{j+1}) \quad (17)$$

for all points $t \in A := [0, w_j - Rh] \cup [w_{j+1} + Lh, 1]$.

Next, we define a new node system $\mathbf{w}'$ by $w'_i := w_i - Rh = w_j - Rh = \alpha$ for all $i \in \mathcal{L}$, $w'_i := w_i + Lh = w_{j+1} + Lh = \beta$ for all $i \in \mathcal{R}$, and the rest unchanged: $w'_i := w_i$ for $i \in \mathcal{I}$. Note that if h is smaller than the distance $\rho > 0$ of the sets $\{w_i \ : \ i \in \mathcal{I}\}$ and I_j, then $0 = w'_0 \le \cdots \le w'_j < w'_{j+1} \le \cdots \le 1$ and hence $\mathbf{w}' \in \overline{S}$. So assume $0 < h < \rho$ from now on, and also that h is chosen so small that $m_j(\mathbf{w}') < \overline{m}(\mathbf{w})$ remains in effect (continuity of m_j, see Proposition 2.1).

With the new node system $\mathbf{w}'$ we have $A = [0,1] \setminus \operatorname{int} I_j(\mathbf{w}')$ and, moreover, inequality (17) can be rewritten as

$$LK(t-w'_j)+RK(t-w'_{j+1}) < LK(t-w_j)+RK(t-w_{j+1}) \quad \text{for } t \in A.$$

Note that adding $J(t) + \sum_{i\in\mathcal{I}} r_i K(t-w_i)$ to both sides, the left-hand side becomes $F(\mathbf{w}', t)$, and the right-hand side becomes $F(\mathbf{w}, t)$.

Applying the Trivial Lemma 3.2 with $A = [0,1] \setminus \operatorname{int} I_j(\mathbf{w}')$, $B := [0,1] \setminus \operatorname{int} I_j(\mathbf{w})$, and $D = [0,1]$ we obtain that $\max_A F(\mathbf{w}', \cdot) < \max_B F(\mathbf{w}, \cdot) \le \overline{m}(\mathbf{w})$, unless the added expression $J(t) + \sum_{i\in\mathcal{I}} r_i K(t-w_i)$ is identically $-\infty$ on A, in which case the left-hand side $\max_A F(\mathbf{w}', \cdot)$ is also $-\infty$. In either case we are led to $\max_A F(\mathbf{w}', \cdot) < \overline{m}(\mathbf{w})$.

Taking into account $\max_{I_j(\mathbf{w}')} F(\mathbf{w}', \cdot) = m_j(\mathbf{w}') < \overline{m}(\mathbf{w})$, we thus infer $\overline{m}(\mathbf{w}') < \overline{m}(\mathbf{w})$, which contradicts the choice of $\mathbf{w}$ as a minimum point of $\overline{m}$. This contradiction proves that $m_j(\mathbf{w}) = \overline{m}(\mathbf{w})$ must hold.

Case 2. Let now $0 = w_j$. Then $I_0 = [0, w_1] \subseteq [0, w_{j+1}] = I_j$, hence $m_0(\mathbf{w}) \le m_j(\mathbf{w}) < \overline{m}(\mathbf{w})$. This implies that $I_0(\mathbf{w}) \subseteq [0,1)$ i.e., $w_1 < 1$, so that there is a maximal index $k \le n$ with $w_1 = w_k$ and $w_k < w_{k+1}$. (Note that $w_1 = \cdots = w_k$ may or may not be in the position 0—this does not matter.)

We consider the new node system $\mathbf{w}'$ with $w'_1 = \cdots = w'_k = w_1 + h$ (and the rest unchanged). With $0 < h < w_{k+1} - w_k$ the new node system $\mathbf{w}'$ also belongs to $\overline{S}$. As above, for small enough h continuity (Proposition 2.1) furnishes $m_0(\mathbf{w}') < \overline{m}(\mathbf{w})$.

Let now $t \in A := [w_1 + h, 1] = [0,1] \setminus \operatorname{rint} I_0(\mathbf{w}')$. Taking into account the strict monotonicity condition (SM) and $h > 0$, we must have

$$\sum_{i=1}^{k} r_i K(t - w_1 - h) < \sum_{i=1}^{k} r_i K(t - w_1) \qquad \text{for all } t \in A.$$

Note that the left-hand side may attain $-\infty$ (at $t = w_1 + h$), but not the right-hand side for $h > 0$. If we add here $J(t) + \sum_{i=k+1}^{n} r_i K(t - w_i)$ to both sides, then the left-hand side becomes $F(\mathbf{w}', t)$, and the right-hand side becomes $F(\mathbf{w}, t)$. Putting $B := [w_1, 1] = [0,1] \setminus \operatorname{rint} I_1(\mathbf{w})$ an application of the Trivial

Lemma 3.2 furnishes $\max_A F(\mathbf{w}', \cdot) < \max_B F(\mathbf{w}, \cdot) \leq \overline{m}(\mathbf{w})$, unless the left-hand side is $-\infty$; in particular, as $\overline{m}(\mathbf{w}) > -\infty$, in either case we obtain $\max_A F(\mathbf{w}', \cdot) < \overline{m}(\mathbf{w})$.

As above, taking into account $\max_{I_1(\mathbf{w}')} F(\mathbf{w}', \cdot) = m_1(\mathbf{w}') < \overline{m}(\mathbf{w})$, we thus infer $\overline{m}(\mathbf{w}') < \overline{m}(\mathbf{w})$, which is a contradiction with the minimality of $\overline{m}(\mathbf{w})$. Therefore, $m_0(\mathbf{w}) = m_j(\mathbf{w}) = \overline{m}(\mathbf{w})$ follows.

Case 3. $w_{j+1} = 1$. It is completely analogous to Case 2.

Cases 1-3 altogether yield that $\mathbf{w}$ is an equioscillation point, as claimed in (16).

Furthermore, if $I_k(\mathbf{w})$ is degenerate, or singular, then by Remark 2.2 $m_k(\mathbf{w}) = -\infty < \overline{m}(\mathbf{w})$ must hold. This contradicts equioscillation, hence is excluded by the above. That is, $\mathbf{w} \in Y$, i.e., no interval $I_k(\mathbf{w})$ can be singular. The proof is complete. □

There exist some maximum points of $\underline{m}$ on $\overline{S}$ by continuity and compactness. Completely analogously to the above, we can as well prove the following about these.

Theorem 3.2 (**Maximin equioscillation**). *Let K be a singular, strictly concave and strictly monotone* (SM) *kernel function and let $J : [0,1] \to \underline{\mathbb{R}}$ be an upper semicontinuous field function.*

For $j = 1, \dots, n$ let $r_j > 0$ and consider the sum of translates function as in (3).

If $\mathbf{y} \in \overline{S}$ is a maximum point of $\underline{m}$ (a maximin point), i.e., $\underline{m}(\mathbf{y}) = m(S) := \max_{\overline{S}} \underline{m}$, then $\mathbf{y} \in Y \subseteq S$, and $\mathbf{y}$ is an equioscillation point.

Proof. By Theorem 3.1 we have a minimax point $\mathbf{w} \in Y$. So that for any maximin point $\mathbf{y} \in \overline{S}$ have $\underline{m}(\mathbf{y}) = m(S) \geq \underline{m}(\mathbf{w}) = \overline{m}(\mathbf{w}) = \max_{[0,1]} F(\mathbf{w}, \cdot) > -\infty$, hence for all $j = 0, 1, \dots, n$ also $m_j(\mathbf{y}) > -\infty$. We conclude $\mathbf{y} \in Y$. By Remark 2.2 no degenerate intervals may exist among the $I_j(\mathbf{y})$. This somewhat simplifies our considerations as compared to the proof of Theorem 3.1.

It remains to prove that once $\mathbf{y} \in \overline{S}$ is a maximin point, we necessarily have that it is an equioscillation point, i.e., $m_j(\mathbf{y}) = \underline{m}(\mathbf{y})$ for $j = 0, 1, \dots, n$. For the proof we assume for a contradiction that there exists some $j \in \{0, 1, \dots, n\}$ with $m_j(\mathbf{y}) > \underline{m}(\mathbf{y})$;

Case 1. First let $I_j(\mathbf{y}) = [y_j, y_{j+1}] \subseteq (0,1)$. Note that then $0 < j < n$, as $y_0 = 0 < y_j < y_{j+1} < 1 = y_{n+1}$ (the inequality $y_j < y_{j+1}$ has been clarified above).

We apply Lemma 3.1 (c) with $\alpha := y_j$, $a := y_j + h/r_j$, $b := y_{j+1} - h/r_{j+1}$, $\beta := y_{j+1}$, $p := r_j$ and $q := r_{j+1}$, where $h > 0$ is so small that $a < b$. We obtain for all $t \in A := [0,1] \setminus \operatorname{int} I_j(\mathbf{y}) = \cup_{i=0, i \neq j}^{n} I_i(\mathbf{y})$ that

$$r_j K(t - y_j) + r_{j+1} K(t - y_{j+1}) < r_j K(t - (y_j + \tfrac{h}{r_j})) + r_{j+1} K(t - (y_j + \tfrac{h}{r_{j+1}})). \quad (18)$$

We define a new node system $\mathbf{y}'$ by setting $y_j' := y_j + h/r_j$ and $y_{j+1}' := y_{j+1} - h/r_{j+1}$ and the rest of the nodes unchanged: $y_i' := y_i$ for $i \neq j$, $j+1$. By the choice of $h > 0$ we have $\mathbf{y}' \in S$. By taking, if necessary, a smaller $h > 0$, we can ensure $m_j(\mathbf{y}') > \underline{m}(\mathbf{y})$ (continuity of m_j, Proposition 2.1). Now adding $J(t) + \sum_{i\neq j,j+1} r_i K(t-y_i)$ to both sides of (18), the left-hand side becomes $F(\mathbf{y},t)$, and the right-hand side becomes $F(\mathbf{y}',t)$.

