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Abstract Heat transfer and cooling solutions play an important role in the design of gas turbine blades. However, the underlying mathematical coupling structures have not been thoroughly investigated. In a previous work, we successfully modeled a simplified version of this problem as an infinite-dimensional system. Here, we construct a spatial discretization for the above problem and investigate its properties. We show that the discrete system is less restrictive than the original infinite-dimensional system, suggesting something like a regularization effect due to discretization.

1 Introduction

The heat transfer within the blade of a gas turbine defines an important task within the simulation of gas turbines [1]. Here, we consider a simplified model system [4], where the metal of the turbine blade itself is reduced to a one-dimensional rod ($a < x_m < b$). One end of the rod is in contact with an external thermal reservoir representing the hot air driving the turbine, and the other end is in contact with the relatively cooler air flowing through the blade's cooling channel ($i < x_c < o$). The heat transfer along the rod is modelled as a simple heat equation (index 'm') with Robin boundary conditions (also known as convective boundary conditions). The cooling channels themselves are modelled as simple transport equations, divided into an incoming channel part (index 'in') and an outgoing channel part (index 'out'), both connected to the rod at the coupling point. Overall, we get a multiphysics model described by three coupled PDE models for the heat equation in the metal and the transport equations for the incoming and outcoming cooling air:

Heat equation of metal

$$\frac{\partial \vartheta_m}{\partial t} = \frac{k}{c_m}\frac{\partial^2 \vartheta_m}{\partial x_m^2}, \quad a < x_m < b, \quad t > 0, \tag{1a}$$

$$-k\,\frac{\partial \vartheta_m}{\partial x_m}(a,t) = \alpha_a\big(T_{\text{ext}}(t) - \vartheta_m(a,t)\big), \quad t > 0, \tag{1b}$$

$$-k\,\frac{\partial \vartheta_m}{\partial x_m}(b,t) = \alpha_b\big(\vartheta_m(b,t) - \vartheta_{\text{in}}(c,t)\big), \quad t > 0, \tag{1c}$$

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Transport of incoming cooling air

$$\frac{\partial \vartheta_{\mathrm{in}}}{\partial t} = -v\frac{\partial \vartheta_{\mathrm{in}}}{\partial x_c}, \quad i < x_c < c, \quad t > 0, \tag{2a}$$

$$\vartheta_{\mathrm{in}}(i,t) = T_{\mathrm{inlet}}(t), \quad t > 0, \tag{2b}$$

Transport of outgoing cooling air

$$\frac{\partial \vartheta_{\mathrm{out}}}{\partial t} = -v\frac{\partial \vartheta_{\mathrm{out}}}{\partial x_c}, \quad c < x_c < o, \quad t > 0, \tag{3a}$$

$$c_c v\big(\vartheta_{\mathrm{out}}(c,t) - \vartheta_{\mathrm{in}}(c,t)\big) = \alpha_b\big(\vartheta_m(b,t) - \vartheta_{\mathrm{in}}(c,t)\big), \quad t > 0. \tag{3b}$$

In a previous work [4] we have shown that this multiphyiscs system can be formulated as an infinite-dimensional Port-Hamiltonian system (pHS) [2, 3] Here, we will show that discretizing the three subsystems separately will define three index-0 port-Hamiltonian descriptor (pHDAE) systems [6] (E is the identity), which can be combined to form a single pHDAE system when properly coupled. pHDAE systems generalize the PHS setting from ODEs to DAEs. For an ease of reference we recall

Definition 1 (port-Hamiltonian descriptor system, pHDAE [6]). Let $\mathscr{X} \subset \mathbb{R}^n$ the state space, $x(t) \in \mathscr{X}$ the state, $u(t),\ y(t) \in \mathbb{R}^m$ the input and output, $E \in \mathbb{R}^{l\times n}$ the flow matrix, $z \in \mathbb{R}^l$ the efforts, $J,R \in \mathbb{R}^{l\times l}$ the structure and dissipation matrices, $B,P \in \mathbb{R}^{l\times m}$ the port matrices and $S,N \in \mathbb{R}^{m\times m}$ the feed-through matrices.

