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Abstract. The control algebraic Riccati equation is studied for a class of systems with unbounded control and observation operators. Using a dichotomy property of the associated Hamiltonian operator matrix, two invariant graph subspaces are constructed which yield a nonnegative and a nonpositive solution of the Riccati equation. The boundedness of the nonnegative solution and the exponential stability of the associated feedback system is proved for the case that the generator of the system has a compact resolvent.

Keywords. algebraic Riccati equation, Hamiltonian matrix, dichotomous operator, invariant subspace, graph subspace.

Mathematics Subject Classification. Primary 47N70; Secondary 47A15, 47A62, 47B44.

1 Introduction

In systems theory, the algebraic Riccati equation

$$A^*X + XA - XBB^*X + C^*C = 0 \tag{1}$$

plays an important role in many areas. One example is the problem of linear quadratic optimal control where a selfadjoint nonnegative solution is of particular interest. For infinite-dimensional systems such a solution is often constructed in parallel to a solution of the optimal control problem. This has been done for different kinds of linear systems, e.g. in [6, 15, 16, 17, 20].

On the other hand, the Riccati equation is closely connected to the so-called Hamiltonian operator matrix

$$T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}. \tag{2}$$

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An operator X is a solution of (1) if and only if its associated graph $\mathcal{R}\left(\begin{smallmatrix} I \\ X \end{smallmatrix}\right)$ is an invariant subspace of the Hamiltonian. In the finite-dimensional case, this connection has lead to a complete characterisation of all solutions of the Riccati equation, see e.g. [3, 13] and the references therein. For infinite-dimensional linear systems, this "Hamiltonian approach" to the Riccati equation has been studied under different boundedness assumptions on the control and observation operators B, C and for different classes of Hamiltonians with respect to their spectral properties. For the case that B, C are bounded and have finite rank, a characterisation of all nonnegative solutions of (1) has been obtained in [5]. In [12] the class of Hamiltonians possessing a Riesz basis of eigenvectors was considered for systems with bounded B and C, and characterisations of solutions and their properties were obtained. In [22, 23] this was extended to unbounded B, C and to more general kinds of Riesz bases. The Riesz basis setting typically leads to the existence of an infinite number of solutions of (1).

However, the existence of a Riesz basis of eigenvectors of T is a strong assumption and might be to restrictive. An often weaker condition is that T is *dichotomous*. This means that the spectrum of T does not contain points in a strip around the imaginary axis and that there exist invariant subspaces corresponding to the parts of the spectrum in the left and right half-plane, respectively. Dichotomous Hamiltonians with bounded B and C were considered in [4, 14] and the existence of a nonnegative and a nonpositive solution of (1) was shown. This result was extended in [18] to a setting where BB^* and C^*C are unbounded closed operators acting on the state space. This however excludes PDE systems with control or observation on the boundary. In this article we will construct a nonnegative and a nonpositive solution of (1) for a class of dichotomous Hamiltonians which allows for systems with boundary control and observation.

In the infinite-dimensional setting the Hamiltonian approach typically leads to unbounded solutions of the Riccati equation in the first instance, see [14, 18, 22, 23]. This means that the boundedness of solutions is an additional question now. Moreover, due to the unboundedness of the operators in (1), additional care has to be taken to exactly determine the domain on which the Riccati equation actually holds.

Our setting is as follows: Let H, U, Y be Hilbert spaces. Let A be a *quasi-sectorial* operator on H, i.e., $A - \mu$ is sectorial for some $\mu \geq 0$. This means that A may have spectrum on and to the right of the imaginary axis up to the line $\operatorname{Re} z = \mu$ and that A generates an analytic semigroup. The operator A determines two scales of Hilbert spaces $\{H_s\}$ and $\{H_s^{(*)}\}$,

$$H_s \subset H \subset H_{-s}, \qquad H_s^{(*)} \subset H \subset H_{-s}^{(*)}, \qquad s > 0,$$

whose norms are given by $\|x\|_s = \|(I+AA^*)^{\frac{s}{2}}x\|$ and $\|x\|_s^{(*)} = \|(I+A^*A)^{\frac{s}{2}}x\|$. If A is a normal operator, then both scales coincide with the usual fractional power spaces, $H_s = H_s^{(*)} = \mathcal{D}(|A|^s)$. In general however, the two scales are different and must be distinguished. Our assumption on the control and observation operators is now

$$B \in L(U, H_{-r}), \qquad C \in L(H_s^{(*)}, Y)$$

where $r, s \geq 0$ and $r + s < 1$. Examples of systems with boundary control and observation which fit into this setting may be found e.g. in [19, 23]. The adjoints of B and C are defined using a duality relation in each of the scales of Hilbert spaces, which is induced by the inner product $(\cdot|\cdot)$ on H: the mapping $y \mapsto (\cdot|y)$, $y \in H$, extends by continuity to isometric isomorphisms $H_{-r} \to (H_r)'$ and $H_{-s}^{(*)} \to (H_s^{(*)})'$. This is also referred to as duality with respect to the pivot space H. With this duality we obtain

$$BB^* : H_r \to H_{-r}, \qquad C^*C : H_s^{(*)} \to H_{-s}^{(*)}.$$

The Hamiltonian is now considered as an unbounded operator

$$T_0 = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}$$

acting on $V_0 = H_{-r} \times H_{-s}^{(*)}$, with appropriate extensions of the operators A and A^*. We prove that if

(a) $\sigma(A) \cap i\mathbb{R} = \varnothing$, or

(b) A has a compact resolvent and

$$\ker(A - it) \cap \ker C = \ker(A^* + it) \cap \ker B^* = \{0\}, \qquad t \in \mathbb{R},$$

then T_0 is dichotomous and hence there is a decomposition $V_0 = V_{0+} \oplus V_{0-}$ into T_0-invariant subspaces such that $\sigma(T_0|_{V_{0\pm}}) \subset \mathbb{C}_\pm$, i.e., V_{0-} corresponds to the spectrum in the open left half-plane $\mathbb{C}_-$ and V_{0+} to the one in the open right half-plane $\mathbb{C}_+$. For the rest of this introduction we assume that (a) or (b) is satisfied.

We derive that $V_{0\pm}$ are graph subspaces in two different situations. In the first we assume that

$$\bigcap_{\lambda \in i\mathbb{R} \cap \varrho(A^*)} \ker B^*(A^* - \lambda)^{-1} = \{0\}. \tag{3}$$

Then $V_{0\pm}$ are graphs, $V_{0\pm} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{0\pm} \end{smallmatrix}\right)$, of closed, possibly unbounded operators $X_{0\pm} : \mathcal{D}(X_{0\pm}) \subset H_{-r} \to H_{-s}^{(*)}$. If in addition

$$\bigcap_{\lambda \in i\mathbb{R} \cap \varrho(A)} \ker C(A - \lambda)^{-1} = \{0\}, \tag{4}$$

then $X_{0\pm}$ are also injective and hence $V_{0\pm} = \mathcal{R}\left(\begin{smallmatrix} Y_{0\pm} \\ I \end{smallmatrix}\right)$ with $Y_{0\pm} = X_{0\pm}^{-1}$. The conditions (3) and (4) were also used in [14, 18, 22, 23], sometimes in different but equivalent forms; (3) amounts to the approximate controllability, (4) to the approximate observability of the system (A, B, C), see [14, 23]. In the second situation, we assume that $\sigma(A) \subset \mathbb{C}_-$. Hence the semigroup generated by A is exponentially stable. In this case we obtain $V_{0-} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{0-} \end{smallmatrix}\right)$ and $V_{0+} = \mathcal{R}\left(\begin{smallmatrix} Y_{0+} \\ I \end{smallmatrix}\right)$ where, again, X_{0-} and Y_{0+} are closed and possibly unbounded, but not necessarily injective.

Under the additional assumption that A has a compact resolvent, we can show that X_{0-} and Y_{0+} are bounded. More precisely, if A has a compact resolvent and either (3) and (4) or $\sigma(A) \subset \mathbb{C}_-$ hold, then $X_{0-} \in L(H_{-r}, H_{-s}^{(*)})$, $Y_{0+} \in L(H_{-s}^{(*)}, H_{-r})$. In this case we also obtain that X_{0-} is a solution of the Riccati equation on the domain $H_{1-r}^{(*)}$ and that the operator $A - BB^*X_{0-}$ associated with the closed loop system generates an exponentially stable semigroup on H_{-r}.

In [14, 18] the two solutions of the Riccati equation are selfadjoint operators on H, one being nonnegative, the other nonpositive. Here the situation is more involved. While $X_{0\pm}$ can be restricted to symmetric operators on H that are nonnegative and nonpositive, respectively, selfadjoint restrictions need not exist in general. More specifically, $X_{0\pm}$ admit restrictions to closed operators $X_{1\pm}$ from $H_s^{(*)}$ to H_r such that

$$X_{1\pm} \subset X_{1\pm}^* = X_{0\pm},$$

where the adjoint is computed with respect to the duality in the scales $\{H_s\}$ and $\{H_s^{(*)}\}$. In particular, $X_{1\pm}$ is symmetric when considered as an operator on H. If $X_{M\pm}$ is the closure of $X_{1\pm}$ as an operator on H and $X_\pm$ is the part of $X_{0\pm}$ in H, then

$$X_{1\pm} \subset X_{M\pm} \subset X_{M\pm}^* = X_\pm \subset X_{0\pm},$$

X_{M-} is symmetric and nonnegative, X_{M+} is symmetric and nonpositive. We can also consider the restriction of the Hamiltonian T_0 to an operator T on $V = H \times H$. Then T has invariant subspaces $V_\pm$ corresponding to the spectrum in $\mathbb{C}_\pm$ and $V_\pm$ is in fact the graph of $X_\pm$. Note here that T will in general not be dichotomous since $V_+ \oplus V_-$ will only be dense in V. Also note that the above statements hold for X_{0-} and its restrictions provided that $V_{0-} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{0-} \end{smallmatrix}\right)$, i.e., if (3) or $\sigma(A) \subset \mathbb{C}_-$ holds. Likewise the statements for the restrictions of X_{0+} hold if $V_{0+} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{0+} \end{smallmatrix}\right)$, i.e., if (3) is true.

Finally assume that $\max\{r, s\} < \frac{1}{2}$. In this case T is in fact dichotomous and we obtain $X_{M\pm} = X_\pm$. Hence X_- is selfadjoint nonnegative, X_+ is selfadjoint nonpositive. If in addition A has a compact resolvent, then X_- is also bounded and a restriction of $A - BB^*X_{0-}$ generates an exponentially stable semigroup on H.

This article is organised as follows: In section 2 we collect some general operator theoretic statements, in particular about dichotomous, sectorial and bisectorial operators. The scales of Hilbert spaces are defined in section 3 and their basic properties are recalled, in particular concerning interpolation. Section 4 contains the definition of the Hamiltonian and basic facts about its spectrum. Moreover we describe the symmetry of the Hamiltonian with respect to two indefinite inner products, which will be essential in sections 6 and 7. In section 5 we prove the bisectoriality and dichotomy of T_0 and T using interpolation in the Hilbert scales. The graph subspace properties of $V_{0\pm}$ and $V_\pm$ are derived in section 6 as well as the boundedness of X_{0-} and Y_{0+}. The symmetry relations between $X_{0\pm}$ and its restrictions are the subject of section 7, while the Riccati equation and the closed loop operator are studied in section 8.

A few remarks on the notation: We denote the domain of a linear operator T by $\mathcal{D}(T)$, its range by $\mathcal{R}(T)$, the spectrum by $\sigma(T)$ and the resolvent set by $\varrho(T)$. The space of all bounded linear operators from a Banach space V to another Banach space W is denoted by $L(V, W)$. For the operator norm of $T \in L(V, W)$ we occasionally write $\|T\|_{V\to W}$ to make the dependence on the spaces V and W explicit.

2 Preliminaries

Lemma 2.1 *Let T be a linear operator on a Banach space V. Let W be another Banach space such that $\mathcal{D}(T) \subset W \subset V$ and such that the imbedding $W \hookrightarrow V$ is continuous. Let $\lambda \in \varrho(T)$.*

(a) The resolvent $(T-\lambda)^{-1}$ yields a bounded operator from V into W, i.e., $(T-\lambda)^{-1} \in L(V, W)$.

(b) If the imbedding $W \hookrightarrow V$ is compact, then the resolvent is compact as an operator from V into V, i.e., $(T-\lambda)^{-1} : V \to V$ is compact.

Proof. (a) The assumption $\mathcal{D}(T) \subset W$ implies that $(T-\lambda)^{-1}$ maps V into W. The operator $(T-\lambda)^{-1} : V \to W$ is thus well defined, and by the closed graph theorem it suffices to show that it is closed. Let $x_n \in V$ with $x_n \to x$ in V and $(T-\lambda)^{-1}x_n \to y$ in W as $n \to \infty$. Then $(T-\lambda)^{-1}x_n \to y$ in V by the continuity of the imbedding $W \hookrightarrow V$, and also $(T-\lambda)^{-1}x_n \to (T-\lambda)^{-1}x$ in V since the resolvent is a bounded operator on V. Consequently $(T-\lambda)^{-1}x = y$ and hence $(T-\lambda)^{-1} : V \to W$ is closed.

(b) This follows immediately from (a) by composing the bounded operator $(T-\lambda)^{-1} : V \to W$ with the compact imbedding $W \hookrightarrow V$.

□

Lemma 2.2 *Let T_0 be a linear operator on a Banach space V_0. Let V be another Banach space satisfying $\mathcal{D}(T_0) \subset V \subset V_0$ with continuous imbedding*

$V \hookrightarrow V_0$. Let T be the part of T_0 in V, i.e., T is the restriction of T_0 to the domain

$$\mathcal{D}(T) = \{x \in \mathcal{D}(T_0) \,|\, T_0 x \in V\},$$

considered as an operator $T : \mathcal{D}(T) \subset V \to V$. Then

(a) $\sigma_p(T) = \sigma_p(T_0)$,

(b) $\varrho(T_0) \subset \varrho(T)$ and $(T-\lambda)^{-1} = (T_0-\lambda)^{-1}|_V$ for all $\lambda \in \varrho(T_0)$,

(c) if $\mathcal{D}(T_0)$ is dense in V, V is dense in V_0 and $\varrho(T_0) \neq \varnothing$, then T is densely defined.