Let now $i \in \{0,1,\ldots,n\}\setminus\{j\}$ be any index and consider $m_i(\mathbf{y}')$ and $m_i(\mathbf{y}) > -\infty$ (recall $\mathbf{y}$ is non-singular). The Trivial Lemma 3.2 with $A := I_i(\mathbf{y}) \subseteq B := I_i(\mathbf{y}')$ yields $-\infty < \underline{m}(\mathbf{y}) \le m_i(\mathbf{y}) = \max_A F(\mathbf{y},t) < \max_B F(\mathbf{y}',\cdot) = m_i(\mathbf{y}')$. As we have already ensured $\underline{m}(\mathbf{y}) < m_j(\mathbf{y}')$, we in fact find $\underline{m}(\mathbf{y}) < m_i(\mathbf{y}')$ for all $i \in \{0,1,\ldots,n\}$, whence we conclude $\underline{m}(\mathbf{y}) < \underline{m}(\mathbf{y}')$, a contradiction with the maximality of $\underline{m}(\mathbf{y})$. We conclude $m_j(\mathbf{y}) = \underline{m}(\mathbf{y})$ in this case.

Case 2. Suppose $y_j = 0$. As $\mathbf{y}$ is a non-degenerate node system, we must have $j = 0$ and $I_j(\mathbf{y}) = I_0(\mathbf{y}) = [0, y_1]$. We will consider the new node system $\mathbf{y}'$ with $y_1' = y_1 - h$ and the rest unchanged: $y_i' = y_i$ for $i = 2,\ldots,n$. With $0 < h < y_1$ the new node system $\mathbf{y}'$ also belongs to S. As above, for small enough $h > 0$ continuity of m_0 (see Proposition 2.1) furnishes $m_0(\mathbf{y}') > \underline{m}(\mathbf{y})$.

Let $i \in \{1,2,\ldots,n\}$ be arbitrary and consider $I_i(\mathbf{y})$ and $I_i(\mathbf{y}')$. Obviously, $I_i(\mathbf{y}) \subseteq I_i(\mathbf{y}')$. Further, $-\infty < \underline{m}(\mathbf{y}) \le m_i(\mathbf{y})$. In view of strict monotonicity of K, we obviously have $r_1K(t-y_1) < r_1K(t-y_1+h)$ for every $t \in I_i(\mathbf{y})$. Adding $J(t) + \sum_{i=2}^n r_iK(t-y_i)$ to this inequality, the left-hand side becomes $F(\mathbf{y},t)$ and the right-hand side becomes $F(\mathbf{y}',t)$. Applying the Trivial Lemma 3.2 with $A := I_i(\mathbf{y})$ and $B := I_i(\mathbf{y}')$ we obtain $-\infty < \underline{m}(\mathbf{y}) \le m_i(\mathbf{y}) < m_i(\mathbf{y}')$ for each $i = 1,2,\ldots,n$. In fact, also $-\infty < \underline{m}(\mathbf{y}) < m_0(\mathbf{y}')$ was guaranteed above, which then furnishes $\underline{m}(\mathbf{y}) < \min_i m_i(\mathbf{y}') = \underline{m}(\mathbf{y}')$, contradicting the maximality of $\underline{m}(\mathbf{y})$. Therefore, $m_0(\mathbf{y}) = \underline{m}(\mathbf{y})$.

Case 3. The case $y_{j+1} = 1$ is completely analogous to Case 2.

Cases 1-3 altogether yield that $\mathbf{y}$ is an equioscillation point, as claimed. □

Corollary 3.1. *Let K be a singular* (∞)*, strictly concave and (strictly) monotone* (SM) *kernel function, and let J be an upper semicontinuous field function.*

Then $M(S) = m(S)$ and there exists a unique equioscillation point $\mathbf{w} \in \overline{S}$, which, in fact, belongs to $Y \subseteq S$. This point $\mathbf{w}$ is the unique minimax point in $\overline{S}$, i.e., $\overline{m}(\mathbf{w}) = M(S)$, and it is the unique maximin point in $\overline{S}$, i.e., $\underline{m}(\mathbf{w}) = m(S)$. In particular, the so-called Sandwich Property holds: for any node system $\mathbf{x} \in S$ we have $\underline{m}(\mathbf{x}) \le M(S) = m(S) \le \overline{m}(\mathbf{x})$.

Proof. The previous two theorems give that both minimax and maximin points must be equioscillation node systems. Now, points in $\overline{S}\setminus Y$ cannot be equioscillation points, as degenerate or singular points $\mathbf{x}$ satisfy $m_i(\mathbf{x}) = -\infty$ for some $i \in \{0,1,\ldots,n\}$ while $\overline{m}(\mathbf{x}) > -\infty$, always. By Theorem 2.1 the difference mapping Φ is a homeomorphism between Y and $\mathbb{R}^n$. In particular, there exists exactly one pre-image of $\mathbf{0}$, i.e., only one equioscillation point in Y, and hence also in $\overline{S}$ in general. As a result, both the maximin and minimax points must coincide with this unique equioscillation point (Theorems 3.1 and 3.2). □

Corollary 3.2. *Let K be a singular* (∞) *and monotone* (M) *kernel function, and let J be an upper semicontinuous field function.*

Then $M(S) = m(S)$ *and there exists some node system* $\mathbf{w} \in \overline{S}$*, also belonging to* Y*, with the three properties that it is an equioscillating point, it attains the simplex maximin and also it attains the simplex minimax:* $\underline{m}(\mathbf{w}) = m(S) = M(S) = \overline{m}(\mathbf{w})$*.*

In particular, the so-called Sandwich Property holds: for any node system $\mathbf{x} \in S$ *we have* $\underline{m}(\mathbf{x}) \le M(S) = m(S) \le \overline{m}(\mathbf{x})$*, and* $M(S) = m(S)$ *is the unique equioscillation value.*

If in addition the kernel K *satisfies* (SM)*, then the point* $\mathbf{w}$ *is the unique equioscillation point.*

Proof. For the proof, we first apply the previous corollary in the situation with the same field function J and the modified kernel functions $K^{(\eta)}(t) := K + \eta\sqrt{|t|}$. If $\eta > 0$, then $K^{(\eta)}$ is strictly concave and strictly monotone, whence Corollary 3.1 applies and provides node systems $\mathbf{w}_\eta$ with the three asserted properties: $m^{(\eta)}(S) = M^{(\eta)}(S) = \underline{m}^{(\eta)}(\mathbf{w}_\eta) = \overline{m}^{(\eta)}(\mathbf{w}_\eta)$, where the notation refers to the corresponding quantities with the use of the kernel $K^{(\eta)}$. With a similar notation for the sum of translates function and putting $R := \sum_{i=1}^n r_i$, it is obvious that $F(\mathbf{x}, t) \le F^{(\eta)}(\mathbf{x}, t) \le F(\mathbf{x}, t) + \eta R$ for all $\mathbf{x} \in \overline{S}$ and $t \in [0, 1]$. Therefore also $m_i(\mathbf{x}) \le m_i^{(\eta)}(\mathbf{x}) \le m_i(\mathbf{x}) + \eta R$ and hence $m_i^{(\eta)}(\mathbf{x}) \to m_i(\mathbf{x})$ for every $i = 0, 1, \ldots, n$ and $\mathbf{x} \in \overline{S}$. By compactness of $\overline{S}$ we can take a convergent subsequence $(\mathbf{w}_{1/k_\ell})$ of $(\mathbf{w}_{1/k})$ with limit $\mathbf{w} := \lim_{\ell\to\infty} \mathbf{w}_{1/k_\ell}$. Moreover, by continuity of m_i (see Proposition 2.1) we obtain

$$\begin{aligned} m_i(\mathbf{w}) &= \lim_{\ell\to\infty} m_i(\mathbf{w}_{1/k_\ell}) \le \liminf_{\ell\to\infty} m_i^{(1/k_\ell)}(\mathbf{w}_{1/k_\ell}) \le \limsup_{\ell\to\infty} m_i^{(1/k_\ell)}(\mathbf{w}_{1/k_\ell}) \\ &\le \limsup_{\ell\to\infty} \Big(m_i(\mathbf{w}_{1/k_\ell}) + R/k_\ell \Big) = m_i(\mathbf{w}), \end{aligned}$$

that is $\lim_{\ell\to\infty} m_i^{(1/k_\ell)}(\mathbf{w}_{1/k_\ell}) = m_i(\mathbf{w})$. Therefore, $\mathbf{w}$ is an equioscillation point in the case of the kernel K, and hence $\mathbf{w} \in Y$. Let $\mathbf{x}$ be a minimum point of $\overline{m}$ on $\overline{S}$. Then

$$M(S) = \overline{m}(\mathbf{x}) = \lim_{\eta\to 0} \overline{m}^{(\eta)}(\mathbf{x}) \ge \limsup_{\eta\to 0} M^{(\eta)}(S) \ge \liminf_{\eta\to 0} M^{(\eta)}(S) \ge M(S),$$

where the last inequality obviously follows from $K^{(\eta)} \ge K$. Therefore, $M(S) = \lim_{\eta\to 0} M^{(\eta)}(S)$, whence we can also conclude

$$M(S) = \lim_{\ell\to\infty} M^{(1/k_\ell)}(S) = \lim_{\ell\to\infty} \overline{m}^{(1/k_\ell)}(\mathbf{w}_{1/k_\ell}) = \overline{m}(\mathbf{w}),$$

i.e., $\mathbf{w}$ is a minimum point of $\overline{m}$. Since for the η-perturbed kernel we have $M^{(\eta)}(S) = m^{(\eta)}(S)$, we infer that

$$\lim_{\eta\to 0} \underline{m}^{(\eta)}(\mathbf{w}_\eta) = \lim_{\eta\to 0} m^{(\eta)}(S) = \lim_{\eta\to 0} M^{(\eta)}(S)$$

exists, and equals $M(S)$.