Then the system of differential (-algebraic) equations

$$E\dot{x} = (J-R)z + (B-P)u, \tag{4a}$$

$$y = (B+P)^\top z + (S-N)u, \tag{4b}$$

associated with the Hamiltonian function $H \in \mathscr{C}^1(\mathscr{X},\mathbb{R})$, is a *port-Hamiltonian descriptor system*, if the following properties hold:

1. The extended structure and dissipation matrices $\Gamma, W \in \mathbb{R}^{l+m\times l+m}$ defined as

$$\Gamma = \begin{pmatrix} J & B \\ -B^\top & N \end{pmatrix}, \quad W = \begin{pmatrix} R & P \\ P^\top & S \end{pmatrix} \tag{5}$$

   satisfy $\Gamma = -\Gamma^\top$ and $W = W^\top \geq 0$, i.e. W is positive semi-definite.
2. $\frac{\partial H}{\partial x} = E^\top z$.

2 Discretization of the Heat Equation

We choose $I_m + 1$ grid points $x_0 = a, \ldots, x_{I_m} = b$ and a step size $h = (b-a)/I_m$. We discretize the spatial derivative in (1a) by the standard second order difference quotient at $x_1, \ldots, x_{I_m - 1}$. Denoting the temperature at the grid points x_i by

$T_i(t) = \vartheta(x_i,t)$, both boundary conditions (1a), (1c) can be solved for T_0 and T_{Im}. Summing up, we get with $T^{(m)} := (T_1,\dots,T_{I_m-1})^\top$

$$\dot{T}^{(m)} = \underbrace{\frac{k}{c_m h^2}\left(\frac{1}{1+\frac{h}{k}\alpha_a}\left(e_1 e_{I_m-1}^\top + e_{I_m-1}e_1^\top\right) + \mathrm{tridiag}(1,-2,1)\right)}_{A_m :=}\underbrace{T^{(m)}}_{z:=} \tag{6}$$
$$+ \underbrace{\frac{k}{c_m h^2}\left(e_1\, e_{I_m-1}\right)}_{B:=}\underbrace{\begin{pmatrix} T_{\mathrm{ext}} \\ \vartheta_{\mathrm{in}}(c,t)\end{pmatrix}}_{u:=}.$$

With A_m, B, z and u defined above, and setting $J=0$, $R=-A_m$, $P=0$, $S=0$, $N=0$, we get the phDAE structure of type (4). Condition (5), i.e. $W \geq 0$, holds as R is positive semi-definite due to the Gershgorin circle theorem for all physically meaningful (i.e. positive) parameters h,k and α_a, α_b.

3 Discretization of the Transport Equations

To discretize the transport equations (2a), (3a) with respect to space, we choose I_c+1 grid points $x_0,\dots,x_{I_c}$ and a first-order upwind discretization (for $v \geq 0$). Replacing T_0 by the inlet boundary condition (2b), we arrive at the following semi-discrete system with $T^{(\mathrm{in})} := (T_1,\dots,T_{I_c})^\top$:

$$\dot{T}^{(\mathrm{in})} = \underbrace{-\frac{v}{h}\,\mathrm{tridiag}(-1,1,0)}_{A_c :=}\underbrace{T^{(\mathrm{in})}}_{z:=} + \frac{v}{h} e_1 \cdot T_{\mathrm{inlet}}. \tag{7}$$

In order to get a pHDAE structure, we split the matrix of (7) into $J = \frac{1}{2}(A_c - A_c^\top)$ and $R = -\frac{1}{2}(A_c + A_c^\top)$, cf. (4) and set

$$B^\top = \frac{v}{h}\left(\tfrac{1}{2}\ 0\ \dots\ 0\ \tfrac{1}{2}\right), \quad P^\top = \frac{v}{h}\left(-\tfrac{1}{2}\ 0\ \dots\ 0\ \tfrac{1}{2}\right), \quad S=\kappa, \quad N=0, \quad u = T_{\mathrm{inlet}},$$

with $\kappa \geq 1$. With these choices, we get

$$W = \begin{pmatrix} R & P \\ P^\top & S \end{pmatrix} = \frac{v}{h}\begin{pmatrix} \mathrm{tridiag}(-\frac{1}{2},1,-\frac{1}{2}) & \frac{1}{2}(-e_1+e_{I_c}) \\ \frac{1}{2}(-e_1+e_{I_c})^\top & \kappa \end{pmatrix}.$$

Again, the Gershgorin circle theorem yields the positive semi-definiteness of W.