Proof. (a) This is clear, since $\mathcal{D}(T_0) \subset V$ implies that all eigenvectors of T_0 belong to V.

(b) Let $\lambda \in \varrho(T_0)$. Then $T-\lambda : \mathcal{D}(T) \to V$ is injective as a restriction of $T_0-\lambda$. Let $y \in V$ and set $x = (T_0-\lambda)^{-1}y$. Then $x \in \mathcal{D}(T_0)$, which implies $x \in V$ and $T_0 x = \lambda x + y \in V$. Therefore $x \in \mathcal{D}(T)$ and $(T-\lambda)x = y$. Hence $T-\lambda : \mathcal{D}(T) \to V$ is bijective with inverse $(T-\lambda)^{-1} = (T_0-\lambda)^{-1}|_V$. Since $(T_0-\lambda)^{-1} \in L(V_0, V)$ by Lemma 2.1 and since $V \hookrightarrow V_0$ is continuous, we obtain $(T-\lambda)^{-1} \in L(V)$ and thus $\lambda \in \varrho(T)$.

(c) Let $\lambda \in \varrho(T_0)$. Since $(T_0-\lambda)^{-1} \in L(V_0, V)$ and since $V \subset V_0$ is dense, we get that $\mathcal{D}(T) = (T_0-\lambda)^{-1}(V)$ is dense in $\mathcal{D}(T_0) = (T_0-\lambda)^{-1}(V_0)$ with respect to the norm in V. As $\mathcal{D}(T_0) \subset V$ is dense, we conclude that $\mathcal{D}(T) \subset V$ is dense.

□

Let us recall the definitions and basic properties of sectorial, bisectorial and dichotomous operators. For more details we refer the reader to [7, 8, 21]. We denote by

$$\Sigma_{\frac{\pi}{2}+\theta} = \left\{\lambda \in \mathbb{C} \setminus \{0\} \,\middle|\, \arg\lambda \in \left[-\frac{\pi}{2} - \theta, \frac{\pi}{2} + \theta\right]\right\} \tag{5}$$

the sector containing the positive real axis with semi-angle $\frac{\pi}{2} + \theta$. We also consider the corresponding bisector around the imaginary axis

$$\Omega_\theta = \Sigma_{\frac{\pi}{2}+\theta} \cap \left(-\Sigma_{\frac{\pi}{2}+\theta}\right) = \left\{\lambda \in \mathbb{C} \setminus \{0\} \,\middle|\, |\arg\lambda| \in \left[\frac{\pi}{2} - \theta, \frac{\pi}{2} + \theta\right]\right\}. \tag{6}$$

For sectorial operators we adopt the convention that the spectrum is contained in a sector in the left half-plane:

Definition 2.3 A densely defined operator S on a Banach space V is called *sectorial* if there exist $\theta \geq 0$ and $M > 0$ such that $\Sigma_{\frac{\pi}{2}+\theta} \subset \varrho(S)$ and

$$\|(S-\lambda)^{-1}\| \leq \frac{M}{|\lambda|} \qquad \text{for all } \lambda \in \Sigma_{\frac{\pi}{2}+\theta}. \tag{7}$$

S is called *quasi-sectorial* if $S - \mu$ is sectorial for some $\mu \in \mathbb{R}$.

If (7) holds for some θ, then it also holds for some $\theta' > \theta$ (with a typically larger constant M). We may therefore always assume that $\theta > 0$. A densely defined operator is sectorial if and only if it is the generator of a bounded analytic semigroup. S is quasi-sectorial if and only if there exist $\theta, M, \rho > 0$ such that[1] $\Sigma_{\frac{\pi}{2}+\theta} \setminus B_\rho(0) \subset \varrho(S)$ and

$$\|(S-\lambda)^{-1}\| \le \frac{M}{|\lambda|} \qquad \text{for all } \lambda \in \Sigma_{\frac{\pi}{2}+\theta},\ |\lambda| \ge \rho. \tag{8}$$

If S is a (quasi-) sectorial operator on a Hilbert space, then its adjoint S^* is also (quasi-) sectorial with the same constants θ, M (and μ, ρ).

Definition 2.4 A linear operator S on V is called *bisectorial* if $i\mathbb{R}\setminus\{0\} \subset \varrho(S)$ and

$$\|(S-\lambda)^{-1}\| \le \frac{M}{|\lambda|} \quad \text{for all } \lambda \in i\mathbb{R} \setminus \{0\} \tag{9}$$

with some constant $M > 0$. S is *almost bisectorial* if $i\mathbb{R}\setminus\{0\} \subset \varrho(S)$ and there exist $0 < \beta < 1$, $M > 0$ such that

$$\|(S-\lambda)^{-1}\| \le \frac{M}{|\lambda|^\beta} \quad \text{for all } \lambda \in i\mathbb{R} \setminus \{0\}. \tag{10}$$

If S is bisectorial, then for some $\theta > 0$ the bisector Ω_θ is contained in the resolvent set $\varrho(S)$, and an estimate (9) holds for all $\lambda \in \Omega_\theta$. Similarly, for an almost bisectorial operator a parabola shaped region around the imaginary axis belongs to $\varrho(S)$. If S is bisectorial and $0 \in \varrho(S)$, then S is almost bisectorial too, for any $0 < \beta < 1$. Note that an almost bisectorial operator always satisfies $0 \in \varrho(S)$, while for a bisectorial operator $0 \in \sigma(S)$ is possible.

Definition 2.5 An operator S on a Banach space V is called *dichotomous* if $i\mathbb{R} \subset \varrho(S)$ and there exist closed S-invariant subspaces $V_\pm$ of V such that $V = V_+ \oplus V_-$ and

$$\sigma(S|_{V_+}) \subset \mathbb{C}_+, \quad \sigma(S|_{V_-}) \subset \mathbb{C}_-.$$

S is *strictly dichotomous* if in addition $\|(S|_{V_\pm} - \lambda)^{-1}\|$ is bounded on $\mathbb{C}_\mp$.

The additional condition for strict dichotomy ensures that the invariant subspaces $V_\pm$ are uniquely determined by the operator.

One of the main results from [21] is that if the resolvent of an operator S is uniformly bounded along the imaginary axis, then S possesses invariant subspaces $V_\pm$ having the same properties as in Definition 2.5, with the exception that $V_+ \oplus V_-$ might be a proper subspace of V, i.e., S need not necessarily be dichotomous. In this case, the corresponding projections are unbounded. We summarise the results for the almost bisectorial situation here.

[1] $B_r(z) \subset \mathbb{C}$ denotes the open disc with radius r centred at z.

Let S be an almost bisectorial operator. Then there exists $h > 0$ such that $\{\lambda \in \mathbb{C} \,|\, |\operatorname{Re}\lambda| \le h\} \subset \varrho(S)$ and the integrals

$$L_\pm = \frac{\pm 1}{2\pi i}\int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda}(S-\lambda)^{-1}\,d\lambda \tag{11}$$

define bounded operators $L_\pm \in L(V)$ which satisfy

$$L_+L_- = L_-L_+ = 0, \qquad L_+ + L_- = S^{-1}, \tag{12}$$

see [21, §5].

Theorem 2.6 *Let S be almost bisectorial on the Banach space V. Then $P_\pm = SL_\pm$ are closed complementary projections, the subspaces $V_\pm = \mathcal{R}(P_\pm)$ are closed, S- and $(S-\lambda)^{-1}$-invariant, and*

(a) $\sigma(S) = \sigma(S|_{V_+}) \cup \sigma(S|_{V_-})$ *with* $\sigma(S|_{V_\pm}) \subset \mathbb{C}_\pm$,

(b) $\|(S|_{V_\pm} - \lambda)^{-1}\|$ *is bounded on* $\mathbb{C}_\mp$,

(c) $\mathcal{D}(S) \subset \mathcal{D}(P_\pm) = V_+ \oplus V_- \subset V$,

(d) $I = P_+ + P_-$ *on* $\mathcal{D}(P_\pm)$.

The projections satisfy the identity

$$P_+x - P_-x = \frac{1}{\pi i}\int_{-i\infty}^{i\infty\,\prime}(S-\lambda)^{-1}x\,d\lambda, \qquad x \in \mathcal{D}(S), \tag{13}$$

where the prime denotes the Cauchy principal value at infinity. Moreover, S is strictly dichotomous if and only if $P_\pm \in L(V)$.

Proof. All assertions follow from Theorem 4.1 and 5.6 as well as Corollary 4.2 and 5.9 in [21]. □

Note that $P_\pm$ are closed complementary projections in the sense that they are closed operators on V and satisfy $\mathcal{R}(P_\pm) \subset \mathcal{D}(P_\pm)$, $P_\pm^2 = P_\pm$, $\mathcal{D}(P_+) = \mathcal{D}(P_-)$ and $I = P_+ + P_-$ on $\mathcal{D}(P_\pm)$. In other words, $P_\pm$ are complementary projections in the algebraic sense acting on the space $\mathcal{D}(P_+) = \mathcal{D}(P_-)$. Since S is invertible, we obtain

$$V_\pm = \mathcal{R}(P_\pm) = \ker P_\mp = \ker L_\mp. \tag{14}$$

The case that $P_\pm$ are unbounded may occur even for bisectorial and almost bisectorial S, see Examples 5.8 and 8.2 in [21].

For use in later sections, we collect some properties of the spaces $\mathcal{R}(L_\pm)$:

Lemma 2.7 *Let S be an almost bisectorial operator. Then the inclusions*

$$\mathcal{D}(S) \cap V_\pm \subset \mathcal{R}(L_\pm) \subset V_\pm \tag{15}$$

hold, in particular $\overline{\mathcal{R}(L_\pm)} \subset V_\pm$. In addition,

(a) if S is also densely defined, then $\overline{\mathcal{D}(S) \cap V_\pm} = \overline{\mathcal{R}(L_\pm)}$;

(b) if S is densely defined and strictly dichotomous, then $\mathcal{D}(S) \cap V_\pm = \mathcal{R}(L_\pm)$ and $\overline{\mathcal{R}(L_\pm)} = V_\pm$.

Proof. From (12) and the invariance properties of $V_\pm$ we get

$$\mathcal{D}(S) \cap V_\pm = S^{-1}(V_\pm) = L_\pm(V_\pm) \subset \mathcal{R}(L_\pm) \subset \ker L_\mp = V_\pm.$$

Since $V_\pm$ are closed, $\overline{\mathcal{R}(L_\pm)} \subset V_\pm$ follows. If S is densely defined, then part (c) of the previous theorem yields $\overline{V_+ \oplus V_-} = V$. Therefore

$$\mathcal{R}(L_\pm) = L_\pm(\overline{V_+ \oplus V_-}) \subset \overline{L_\pm(V_+ \oplus V_-)} = \overline{L_\pm(V_\pm)} = \overline{\mathcal{D}(S) \cap V_\pm},$$

and hence the inclusion "$\supset$" in (a) holds. The other inclusion is clear by (15). If now S is also strictly dichotomous, then $P_\pm$ are bounded. In particular $\mathcal{R}(L_\pm) \subset \mathcal{D}(S)$ and hence $\mathcal{R}(L_\pm) = \mathcal{D}(S) \cap V_\pm$. Using that S and $L_\pm$ commute, we obtain

$$V_\pm = \mathcal{R}(P_\pm) = P_\pm(\overline{\mathcal{D}(S)}) \subset \overline{P_\pm(\mathcal{D}(S))} = \overline{L_\pm S(\mathcal{D}(S))} = \overline{\mathcal{R}(L_\pm)}$$

and hence $\overline{\mathcal{R}(L_\pm)} = V_\pm$ by (15). □

We remark that the inclusion $\overline{\mathcal{R}(L_\pm)} \subset V_\pm$ is strict in general, see [21, §6] and Examples 8.3 and 8.5 in [21].

3 Two scales of Hilbert spaces associated with a closed operator

In this section we construct two scales of Hilbert spaces $\{H_s\}$ and $\{H_s^{(*)}\}$ associated with a closed, densely defined operator A. Although the results are well known, the presentations found in the literature often cover only parts of the full theory or are restricted to certain special cases: The construction of the spaces $H_{\pm 1}$ and $H_{\pm 1}^{(*)}$ for general A can be found e.g. in [9, 19]. The intermediate spaces for $s = \pm\frac{1}{2}$ are defined in [9] for general, and in [19] for selfadjoint positive A. The spaces H_s with arbitrary s are constructed in [10] for selfadjoint A, while a general theory of scales of Hilbert spaces including interpolation results is contained in [2]. Note that in [19] a different naming convention and different but equivalent definitions of the spaces are used. Our presentation follows [2, 9].