On the other hand, $m(S) \le m^{(\eta)}(S) = \underline{m}^{(\eta)}(\mathbf{w}_\eta)$, so that

$$m(S) \le \lim_{\eta\to 0} m^{(\eta)}(S) = \lim_{\eta\to 0} \underline{m}^{(\eta)}(\mathbf{w}_\eta) = \lim_{\ell\to\infty} \underline{m}^{(1/k_\ell)}(\mathbf{w}_{1/k_\ell}) = \underline{m}(\mathbf{w}) \le m(S).$$

Hence $\lim_{\eta\to 0} m^{(\eta)}(S) = m(S)$ and $\underline{m}(\mathbf{w}) = m(S)$, i.e., $\mathbf{w}$ is a maximum point of $\underline{m}$.

Finally, uniqueness of the equioscillation point under condition (SM) follows from Theorem 2.1. □

4 Intertwining

In this section we present the intertwining assertion (see Theorem 4.1) which says, in the terminology of [6], that no strict majorization is possible. For the proof, we need the following perturbation type lemma which is interesting on its own.

Lemma 4.1 (**General maximum perturbation lemma**). *Let $n \in \mathbb{N}$ be a natural number, let $r_1, \ldots, r_n > 0$, let $J : [0, 1] \to \underline{\mathbb{R}}$ be an upper semicontinuous n-field function, and let K be a kernel function satisfying the monotonicity condition* (M)*. Consider the sum of translates function F as in* (3)*.*

Let $\mathbf{w} \in S$ be a non-degenerate node system, and let $\mathcal{I} \cup \mathcal{J} = \{0, 1, \ldots, n\}$ be a non-trivial partition. Then there exists $\mathbf{w}' \in S \setminus \{\mathbf{w}\}$ arbitrarily close to $\mathbf{w}$ with

$$F(\mathbf{w}', t) \le F(\mathbf{w}, t) \text{ for all } t \in I_i(\mathbf{w}') \quad \text{and} \quad I_i(\mathbf{w}') \subseteq I_i(\mathbf{w}) \quad \text{for all } i \in \mathcal{I}; \tag{19}$$

$$F(\mathbf{w}', t) \ge F(\mathbf{w}, t) \text{ for all } t \in I_j(\mathbf{w}) \quad \text{and} \quad I_j(\mathbf{w}') \supseteq I_j(\mathbf{w}) \quad \text{for all } j \in \mathcal{J}. \tag{20}$$

As a result, we also have

$$m_i(\mathbf{w}') \le m_i(\mathbf{w}) \text{ for } i \in \mathcal{I} \quad \text{and} \quad m_j(\mathbf{w}') \ge m_j(\mathbf{w}) \text{ for } j \in \mathcal{J} \tag{21}$$

for the corresponding interval maxima.

Moreover, if K is strictly concave (and hence by condition (M) *also strictly monotone), then the inequalities in* (19) *and* (20) *are strict for all points in the respective intervals where $J(t) \ne -\infty$.*

Furthermore, the inequalities in (21) *are also strict for all indices k with non-singular $I_k(\mathbf{w})$; in particular, for all indices if $\mathbf{w} \in Y$.*

Proof. Before the main argument, we observe that the assertion in (21) is indeed a trivial consequence of the previous inequalities (20) and (19), so that we need not give a separate proof for that.

A second important observation is the following. With the pure sum of translates function f we write $F(\mathbf{w}, t) = f(\mathbf{w}, t) + J(t)$, and so the inequalities

(19) and (20) follow from

$$f(\mathbf{w}',t) \leq f(\mathbf{w},t) \ (\forall t \in I_i(\mathbf{w}')) \quad \text{and} \quad I_i(\mathbf{w}') \subseteq I_i(\mathbf{w}) \quad \text{for all } i \in \mathcal{I}; \tag{22}$$

$$f(\mathbf{w}',t) \geq f(\mathbf{w},t) \ (\forall t \in I_j(\mathbf{w})) \quad \text{and} \quad I_j(\mathbf{w}') \supseteq I_j(\mathbf{w}) \quad \text{for all } j \in \mathcal{J}. \tag{23}$$

Moreover, strict inequalities for all points t with $J(t) \neq -\infty$ will follow in (19) and (20) if we can prove strict inequalities in (22) and (23) *for all values of* t in the said intervals.

Furthermore, in case we have strict inequalities in (22) and (23) for all points t, then for non-singular $I_k(\mathbf{w})$ this entails strict inequalities also in (21) (for the corresponding k; and for all k if $\mathbf{w} \in Y$). To see this, one may refer back to the Trivial Lemma 3.2 with $\{f,g\} = \{f(\mathbf{w},\cdot), f(\mathbf{w}',\cdot)\}$, $h = J$, $\{A,B\} = \{I_k(\mathbf{w}), I_k(\mathbf{w}')\}$.

In the next, main part of the argument we prove (19), (20), (22), and (23) by induction on n for any n-field function and any kernel function.

If $n = 1$ and $\mathcal{I} = \{0\}$, $\mathcal{J} = \{1\}$, then $\mathbf{w}' = (w_1') = (w_1 + h)$ and if $\mathcal{J} = \{0\}$, $\mathcal{I} = \{1\}$, then $\mathbf{w}' = (w_1') = (w_1 - h)$ works with any $0 < h < \min(w_1, 1 - w_1)$. For this only monotonicity resp. strict monotonicity of the kernel is needed, and hence (22), (23) follow readily, while (19), (20) follow by the preliminary observation made above.

Let now $n > 1$ and assume, as inductive hypothesis, the validity of the assertions for $\widetilde{n} := n - 1$ for any choice of kernel and n-field functions.

Case 1. If some of the partition sets $\mathcal{I}, \mathcal{J}$ contain neighboring indices $k, k+1$, then we consider the kernel function $\widetilde{K} := K$, and the $\widetilde{n}$-field function $\widetilde{J} := K(\cdot - w_k)$ with now the sum of translates function $\widetilde{F}$ formed by using $\widetilde{n} = n - 1$ translates with respect to the node system

$$\widetilde{\mathbf{w}} := (w_1, w_2, \ldots, w_{k-1}, w_{k+1}, \ldots, w_n).$$

Formally, the indices change: $\widetilde{w}_\ell = w_\ell$ for $\ell = 1, \ldots, k-1$, but $\widetilde{w}_\ell = w_{\ell+1}$ for $\ell = k, \ldots, n$, the kth coordinate being left out.

We apply the same change of indices in the partition: k is dropped out (but the corresponding index set $\mathcal{I}$ or $\mathcal{J}$ will not become empty, for it contains $k+1$); and then shift indices one left for $\ell > k$: so that

$$\widetilde{\mathcal{I}} := \{i \in \mathcal{I} : i < k\} \cup \{i - 1 \in \mathcal{I} : i > k\}$$

and

$$\widetilde{\mathcal{J}} := \{j \in \mathcal{J} : j < k\} \cup \{j - 1 \in \mathcal{J} : j > k\}.$$

Observe that $\widetilde{F}(\widetilde{\mathbf{w}}, t) = f(\mathbf{w}, t)$ for all $t \in [0,1]$, while

$$I_\ell(\widetilde{\mathbf{w}}) = \begin{cases} I_\ell(\mathbf{w}) & \text{if} \quad \ell < k, \\ I_k(\mathbf{w}) \cup I_{k+1}(\mathbf{w}) & \text{if} \quad \ell = k, \\ I_{\ell+1}(\mathbf{w}) & \text{if} \quad \ell > k. \end{cases}$$

If $\mathbf{w}'$ is close enough to $\mathbf{w}$, then a similar correspondence holds for $\mathbf{w}' \in S^{(n)}$ and $\widetilde{\mathbf{w}}' \in S^{(\widetilde{n})}$ (where $S^{(n)}$ and $S^{(\widetilde{n})}$ denote the simplices of the annotated dimension). We will use this only with $w'_k = w_k$ remaining the same. In this case using that k and $k+1$ belong to the same index set $\mathcal{I}$ or $\mathcal{J}$, it is easy to check that $I_i(\widetilde{\mathbf{w}}') \subseteq I_i(\widetilde{\mathbf{w}})$ for all $i \in \widetilde{\mathcal{I}}$ is equivalent to $I_i(\mathbf{w}') \subseteq I_i(\mathbf{w})$ for all $i \in \mathcal{I}$, and $I_j(\widetilde{\mathbf{w}}') \supseteq I_j(\widetilde{\mathbf{w}})$ for all $j \in \widetilde{\mathcal{J}}$ is equivalent to $I_j(\mathbf{w}') \supseteq I_j(\mathbf{w})$ for all $j \in \mathcal{J}$. Therefore an application of the inductive hypothesis yields the assertions (22), (23) in this case. Whence, by the preliminary observations also (19), (20) follow.

Case 2. It remains to prove the assertion when $\mathcal{I}, \mathcal{J}$ contain no neighboring indices: so that $\mathcal{I} \cup \mathcal{J}$ partitions $\{0, 1, \ldots, n\}$ to the subsets of odd and even natural numbers up to n. We can assume that $\mathcal{I} = (2\mathbb{N}_0 + 1) \cap \{0, 1, \ldots, n\}$ and $\mathcal{J} = 2\mathbb{N}_0 \cap \{0, 1, \ldots, n\}$, the other case can be proved analogously.