For the outgoing cooling air (3a) we proceed analogously, but replace the coupling condition (3b) with a simple input similar to equation (2b). Equation (3b) is later included as a coupling condition in the coupled system in Section 4. We then arrive at the semi-discrete system

$$\dot{T}^{(\text{out})} = -\frac{v}{h}\,\text{tridiag}(-1,1,0)T^{(\text{out})} + \frac{v}{h}e_1 \cdot T^{(\text{out})}_{\text{inlet}}, \tag{8}$$

with $T^{(\text{out})} := (T_1, \dots, T_{I_c})^\top$. Making the same choices as above, it is obvious that this is also a pHDAE.

4 The Coupled Discrete System

In the previous sections we have formulated the semi-discretized subsystems as three port-Hamiltonian systems of the type (with $x \in \{\text{m}, \text{in}, \text{out}\}$):

$$\begin{aligned} \dot{T}^{(x)} &= (J^{(x)} - R^{(x)})T^{(x)} + (B^{(x)} - P^{(x)})u^{(x)}, \\ y^{(x)} &= (B^{(x)} + P^{(x)})^\top T^{(x)} + (S^{(x)} - N^{(x)})u^{(x)}. \end{aligned}$$

According to Mehrmann and Morandin [6], an interconnection of port-Hamiltonian descriptor systems (pHDAEs) (see Definition 1) is again a pHDAE if we can find an interconnection relation satisfying

$$Mu + Ny = 0, \tag{9}$$

with any matrices M and N. Note, however, that this does not reduce the number of inputs and outputs in general.

The resulting pHDAE then has the form, cf. (5)

$$\begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{T} \\ \dot{\hat{u}} \\ \dot{\hat{y}} \end{pmatrix} = \begin{pmatrix} \Gamma - W & 0 & 0 \\ & I & -M^\top \\ 0 & -I & 0 & -N^\top \\ 0 & M & N & 0 \end{pmatrix} \begin{pmatrix} T \\ \hat{u} \\ \hat{y} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ I \\ 0 \end{pmatrix} u, \tag{10}$$

$$y = \hat{y}, \tag{11}$$

with

$$T(t) = \begin{pmatrix} T^{(m)}(t) \\ T^{(\text{in})}(t) \\ T^{(\text{out})}(t) \end{pmatrix} \in \mathbb{R}^{I_m - 1 + 2I_c}, \qquad \hat{u}(t), \hat{y}(t) \in \mathbb{R}^4,$$

$$\Gamma - W = \Pi \ \text{diag}\big(\Gamma^{(m)} - W^{(m)}, \Gamma^{(\text{in})} - W^{(\text{in})}, \Gamma^{(\text{out})} - W^{(\text{out})}\big)\Pi^\top,$$

as in Definition 1 with a permutation matrix

$$\Pi = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}.$$

It is worth mentioning that the additionally introduced variables $\hat{u}$ and $\hat{y}$ are just copies of the inputs u and outputs y.

Note that the above property makes no statement about the index of the resulting pHDAE. While this is common also for coupling "regular" ODEs, it is important to keep in mind, since even when all subsystems are index-0 (i.e. ODEs), the coupled system can have a higher index.