Let A be a closed, densely defined linear operator on a separable Hilbert space H. We denote by $\|\cdot\|$ the norm on H and consider the positive selfadjoint operator $\Lambda = (I + AA^*)^{\frac{1}{2}}$. For $s > 0$ let $H_s = \mathcal{D}(\Lambda^s)$ be equipped with the norm $\|x\|_s = \|\Lambda^s x\|$, and let H_{-s} be the completion of H with respect to the norm $\|x\|_{-s} = \|\Lambda^{-s} x\|$. Then H_s and H_{-s} are Hilbert spaces,

$$H_s \subset H \subset H_{-s},$$

and the imbeddings are continuous and dense. The family of spaces $\{H_s\}$ is called a *scale of Hilbert spaces.* In particular we obtain $H_1 = \mathcal{D}(A^*)$ and

$$\|x\|_1 = (\|x\|^2 + \|A^* x\|^2)^{\frac{1}{2}}, \quad x \in H_1.$$

For any $s > 0$, the spaces H_s and H_{-s} are dual to each other with respect to the inner product $(\cdot|\cdot)$ of H. More precisely, the norm on H_s satisfies

$$\|y\|_{-s} = \sup\big\{|(x|y)| \,\big|\, x \in H_s,\ \|x\|_s = 1\big\}, \quad y \in H,$$

which implies that the inner product of H extends by continuity to a bounded sesquilinear form on $H_s \times H_{-s}$, which we denote by $(\cdot|\cdot)_{s,-s}$. In fact,

$$(x|y)_{s,-s} = (\Lambda^s x | \Lambda^{-s} y), \quad x \in H_s,\ y \in H.$$

The space H_{-s} can now be identified with the dual space of H_s by means of the isometric isomorphism $H_{-s} \to (H_s)'$, $y \mapsto (\cdot|y)_{s,-s}$. For convenience, we also define a sesquilinear form on $H_{-s} \times H_s$ by

$$(y|x)_{-s,s} = \overline{(x|y)_{s,-s}}, \quad x \in H_s,\ y \in H_{-s}.$$

With respect to the duality in the scale $\{H_s\}$ we obtain the following notion of adjoint operators:

Definition 3.1 Let W be a Hilbert space and $C \in L(H_s, W)$. Then the operator $C^* \in L(W, H_{-s})$ satisfying

$$(Cx|w)_W = (x|C^* w)_{s,-s}, \quad x \in H_s,\ w \in W, \tag{16}$$

where $(\cdot|\cdot)_W$ denotes the inner product of W, is called the *adjoint of C with respect to the scale $\{H_s\}$*. Similarly the adjoint of $B \in L(W, H_{-s})$ with respect to $\{H_s\}$ is the operator $B^* \in L(H_s, W)$ such that

$$(x|Bw)_{s,-s} = (B^* x|w)_W, \quad x \in H_s,\ w \in W. \tag{17}$$

The adjoints exist, are uniquely determined and satisfy $B = B^{**}$, $C = C^{**}$, $\|B\| = \|B^*\|$ and $\|C\| = \|C^*\|$. The adjoints of $\tilde{C} \in L(W, H_s)$ and $\tilde{B} \in L(H_{-s}, W)$ are defined in a similar way. If $C \in L(H_s, W)$ is an isomorphism, then C^* is an isomorphism too and $(C^*)^{-1} = (C^{-1})^*$.

Remark 3.2 The notion of adjoints with respect to the scale $\{H_s\}$ generalises the usual definition of adjoints of unbounded operators on Hilbert spaces: Let $C \in L(H_s, W)$. Then C can be regarded as a densely defined unbounded operator $C_1 : \mathcal{D}(C_1) \subset H \to W$ with domain $\mathcal{D}(C_1) = H_s$. The adjoint of C_1 in the usual sense of unbounded operators is an operator $C_1^* : \mathcal{D}(C_1^*) \subset W \to H$. Observe that C_1 and C_1^* satisfy (16) provided that $w \in \mathcal{D}(C_1^*)$. Consequently C_1^* is a restriction of $C^* : W \to H_{-s}$. In fact

$$\mathcal{D}(C_1^*) = \{w \in W \,|\, C^* w \in H\}\,.$$

Note here that $C \in L(H_s, W)$ does not imply that C_1 is closable. Hence C_1^* need not be densely defined and even $\mathcal{D}(C_1^*) = \{0\}$ is possible.

Since $H_1 = \mathcal{D}(A^*)$ and since $\|\cdot\|_1$ is equal to the graph norm of A^*, we can consider A^* as a bounded operator $A^* : H_1 \to H$. The adjoint with respect to $\{H_s\}$ is a bounded operator $A^{**} : H \to H_{-1}$ and in view of the last remark A^{**} is an extension of the original operator A. We will denote this extension by A again,

$$A : H \to H_{-1}.$$

Now for any $\lambda \in \varrho(A)$, the operator $A^* - \bar{\lambda} : H_1 \to H$ is an isomorphism. Hence its adjoint $A - \lambda : H \to H_{-1}$ is an isomorphism too. In particular $\|(A-\lambda)^{-1}\cdot\|$ is an equivalent norm on H_{-1}.

Consider now the positive selfadjoint operator $\Lambda_* = (I + A^*A)^{\frac{1}{2}}$, and let $\{H_s^{(*)}\}$ be the scale of Hilbert spaces associated with it. In other words, we repeat the above construction with the roles of A and A^* interchanged. We denote the respective norms and the extension of the inner product by $\|\cdot\|_s^{(*)}$, $\|\cdot\|_{-s}^{(*)}$ and $(\cdot|\cdot)_{s,-s}^{(*)}$. Moreover $H_1^{(*)} = \mathcal{D}(A)$, the norm on $H_1^{(*)}$ is equal to the graph norm of A, the norm on $H_{-1}^{(*)}$ is equivalent to $\|(A^* - \lambda)^{-1}\cdot\|$ for $\lambda \in \varrho(A^*)$, and we get bounded operators

$$A : H_1^{(*)} \to H, \qquad A^* : H \to H_{-1}^{(*)}.$$

Lemma 3.3 *If A has a compact resolvent, then the imbeddings $H_s \hookrightarrow H$ and $H_s^{(*)} \hookrightarrow H$ are compact for all $s > 0$.*

Proof. Let $\lambda \in \varrho(A)$. So $(A - \lambda)^{-1}$ and $(A^* - \bar{\lambda})^{-1}$ are compact operators in $L(H)$. The imbedding $H_1 \hookrightarrow H$ can be written as the composition

$$H_1 \xrightarrow{A^*-\bar{\lambda}} H \xrightarrow{(A^*-\bar{\lambda})^{-1}} H.$$

Since $A^* - \bar{\lambda} : H_1 \to H$ is bounded, it follows that $H_1 \hookrightarrow H$ is compact. Since $\Lambda^{-1} : H \to H_1$ is bounded, the sequence

$$H \xrightarrow{\Lambda^{-1}} H_1 \hookrightarrow H$$

implies that the operator $\Lambda^{-1} : H \to H$ is compact. Consequently $\Lambda^{-s} : H \to H$ is also compact for all $s > 0$. Decomposing $H_s \hookrightarrow H$ as

$$H_s \xrightarrow{\Lambda^s} H \xrightarrow{\Lambda^{-s}} H$$

where $\Lambda^s : H_s \to H$ is bounded, we conclude that $H_s \hookrightarrow H$ is compact. The proof for $H_s^{(*)} \hookrightarrow H$ is analogous. □

For operators acting between two scales of Hilbert spaces, there is the following interpolation result, which is also known as Heinz' inequality, see [11, Theorem I.7.1]. Let H and G be Hilbert spaces. Consider the scales of Hilbert spaces $\{H_s\}$ and $\{G_r\}$ with corresponding positive selfadjoint operators Λ and Δ on H and G, respectively.

Theorem 3.4 ([2, Theorem III.6.10]) *Let $r_1 < r_2$, $s_1 < s_2$ and let $B : G_{r_1} \to H_{s_1}$ be a bounded linear operator which restricts to a bounded operator $B : G_{r_2} \to H_{s_2}$. Let $0 < \lambda < 1$ and*

$$r = \lambda r_1 + (1-\lambda) r_2, \quad s = \lambda s_1 + (1-\lambda) s_2.$$

Then B also restricts to a bounded operator $B : G_r \to H_s$ and

$$\|B\|_{G_r \to H_s} \le \|B\|_{G_{r_1} \to H_{s_1}}^{\lambda} \|B\|_{G_{r_2} \to H_{s_2}}^{1-\lambda}.$$

We remark that if B restricts to an operator $B : G_{r_2} \to H_{s_2}$, i.e., if B maps G_{r_2} into H_{s_2}, then the boundedness of the restriction already follows from the closed graph theorem.

Applying interpolation to $A : H_1^{(*)} \to H$ and its extension $A : H \to H_{-1}$, we obtain that A also acts as a bounded operator

$$A : H_{1-s}^{(*)} \to H_{-s}, \qquad s \in [0,1].$$

Similarly,

$$A^* : H_{1-s} \to H_{-s}^{(*)}, \qquad s \in [0,1].$$

Moreover, if $\lambda \in \varrho(A)$ then $A - \lambda : H_{1-s}^{(*)} \to H_{-s}$ and $A^* - \bar\lambda : H_{1-s} \to H_{-s}^{(*)}$ are both isomorphisms. Here surjectivity follows from the fact that for example the resolvent $(A-\lambda)^{-1}$ is an operator in $L(H, H_1^{(*)})$ and $L(H_{-1}, H)$ and hence by interpolation also in $L(H_{-s}, H_{1-s}^{(*)})$.

The extensions of A and A^* satisfy the identity

$$(Ax|y)_{-s,s} = (x|A^*y)_{1-s,s-1}^{(*)}, \quad x \in H_{1-s}^{(*)},\ y \in H_s. \tag{18}$$

This follows from an extension by continuity of the relation $(Ax|y) = (x|A^*y)$, $x \in \mathcal{D}(A)$, $y \in \mathcal{D}(A^*)$.

In view of the above, using appropriate restrictions and extensions, the resolvent $(A-\lambda)^{-1}$ belongs to $L(H)$ as well as $L(H_{-1})$ and $L(H_1^{(*)})$. Similarly, $(A^* - \bar\lambda)^{-1}$ belongs to $L(H)$, $L(H_{-1}^{(*)})$ and $L(H_1)$. The corresponding operator norms can be estimated as follows:

Lemma 3.5 *For any $\lambda \in \varrho(A)$ the estimates*

$$\|(A-\lambda)^{-1}\|_{L(H_{-1})} \le \|(A^*-\bar{\lambda})^{-1}\|_{L(H_1)} \le \|(A-\lambda)^{-1}\|_{L(H)}$$

and

$$\|(A^*-\bar{\lambda})^{-1}\|_{L(H_{-1}^{(*)})} \le \|(A-\lambda)^{-1}\|_{L(H_1^{(*)})} \le \|(A^*-\bar{\lambda})^{-1}\|_{L(H)}$$

hold.

Proof. From

$$\begin{aligned}\|(A^*-\bar{\lambda})^{-1}x\|_1^2 &= \|(A^*-\bar{\lambda})^{-1}x\|^2 + \|A^*(A^*-\bar{\lambda})^{-1}x\|^2\\ &= \|(A^*-\bar{\lambda})^{-1}x\|^2 + \|(A^*-\bar{\lambda})^{-1}A^*x\|^2\\ &\le \|(A^*-\bar{\lambda})^{-1}\|_{L(H)}^2\|x\|_1^2\end{aligned}$$

for $x \in H_1$ we obtain

$$\|(A^*-\bar{\lambda})^{-1}\|_{L(H_1)} \le \|(A^*-\bar{\lambda})^{-1}\|_{L(H)} = \|(A-\lambda)^{-1}\|_{L(H)}.$$

Moreover for $x \in H_1$, $y \in H_{-1}$,

$$\begin{aligned}|(x|(A-\lambda)^{-1}y)| &= |((A^*-\bar{\lambda})^{-1}x|y)_{1,-1}|\\ &\le \|(A^*-\bar{\lambda})^{-1}\|_{L(H_1)}\|x\|_1\|y\|_{-1},\end{aligned}$$

which implies $\|(A-\lambda)^{-1}y\|_{-1} \le \|(A^*-\bar{\lambda})^{-1}\|_{L(H_1)}\|y\|_{-1}$ and hence

$$\|(A-\lambda)^{-1}\|_{L(H_{-1})} \le \|(A^*-\bar{\lambda})^{-1}\|_{L(H_1)}.$$

The other estimates are analogous. □

Interpolation now yields the following:

Corollary 3.6 *For $\lambda \in \varrho(A)$, $s \in [0,1]$,*

$$\begin{aligned}\|(A-\lambda)^{-1}\|_{L(H_{-s})} &\le \|(A-\lambda)^{-1}\|_{L(H)},\\ \|(A^*-\bar{\lambda})^{-1}\|_{L(H_{-s}^{(*)})} &\le \|(A^*-\bar{\lambda})^{-1}\|_{L(H)}.\end{aligned}$$

4 The Hamiltonian

Let A be a closed, densely defined operator on a Hilbert space H and let $\{H_s\}$ and $\{H_s^{(*)}\}$ be the associated scales of Hilbert spaces defined in Section 3. Let

$$B \in L(U, H_{-r}), \qquad C \in L(H_s^{(*)}, Y)$$

where U, Y are additional Hilbert spaces and $r, s \in [0,1]$ satisfy $r + s \le 1$. The adjoints of B and C with respect to the scales of Hilbert spaces are

$$B^* \in L(H_r, U), \qquad C^* \in L(Y, H^{(*)}_{-s}).$$

We define the *Hamiltonian* as the operator matrix

$$T_0 = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}.$$

Then T_0 is a well-defined linear operator from $\mathcal{D}(T_0) = H^{(*)}_{1-r} \times H_{1-s}$ to the product Hilbert space

$$V_0 = H_{-r} \times H^{(*)}_{-s}.$$

Indeed we have

$$\begin{aligned} A : H^{(*)}_{1-r} \to H_{-r}, &\qquad BB^* : H_r \to H_{-r}, \\ C^*C : H^{(*)}_{s} \to H^{(*)}_{-s}, &\qquad A^* : H_{1-s} \to H^{(*)}_{-s}, \end{aligned}$$

and the assumption $r + s \le 1$ implies

$$H^{(*)}_{1-r} \subset H^{(*)}_{s}, \qquad H_{1-s} \subset H_r.$$

We consider T_0 as an unbounded operator on V_0 with domain $\mathcal{D}(T_0)$ as above. In particular, T_0 is densely defined.

Alongside V_0 we will also consider the two product Hilbert spaces

$$V_1 = H^{(*)}_{s} \times H_r \qquad \text{and} \qquad V = H \times H.$$

Thus

$$\mathcal{D}(T_0) \subset V_1 \subset V \subset V_0.$$

Let T be the part of T_0 in V. Then $\sigma_p(T) = \sigma_p(T_0)$. Moreover T will be densely defined as soon as $\varrho(T_0) \neq \varnothing$. This follows from Lemma 2.2 since both inclusions $\mathcal{D}(T_0) \subset V$ and $V \subset V_0$ are dense.