We emphasize here that it is important that $\mathbf{w} \in S$ is non-degenerate. This allows, for sufficiently small $\delta > 0$, to move any w_ℓ within a distance $\delta > 0$ still keeping that the perturbed node system $\mathbf{w}'$ belongs to S. We fix such a $\delta > 0$ at the outset and consider perturbations $\mathbf{w}'$ of $\mathbf{w}$ only within distance δ from now on. Our new perturbed node system $\mathbf{w}'$ will be, with an arbitrary $0 < h < \delta / \max\{r_1, \ldots, r_n\}$, the system

$$\mathbf{w}' := (w'_1, \ldots, w'_n) \quad \text{with} \quad w'_\ell := w_\ell - (-1)^\ell \frac{1}{r_\ell} h, \quad \ell = 1, 2, \ldots, n. \tag{24}$$

Obviously, $I_j(\mathbf{w}') \supseteq I_j(\mathbf{w})$ holds for all $j \in \mathcal{J}$, and $I_i(\mathbf{w}') \subseteq I_i(\mathbf{w})$ for all $i \in \mathcal{I}$.

Take now an even index interval $I_{2k}(\mathbf{w}) = [w_{2k}, w_{2k+1}]$, so that $2k \in \mathcal{J}$. Our change of nodes can now be grouped as *pairs of changing nodes* $w_{2\ell-1}, w_{2\ell}$ among $w_1, \ldots w_{2k}$, and then again among $w_{2k+1}, \ldots, w_{2\lfloor n/2 \rfloor}$, plus a left-over change of w_n in case n is odd. Now, *the pairs* are always changed so that the intervals in between shrink, and shrink exactly as is described in Lemma 3.1. We apply this lemma for each pair of such nodes with the choices $a = w'_{2\ell-1}$, $b = w'_{2\ell}$, $\alpha = w_{2\ell-1}$, $\beta = w_{2\ell}$, $p = r_{2\ell-1}$, $q = r_{2\ell}$. This gives that for each such pair of changes, for t *outside of the enclosed interval* $(w_{2\ell-1}, w_{2\ell})$ we have

$$r_{2\ell-1} K(t - w'_{2\ell-1}) + r_{2\ell} K(t - w'_{2\ell}) \geq r_{2\ell-1} K(t - w_{2\ell-1}) + r_{2\ell} K(t - w_{2\ell}). \tag{25}$$

Note that $I_{2k}(\mathbf{w})$, hence any $t \in I_{2k}(\mathbf{w})$, is *always outside of the intervals*, therefore (25) holds. If there is a left-over, unpaired change, then n is odd, the respective node w_n is increased, and *now by monotonicity* we conclude for $t \in I_{2k}(\mathbf{w})$ that $K(t - w'_n) = K(t - w_n - h/r_n) \geq K(t - w_n)$. Altogether, we find with $\eta := 1$ for n odd and $\eta := 0$ for n even that

$$\begin{aligned} f(\mathbf{w}, t) &= \sum_{\ell=1}^{\lfloor n/2 \rfloor} \left(r_{2\ell-1} K(t - w_{2\ell-1}) + r_{2\ell} K(t - w_{2\ell}) \right) + \eta K(t - w_n) \\ &\leq \sum_{\ell=1}^{\lfloor n/2 \rfloor} \left(r_{2\ell-1} K(t - w'_{2\ell-1}) + r_{2\ell} K(t - w'_{2\ell}) \right) + \eta K(t - w'_n) = f(\mathbf{w}', t). \end{aligned} \tag{26}$$

Furthermore, all the appearing inequalities are strict in case K is strictly monotone (and hence is strictly concave). We have proved (23), even with strict inequality under appropriate assumptions.

The proof of (22) runs analogously by grouping the change of nodes as a change of a singleton w_1, and then of pairs $w_{2\ell}, w_{2\ell+1}$ for $\ell = 1, \ldots, \lfloor (n-1)/2 \rfloor$, and of another singleton w_n if n is even. $\square$

Theorem 4.1 (**Intertwining theorem**). *Let K be a singular* (∞)*, strictly concave and (strictly) monotone* (SM) *kernel function and let $J : [0,1] \to \underline{\mathbb{R}}$ be an upper semicontinuous field function.*

Then for nodes $\mathbf{x}, \mathbf{y} \in Y$ majorization cannot hold, i.e., the coordinatewise inequality $\mathbf{m}(\mathbf{x}) \leq \mathbf{m}(\mathbf{y})$ can only hold if $\mathbf{x} = \mathbf{y}$.

Proof. Take two node systems $\mathbf{x}, \mathbf{y} \in Y$ and assume that majorization holds between them: say $\mathbf{m}(\mathbf{x}) \leq \mathbf{m}(\mathbf{y})$ in the sense that $m_i(\mathbf{x}) \leq m_i(\mathbf{y})$ for $i = 0, 1, \ldots, n$. We need to show that in fact $\mathbf{x} = \mathbf{y}$.

First, if $\mathbf{m}(\mathbf{x}) = \mathbf{m}(\mathbf{y})$, then of course $\Phi(\mathbf{x}) = \Phi(\mathbf{y})$, hence in view of the Homeomorphism Theorem 2.1 (which requires condition (∞)) only $\mathbf{x} = \mathbf{y}$ is possible.

So assume that $\mathbf{m}(\mathbf{x}) \neq \mathbf{m}(\mathbf{y})$, so that there exists i with $m_i(\mathbf{x}) < m_i(\mathbf{y})$. Let us introduce the following two ("maximal" and "minimal") distance functions

$$d(\mathbf{z}, \mathbf{y}) := \max_{i=0,1,\ldots,n} (m_i(\mathbf{y}) - m_i(\mathbf{z})),$$
$$\rho(\mathbf{z}, \mathbf{y}) := \min_{i=0,1,\ldots,n} (m_i(\mathbf{y}) - m_i(\mathbf{z})).$$

Then $\rho(\mathbf{z}, \mathbf{y}) \leq d(\mathbf{z}, \mathbf{y})$. Moreover, $0 \leq \rho(\mathbf{z}, \mathbf{y})$ if and only if $\mathbf{m}(\mathbf{z}) \leq \mathbf{m}(\mathbf{y})$, and for $d_0 := d(\mathbf{x}, \mathbf{y})$ we have $d_0 > 0$. Consider the set

$$Z := \left\{ \mathbf{z} \in \overline{S} \ : \ \mathbf{m}(\mathbf{z}) \leq \mathbf{m}(\mathbf{y}), d(\mathbf{z}, \mathbf{y}) \leq d_0 \right\} \subseteq \overline{S}.$$

Obviously, $\mathbf{x} \in Z$, whence $Z \neq \emptyset$. By Proposition 2.1 the distance functions $d(\cdot, \mathbf{y}), \rho(\cdot, \mathbf{y})$ are (extended) continuous on $\overline{S}$ and continuous on Y. In fact, $\mathbf{z} \in Z$ cannot be singular, whence $Z \subseteq Y$, where the distance functions are continuous. Further, as the intersection of $\leq$ level sets of continuous functions (see Proposition 2.1) Z is closed, and therefore compact.

Here we arrive at the key of our argument: We now maximize $\rho(\cdot, \mathbf{y})$ on the compact set Z. Surely, $\rho(\cdot, \mathbf{y})$ can be at most d_0 on Z. Let $\mathbf{z}_0 \in Z$ be a maximum point for $\rho(\cdot, \mathbf{y})$, achieving $\rho(\mathbf{z}_0, \mathbf{y}) =: \rho_0 \leq d_0$. We claim that the difference $m_i(\mathbf{y}) - m_i(\mathbf{z}_0)$ is constant ρ_0 for all $i = 0, 1, \ldots, n$.

Indeed, if this is not the case, then by means of Lemma 4.1, we can perturb $\mathbf{z}_0$ to another node system $\mathbf{w}$ with a larger ρ value. In detail: assume for a contradiction that $\mathbf{m}(\mathbf{y}) - \mathbf{m}(\mathbf{z}_0) \neq \rho_0 \mathbf{1}$. Note that then we also have $\rho_0 < d_0$, for in case $\rho_0 = d_0$ we must have $\mathbf{m}(\mathbf{z}_0) = \mathbf{m}(\mathbf{y}) - d_0 \mathbf{1} = \mathbf{m}(\mathbf{y}) - \rho_0 \mathbf{1}$, contradicting the assumption.