We can now check whether the coupled system (10) exhibits the form (9). The inputs u and outputs y of the coupled system (10) are

$$\text{inputs:} \quad u = \begin{pmatrix} u_1^{(m)} \\ u_2^{(m)} \\ u^{(\text{in})} \\ u^{(\text{out})} \end{pmatrix} = \begin{pmatrix} T_{\text{ext}}^{(m)} \\ \vartheta_{\text{in}}(c,t) \\ T_{\text{inlet}}^{(\text{in})} \\ T_{\text{inlet}}^{(\text{out})} \end{pmatrix}, \tag{12}$$

$$\text{outputs:} \quad y = \begin{pmatrix} y_1^{(m)} \\ y_2^{(m)} \\ y^{(\text{in})} \\ y^{(\text{out})} \end{pmatrix} = \begin{pmatrix} \frac{k}{c_m h^2} \frac{1}{1+\frac{1}{\frac{h}{k}\alpha_a}} T_1^{(m)} \\ \frac{k}{c_m h^2} \frac{1}{1+\frac{1}{\frac{h}{k}\alpha_b}} T_{I_m-1}^{(m)} \\ \frac{v}{h} T_{I_c}^{(\text{in})} + \kappa T_{\text{inlet}}^{(\text{in})} \\ \frac{v}{h} T_{I_c}^{(\text{out})} + \kappa T_{\text{inlet}}^{(\text{out})} \end{pmatrix}, \qquad y^{(m)} = \begin{pmatrix} y_1^{(m)} \\ y_2^{(m)} \end{pmatrix}. \tag{13}$$

The input of the heat equation still references $\vartheta_{\text{in}}(c,t)$, a quantity of the continuous system. From equation (2a) as well as equation (7), we can see that it is equivalent to $T_{I_c}^{(\text{in})}$ of the discrete cooling channel:

$$u_2^{(m)} = \vartheta_{\text{in}}(c,t) = T_{I_c}^{(\text{in})} = \frac{h}{v} y^{(\text{in})} - \frac{h\kappa}{v} u^{(\text{in})}.$$

Equation (3b) yields the coupling condition

$$c_c v T_0^{(\text{out})} - c_c v T_{I_c}^{(\text{in})} = \alpha_b T_{I_m}^{(m)} - \alpha_b T_{I_c}^{(\text{in})}$$

to the outgoing cooling channel, i.e. using the notation (12), (13) and the explicit formula of T_{I_m}

$$c_c v u^{(\text{out})} - c_c v \frac{h}{v}\left(y^{(\text{in})} - \kappa u^{(\text{in})}\right) = c_m h y_2^{(m)} + \frac{\alpha_b}{1+\frac{1}{\frac{h}{k}\alpha_b}} u_2^{(m)} - \alpha_b \frac{h}{v}\left(y^{(\text{in})} - \kappa u^{(\text{in})}\right).$$

Together this leads to an interconnection relation of the form (9)

$$\underbrace{\begin{pmatrix} 0 & 1 & \frac{h\kappa}{v} & 0 \\ 0 & -\frac{\alpha_b}{1+\frac{1}{\frac{h}{k}\alpha_b}} & (c_c h - \frac{h}{v}\alpha_b)\kappa & c_c v \end{pmatrix}}_{M} u + \underbrace{\begin{pmatrix} 0 & 0 & -\frac{h}{v} & 0 \\ 0 & -c_m h & \alpha_b \frac{h}{v} - c_c h & 0 \end{pmatrix}}_{N} y = 0,$$

and therefore, the considered coupled system is a pHDAE.

Remark 1. However, the above model does not define a Dirac structure for (y,u) and is therefore not an energy-conserving coupling in terms of the quantity acting as energy in the Hamiltonian under consideration, i.e., not the physical energy. Consequently, the criteria given in [6] for index reduction and row operations to reduce the system are not satisfied.

5 Conclusion

We have found that the multiphysics approach to discretization before coupling works quite well and requires only a small change in our transport equations. Interestingly, unlike the continuous system, it has no constraints on the parameters, but leads to a pHDAE that potentially has a nonzero index.

In future work, following the ideas of Kotyczka and Lefèvre [5], we will consider our multiphysics problem as a discrete-time port Hamiltonian system arising from a discrete-time Dirac structure, that is obtained by a symplectic Gauss-Legendre collocation method.

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