Lemma 4.1 *The Hamiltonian satisfies*

$$\sigma_p(T_0) \cap i\mathbb{R} = \varnothing$$

if and only if

$$\ker(A - it) \cap \ker C = \ker(A^* + it) \cap \ker B^* = \{0\} \quad \text{for all} \quad t \in \mathbb{R}. \tag{19}$$

Proof. Suppose first that (19) holds and that

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}(T_0), \qquad T_0 \begin{pmatrix} x \\ y \end{pmatrix} = it \begin{pmatrix} x \\ y \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Then

$$(A - it)x - BB^*y = 0, \qquad -C^*Cx - (A^* + it)y = 0$$

where $x \in H_{1-r}^{(*)} \subset H_s^{(*)}$, $y \in H_{1-s} \subset H_r$. Using the extended inner products of the scales $\{H_s\}$ and $\{H_s^{(*)}\}$, we find

$$\begin{aligned} 0 &= ((A - it)x - BB^*y|y) = ((A - it)x|y)_{-r,r} - (BB^*y|y)_{-r,r}, \\ 0 &= (-C^*Cx - (A^* + it)y|x) = -(C^*Cx|x)_{-s,s}^{(*)} - ((A^* + it)y|x)_{-s,s}^{(*)}. \end{aligned} \tag{20}$$

From (18) we see that

$$(Ax|y)_{-r,r} = (x|A^*y)_{s,-s}^{(*)}, \quad x \in H_{1-r}^{(*)}, \, y \in H_{1-s}.$$

Adding the two equations in (20) and taking the real part, we thus obtain

$$0 = -(BB^*y|y)_{-r,r} - (C^*Cx|x)_{-s,s}^{(*)} = -\|B^*y\|_U^2 - \|Cx\|_Y^2.$$

Consequently $B^*y = Cx = 0$ and hence also $(A - it)x = (A^* + it)y = 0$. Now (19) implies $x = y = 0$ and so $it \notin \sigma_p(T_0)$. For the reverse implication note that if for example $x \in \ker(A - it) \cap \ker C$ and $x \neq 0$, then $(x, 0)$ is an eigenvector of T_0 with eigenvalue it. □

Lemma 4.2 *The Hamiltonian satisfies*

$$\sigma_{\text{app}}(T_0) \cap i\mathbb{R} \subset \sigma(A). \tag{21}$$

Proof. Let $t \in \mathbb{R}$, $it \in \sigma_{\text{app}}(T_0)$. Then there exist $v_n \in \mathcal{D}(T_0)$ such that $\|v_n\|_{V_0} = 1$ and

$$\lim_{n\to\infty} (T_0 - it)v_n = 0 \quad \text{in} \quad V_0.$$

By the continuity of the imbedding $V_1 \hookrightarrow V_0$ there is a constant $c > 0$ such that

$$1 = \|v_n\|_{V_0} \le c\|v_n\|_{V_1}.$$

Thus also

$$\lim_{n\to\infty} (T_0 - it)\frac{v_n}{\|v_n\|_{V_1}} = 0 \quad \text{in} \quad V_0.$$

Setting $(x_n, y_n) = v_n/\|v_n\|_{V_1}$ we obtain $\|x_n\|_s^{(*)2} + \|y_n\|_r^2 = 1$ and

$$\lim_{n\to\infty} (T_0 - it) \begin{pmatrix} x_n \\ y_n \end{pmatrix} = 0 \quad \text{in} \quad V_0,$$

or

$$\begin{aligned} (A - it)x_n - BB^*y_n \to 0 \quad &\text{in} \quad H_{-r} \\ -C^*Cx_n - (A^* + it)y_n \to 0 \quad &\text{in} \quad H_{-s}^{(*)} \end{aligned} \tag{22}$$

as $n \to \infty$. Since the sequences (x_n) and (y_n) are bounded in $H_s^{(*)}$ and H_r, respectively, this implies that

$$((A-it)x_n - BB^*y_n|y_n)_{-r,r} \to 0$$
$$(-C^*Cx_n - (A^*+it)y_n|x_n)_{-s,s}^{(*)} \to 0$$

Similarly to the previous proof, we add these identities and take the real part to obtain

$$-(BB^*y_n|y_n)_{-r,r} - (C^*Cx_n|x_n)_{-s,s}^{(*)} = -\|B^*y_n\|_U^2 - \|Cx_n\|_Y^2 \to 0.$$

Consequently $B^*y_n \to 0$ and $Cx_n \to 0$.

Now suppose in addition that $it \in \varrho(A)$. Then $A - it$ is an isomorphism from $H_{1-r}^{(*)}$ to H_{-r}, see section 3. Therefore $(A-it)^{-1} \in L(H_{-r}, H_s^{(*)})$ and analogously $(A^*+it)^{-1} \in L(H_{-s}^{(*)}, H_r)$. It follows that

$$(A-it)^{-1}BB^*y_n \to 0 \quad \text{in} \quad H_s^{(*)}, \qquad (A^*+it)^{-1}C^*Cx_n \to 0 \quad \text{in} \quad H_r.$$

On the other hand, we infer from (22) that

$$x_n - (A-it)^{-1}BB^*y_n \to 0 \quad \text{in} \quad H_s^{(*)},$$
$$-(A^*+it)^{-1}C^*Cx_n - y_n \to 0 \quad \text{in} \quad H_r.$$

Therefore $x_n \to 0$ in $H_s^{(*)}$ and $y_n \to 0$ in H_r, which contradicts $\|x_n\|_s^{(*)2} + \|y_n\|_r^2 = 1$. □

Lemma 4.3 *If A has a compact resolvent, $r + s < 1$ and $\varrho(T_0) \neq \varnothing$, then both T and T_0 have a compact resolvent too.*

Proof. First we have $\varrho(T) \neq \varnothing$ by Lemma 2.2. Lemma 3.3 shows that the imbeddings $H_{1-r}^{(*)} \times H_{1-s} \hookrightarrow V$ and $H_{1-r}^{(*)} \times H_{1-s} \hookrightarrow V_0$ are compact. Since $\mathcal{D}(T) \subset \mathcal{D}(T_0) = H_{1-r}^{(*)} \times H_{1-s}$, Lemma 2.1 implies that the resolvents of T and T_0 are compact. □

On $V = H \times H$ we consider the two indefinite inner products

$$[v|w] = (Jv|w), \quad [v|w]_\sim = (\tilde{J}v|w), \quad v, w \in H \times H, \tag{23}$$

with fundamental symmetries

$$J = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \qquad \tilde{J} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

For $v = (x, y), w = (\tilde{x}, \tilde{y})$ this yields

$$[v|w] = i(x|\tilde{y}) - i(y|\tilde{x}), \qquad [v|w]_\sim = (x|\tilde{y}) + (y|\tilde{x}).$$

For the first inner product, we also consider its extension to $v \in V_1 = H_s^{(*)} \times H_r$ and $w \in V_0 = H_{-r} \times H_{-s}^{(*)}$ which we denote again by $[\cdot|\cdot]$ and which is given by

$$\begin{aligned} [v|w] &= i(x|\tilde{y})_{s,-s}^{(*)} - i(y|\tilde{x})_{r,-r}, \\ [w|v] &= i(\tilde{x}|y)_{-r,r} - i(\tilde{y}|x)_{-s,s}^{(*)} = \overline{[v|w]}. \end{aligned} \tag{24}$$

Note that the extended inner product is non-degenerate in the sense that if $w \in V_0$ is such that $[v|w] = 0$ for all $v \in V_1$, then $w = 0$. Analogously $v \in V_1$ with $[v|w] = 0$ for all $w \in V_0$ implies $v = 0$.

The Hamiltonian has the following properties with respect to the inner products defined above:

Lemma 4.4

$$\begin{aligned} [T_0 v|w] &= -[v|T_0 w], \qquad & v, w \in \mathcal{D}(T_0), \\ \mathrm{Re}[Tv|v]_\sim &\le 0, \qquad & v \in \mathcal{D}(T). \end{aligned}$$

Proof. Let $v, w \in \mathcal{D}(T_0) = H_{1-r}^{(*)} \times H_{1-s}$ and $v = (x, y), w = (\tilde{x}, \tilde{y})$. Then

$$x, \tilde{x} \in H_{1-r}^{(*)} \subset H_s^{(*)}, \quad y, \tilde{y} \in H_{1-s} \subset H_r, \quad T_0 v, T_0 w \in V_0 = H_{-r} \times H_{-s}^{(*)}.$$

We obtain

$$\begin{aligned} [T_0 v|w] &= i(Ax - BB^* y|\tilde{y})_{-r,r} - i(-C^*Cx - A^* y|\tilde{x})_{-s,s}^{(*)} \\ &= i(Ax|\tilde{y})_{-r,r} - i(BB^* y|\tilde{y})_{-r,r} + i(C^*Cx|\tilde{x})_{-s,s}^{(*)} + i(A^* y|\tilde{x})_{-s,s}^{(*)} \\ &= i(x|A^*\tilde{y})_{s,-s}^{(*)} - i(y|BB^*\tilde{y})_{r,-r} + i(x|C^*C\tilde{x})_{s,-s}^{(*)} + i(y|A\tilde{x})_{r,-r} \\ &= i(x|C^*C\tilde{x} + A^*\tilde{y})_{s,-s}^{(*)} - i(y| - A\tilde{x} + BB^*\tilde{y})_{r,-r} \\ &= [v| - T_0 w]. \end{aligned}$$

Let now $v = (x, y) \in \mathcal{D}(T)$. Then

$$\begin{aligned} [Tv|v]_\sim &= (Ax - BB^* y|y) + (-C^*Cx - A^* y|x) \\ &= (Ax|y)_{-r,r} - (BB^* y|y)_{-r,r} - (C^*Cx|x)_{-s,s}^{(*)} - (A^* y|x)_{-s,s}^{(*)} \\ &= (Ax|y)_{-r,r} - \|B^* y\|_U^2 - \|Cx\|_Y^2 - (y|Ax)_{r,-r} \end{aligned}$$

and hence

$$\mathrm{Re}[Tv|v]_\sim = -\|B^* y\|_U^2 - \|Cx\|_Y^2 \le 0.$$

□

Corollary 4.5 *(a) If there exists $\lambda \in \mathbb{C}$ such that $\lambda, -\bar{\lambda} \in \varrho(T_0)$, then T is J-skew-selfadjoint and $\sigma(T)$ is symmetric with respect to the imaginary axis.*

(b) *If both T and T_0 have a compact resolvent, then $\sigma(T_0)$ is symmetric with respect to the imaginary axis.*

Proof. The previous lemma yields $[Tv|w] = -[v|Tw]$ for $v, w \in V$. Also recall that T is densely defined since $\varrho(T_0) \neq \varnothing$. Lemma 2.2 implies $\varrho(T_0) \subset \varrho(T)$ and hence $\lambda, -\bar{\lambda} \in \varrho(T)$. By the theory of operators in Krein spaces, we conclude that T is skew-selfadjoint with respect to the J-inner product, which in turn implies the symmetry of the spectrum. If now both resolvents are compact, then $\sigma(T) = \sigma_p(T) = \sigma_p(T_0) = \sigma(T_0)$ and the symmetry of the spectrum follows from part (a). □

Remark 4.6 The symmetries of the Hamiltonian with respect to the two indefinite inner products on $H \times H$ have been used already in [14, 18, 22, 23]. The use of the Hamiltonian T_0 on the extended space V_0 as well as the extended indefinite inner product is new here and is motivated by the better properties of T_0 compared to T.

5 Bisectorial Hamiltonians

Starting from this section we consider Hamiltonians whose operator A is quasi-sectorial, see Definition 2.3. Recall from Section 4 that

$$V_1 = H_s^{(*)} \times H_r, \qquad V_0 = H_{-r} \times H_{-s}^{(*)}$$

and

$$BB^* \in L(H_r, H_{-r}), \qquad C^*C \in L(H_s^{(*)}, H_{-s}^{(*)}),$$

We consider the following decomposition of T_0 on V_0:

$$T_0 = S_0 + R, \qquad S_0 = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -BB^* \\ -C^*C & 0 \end{pmatrix}. \tag{25}$$

Here S_0, like T_0, is an unbounded operator on V_0 with domain $\mathcal{D}(S_0) = \mathcal{D}(T_0) = H_{1-r}^{(*)} \times H_{1-s}$. On the other hand, R is a bounded operator $R \in L(V_1, V_0)$.

By Corollary 3.6 the extensions of A and A^* to unbounded operators on H_{-r} and $H_{-s}^{(*)}$, respectively, are quasi-sectorial and satisfy

$$\|(A-\lambda)^{-1}\|_{L(H_{-r})} \le \frac{M}{|\lambda|}, \qquad \|(A^*-\lambda)^{-1}\|_{L(H_{-s}^{(*)})} \le \frac{M}{|\lambda|}$$

for all $\lambda \in \Sigma_{\frac{\pi}{2}+\theta}$, $|\lambda| \ge \rho$ where θ, M, ρ are the constants from (8). Consequently

$$\|(S_0-\lambda)^{-1}\|_{L(V_0)} \le \frac{M}{|\lambda|}, \qquad \lambda \in \Omega_\theta, |\lambda| \ge \rho, \tag{26}$$

with Ω_θ the bisector from (6).

We derive a few estimates for the resolvents of A and A^* with respect to the scales of Hilbert spaces $\{H_s\}$ and $\{H_s^{(*)}\}$.

Lemma 5.1 *Let A be quasi-sectorial and let $\theta, M, \rho > 0$ be the corresponding constants from* (8)*. Then for all $\lambda \in \Sigma_{\frac{\pi}{2}+\theta}$ with $|\lambda| \geq \rho$ the estimates*

$$\|(A-\lambda)^{-1}\|_{H\to H_1^{(*)}} \leq M_1, \qquad \|(A-\lambda)^{-1}\|_{H_{-1}\to H} \leq M_1,$$
$$\|(A^*-\lambda)^{-1}\|_{H\to H_1} \leq M_1, \qquad \|(A^*-\lambda)^{-1}\|_{H_{-1}^{(*)}\to H} \leq M_1$$

hold where $M_1 = M\left(\frac{1}{\rho}+1\right)+1$.