Now let us define the index sets

$$\mathcal{I} := \{ i \in \{0, 1, \ldots, n\} \ : \ m_i(\mathbf{z}_0) = m_i(\mathbf{y}) - \rho_0 \},$$
$$\mathcal{J} := \{ j \in \{0, 1, \ldots, n\} \ : \ m_j(\mathbf{y}) - d_0 \leq m_j(\mathbf{z}_0) < m_j(\mathbf{y}) - \rho_0 \}.$$

As $m_i(\mathbf{y}) - m_i(\mathbf{z}_0)$ is not constant in i, we certainly have indices in both index sets $\mathcal{I}, \mathcal{J}$. Moreover, in view of $\mathbf{z}_0 \in Z$ we have $m_k(\mathbf{z}_0) \in [m_k(\mathbf{y}) - d_0, m_k(\mathbf{y}) - \rho_0]$ for all $k = 0, 1, \ldots, n$. Therefore, $\mathcal{I} \cup \mathcal{J}$ is in fact a non-trivial partition of $\{0, 1, \ldots, n\}$. Thus, Lemma 4.1 applies for these indices and to the non-singular, non-degenerate point $\mathbf{z}_0$, resulting in another node system $\mathbf{w} \in Y \setminus \{\mathbf{z}_0\}$ arbitrarily close to $\mathbf{z}_0$ with $m_i(\mathbf{w}) < m_i(\mathbf{z}_0)$ for $i \in \mathcal{I}$ and $m_j(\mathbf{w}) > m_j(\mathbf{z}_0)$ for $j \in \mathcal{J}$. Since for $j \in \mathcal{J}$ we have $m_j(\mathbf{z}_0) < m_j(\mathbf{y}) - \rho_0$, by the continuity of the functions m_i (Proposition 2.1) if $\mathbf{w}$ is sufficiently near to $\mathbf{z}_0$ we have that $m_j(\mathbf{w}) < m_j(\mathbf{y}) - \rho_0$ for all $j \in \mathcal{J}$. Of course, for these indices $j \in \mathcal{J}$ also the inequality $m_j(\mathbf{w}) \geq m_j(\mathbf{y}) - d_0$ remains valid, since $m_j(\mathbf{w}) > m_j(\mathbf{z}_0) \geq m_j(\mathbf{y}) - d_0$ for $j \in \mathcal{J}$.

Similarly, after perturbation we find $m_i(\mathbf{w}) < m_i(\mathbf{z}_0) = m_i(\mathbf{y}) - \rho_0$ for all $i \in \mathcal{I}$, and, by continuity, in a sufficiently small neighborhood of $\mathbf{z}_0$ also the inequality $m_i(\mathbf{w}) \geq m_i(\mathbf{y}) - d_0$ holds. (Here of course we need that $0 < d_0 - \rho_0$, and use continuity.)

Altogether we find $\mathbf{w} \in Z$, but $m_k(\mathbf{w}) < m_k(\mathbf{y}) - \rho_0$ for all $k = 0, 1, \ldots, n$, whence $\rho(\mathbf{w}, \mathbf{y}) > \rho_0$ follows, a contradiction with the choice of $\mathbf{z}_0$ as maximizing $\rho(\cdot, \mathbf{y})$ on Z. This proves that $\mathbf{z}_0 \in Z$ can only be a point with coordinates of $\mathbf{m}(\mathbf{z}_0)$ having constant distance ρ_0 from the respective coordinates of $\mathbf{m}(\mathbf{y})$: $m_k(\mathbf{z}_0) = m_k(\mathbf{y}) - \rho_0$, for $k = 0, 1, \ldots, n$.

It follows that $\mathbf{y}, \mathbf{z}_0$ are two points of Y with equal difference vectors: $\Phi(\mathbf{y}) = \Phi(\mathbf{z}_0)$. By Theorem 2.1 Φ is, in particular, injective, hence $\mathbf{z}_0 = \mathbf{y}$. It follows that $\rho_0 = 0$, and the maximum of ρ-distances between points of Z to $\mathbf{y}$—and therefore ρ-distances of *any node system* $\mathbf{z} \in Z$ from $\mathbf{y}$—can only be zero (since $\rho(\cdot, \mathbf{y}) \geq 0$ on Z). That is, all $\mathbf{z} \in Z$ are maximum points for the ρ-distance: $\rho(\mathbf{z}, \mathbf{y}) = \rho_0 = 0$ for all $\mathbf{z} \in Z$. It follows that for any $\mathbf{z} \in Z$ the same applies as for the selected $\mathbf{z}_0$ and we conclude that $Z = \{\mathbf{y}\}$. Since $\mathbf{x} \in Z$, it follows that $\mathbf{x} = \mathbf{y}$, and that was to be proved. $\square$

Remark 4.1. Similar non-majorization results are rare, but we may compare to, e.g., Theorem 1, on p. 17 of [18]. If the kernel function K and also the external field function J are smooth, then we may consider the Jacobi matrix $\mathrm{Jac}\Phi$ of Φ (the interval maxima difference function) and the proof of the Homeomorphism Theorem 2.1–i.e., the proof of Theorem 2.1 in [7]–where it is shown that this Jacobi matrix is diagonally dominant. It is known that diagonally dominant matrices are so-called "P-matrices" (for more, we refer to pp. 134-137 of [2]), hence the condition of the cited Theorem 1 of [18] is satisfied. However, the conclusion of that result is far weaker than ours: it excludes majorization only in case the nodes $\mathbf{x}, \mathbf{y}$ are ordered similarly coordinatewise: $x_i \leq y_i$ $(i = 1, \ldots, n)$. The generality that we get non-majorization for *all node systems* $\mathbf{x}, \mathbf{y} \in Y$ can be attributed to the special setup, where Φ is formed from differences of interval maxima of sum of translates functions satisfying our assumptions.

Corollary 4.1. *Consider the (almost) two centuries old classical Chebyshev problem, where in our terminology* $K(t) := \log|t|$*,* $J(t) \equiv 0$*, and so we have strict concavity and monotonicity. Then for any two node systems* $\mathbf{x}, \mathbf{y} \in S$ *we*

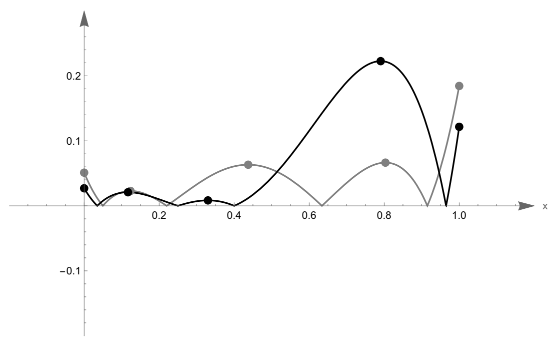


Figure 1: Graphs of $|(x-0.915)(x-0.634)(x-0.22)(x-0.05)|$ and $|(x-0.965)(x-0.4)(x-0.25)(x-0.035)|$ with dots at
local maxima over $[0,1]$ in gray and black, respectively.

necessarily have some indices $0 \le i \ne j \le n$ *such that*

$$\max_{t\in I_i(\mathbf{x})}\Big|\prod_{k=1}^{n}(t-x_k)\Big| < \max_{t\in I_i(\mathbf{y})}\Big|\prod_{k=1}^{n}(t-y_k)\Big|,$$
$$\max_{t\in I_j(\mathbf{x})}\Big|\prod_{k=1}^{n}(t-x_k)\Big| > \max_{t\in I_j(\mathbf{y})}\Big|\prod_{k=1}^{n}(t-y_k)\Big|.$$

Remark 4.2. It seems that even in this very classical situation the above general statement has not been observed thus far. The special case when one of the node systems say $\mathbf{x}$ is the extremal (equioscillating) node system $\mathbf{w}$, is well known and seems to be folklore. However, comparison of arbitrary two node systems looks more complicated and nothing was written about it in the literature what we could page through.

Corollary 4.2 (Non-majorization theorem). *Let K be a singular* (∞) *and monotone* (M) *kernel function, and let J be an upper semicontinuous field function.*

Then strict majorization $m_i(\mathbf{x}) > m_i(\mathbf{y})$ *for every* $i = 0, 1, \ldots, n$ *cannot hold between any two node systems* $\mathbf{x}, \mathbf{y} \in Y$.

Proof. As in the proof of Corollary 3.2, consider the modified kernel functions $K^{(\eta)}(t) := K(t)+\eta\sqrt{|t|}$, which are strictly concave and strictly monotone kernel functions. Let $\mathbf{x}, \mathbf{y} \in \overline{S}$. Then, as in the mentioned proof, for $\eta \downarrow 0$ we have $\mathbf{m}^{(\eta)}(\mathbf{x}) \to \mathbf{m}(\mathbf{x})$ and $\mathbf{m}^{(\eta)}(\mathbf{y}) \to \mathbf{m}(\mathbf{y})$. This implies that once $\mathbf{m}(\mathbf{x}) > \mathbf{m}(\mathbf{y})$, we must have $\mathbf{m}^{(\eta)}(\mathbf{x}) > \mathbf{m}^{(\eta)}(\mathbf{y})$ for every sufficiently small $\eta > 0$, which is impossible by Theorem 4.1, given that by condition $\mathbf{x} \ne \mathbf{y}$. Whence we conclude that $\mathbf{m}(\mathbf{x}) > \mathbf{m}(\mathbf{y})$ is for no pair $\mathbf{x}, \mathbf{y} \in \overline{S}$ possible. $\square$

Corollary 4.3. *Let K be a singular* (∞) *and monotone* (M) *kernel function and let J be a field function. Let $\mathbf{x}, \mathbf{y} \in \overline{S}$ with $\underline{m}(\mathbf{x}) = m(S) = M(S) = \overline{m}(\mathbf{y})$.*

Then there is a $j \in \{0, 1, \ldots, n\}$ such that

$$m_j(\mathbf{x}) = m(S) = M(S) = m_j(\mathbf{y}).$$

5 On the necessity of conditions

In this section we show by examples that dropping conditions from our results entail that the conclusions may not hold true any more. These justify using the given conditions, even if at first glance assuming, e.g., monotonicity or strict concavity may not seem to be natural or necessary. The examples here also highlight the generality of our statements, where further, e.g., smoothness conditions were not supposed. In our results only conditions which are shown to be necessary here, were assumed throughout.

Example 5.1 (**Necessity of singularity).** Let $n = 2$, $J(t) := 8\sqrt{1-t}$ and $K(t) := \sqrt{t+4}$ if $t \in [0, 1]$ and $K(t) := K(-t)$ if $t \in [-1, 0)$. Then, J is a concave field function, $J \in C^\infty([0, 1])$, and $K \in C^\infty([-1, 1] \setminus \{0\})$ is a strictly concave kernel function, further, K is monotone as in (M) and J and K do not satisfy the condition (∞).