Proof. For $x \in H$ we have

$$\begin{aligned}\|(A-\lambda)^{-1}x\|_1^{(*)} &\leq \|(A-\lambda)^{-1}x\| + \|A(A-\lambda)^{-1}x\| \\ &\leq \|(A-\lambda)^{-1}x\| + \|x\| + |\lambda|\|(A-\lambda)^{-1}x\| \\ &\leq \left(\frac{M}{|\lambda|}+1+M\right)\|x\| \leq \left(\frac{M}{\rho}+1+M\right)\|x\|\end{aligned}$$

and hence $\|(A-\lambda)^{-1}\|_{H\to H_1^{(*)}} \leq M_1$. Since the adjoint of $(A-\bar{\lambda})^{-1} : H \to H_1^{(*)}$ with respect to the scale $\{H_s^{(*)}\}$ is $(A^*-\lambda)^{-1} : H_{-1}^{(*)} \to H$, see Section 3, we also get

$$\|(A^*-\lambda)^{-1}\|_{H_{-1}^{(*)}\to H} = \|(A-\bar{\lambda})^{-1}\|_{H\to H_1^{(*)}} \leq M_1.$$

Note here that if λ belongs to $\Sigma_{\frac{\pi}{2}+\theta}$ then so does $\bar{\lambda}$. The other estimates follow by interchanging the roles of A and A^*. □

Corollary 5.2 *Let A be quasi-sectorial, θ, M, ρ as above. Let $r, s \geq 0$ with $r+s \leq 1$. Then for $\lambda \in \Sigma_{\frac{\pi}{2}+\theta}$, $|\lambda| \geq \rho$:*

$$\|(A-\lambda)^{-1}\|_{H_{-r}\to H_s^{(*)}} \leq \frac{M_2}{|\lambda|^{1-r-s}}, \qquad \|(A^*-\lambda)^{-1}\|_{H_{-s}^{(*)}\to H_r} \leq \frac{M_2}{|\lambda|^{1-r-s}}.$$

The constant M_2 depends on M, ρ, r, s only.

Proof. We apply interpolation to the results of Lemma 5.1. As a first step we get

$$\begin{aligned}\|(A-\lambda)^{-1}\|_{H\to H_{r+s}^{(*)}} &\leq \|(A-\lambda)^{-1}\|_{H\to H_1^{(*)}}^{r+s}\|(A-\lambda)^{-1}\|_{H\to H}^{1-r-s} \\ &\leq M_1^{r+s}\left(\frac{M}{|\lambda|}\right)^{1-r-s} = \frac{M_2}{|\lambda|^{1-r-s}}\end{aligned}$$

with $M_2 = M_1^{r+s}M^{1-r-s}$ and similarly

$$\|(A-\lambda)^{-1}\|_{H_{-r-s}\to H} \leq \frac{M_2}{|\lambda|^{1-r-s}}.$$

From this we obtain with $\tau = \frac{r}{r+s}$

$$\begin{aligned}\|(A-\lambda)^{-1}\|_{H_{-r}\to H_s^{(*)}} &\leq \|(A-\lambda)^{-1}\|^{\tau}_{H_{-r-s}\to H}\|(A-\lambda)^{-1}\|^{1-\tau}_{H\to H^{(*)}_{r+s}}\\ &\leq \frac{M_2}{|\lambda|^{1-r-s}}.\end{aligned}$$

The estimates for $\|(A^*-\lambda)^{-1}\|_{H^{(*)}_{-s}\to H_r}$ are again analogous. □

Lemma 5.3 *Let A be quasi-sectorial, let θ, ρ be the constants from* (8)*. Suppose that $r+s<1$. Then there exists $\rho_1 \geq \rho$ and $c_0, c_1 > 0$ such that $\Omega_\theta \setminus B_{\rho_1}(0) \subset \varrho(T_0)$ and*

$$\|(T_0-\lambda)^{-1}\|_{L(V_0)} \leq \frac{c_0}{|\lambda|}, \tag{27}$$

$$\|(T_0-\lambda)^{-1}-(S_0-\lambda)^{-1}\|_{L(V_0)} \leq \frac{c_1}{|\lambda|^{2-r-s}} \tag{28}$$

for all $\lambda \in \Omega_\theta$, $|\lambda| \geq \rho_1$.

Proof. This is a standard perturbation argument for $T_0 = S_0 + R$ on V_0: For $\lambda \in \varrho(S_0)$, the identity

$$T_0 - \lambda = \big(I - R(S_0-\lambda)^{-1}\big)(S_0-\lambda)$$

holds. Corollary 5.2 implies that

$$\|(S_0-\lambda)^{-1}\|_{L(V_0,V_1)} \leq \frac{M_2}{|\lambda|^{1-r-s}}, \qquad \lambda \in \Omega_\theta, |\lambda| \geq \rho.$$

Since $\|R(S_0-\lambda)^{-1}\|_{L(V_0)} \leq \|R\|\|(S_0-\lambda)^{-1}\|_{L(V_0,V_1)}$ and $1-r-s>0$, it follows that there exists $\rho_1 \geq \rho$ such that

$$\|R(S_0-\lambda)^{-1}\|_{L(V_0)} \leq \frac{1}{2} \qquad \text{for all } \lambda \in \Omega_\theta,\ |\lambda| \geq \rho_1.$$

Hence $I - R(S_0-\lambda)^{-1}$ is an isomorphism on V_0 and thus $\lambda \in \varrho(T_0)$ with

$$(T_0-\lambda)^{-1} = (S_0-\lambda)^{-1}\big(I - R(S_0-\lambda)^{-1}\big)^{-1} \tag{29}$$

and

$$\|(T_0-\lambda)^{-1}\|_{L(V_0)} \leq \|(S_0-\lambda)^{-1}\|_{L(V_0)}\big\|\big(I-R(S_0-\lambda)^{-1}\big)^{-1}\big\|_{L(V_0)} \leq \frac{2M}{|\lambda|}$$

for $\lambda \in \Omega_\theta$, $|\lambda| \geq \rho_1$. Moreover

$$(S_0-\lambda)^{-1} - (T_0-\lambda)^{-1} = (T_0-\lambda)^{-1}R(S_0-\lambda)^{-1} \tag{30}$$

which implies

$$\begin{aligned}\|(S_0-\lambda)^{-1}-(T_0-\lambda)^{-1}\|_{L(V_0)} &\le \|(T_0-\lambda)^{-1}\|_{L(V_0)}\|R\|\|(S_0-\lambda)^{-1}\|_{L(V_0,V_1)}\\ &\le \frac{2M\|R\|M_2}{|\lambda|^{2-r-s}}.\end{aligned}$$

□

Lemma 5.4 *Let A be quasi-sectorial and let $Q_{0\pm}\in L(V_0)$ be the projections*

$$Q_{0-}=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix},\qquad Q_{0+}=\begin{pmatrix} 0 & 0\\ 0 & I\end{pmatrix}. \tag{31}$$

Consider the integration contours $\gamma_1(t)=it$, $t\in]-\infty,-\rho]\cup[\rho,\infty[$ as well as $\gamma_{0+}(t)=\rho e^{it}$, $t\in[-\frac{\pi}{2},\frac{\pi}{2}]$ and $\gamma_{0-}(t)=\rho e^{-it}$, $t\in[\frac{\pi}{2},\frac{3\pi}{2}]$ where ρ is the constant from (8) *for A. Then*

$$Q_{0+}v-Q_{0-}v=\frac{1}{\pi i}\int_{\gamma_1}{}' (S_0-\lambda)^{-1}v\,d\lambda+Kv,\qquad v\in V_0,$$

where the prime denotes the Cauchy principal value at infinity and $K\in L(V_0)$ is given by $K=\left(\begin{smallmatrix} K_1 & 0\\ 0 & K_2\end{smallmatrix}\right)$ with

$$K_1=\frac{1}{\pi i}\int_{\gamma_{0+}}(A-\lambda)^{-1}\,d\lambda,\quad K_2=\frac{1}{\pi i}\int_{\gamma_{0-}}(-A^*-\lambda)^{-1}\,d\lambda.$$

Proof. We consider A as an operator on H_{-r}. Since $A-\rho$ is sectorial and $0\in\varrho(A-\rho)$,

$$\frac{1}{\pi i}\int_{-i\infty}^{i\infty}{}'(A-\rho-\lambda)^{-1}x\,d\lambda=-x,\qquad x\in H_{-r},$$

holds by [14, Lemma 6.1]. Using Cauchy's theorem in conjunction with the resolvent decay of A to alter the integration contour, we obtain

$$\begin{aligned}-x&=\frac{1}{\pi i}\int_{\rho-i\infty}^{\rho+i\infty}{}'(A-\lambda)^{-1}x\,d\lambda\\ &=\frac{1}{\pi i}\int_{\gamma_1}{}'(A-\lambda)^{-1}x\,d\lambda+\frac{1}{\pi i}\int_{\gamma_{0+}}(A-\lambda)^{-1}x\,d\lambda,\qquad x\in H_{-r}.\end{aligned}$$

Looking at $-A^*$, we get

$$\frac{1}{\pi i}\int_{-i\infty}^{i\infty}{}'(-A^*+\rho-\lambda)^{-1}y\,d\lambda=y,\qquad y\in H_{-s}^{(*)},$$

and hence

$$y=\frac{1}{\pi i}\int_{\gamma_1}{}'(-A^*-\lambda)^{-1}y\,d\lambda+\frac{1}{\pi i}\int_{\gamma_{0-}}(-A^*-\lambda)^{-1}y\,d\lambda,\qquad y\in H_{-s}^{(*)}.$$

Combining both identities and noting that $Q_{0+}v-Q_{0-}v=(-x,y)$ for $v=(x,y)$, we obtain the claim. □

Theorem 5.5 *Let A be quasi-sectorial and let $r+s<1$. If $\sigma(A)\cap i\mathbb{R}=\varnothing$ or if A has a compact resolvent and*

$$\ker(A-it)\cap\ker C=\ker(A^*+it)\cap\ker B^*=\{0\}\quad \textit{for all}\quad t\in\mathbb{R},\tag{32}$$

then the Hamiltonian T_0 is bisectorial and strictly dichotomous.

Proof. We first show that $i\mathbb{R}\subset\varrho(T_0)$. If $\sigma(A)\cap i\mathbb{R}=\varnothing$, then Lemma 4.2 implies $\sigma_{\mathrm{app}}(T_0)\cap i\mathbb{R}=\varnothing$. Since $\partial\sigma(T_0)\subset\sigma_{\mathrm{app}}(T_0)$ and since $i\mathbb{R}\cap\varrho(T_0)\neq\varnothing$ by Lemma 5.3 it follows that $i\mathbb{R}\subset\varrho(T_0)$. Suppose on the other hand that A has a compact resolvent and that (32) holds. By Lemma 4.3 T_0 has a compact resolvent too and therefore $\sigma(T_0)=\sigma_p(T_0)$. Lemma 4.1 then implies $\sigma(T_0)\cap i\mathbb{R}=\varnothing$.

From $i\mathbb{R}\subset\varrho(T_0)$ and the estimate (27) we obtain that T_0 is bisectorial. In particular Theorem 2.6 can be applied to T_0 and yields corresponding closed projections on V_0, which we denote by $P_{0\pm}$. By Lemma 5.4 the mapping

$$v\mapsto\frac{1}{\pi i}\int_{\gamma_1}^{\prime}(S_0-\lambda)^{-1}v\,d\lambda,\qquad v\in V_0,$$

defines a bounded operator in $L(V_0)$. In view of (28) the integral

$$\int_{\gamma_1}(T_0-\lambda)^{-1}-(S_0-\lambda)^{-1}\,d\lambda$$

converges in $L(V_0)$. Consequently $v\mapsto\frac{1}{\pi i}\int_{\gamma_1}^{\prime}(T_0-\lambda)^{-1}v\,d\lambda$ and hence also

$$v\mapsto\frac{1}{\pi i}\int_{-i\infty}^{i\infty\,\prime}(T_0-\lambda)^{-1}v\,d\lambda,\qquad v\in V_0,$$

defines a bounded operator in $L(V_0)$. By (13) this last operator coincides with $P_{0+}-P_{0-}$ on $\mathcal{D}(T_0)$. Since $P_{0+}-P_{0-}$ is closed and $\mathcal{D}(T_0)$ is dense in V_0, we conclude that $\mathcal{D}(P_{0\pm})=V_0$ and hence $P_{0\pm}\in L(V_0)$ by the closed graph theorem. Therefore T_0 is strictly dichotomous. □

Remark 5.6 Combining the results from Lemma 5.3 with the dichotomy of T_0 from Theorem 5.5 we find that in fact

$$\big(\Omega_\theta\setminus B_{\rho_1}(0)\big)\cup\big\{\lambda\in\mathbb{C}\,\big|\,|\lambda|\le h\big\}\subset\varrho(T_0)$$

where $\rho_1\ge\rho$, $h>0$, and θ,ρ are the constants from (8) corresponding to the quasi-sectoriality of A. Also note that the last proof shows that T_0 is bisectorial and strictly dichotomous whenever $r+s<1$ and $i\mathbb{R}\subset\varrho(T_0)$.

We close this section by investigating the dichotomy properties of the Hamiltonian on $V=H\times H$, i.e., of the operator T. Let

$$S=\begin{pmatrix}A&0\\0&-A^*\end{pmatrix}$$

with domain $\mathcal{D}(S) = H_1^{(*)} \times H_1$, considered as an unbounded operator on V, i.e., S is the part of S_0 in V. Note that a decomposition similar to (25) does not hold for the operators T and S since R maps out of V into the larger space V_0. In particular we have $\mathcal{D}(T) \neq \mathcal{D}(S)$ in general.

Lemma 5.7 *Let A be quasi-sectorial with constants θ, ρ as in* (8). *Let $r + s < 1$. Then there exist $\rho_1 \geq \rho$ and $c_0, c_1 > 0$ such that $\Omega_\theta \setminus B_{\rho_1}(0) \subset \varrho(T)$ and*

$$\|(T-\lambda)^{-1}\|_{L(V)} \leq \frac{c_0}{|\lambda|^\beta}, \tag{33}$$

$$\|(T-\lambda)^{-1} - (S-\lambda)^{-1}\|_{L(V)} \leq \frac{c_1}{|\lambda|^{2(1-\max\{r,s\})}}, \tag{34}$$

for all $\lambda \in \Omega_\theta$, $|\lambda| \geq \rho_1$ where

$$\beta = \begin{cases} 1, & \max\{r,s\} \leq \frac{1}{2}, \\ 2(1-\max\{r,s\}), & \max\{r,s\} > \frac{1}{2}. \end{cases}$$

Proof. By Corollary 5.2 there exist $M_2, M_2' > 0$ with

$$\|(A-\lambda)^{-1}\|_{L(H_{-r},H)} \leq \frac{M_2}{|\lambda|^{1-r}}, \qquad \|(-A^*-\lambda)^{-1}\|_{L(H_{-s}^{(*)},H)} \leq \frac{M_2'}{|\lambda|^{1-s}}$$

for all $\lambda \in \Omega_\theta$, $|\lambda| \geq \rho$. Since $\rho > 0$ we can thus find $c > 0$ such that

$$\|(S_0-\lambda)^{-1}\|_{L(V_0,V)} \leq \frac{c}{|\lambda|^{1-\max\{r,s\}}} \qquad \text{for } \lambda \in \Omega_\theta,\ |\lambda| \geq \rho.$$

Similarly there exists $c' > 0$ with

$$\|(S-\lambda)^{-1}\|_{L(V,V_1)} \leq \frac{c'}{|\lambda|^{1-\max\{r,s\}}} \qquad \text{for } \lambda \in \Omega_\theta,\ |\lambda| \geq \rho.$$

Let now $\rho_1 \geq \rho$ be chosen as in Lemma 5.3 and let $\lambda \in \Omega_\theta$, $|\lambda| \geq \rho_1$. Then $\lambda \in \varrho(T_0)$ and we obtain from (29) that

$$\begin{aligned} \|(T_0-\lambda)^{-1}\|_{L(V_0,V)} &\leq \|(S_0-\lambda)^{-1}\|_{L(V_0,V)} \big\|\big(I - R(S_0-\lambda)^{-1}\big)^{-1}\big\|_{L(V_0)} \\ &\leq \frac{2c}{|\lambda|^{1-\max\{r,s\}}} \end{aligned} \tag{35}$$

and consequently

$$\begin{aligned} \|(T_0-\lambda)^{-1}R(S-\lambda)^{-1}\|_{L(V)} &\leq \|(T_0-\lambda)^{-1}\|_{L(V_0,V)}\|R\|\|(S-\lambda)^{-1}\|_{L(V,V_1)} \\ &\leq \frac{2cc'\|R\|}{|\lambda|^{2(1-\max\{r,s\})}}. \end{aligned} \tag{36}$$

Lemma 2.2 implies that $\lambda \in \varrho(T)$ and $(T-\lambda)^{-1} = (T_0-\lambda)^{-1}|_V$. Restricting (30) to the space V, we get

$$(S-\lambda)^{-1} - (T-\lambda)^{-1} = (T_0-\lambda)^{-1}R(S-\lambda)^{-1}. \tag{37}$$

Combining this with (36) and $\|(S-\lambda)^{-1}\|_{L(V)} \leq M/|\lambda|$, we obtain the desired estimates. □

Remark 5.8 The statement of Lemma 5.4 remains true if all involved operators are restricted to V. This means that V_0, S_0 and $Q_{0\pm}$ are replaced by V, S and $Q_\pm$, respectively, where $Q_\pm$ are the restrictions of $Q_{0\pm}$ to V. The proof remains unchanged except for an adaption of the spaces.

Theorem 5.9 *Let A be quasi-sectorial and let $r+s<1$. If $\sigma(A)\cap i\mathbb{R}=\varnothing$ or if A has a compact resolvent and*

$$\ker(A-it)\cap\ker C=\ker(A^*+it)\cap\ker B^*=\{0\}\quad \textit{for all}\quad t\in\mathbb{R},$$

then T is almost bisectorial; in particular there exist closed, T- and $(T-\lambda)^{-1}$-invariant subspaces $V_\pm\subset V$ such that $\sigma(T|_{V_\pm})\subset\mathbb{C}_\pm$. If in addition $\max\{r,s\}<\frac{1}{2}$, then T is even bisectorial and strictly dichotomous.

Proof. From Theorem 5.5 we know that $i\mathbb{R}\subset\varrho(T_0)$. Hence also $i\mathbb{R}\subset\varrho(T)$ by Lemma 2.2. From (33) in Lemma 5.7 we thus conclude that T is almost bisectorial with $0<\beta<1$ if $\max\{r,s\}>\frac{1}{2}$ and bisectorial if $\max\{r,s\}\le\frac{1}{2}$. Note that bisectoriality implies almost bisectoriality here since $0\in\varrho(T)$. The existence of $V_\pm$ follows by Theorem 2.6. If now $\max\{r,s\}<\frac{1}{2}$ then (34) yields

$$\|(T-\lambda)^{-1}-(S-\lambda)^{-1}\|\le\frac{c_1}{|\lambda|^{1+\varepsilon}},\qquad \lambda\in\Omega_\theta,\ |\lambda|\ge\rho_1,$$

with some $\varepsilon>0$. In view of Remark 5.8 we can then derive in the same way as in the proof of Theorem 5.5 that T is dichotomous. □

6 Graph and angular subspaces

In this section we consider a Hamiltonian with quasi-sectorial A, $r+s<1$, and $i\mathbb{R}\subset\varrho(T_0)$. From the last section we know that then T_0 is bisectorial and strictly dichotomous and T is almost bisectorial. We denote by $V_{0\pm}$ and $V_\pm$ the corresponding invariant subspaces of T_0 and T, respectively, and by $P_{0\pm}$ and $P_\pm$ the associated projections; see Theorem 2.6. In particular $P_{0\pm}\in L(V_0)$ while $P_\pm$ are closed operators on V. The projections $P_{0\pm}$ are given by $P_{0\pm}=TL_{0\pm}$ where $L_{0\pm}\in L(V_0)$,

$$L_{0\pm}=\frac{\pm1}{2\pi i}\int_{\pm h-i\infty}^{\pm h+i\infty}\frac{1}{\lambda}(T_0-\lambda)^{-1}\,d\lambda. \tag{38}$$

Recall from (24) the extended indefinite inner product $[\cdot|\cdot]$ defined on $V_1\times V_0$ as well as $V_0\times V_1$.

Lemma 6.1 *The operators $L_{0\pm}$ satisfy $L_{0\pm}\in L(V_0,V_1)$ and*

$$[L_{0+}v|w]=-[v|L_{0-}w]\qquad \textit{for all } v,w\in V_0.$$

Proof. In the proof of Lemma 5.3 we have seen that there exists $\rho_1 > 0$ such that

$$(T_0 - \lambda)^{-1} = (S_0 - \lambda)^{-1}\big(I - R(S_0 - \lambda)^{-1}\big)^{-1}$$

for $\lambda \in \Omega_\theta$, $|\lambda| > \rho_1$, and the estimates

$$\|(S_0 - \lambda)^{-1}\|_{L(V_0,V_1)} \leq \frac{M_2}{|\lambda|^{1-r-s}}, \qquad \|R(S_0 - \lambda)^{-1}\|_{L(V_0)} \leq \frac{1}{2}$$

hold. It follows that

$$\|(T_0 - \lambda)^{-1}\|_{L(V_0,V_1)} \leq \frac{2M_2}{|\lambda|^{1-r-s}}. \tag{39}$$

Since $1-r-s > 0$ this implies that the integral in (38) converges in $L(V_0, V_1)$; in particular $L_{0\pm} \in L(V_0, V_1)$. For $v, w \in V_0$ we can now derive, using Lemma 4.4,

$$\begin{aligned}
[L_{0+}v|w] &= \Big[\frac{1}{2\pi i}\int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda}(T_0 - \lambda)^{-1} v\, d\lambda \Big| w\Big] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big[\frac{1}{h+it}(T_0 - h - it)^{-1} v \Big| w\Big]\, dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big[v \Big| \frac{1}{h-it}(-T_0 - h + it)^{-1} w\Big]\, dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big[v \Big| \frac{1}{-h+it}(T_0 + h - it)^{-1} w\Big]\, dt \\
&= \Big[v \Big| \frac{1}{2\pi i}\int_{-h-i\infty}^{-h+i\infty} \frac{1}{\lambda}(T_0 - \lambda)^{-1} w\, d\lambda\Big] = -[v|L_{0-}w].
\end{aligned}$$

□

Corollary 6.2

$$[v|w] = 0 \qquad \textit{for all} \quad v \in V_{0\pm},\ w \in \mathcal{R}(L_{0\pm}).$$

Proof. This is immediate since $V_{0\pm} = \ker L_{0\mp}$. □

We can now establish conditions for the subspaces $V_{0\pm}$ to be graphs of operators. We say that a subspace $U \subset V_0 = H_{-r} \times H_{-s}^{(*)}$ is the graph of a (possibly unbounded) operator $X : \mathcal{D}(X) \subset H_{-r} \to H_{-s}^{(*)}$ if

$$U = \left\{ \begin{pmatrix} x \\ Xx \end{pmatrix} \,\middle|\, x \in \mathcal{D}(X) \right\} = \mathcal{R}\begin{pmatrix} I \\ X \end{pmatrix}.$$

We also consider the inverse situation where $U \subset H_{-r} \times H_{-s}^{(*)}$ is the graph of an operator $Y : \mathcal{D}(Y) \subset H_{-s}^{(*)} \to H_{-r}$, i.e.,

$$U = \left\{ \begin{pmatrix} Yy \\ y \end{pmatrix} \,\middle|\, y \in \mathcal{D}(Y) \right\} = \mathcal{R}\begin{pmatrix} Y \\ I \end{pmatrix}.$$

Proposition 6.3 *If*

$$\bigcap_{\lambda\in i\mathbb{R}\cap\varrho(A^*)} \ker B^*(A^*-\lambda)^{-1} = \{0\} \qquad \text{on} \quad H_{-s}^{(*)}, \tag{40}$$

then $V_{0\pm} = \mathcal{R}\begin{pmatrix} I \\ X_{0\pm}\end{pmatrix}$ *with closed operators* $X_{0\pm} : \mathcal{D}(X_{0\pm}) \subset H_{-r} \to H_{-s}^{(*)}$. *If*

$$\bigcap_{\lambda\in i\mathbb{R}\cap\varrho(A)} \ker C(A-\lambda)^{-1} = \{0\} \qquad \text{on} \quad H_{-r}, \tag{41}$$

then $V_{0\pm} = \mathcal{R}\begin{pmatrix} Y_{0\pm} \\ I\end{pmatrix}$ *with closed operators* $Y_{0\pm} : \mathcal{D}(Y_{0\pm}) \subset H_{-s}^{(*)} \to H_{-r}$. *If both* (40) *and* (41) *hold then* $X_{0\pm}$ *are injective and* $X_{0\pm}^{-1} = Y_{0\pm}$.

Proof. For the first assertion, since $V_{0\pm}$ are closed linear subspaces of V_0, it suffices to show that $(0, w) \in V_{0\pm}$ implies $w = 0$. Let $(0, w) \in V_{0\pm}$ and $t \in \mathbb{R}$ such that $-it \in \varrho(A^*)$. Set

$$\begin{pmatrix} x \\ y \end{pmatrix} = (T_0 - it)^{-1}\begin{pmatrix} 0 \\ w \end{pmatrix}.$$

Then $(x, y) \in \mathcal{D}(T_0) \cap V_{0\pm}$ by the invariance of $V_{0\pm}$. By Lemma 2.7 it follows that $(x, y) \in \mathcal{R}(L_{0\pm})$. Using Corollary 6.2, we get

$$0 = \left[\begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} 0 \\ w \end{pmatrix}\right] = i(x|w)_{s,-s}^{(*)}.$$

From

$$\begin{pmatrix} 0 \\ w \end{pmatrix} = (T_0 - it)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (A-it)x - BB^*y \\ -C^*Cx - (A^*+it)y \end{pmatrix}$$

we thus obtain

$$\begin{aligned} 0 = (x|w)_{s,-s}^{(*)} &= -(x|C^*Cx)_{s,-s}^{(*)} - (x|(A^*+it)y)_{s,-s}^{(*)} \\ &= -\|Cx\|^2 - ((A-it)x|y)_{-r,r} = -\|Cx\|^2 - (BB^*y|y)_{-r,r} \\ &= -\|Cx\|^2 - \|B^*y\|^2 \end{aligned}$$

and therefore $Cx = B^*y = 0$. This implies $w = -(A^*+it)y$ and hence $-B^*y = B^*(A^*+it)^{-1}w = 0$. Since $t \in \mathbb{R}$ with $-it \in \varrho(A^*)$ was arbitrary, (40) implies that $w = 0$. For the second assertion, we show in an analogous way that $(w, 0) \in V_{0\pm}$ implies $w = 0$ provided that (41) holds. The final statement is then clear. $\square$

Proposition 6.4 *Suppose that* A *is sectorial with* $0 \in \varrho(A)$. *Then*

$$V_{0-} = \mathcal{R}\begin{pmatrix} I \\ X_{0-}\end{pmatrix}, \qquad V_{0+} = \mathcal{R}\begin{pmatrix} Y_{0+} \\ I\end{pmatrix}$$

with closed operators $X_{0-} : \mathcal{D}(X_{0-}) \subset H_{-r} \to H_{-s}^{(*)}$ *and* $Y_{0+} : \mathcal{D}(Y_{0+}) \subset H_{-s}^{(*)} \to H_{-r}$.

Proof. Let $(0,w) \in V_{0-}$ and $t \in \mathbb{R}$. Proceeding as in the previous proof, we set

$$\begin{pmatrix} x \\ y \end{pmatrix} = (T_0 - it)^{-1} \begin{pmatrix} 0 \\ w \end{pmatrix}$$

and obtain $Cx = B^*y = 0$ and hence $(A - it)x = 0$ and $w = -(A^* + it)y$. Since $i\mathbb{R} \subset \varrho(A)$ it follows that

$$(T_0 - it)^{-1} \begin{pmatrix} 0 \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ (-A^* - it)^{-1}w \end{pmatrix}.$$

We consider now the two functions

$$\varphi(\lambda) = (T_0 - \lambda)^{-1} \begin{pmatrix} 0 \\ w \end{pmatrix}, \qquad \psi(\lambda) = \begin{pmatrix} 0 \\ (-A^* - \lambda)^{-1}w \end{pmatrix}.$$

φ is analytic on a strip $\{\lambda \in \mathbb{C} \,|\, |\operatorname{Re}\lambda| < \varepsilon\}$ while ψ is analytic on a half-plane $\{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda < \varepsilon\}$ where $\varepsilon > 0$ is sufficiently small. The above derivation shows that φ and ψ coincide on $i\mathbb{R}$. Hence they coincide for $|\operatorname{Re}\lambda| < \varepsilon$ by the identity theorem. Moreover ψ is bounded on $\overline{\mathbb{C}_-}$ since A is sectorial with $0 \in \varrho(A)$. On the other hand φ extends to a bounded analytic function on $\overline{\mathbb{C}_+}$ since $(0,w) \in V_{0-}$, see Theorem 2.6. Therefore φ extends to a bounded entire function and is thus constant by Liouville's theorem. This implies $w = 0$.