We have $M(S) = m(S) = -4$, in $\overline{S}$ there are unique equioscillation, unique minimax and unique maximin node systems and all these are $(0, 0) \in \partial S$. In other words, almost all conclusions of the above theorems hold true, except that this point of extrema and equioscillation is not in S, but on the boundary ∂S.

The key observation is that $\frac{d}{dt}F(\mathbf{y}, t) < 0$ for any $\mathbf{y} \in S$ at every $t \in [0, 1]$. Hence

$$\begin{aligned}
m_0(\mathbf{y}) &= \max\{F(\mathbf{y}, t) : \ 0 \le t \le y_1\} = F(\mathbf{y}, 0) = 8 + \sqrt{4 + y_1} + \sqrt{4 + y_2}, \\
m_1(\mathbf{y}) &= F(\mathbf{y}, y_1) = 8\sqrt{1 - y_1} + 2 + \sqrt{4 + y_2 - y_1}, \\
m_2(\mathbf{y}) &= F(\mathbf{y}, y_2) = 8\sqrt{1 - y_2} + \sqrt{4 + y_2 - y_1} + 2.
\end{aligned}$$

By the observation, $m_0(\mathbf{y}) \ge m_1(\mathbf{y})$ with equality if and only if $y_1 = 0$ and similarly, $m_1(\mathbf{y}) \ge m_2(\mathbf{y})$ with equality if and only if $y_1 = y_2$. Also, $\overline{m}(\mathbf{y}) = m_0(\mathbf{y})$ and $\underline{m}(\mathbf{y}) = m_2(\mathbf{y})$. Obviously, $m_0(\mathbf{y})$, $y \in \overline{S}$ is minimal if and only if $\mathbf{y} = (0, 0)$, and $m_2(\mathbf{y})$ is maximal if and only if $\mathbf{y} = (0, 0)$. Therefore we obtain that $M(\overline{S}) = m(\overline{S}) = -4$ and these are attained at $\mathbf{y} = (0, 0)$ only and there is a unique equioscillation configuration in $\overline{S}$, namely $\mathbf{y} = (0, 0)$.

For convenience, we introduce the kernel function $L_a(t) := \min(0, \log|t/a|)$.

Example 5.2 (**Necessity of monotonicity).** Let $n = 1$, $J(t) := \sqrt{t}$ and $K_1(t) := L_{0.1}(t) + 1 - 2t^2$. Note that J is a strictly concave field function and K_1 is a strictly concave kernel function and K is singular, but it does not satisfy any of the monotonicity conditions (M) and (SM).

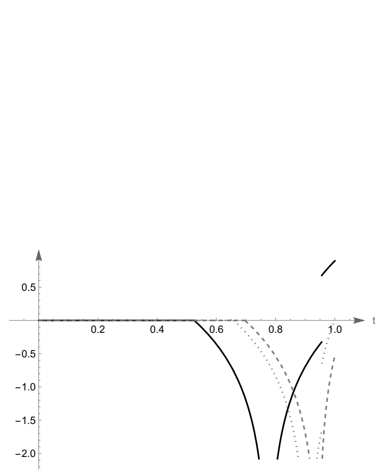


Figure 2: The graphs of sums of translates functions $J(t) + K(t-x)$ when $a = 1/4$, $b = 0.955671$ and $x = 0.775$ (black), $x = 0.907$ (grey, dotted) and $x = 0.946$ (grey, dashed)

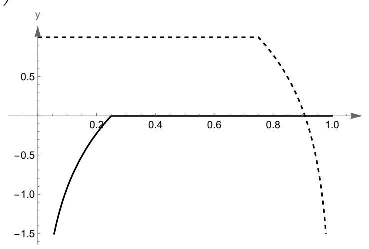


Figure 3: The graphs of the interval maxima functions $m_0(x)$ (black) and $m_1(x)$ (black, dashed).

Then the global minimum $M(S)$ of $\overline{m}(.)$ is attained only at $\mathbf{y} = (y_1) = (0)$, so I_0 is degenerate and $\mathbf{y} \in \partial S$. Also, $m_0(\mathbf{y}) = -\infty$ and $m_1(\mathbf{y}) = F(0, 1/4) = 11/8$. Obviously, $F(\mathbf{y}, \cdot)$ does not equioscillate.

Indeed, if $0 < y_1 \le 1/2$, then $\overline{m}(\mathbf{y}) \ge F(y_1, y_1 + 1/4) = \sqrt{y_1 + 1/4} + 7/8 > 1/2 + 7/8 = 11/8$ and if $1/2 < y_1 \le 1$ then $\overline{m}(\mathbf{y}) \ge F(y_1, y_1 - 1/4) = \sqrt{y_1 - 1/4} + 7/8 > 11/8$. So $\overline{m}(\mathbf{y}) = 11/8$ is attained at $\mathbf{y} = (y_1) = (0)$ only.

Example 5.3 (**Necessity of strict monotonicity and concavity).** Let $0 < a < \frac{e}{1+e}$ be arbitrary and $K(t) := L_a(t)$. Set $n = 1$. Let $b \in (0, 1)$ satisfy $(a <)1 - a/e < b < 1$ and let J be the characteristic function of the interval $[b, 1]$. Then

(a) $\mathbf{m}(\mathbf{x}) = (0, 0)$ if and only if $\mathbf{x} = (x)$, $x = 1 - a/e$, and this is the unique equioscillation point of F, moreover, $m(S) = M(S) = 0$;

(b) however, $\underline{m}(\mathbf{x}) = 0$ for all $a \le x \le 1 - a/e$ and thus $\underline{m}(\mathbf{x}) = m(S) = 0$ is attained not only for the equioscillating node, but for several other, non-equioscillating ones.

Note that K is a concave and monotone kernel function, but not strictly concave or strictly monotone.

Indeed[7], $F(x,\cdot)$ cannot attain positive values on $[0,x]$. If $x \le b$, then $J(x) = 0$ and $L_a(x) \le 0$. If $b \le x \le 1$, then by monotonicity of $L_a(.)$, $F(x,\cdot)$ is maximal on $[0,x]$ either at b or at 0. The value at b is $L_a(b-x)+1 \le L_a(b-1)+1 < \log|(a/e)/a|+1 = 0$. The value at 0 is $F(x,0) \le 0$, moreover, taking into account that $x \ge a$ also holds, we have $L_a(0-x) = 0$, so that $F(x,0) = 0$.

It follows that $m_0(x) \le 0$. Note that $m_0(0) = -\infty$ and $m_0(.)$ is strictly increasing on $[0,a]$. If $a \le x \le b$, then $F(x,t)$ is monotone decreasing on $t \in [0,x]$, and $F(x,0) = 0$ so $m_0(x) = 0$. If $b \le x \le 1$, then $F(x,t)$ is monotone decreasing on $t \in [0,b]$ and is also monotone decreasing on $t \in [b,x]$. Moreover $F(x,0) = 0$ and $F(x,b) = 1+L_a(b-x) \le 1+L_a(b-1) \le 1+\log|(1-b)/a| \le 0$, so that $m_0(x) = F(x,0) = 0$ in this case, too. In all, $m_0(x) \le 0$ for all x, and $m_0(x) = 0$ precisely for $a \le x \le 1$.

Regarding $m_1(.)$, note that $J(t)$ is monotone increasing and $L_a(t-x)$ is also monotone increasing on $t \in [x,1]$, so that $m_1(x) = F(x,1) = 1 + L_a(1-x)$. If $0 \le x \le 1-a$, then $L_a(1-x) = 0$, so $m_1(x) = 1$; and if $1-a < x \le 1$, then $L_a(1-x) = \log((1-x)/a)$, so $m_1(x) = \log((1-x)/a)+1 < 1$. In sum, $m_1(x) = 1$ for $x \in [0,1-a]$ and then it is strictly decreasing in $[1-a,1]$ from 1 to $-\infty$, attaining 0 exactly for $x = 1-a/e$.

Comparing these cases for different ranges of x, we see that $m_0(x) = m_1(x)$ holds if and only if $m_0(x) = 0$ and $m_1(x) = 0$ which occurs precisely for $x = 1-a/e$.

Therefore there is a unique equioscillation point. Also, if $0 \le x \le 1-a/e$, then $\underline{m}(x) = 0$.

Example 5.4 (Necessity of monotonicity). Let

$$K(t) := \min(\log|10t|, \log(\frac{1-|t|}{0.9}))$$

and $J(t) := 0$. Then K is strictly concave, but not monotone. Set $n = 2$. Then

(a) there are several equioscillating node systems, but their $\overline{m}$ values are the same;

(b) there is a unique minimax node system;

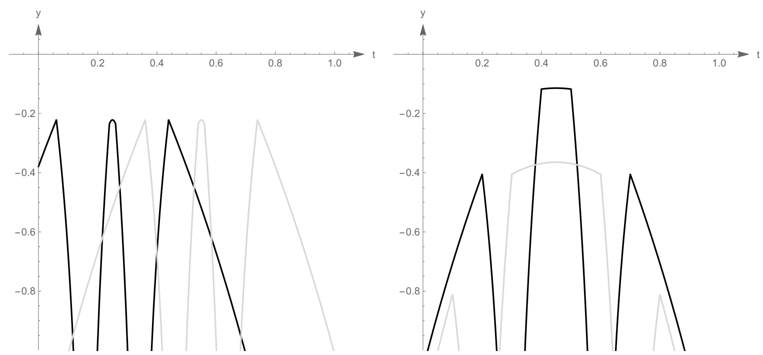
(c) there are several maximin node systems; and

(d) strict majorization occurs.