Similarly for $(w,0) \in V_{0+}$, $t \in \mathbb{R}$ and

$$\begin{pmatrix} x \\ y \end{pmatrix} = (T_0 - it)^{-1} \begin{pmatrix} w \\ 0 \end{pmatrix}$$

we derive $Cx = B^*y = 0$, $w = (A - it)x$ and $(A^* + it)y = 0$; hence

$$(T_0 - it)^{-1} \begin{pmatrix} w \\ 0 \end{pmatrix} = \begin{pmatrix} (A - it)^{-1}w \\ 0 \end{pmatrix}.$$

In this case the analytic functions

$$\varphi(\lambda) = (T_0 - \lambda)^{-1} \begin{pmatrix} w \\ 0 \end{pmatrix}, \qquad \psi(\lambda) = \begin{pmatrix} (A - \lambda)^{-1}w \\ 0 \end{pmatrix}$$

coincide on $i\mathbb{R}$, φ is bounded on $\overline{\mathbb{C}_-}$ since $(w,0) \in V_{0+}$, and ψ is bounded on $\overline{\mathbb{C}_+}$. Therefore φ is again constant and hence $w = 0$. $\square$

We turn to the question of the boundedness of the operators $X_{0\pm}$, $Y_{0\pm}$. To this end we recall the concept of angular subspaces, see [1, §5.1], [23, Lemma 7.1]: Let $Q_{0\pm} \in L(V_0)$ be the projections defined in (31),

$$Q_{0-} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q_{0+} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

A closed subspace $U \subset V_0$ is the graph $U = \mathcal{R}\left(\begin{smallmatrix} I \\ X \end{smallmatrix}\right)$ of a *bounded* operator $X \in L(H_{-r}, H_{-s}^{(*)})$ if and only if

$$V_0 = U \oplus \ker Q_{0-}.$$

In this case U is called *angular* with respect to Q_{0-} and X is called the *angular operator* for U. Similarly $U = \mathcal{R}\left(\begin{smallmatrix} Y \\ I \end{smallmatrix}\right)$ with $Y \in L(H_{-s}^{(*)}, H_{-r})$ if and only if $V_0 = U \oplus \ker Q_{0+}$, i.e., U is angular with respect to Q_{0+}. On the other hand, we know that $U = \mathcal{R}\left(\begin{smallmatrix} I \\ X \end{smallmatrix}\right)$ with a possibly unbounded operator $X : \mathcal{D}(X) \subset H_{-r} \to H_{-s}^{(*)}$ if and only if

$$U \cap \ker Q_{0-} = \{0\},$$

and $U = \mathcal{R}\left(\begin{smallmatrix} Y \\ I \end{smallmatrix}\right)$ with possibly unbounded Y if and only if $U \cap \ker Q_{0+} = \{0\}$.

The idea for the proof of the next lemma goes back to [4, Theorem 2.3], see also [1, §6.4], where instead of F_1 and F_2 the operator $Q_{0-}P + Q_{0+}\tilde{P}$ is used.

Lemma 6.5 *Suppose $V_0 = U \oplus \tilde{U}$ with closed subspaces $U, \tilde{U} \subset V_0$. Let $P, \tilde{P} \in L(V_0)$ be the associated complementary projections, $U = \mathcal{R}(P)$, $\tilde{U} = \mathcal{R}(\tilde{P})$, $I = P + \tilde{P}$. Let $F_1 = I - Q_{0-} + P$ and $F_2 = I - P + Q_{0-}$.*

(a) If

$$U = \mathcal{R}\begin{pmatrix} I \\ X \end{pmatrix}, \qquad \tilde{U} = \mathcal{R}\begin{pmatrix} Y \\ I \end{pmatrix} \tag{42}$$

with some $X : \mathcal{D}(X) \subset H_{-r} \to H_{-s}^{(*)}$ and $Y : \mathcal{D}(Y) \subset H_{-s}^{(*)} \to H_{-r}$, then F_1 and F_2 are injective.

(b) If F_1 and F_2 are bijective, then (42) *holds with bounded operators $X \in L(H_{-r}, H_{-s}^{(*)})$, $Y \in L(H_{-s}^{(*)}, H_{-r})$.*

Proof. (a) The identity (42) implies that $U \cap \ker Q_{0-} = \tilde{U} \cap \ker Q_{0+} = \{0\}$. Let $F_1 v = 0$. Then $(I - Q_{0-})v = -Pv \in U \cap \ker Q_{0-}$, which implies $(I - Q_{0-})v = Pv = 0$. It follows that $v \in \mathcal{R}(Q_{0-}) \cap \ker P = \ker Q_{0+} \cap \tilde{U}$ and hence $v = 0$. The injectivity of F_2 is analogous.

(b) Let $v \in U \cap \ker Q_{0-}$. Then $(I - P)v = Q_{0-}v = 0$, which yields $F_2 v = 0$ and thus $v = 0$. On the other hand we can write $w \in V_0$ as $w = F_1 v = (I - Q_{0-})v + Pv$ and so $w \in U + \ker Q_{0-}$. This shows that $V_0 = U \oplus \ker Q_{0-}$, i.e., U is angular with respect to Q_{0-}. Since $F_1 = I - \tilde{P} + Q_{0+}$ and $F_2 = I - Q_{0+} + \tilde{P}$, we get by symmetry that $\tilde{U}$ is angular to Q_{0+}.

□

Corollary 6.6 *Suppose that $P_{0-} - Q_{0-}$ is compact. If*

$$V_{0-} = \mathcal{R}\begin{pmatrix} I \\ X_{0-} \end{pmatrix}, \qquad V_{0+} = \mathcal{R}\begin{pmatrix} Y_{0+} \\ I \end{pmatrix},$$

with some operators X_{0-}, Y_{0+}, then these operators are in fact bounded, $X_{0-} \in L(H_{-r}, H_{-s}^{(*)})$, $Y_{0+} \in L(H_{-s}^{(*)}, H_{-r})$.

Proof. We use the previous lemma with $U = V_{0-}$, $\tilde{U} = V_{0+}$, $P = P_{0-}$, $\tilde{P} = P_{0+}$. Then $F_1 = I + (P_{0-} - Q_{0-})$ and $F_2 = I - (P_{0-} - Q_{0-})$, and the assertion follows from Fredholm's alternative. □

Theorem 6.7 *Suppose that A has a compact resolvent. If*

$$\bigcap_{\lambda \in i\mathbb{R} \cap \varrho(A^*)} \ker B^*(A^* - \lambda)^{-1} = \{0\} \qquad \text{on} \quad H_{-s}^{(*)}, \tag{43}$$

and

$$\bigcap_{\lambda \in i\mathbb{R} \cap \varrho(A)} \ker C(A - \lambda)^{-1} = \{0\} \qquad \text{on} \quad H_{-r}, \tag{44}$$

then $V_{0\pm} = \mathcal{R}\begin{pmatrix} I \\ X_{0\pm} \end{pmatrix}$ where the operators X_{0-} and X_{0+} are injective, $X_{0-} \in L(H_{-r}, H_{-s}^{(*)})$ and $X_{0+}^{-1} \in L(H_{-s}^{(*)}, H_{-r})$.

Proof. If A has a compact resolvent, then the same is true for S_0 and T_0, compare Lemma 4.3. From Theorem 2.6 and Lemma 5.4 we know that

$$P_{0+}v - P_{0-}v = \frac{1}{\pi i} \int_{-i\infty}^{i\infty}{}' (T_0 - \lambda)^{-1} v \, d\lambda, \qquad v \in \mathcal{D}(T_0),$$

$$Q_{0+}v - Q_{0-}v = \frac{1}{\pi i} \int_{\gamma_1}{}' (S_0 - \lambda)^{-1} v \, d\lambda + Kv, \qquad v \in V_0,$$

where $K \in L(V_0)$. Since

$$Q_{0+} - Q_{0-} - (P_{0+} - P_{0-}) = I - 2Q_{0-} - (I - 2P_{0-}) = 2(P_{0-} - Q_{0-})$$

we find

$$2(P_{0-} - Q_{0-})v = \frac{1}{\pi i} \int_{\gamma_1} (S_0 - \lambda)^{-1} - (T_0 - \lambda)^{-1} \, d\lambda \, v \\ - \frac{1}{\pi i} \int_{-i\rho}^{i\rho} (T_0 - \lambda)^{-1} \, d\lambda \, v + Kv$$

for $v \in \mathcal{D}(T_0)$. Note here that because of (28) the first integral converges in the operator norm topology of $L(V_0)$. In particular, both integrals on the right-hand side define bounded operators in $L(V_0)$ and hence the above identity holds for all $v \in V_0$. Since $(T_0 - \lambda)^{-1}$ and $(S_0 - \lambda)^{-1}$ are compact, both integrals yield in fact compact operators. The expression for K in Lemma 5.4 implies that K is compact too. Consequently $P_{0-} - Q_{0-}$ is compact. The assertion is now a consequence of Proposition 6.3 and Corollary 6.6. □

Theorem 6.8 *Suppose that A has a compact resolvent, is sectorial and $0 \in \varrho(A)$. Then*

$$V_{0-} = \mathcal{R}\begin{pmatrix} I \\ X_{0-} \end{pmatrix}, \qquad V_{0+} = \mathcal{R}\begin{pmatrix} Y_{0+} \\ I \end{pmatrix}$$

with $X_{0-} \in L(H_{-r}, H_{-s}^{(*)})$, $Y_{0+} \in L(H_{-s}^{(*)}, H_{-r})$.

Proof. As in the previous theorem we obtain that $P_{0-} - Q_{0-}$ is compact. Hence Proposition 6.4 and Corollary 6.6 complete the proof. □

Next we investigate the graph properties of the invariant subspaces $V_\pm$ of T. We know that $V_\pm = \mathcal{R}(P_\pm)$ where $P_\pm$ are the closed projections on V given by $P_\pm = TL_\pm$ with $L_\pm \in L(V)$,

$$L_\pm = \frac{\pm 1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda}(T-\lambda)^{-1}\, d\lambda.$$

In particular $L_\pm$ are the restrictions of $L_{0\pm}$ to V. Since $V_\pm = \ker L_\mp$ and $\ker L_\mp = \ker L_{0\mp} \cap V$ it follows that

$$V_\pm = V_{0\pm} \cap V. \tag{45}$$

This implies that graph subspace structures of $V_{0\pm}$ are inherited by the spaces $V_\pm$:

Lemma 6.9 *If*

$$V_{0+} = \mathcal{R}\begin{pmatrix} I \\ X_{0+} \end{pmatrix}$$

with a closed operator $X_{0+} : \mathcal{D}(X_{0+}) \subset H_{-r} \to H_{-s}^{(*)}$, then also

$$V_{+} = \mathcal{R}\begin{pmatrix} I \\ X_{+} \end{pmatrix}$$

where $X_+ : \mathcal{D}(X_+) \subset H \to H$ is closed and is the part of X_{0+} in H, i.e. $\mathcal{D}(X_+) = \{x \in \mathcal{D}(X_{0+}) \cap H \,|\, X_{0+}x \in H\}$. Similarly, if

$$V_{0+} = \mathcal{R}\begin{pmatrix} Y_{0+} \\ I \end{pmatrix}$$

with a closed operator $Y_{0+} : \mathcal{D}(Y_{0+}) \subset H_{-s}^{(*)} \to H_{-r}$, then

$$V_{+} = \mathcal{R}\begin{pmatrix} Y_{+} \\ I \end{pmatrix}$$

where $Y_+ : \mathcal{D}(Y_+) \subset H \to H$ is closed and is the part of Y_{0+} in H. The corresponding statements hold for V_{0-} and V_-.

Proof. This is immediate from (45) and the fact that $V_\pm$ are closed subspaces of $V = H \times H$. □

Remark 6.10 A result analogous to Corollary 6.6 holds for the subspaces $V_\pm$ of V in the case that T is strictly dichotomous, i.e. if $P_\pm \in L(V)$. In particular if $P_- - Q_-$ is compact where $Q_- = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right) \in L(V)$ and

$$V_- = \mathcal{R}\begin{pmatrix} I \\ X_- \end{pmatrix}, \qquad V_+ = \mathcal{R}\begin{pmatrix} Y_+ \\ I \end{pmatrix},$$

then $X_-, Y_+ \in L(H)$.

Theorem 6.11 *Suppose that A has a compact resolvent and that* $\max\{r, s\} < \frac{1}{2}$.

(a) If (43) and (44) hold, then $V_\pm = \mathcal{R}\left(\begin{smallmatrix} I \\ X_\pm \end{smallmatrix}\right)$ where $X_\pm$ are the parts of $X_{0\pm}$ in H. The operators $X_\pm$ are injective and satisfy $X_-, X_+^{-1} \in L(H)$.

(b) If A is sectorial and $0 \in \varrho(A)$, then $V_- = \mathcal{R}\left(\begin{smallmatrix} I \\ X_- \end{smallmatrix}\right)$, $V_+ = \mathcal{R}\left(\begin{smallmatrix} Y_+ \\ I \end{smallmatrix}\right)$ where X_- and Y_+ are the parts of X_{0-} and Y_{0+} in H, respectively, and $X_-, Y_+ \in L(H)$.