Observe first that $K(t) \le 0$ for $t \in [-1,1]$. This immediately comes from that $\log|10t| = \log((1-|t|)/0.9)$ holds if and only if $t = 0.1$, and $K(0.1) = 0$ and $K(.)$ is strictly monotone increasing on $[0,0.1]$ and is strictly monotone decreasing on $[0.1,1]$.

Write the nodes as $\mathbf{x} = (a-\delta, a+\delta)$, with $a := \frac{x_1+x_2}{2}$ and $\delta := \frac{x_2-x_1}{2}$. By condition $0 \le a-\delta \le a+\delta \le 1$, in other words, $0 \le a \le 1$ and $0 \le \delta \le a, 1-a$.

[7] For convenience, we write $F(x,t)$, $m_j(x)$, etc in place of $F(\mathbf{x},t)$, $m_j(\mathbf{x})$, etc. respectively.



First, observe that

$$m_1(\mathbf{x}) = \sup\{F(\mathbf{x}, t) : a - \delta \le t \le a + \delta\} = F(\mathbf{x}, a), \text{ so}$$

$$m_1(\mathbf{x}) = \begin{cases} 2\log(10\delta) & \text{if } 0 \le \delta \le 0.1, \\ 2\log\frac{10(1-\delta)}{9} & \text{if } 0.1 \le \delta, \end{cases}$$

Also $m_1(\mathbf{x}) \le 0$, and $m_1(\mathbf{x}) = 0$ if and only if $\delta = 1/10$.

If $t \in [a + \delta, 1]$, then we have the following three cases depending on t ($a + \delta \le t \le a - \delta + 0.1$ may be empty as well) with $u = t - a$, $\delta \le u \le 1 - a$

$$F(\mathbf{x}, t) = \begin{cases} \log 100(u-\delta)(u+\delta) & \text{if } \delta \le 1/20, \delta \le u \le 1/10 - \delta, \\ \log \frac{100}{9}(1-(u+\delta))(u-\delta) & \text{if } \delta, 1/10 - \delta \le u \le \delta + 1/10, \\ \log \frac{100}{81}(1-(u+\delta))(1-(u-\delta)) & \text{if } \delta + 1/10 \le u \le 1 - a. \end{cases}$$

Here it is clear that the first expression is strictly increasing in u and the third expression is strictly decreasing in u, so that $m_2(\mathbf{x})$ equals the maximum of the second expression. Now elementary calculus shows that the second expression is strictly increasing in u if $\delta + 0.1 \le 1/2$, and if $\delta + 0.1 > 1/2$, then it has strict local maximum at $u = 1/2$, that is, when $t = a + 1/2$. Therefore

$$m_2(\mathbf{x}) = \begin{cases} F(\mathbf{x}, a + \delta + 1/10) = \log\left(1 - \frac{20}{9}\delta\right) & \text{if } \delta + 1/10 \le 1/2, \\ F(\mathbf{x}, 1/2 + a) = \log\frac{100}{9}\left(\frac{1}{2} - \delta\right)^2 & \text{if } 4/10 \le \delta \le 1/2. \end{cases}$$

To determine the equioscillating configurations, we compare the values of $m_1(\mathbf{x})$ and $m_2(\mathbf{x})$ in three cases depending on δ. They can be equal if and only if $\delta = \delta_0 := (\sqrt{82} - 1)/90 \approx 0.0895$. Therefore, any $\mathbf{x} = (a - \delta_0, a + \delta_0)$, $a \in [\delta_0 + 1/10, 1 - \delta_0 - 1/0]$ is an equioscillating configuration.

Also, $\overline{m}(\mathbf{x}) \le 0$ and $\overline{m}(\mathbf{x}) = 0$ if and only if $\mathbf{x} = (a - \delta, a + \delta)$ and $\delta = 0$ or $\delta = 1/10$.

Moreover, both $m_1(\mathbf{x})$ and $m_2(\mathbf{x})$ are strictly decreasing if $\delta \ge 1/10$ (and δ is not too large) showing that strict majorization holds (for some configurations).

6 Applications

6.1 Bojanov's problem on the interval

Consider now the set of *monic* "generalized algebraic polynomials" (GAP, cf. Appendix A4, page 392 of [4]), with given degree $\mathbf{r} := (r_1, \ldots, r_n)$, where $r_1, \ldots, r_n > 0$ are given positive exponents:

$$\mathcal{P}_{\mathbf{r}}[a,b] := \left\{ P \ : \ P(t) = \prod_{j=1}^{n} |t - x_j|^{r_j} \ (t \in [a,b]), \ a \le x_1 \le \ldots \le x_n \le b \right\}.$$

Take an upper semicontinuous weight function $w : I \to \mathbb{R}_+ := [0, \infty)$, satisfying the condition that it is non-zero at least at $n+2$ points within the interval $I := [a,b]$. Consider the w-weighted uniform norm $\| \cdot \|_w$ defined by $\|f\|_w := \|fw\|_\infty := \sup_I |f|w$. Then Bojanov's Extremal Problem, extended to GAP-s, is to find the GAP $P \in \mathcal{P}_{\mathbf{r}}[a,b]$ with the least possible $\|P\|_w$. If such an extremal polynomial exists, it will be called a *Bojanov-Chebyshev polynomial*, so that $\|P\|_w = \min_{Q \in \mathcal{P}_{\mathbf{r}}[a,b]} \|Q\|_w$.

Actually, similarly to the classical Chebyshev problem, there are two possible formulations of the extremal problem: there is an "unrestricted" version, where we do not assume that the zeroes of the polynomial belong to $[a,b]$. However, as said in the Introduction, here it is of importance that the order of the arising zero factors follow the order of the given exponents r_j: if the ordering is left free, then one must look for the minimal case–among the in general 2^n possible different ordering–of these various orderings. This is a finite minimization of extrema belonging to the various given orderings, so that we do not further discuss this question, but what we should interpret in this context is the suitable description of all the monic polynomials when the restriction $x_j \in [a,b]$ is relaxed. For real zeroes this is natural, but if we allow complex zeroes–as is well possible in general for GAPs–then the ordering restriction must somehow be preserved. The right interpretation is thus that we take

$$\mathcal{P}_{\mathbf{r}} := \left\{ \prod_{j=1}^{n} |t - (x_j + iy_j)|^{r_j} \ : \ -\infty < x_1 \le \ldots \le x_n < \infty, \ y_1, \ldots, y_n \in \mathbb{R} \right\}.$$

We can thus define the corresponding Chebyshev constants as the *(restricted)* Bojanov-Chebyshev constant $R_{\mathbf{r}}^w[a,b] := \min_{Q \in \mathcal{P}_{\mathbf{r}}[a,b]} \|Q\|_w$, and the *unrestricted* Bojanov-Chebyshev constant $C_{\mathbf{r}}^w[a,b] := \min_{Q \in \mathcal{P}_{\mathbf{r}}} \|Q\|_w$. As in the classical case, we easily see that if $Q \in \mathcal{P}_{\mathbf{r}} \setminus \mathcal{P}_{\mathbf{r}}[a,b]$, then for any given $t \in I$ the polynomial $\widetilde{Q}(t) := \prod_{j=1}^{n} |t - \widetilde{x}_j|^{r_j}$ assumes strictly smaller values than the original $Q(t) := \prod_{j=1}^{n} |t - (x_j + iy_j)|^{r_j}$, if we take $\widetilde{x}$ to be the projection of $x + iy$ to the interval I, i.e., $\widetilde{x}_j := \max(a, \min(x_j, b))$ (and $\widetilde{y}_j$ is taken to be 0) for all $j = 1, \ldots, n$. Using that w is upper semicontinuous and is non-zero at more than n points (in the above weighted sense), it follows that on the compact interval I we have $\|Q\|_w \ge Q(t_0)w(t_0) > \widetilde{Q}(t_0)w(t_0) \ge R_{\mathbf{r}}^w[a,b]$

with the point $t_0 \in I$ where the maximum of Qw is attained. (Note that for $Q(t_0)w(t_0) > \widetilde{Q}(t_0)w(t_0)$ it is indeed used that $\|Q\|_w > 0$, whence $w(t_0) > 0$, too.) Therefore, although formally $C^w_{\mathbf{r}}[a,b]$ is an infimum over a larger set, we still have $C^w_{\mathbf{r}}[a,b] = R^w_{\mathbf{r}}[a,b]$, furthermore, extremizers exist only in $\mathcal{P}_{\mathbf{r}}[a,b]$ (if anywhere). With this preliminary discussion we can now give a somewhat more precise statement than Theorem 1.2.

Theorem 6.1. *Let $r_1, r_2, \ldots, r_n$ be positive numbers, $[a,b]$ a non-degenerate, compact interval, and w be an upper semicontinuous, non-negative weight function on $[a,b]$, assuming non-zero values more than n (weighted) points of the interval $[a,b]$.*

Then $C^w_{\mathbf{r}}[a,b] = R^w_{\mathbf{r}}[a,b]$, and there exists one, unique Chebyshev-Bojanov generalized extremal polynomial P, belonging to $\mathcal{P}_{\mathbf{r}}[a,b]$. This GAP has the form

$$P(t) = \prod_{j=1}^{n} |t - x_j^*|, \tag{27}$$

with the node system $\mathbf{x}^* := (x_1^*, \ldots, x_n^*)$ satisfying $a < x_1^* < \ldots < x_n^* < b$ and uniquely determined by the following equioscillation property: there exists an array of $n+1$ points $a \le t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n \le b$ (actually interlacing with the x_i^* so that $a \le t_0 < x_1^* < t_1 < x_2^* < \ldots < x_n < t_n \le b$) such that

$$P(t_k) = \|P\|_w \qquad (k = 0, 1, \ldots, n).$$

Remark 6.1. Note that Theorem 13.7 from [6] is the unweighted case. Also note that here we departed from considering the signatures, but in case $r_j \in \mathbb{N}$ the analogous signed version can be seen easily. Also in fact one can assign signs to the factors of type $|t|^r$ arbitrarily, e.g. considering $|t|^r \operatorname{sign} t$ or, in case $r \in \mathbb{N}$, $|t|^r (\operatorname{sign} t)^r$. Then the arising signed problem can be easily seen to become equivalent to the absolute value version. This shows that not the sign changes, but the attainment of minimal norm, are the decisive properties of an extremizer.

Let us write $Q_{\mathbf{x}}$ for the GAP with $Q_{\mathbf{x}}(t) := \prod_{j=1}^{n} |t - x_j|^{r_j}$. According to the above there is an even more precise understanding of the situation with the Bojanov-Chebyshev Problem. Writing $M_j(\mathbf{x}) := \max_{I_j(\mathbf{x})} |Qw|$, we have the intertwining property that for *any two admissible node systems* $\mathbf{x}, \mathbf{y} \in \overline{S}$ there exist indices i, k with $M_i(\mathbf{x}) < M_i(\mathbf{y})$ and $M_k(\mathbf{x}) > M_k(\mathbf{y})$; in particular, for any node system $\mathbf{x} \neq \mathbf{x}^*$ there exist indices i, k with $M_i(\mathbf{x}) < C^w_{\mathbf{r}}[a,b]$ and $M_k(\mathbf{x}) > C^w_{\mathbf{r}}[a,b]$, so that these Chebyshev constants are bounded from both sides by interval maxima of an arbitrary node system: $\min_{i=0,\ldots n} M_i(\mathbf{x}) < C^w_{\mathbf{r}}[a,b] < \min_{k=0,\ldots n} M_k(\mathbf{x})$.

Let us emphasize that the above discussion not only generalizes the Bojanov-Chebyshev Problem to weighted GAPs, but is much more general, given that we can take any log-concave, monotone factors $G(t)$ in place of $|t|$ (corresponding to $K(t) := \log G(t)$ being more general than $\log |t|$).

6.2 Comparison of Chebyshev Constants of Union of Intervals

The above discussion of various versions of the Chebyshev constant might have been considered trivial, but if we move to non-convex sets, then the distinction between restricted and non-restricted Chebyshev constants becomes essential.

Let $E \subset \mathbb{R}$ be a compact set and $w \geq 0$ be a weight. As above, define the *restricted Bojanov-Chebyshev constant* $R^w_{\mathbf{r}}(E) := \min_{Q \in \mathcal{P}_{\mathbf{r}}(E)} \|Q\|_w$, and the *unrestricted Bojanov-Chebyshev constant* $C^w_{\mathbf{r}}(E) := \min_{Q \in \mathcal{P}_{\mathbf{r}}} \|Q\|_w$.

What we can do here is the following.

Theorem 6.2. *Let $k, n \in \mathbb{N}$, $a_1 < b_1 < a_2 < b_2 < \ldots < a_k < b_k$ be arbitrary real numbers, $E := \cup_{\ell=1}^{k} [a_\ell, b_\ell]$, and $\mathbf{r} \in (0,\infty)^n$ be an arbitrary exponent system. Then the restricted and unrestricted Chebyshev constants satisfy the inequality*

$$C^w_{\mathbf{r}}(E) \leq R^w_{\mathbf{r}}(E) \leq C(k,\mathbf{r}) C^w_{\mathbf{r}}(E), \tag{28}$$

where $C(k,\mathbf{r}) := 2^{\max\{r_{i_1}+\ldots+r_{i_{k-1}} \,:\, 1 \leq i_1 < \ldots < i_{k-1} \leq n\}}$. In particular, if $\mathbf{r} := \mathbf{1}$, i.e., $\mathcal{P}_{\mathbf{r}} = \mathcal{P}^1_n$, the family of the (absolute value of the) ordinary monic degree n algebraic polynomials, then $C^w_n(E) \leq R^w_n(E) \leq 2^{k-1} C^w_n(E)$, independently of the value of n.

Proof. As above, it is easy to see that for the unrestricted Chebyshev constant it suffices to consider polynomials with roots x_j all in the closed convex hull $E^* := \operatorname{con} E = [a_1, b_k]$ of E. So by compactness of E^* and upper semicontinuity of w there exists an extremizer, so that for this $P \in \mathcal{P}_{\mathbf{r}}(E^*)$ we have $C^w_{\mathbf{r}}(E) = \|P\|_w (:= \max_E |Pw|)$. We will construct a $Q \in \mathcal{P}_{\mathbf{r}}(E)$ with $\|Q\|_w \leq C(k,\mathbf{r})\|P\|_w$, so that minimizing over $\mathcal{P}_{\mathbf{r}}(E)$ would be seen to yield $R^w_{\mathbf{r}}(E) \leq C(k,\mathbf{r})\|P\|_w = C(k,\mathbf{r}) C^w_{\mathbf{r}}(E)$, as needed.

For convenience assume $E \subset [0,1]$, or for technical ease, even $a_1 = 0$ and $b_k = 1$. Take $K(t) := \log|t|$, $J(t) := \log(w\chi_E)$, where w is understood as defined all over $\mathbb{R}$ and χ_E being the indicator function of E (which is upper semicontinuous). Obviously, then we have for any $Q_{\mathbf{x}}(t) \in \mathcal{P}_{\mathbf{r}}(E^*)$ with root system $\mathbf{x} \in E^* = [0,1]$ that

$$\log \|Q_{\mathbf{x}}\|_w = \max_{t\in[0,1]} J(t) + \sum_{i=1}^{n} r_j K(t - x_j) = \max_{t \in [0,1]} F(\mathbf{x}, t) = \overline{m}(\mathbf{x}).$$

So in particular the minimality of $\|P\|_w$ over choices of zeroes $\mathbf{x} \in E^* = [0,1]$ translates to the statement that $P = Q_{\mathbf{w}}$ with some $\mathbf{w} \in \overline{S}$ being a minimax point of F. As has been shown above in Theorem 3.1, for the strictly concave and singular kernel function K, satisfying strict monotonicity (SM), we have that $\mathbf{w} \in Y$. As in the complementary intervals $J_\ell := (b_\ell, a_{\ell+1})$ $(\ell = 1, k-1)$ the indicator function χ_E vanishes, also $J = -\infty$, and subintervals of that complementary intervals are singular. So, no $I_i(\mathbf{w})$ can be subinterval of those complementary intervals, because $\mathbf{w}$ is non-singular. In other words, in such a complementary interval either there is no $w_i \in J_\ell$, or is at most one $w_i \in J_\ell$.

To construct our $Q_{\mathbf{x}}$, i.e., the corresponding $\mathbf{x}$ (with all $x_i \in E$) and $F(\mathbf{x}\cdot)$, we choose the node system $\mathbf{x}$ so that $x_i = w_i$ whenever $w_i \in E$, and $x_i = b_\ell$ or $x_i = a_{\ell+1}$, whichever is closer to w_i, in case $w_i \in J_\ell$ (and say $x_i := b_\ell$ if they are of equal distance, i.e., $w_i = \frac{b_\ell + a_{\ell+1}}{2}$).

Let us compare the pure sum of translates functions for $\mathbf{w}$ and $\mathbf{x}$. We get

$$f(\mathbf{x}, t) - f(\mathbf{w}, t) = \sum_{i \ w_i \notin E} \left(r_i K(t - x_i) - r_i K(t - w_i) \right).$$

If i is such that $w_i \notin E$, then $w_i \in J_\ell = (b_\ell, a_{\ell+1})$ for some $1 \le \ell \le k-1$), while for $t \in E$ we have either $t \le b_\ell$, or $t > a_{\ell+1}$. In case $x_i = b_\ell$ we have for all $0 \le t \le b_\ell$ that $K(t - x_i) < K(t - w_i)$ by monotonicity. Let now $a_{\ell+1} \le t \le 1$. Then $K(t - x_i) = \log(t - x_i) = \log(t - w_i + (w_i - x_i)) \le \log(2(t - w_i)) = \log 2 + K(t - w_i)$, for $t - w_i \ge a_{\ell+1} - w_i \ge \frac{a_{\ell+1} - b_\ell}{2} \ge w_i - x_i$ by construction. It follows that $K(t - x_i) \le \log 2 + K(t - w_i)$ for all $t \in E$. Similarly, it is easy to see that the same holds whenever $x_i = a_{\ell+1}$. Adding this for all indices i with $w_i \notin E$ we find

$$f(\mathbf{x}, t) - f(\mathbf{w}, t) = \sum_{i \ w_i \notin E} r_i \log 2 \le \log C(k, \mathbf{r}) \qquad (\forall t \in E).$$

Therefore, we also have $F(\mathbf{x}, t) \le \log C(k, \mathbf{r}) + F(\mathbf{w}, t)$ for all points $t \in [0, 1]$, where $J(t) \ne -\infty$. However, if $t \notin E$, then adding $J(t) = -\infty$ makes both sides $F(\mathbf{x}, t) = F(\mathbf{w}, t) = -\infty$, so that the same inequality reads as $-\infty \le -\infty$, and it remains in effect. Finally, taking maximums we thus get $\overline{m}(\mathbf{x}) \le \overline{m}(\mathbf{w}) + \log C(k, \mathbf{r})$.

The assertion follows. $\square$

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