Proof. The proof is analogous to the ones of Theorem 6.7 and 6.8, where it is shown that $V_{0\pm}$ are angular subspaces. First note that S and T have a compact resolvent, see Lemma 4.3. Second, since $\max\{r, s\} < \frac{1}{2}$ and since $i\mathbb{R} \subset \varrho(T)$ by our general assumption in this section, Theorem 5.9 in conjunction with Lemma 4.1 implies that T is strictly dichotomous. Consequently the projections $P_\pm$ are bounded and satisfy

$$P_+ v - P_- v = \frac{1}{\pi i} \int_{-i\infty}^{i\infty\,\prime} (T - \lambda)^{-1} v \, d\lambda, \qquad v \in \mathcal{D}(T).$$

On the other hand, for $Q_\pm \in L(V)$ given by $Q_- = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $Q_+ = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$ the identity

$$Q_+ v - Q_- v = \frac{1}{\pi i} \int_{\gamma_1}' (S - \lambda)^{-1} v \, d\lambda + K v, \qquad v \in V,$$

holds with some $K \in L(V)$, see Lemma 5.4 and Remark 5.8. Consequently

$$2(P_- - Q_-) v = \frac{1}{\pi i} \int_{\gamma_1} (S - \lambda)^{-1} - (T - \lambda)^{-1} \, d\lambda \, v \\ - \frac{1}{\pi i} \int_{-i\rho}^{i\rho} (T - \lambda)^{-1} \, d\lambda \, v + K v$$

for $v \in V$, where we have used that in view of $\max\{r, s\} < \frac{1}{2}$ and (34) all terms on the right-hand side yield bounded operators from $L(V)$. Since the resolvents of S and T are compact, we conclude that $P_- - Q_-$ is compact too. The assertion now follows from Theorems 6.7 and 6.8, Lemma 6.9 and Remark 6.10. □

7 Symmetries of the angular operators

The aim of this section is to derive symmetry properties for the operators $X_{0\pm}$ and $X_\pm$. We keep our general assumptions on the Hamiltonian: A is quasi-sectorial, $r+s<1$ and $i\mathbb{R}\subset\varrho(T_0)$. Hence T_0 is bisectorial, strictly dichotomous and the invariant subspaces are given by

$$V_{0\pm}=\mathcal{R}(P_{0\pm})=\ker L_{0\mp}$$

where $P_{0\pm}=TL_{0\pm}$, $L_{0\pm}\in L(V_0,V_1)$ and

$$[L_{0+}v|w]=-[v|L_{0-}w],\qquad v,w\in V_0, \tag{46}$$

with the extended indefinite inner product defined in (24), see Lemma 6.1.

For a subspace $U\subset V_1$ we consider its orthogonal complement $U^{[\perp]}\subset V_0$ with respect to the extended inner product:

$$U^{[\perp]}=\{w\in V_0\,|\,[v|w]=0\text{ for all }v\in V_1\}\,.$$

For $\tilde U\subset V_0$ the orthogonal complement $\tilde U^{[\perp]}\subset V_1$ is defined analogously. Then, as in the usual Hilbert or Krein space setting, orthogonal complements are closed and $U^{[\perp][\perp]}=\overline{U}$. Let $V_{1\pm}$ be the closure of $\mathcal{R}(L_{0\pm})$ in V_1,

$$V_{1\pm}=\overline{\mathcal{R}(L_{0\pm})}^{V_1}. \tag{47}$$

Lemma 7.1 *The following identities hold:*

(a) $V_{1\pm}^{[\perp]}=V_{0\pm}$,

(b) $V_{1\pm}=V_{0\pm}\cap V_1$.

Proof. (a) From (46) we get

$$V_{0\pm}=\ker L_{0\mp}\subset\mathcal{R}(L_{0\pm})^{[\perp]}=V_{1\pm}^{[\perp]}.$$

If on the other hand $w\in V_{1\pm}^{[\perp]}$, then $[v|L_{0\mp}w]=-[L_{0\pm}v|w]=0$ for all $v\in V_0$. Since the inner product is non-degenerate, this implies $L_{0\mp}w=0$ and thus $w\in V_{0\pm}$.

(b) By Lemma 2.7 we have $\mathcal{R}(L_{0\pm})\subset V_{0\pm}$. By the continuity of the imbedding $V_1\hookrightarrow V_0$, the subspace $V_{0\pm}\cap V_1$ is closed in V_1, and hence the inclusion from left to right follows. For the reverse inclusion let $v\in V_{0\pm}\cap V_1$. Then

$$[w|v]=0\qquad\text{for all }w\in V_{1\pm}$$

by (a). Since T_0 is densely defined and strictly dichotomous, Lemma 2.7 implies $\overline{\mathcal{R}(L_{0\pm})}^{V_0}=V_{0\pm}$. Hence $\overline{V_{1\pm}}^{V_0}=V_{0\pm}$ and therefore

$$[w|v]=0\qquad\text{for all }w\in V_{0\pm}.$$

Consequently $v\in V_{0\pm}^{[\perp]}=V_{1\pm}^{[\perp][\perp]}=V_{1\pm}$.

□

Let $X_1 : \mathcal{D}(X_1) \subset H_s^{(*)} \to H_r$ be a densely defined operator. We define its adjoint with respect to the scales of Hilbert spaces $\{H_r\}$ and $\{H_s^{(*)}\}$ as the operator $X_1^* : \mathcal{D}(X_1^*) \subset H_{-r} \to H_{-s}^{(*)}$ with maximal domain such that

$$(X_1 x|y)_{r,-r} = (x|X_1^* y)_{s,-s}^{(*)}, \qquad x \in \mathcal{D}(X_1),\, y \in \mathcal{D}(X_1^*). \tag{48}$$

Then X_1^* is uniquely determined and closed.

Lemma 7.2 *If $V_{0-} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{0-} \end{smallmatrix}\right)$ with a closed operator*

$$X_{0-} : \mathcal{D}(X_{0-}) \subset H_{-r} \to H_{-s}^{(*)},$$

then also $V_{1-} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{1-} \end{smallmatrix}\right)$ with a closed operator

$$X_{1-} : \mathcal{D}(X_{1-}) \subset H_s^{(*)} \to H_r.$$

In this case:

(a) $\mathcal{D}(X_{1-}) = \left\{x \in \mathcal{D}(X_{0-}) \cap H_s^{(*)} \,\middle|\, X_{0-}x \in H_r\right\}$, i.e., X_{1-} is the part of X_{0-} in the space of operators from $H_s^{(*)}$ to H_r;

(b) X_{1-} and X_{0-} are densely defined and $X_{1-}^* = X_{0-}$;

(c) the set $\left\{x \in \mathcal{D}(X_{0-}) \cap H_{1-r}^{(*)} \,\middle|\, X_{0-}x \in H_{1-s}\right\}$ is a core for X_{1-} and X_{0-}.

Analogous statements hold for the spaces V_{0+}, V_{1+} and the operators X_{0+}, X_{1+}.

Proof. The inclusion $V_{1-} \subset V_{0-}$ implies that if V_{0-} is a graph, then so is V_{1-} and that X_{1-} is a restriction of X_{0-}. X_{1-} is closed since V_{1-} is closed in $V_1 = H_s^{(*)} \times H_r$. (a) is now immediate from $V_{1-} = V_{0-} \cap V_1$.

To show (b), suppose $x \in H_{-r}$, $y \in H_{-s}^{(*)}$ are such that

$$(X_{1-}u|x)_{r,-r} = (u|y)_{s,-s}^{(*)} \qquad \text{for all } u \in \mathcal{D}(X_{1-}). \tag{49}$$

Then

$$\left[\begin{pmatrix} u \\ X_{1-}u \end{pmatrix} \,\middle|\, \begin{pmatrix} x \\ y \end{pmatrix}\right] = 0, \qquad u \in \mathcal{D}(X_{1-}),$$

i.e., $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in V_{1-}^{[\perp]} = V_{0-}$ and thus $x \in \mathcal{D}(X_{0-})$, $X_{0-}x = y$. This implies that $\mathcal{D}(X_{1-})$ is dense in $H_s^{(*)}$. Indeed if $y \in H_{-s}^{(*)}$ with $(u|y)_{s,-s} = 0$ for all $u \in \mathcal{D}(X_{1-})$, then (49) holds with $x = 0$ and it follows that $y = 0$. On the other hand $V_{1-}^{[\perp]} = V_{0-}$ implies

$$i(u|X_{0-}x)_{s,-s}^{(*)} - i(X_{1-}u|x)_{r,-r} = \left[\begin{pmatrix} u \\ X_{1-}u \end{pmatrix} \,\middle|\, \begin{pmatrix} x \\ X_{0-}x \end{pmatrix}\right] = 0$$

for all $u \in \mathcal{D}(X_{1-})$, $x \in \mathcal{D}(X_{0-})$ and therefore $X_{0-} \subset X_{1-}^*$. Moreover if $x \in \mathcal{D}(X_{1-}^*)$ and $y = X_{1-}^* x$, then x, y satisfy (49) and we obtain $x \in \mathcal{D}(X_{0-})$. Consequently $X_{0-} = X_{1-}^*$. Finally X_{0-} is densely defined since $\mathcal{D}(X_{1-})$ is dense in $H_s^{(*)}$ and the imbedding $H_s^{(*)} \hookrightarrow H_{-r}$ is continuous and dense.

Finally (c) follows from the equivalence

$$u \in \mathcal{D}(X_{0-}) \cap H_{1-r}^{(*)} \wedge X_{0-}u \in H_{1-s} \quad \Longleftrightarrow \quad \begin{pmatrix} u \\ X_{0-}u \end{pmatrix} \in V_{0-} \cap \mathcal{D}(T_0)$$

in conjunction with $\mathcal{R}(L_{0-}) = V_{0-} \cap \mathcal{D}(T_0)$, $V_{0-} = \overline{\mathcal{R}(L_{0-})}^{V_0}$, see Lemma 2.7, and $V_{1-} = \overline{\mathcal{R}(L_{0-})}^{V_1}$. □

Remark 7.3 The previous lemma implies $X_{1\pm} \subset X_{0\pm} = X_{1\pm}^*$. From this identity and (48) we obtain

$$(X_{1\pm}x|y) = (x|X_{1\pm}y), \qquad x, y \in \mathcal{D}(X_{1\pm}).$$

Consequently, if we consider $X_{1\pm}$ as an unbounded operator on H, then it is densely defined and symmetric and hence closable. The corresponding closure will be determined in Lemma 7.5.

Now we turn to the symmetry properties of the operators $X_{\pm}$. To this end, we look at the subspaces

$$M_{\pm} = \overline{\mathcal{R}(L_{\pm})}^{V} \tag{50}$$

of V. By Lemma 2.7 we have $M_{\pm} \subset V_{\pm}$ and this inclusion may be strict. The next lemma shows that $M_{\pm}^{[\perp]}$ coincides with $V_{\pm}$. Note here that since $M_{\pm} \subset V$, $M_{\pm}^{[\perp]}$ is the orthogonal complement with respect to the inner product $[\cdot|\cdot]$ *in* V, i.e. $M_{\pm}^{[\perp]} \subset V$ in the usual Krein space sense.

Lemma 7.4 *The following identities hold:*

(a) $V_{1\pm} \subset M_{\pm}$ *and* $\overline{V_{1\pm}}^{V} = M_{\pm}$*;*

(b) $M_{\pm}^{[\perp]} = V_{\pm}$*.*

Proof. (a) Since $\mathcal{D}(T_0)$ is dense in V_0 and $L_{0\pm} \in L(V_0, V_1)$, we have

$$V_{1\pm} = \overline{\mathcal{R}(L_{0\pm})}^{V_1} \subset \overline{L_{0\pm}(\mathcal{D}(T_0))}^{V_1} \subset \overline{L_{0\pm}(\mathcal{D}(T_0))}^{V} \subset \overline{L_{0\pm}(V)}^{V} = M_{\pm}.$$

On the other hand $\mathcal{R}(L_{\pm}) \subset \mathcal{R}(L_{0\pm}) \subset V_{1\pm}$, which implies $M_{\pm} \subset \overline{V_{1\pm}}^{V}$ and thus equality.

(b) Lemma 6.1 implies $[L_+v|w] = -[v|L_-w]$ for all $v, w \in V$. Using this and the definitions of $V_{\pm}$ and $M_{\pm}$, the proof is completely analogous to Lemma 7.1(a).

□

Lemma 7.5 *Suppose V_{0-} is a graph subspace $V_{0-} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{0-} \end{smallmatrix}\right)$. Then $V_- = \mathcal{R}\left(\begin{smallmatrix} I \\ X_- \end{smallmatrix}\right)$ and $M_- = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{M-} \end{smallmatrix}\right)$ where X_-, X_{M-} are closed operators on H. Moreover*

(a) $X_{M-} \subset X_-$,

(b) X_- is the part of X_{0-} in H,

(c) X_{M-} is the closure of X_{1-} when considered as an operator on H,

(d) $\left\{x \in \mathcal{D}(X_{0-}) \cap H_{1-r}^{(*)} \,\middle|\, X_{0-}x \in H_{1-s}\right\}$ is a core for X_{M-},

(e) X_{M-} and X_- are densely defined and $X_{M-}^* = X_-$. In particular X_{M-} is symmetric.

Again, analogous statements hold for V_{0+}, V_+ and M_+ and the respective operators.

Proof. The first assertions up to (c) follow readily from $M_- \subset V_- \subset V_{0-}$, $V_- = V_{0-} \cap V$, $\overline{V_{1-}}^V = M_-$ and the closedness of M_- and V_- in V. (d) is a consequence of (c) and Lemma 7.2(c), and (e) follows from $M_-^{[\perp]} = V_-$ in an analogous way to the proof of Lemma 7.2(b). □

Lemma 7.6 *The symmetric operators X_{M-} and X_{M+} are nonnegative and nonpositive, respectively.*

Proof. Here we employ the indefinite inner product $[\cdot|\cdot]_\sim$ defined in (23). Observe that X_{M-} is nonnegative, i.e., $(X_{M-}x|x) \geq 0$ for all $x \in \mathcal{D}(X_{M-})$, if and only if $[v|v]_\sim \geq 0$ for all $v \in M_-$. Likewise $(X_{M+}x|x) \leq 0$ for all $x \in \mathcal{D}(X_{M+})$ if and only if $[v|v]_\sim \leq 0$ for all $v \in M_+$. Consider first $v \in \mathcal{D}(T)$. Using (13) and Lemma 4.4, we calculate

$$\begin{aligned}
\operatorname{Re}[P_+v - P_-v|v]_\sim &= \frac{1}{\pi}\int_{-\infty}^{\infty\,\prime} \operatorname{Re}[(T-it)^{-1}v|v]_\sim \, dt \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty\,\prime} \operatorname{Re}[(T-it)^{-1}v|(T-it)(T-it)^{-1}v]_\sim \, dt \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty\,\prime} \operatorname{Re}[T(T-it)^{-1}v|(T-it)^{-1}v]_\sim \, dt \leq 0.
\end{aligned}$$

If now $v \in \mathcal{D}(T) \cap V_-$ then $P_+v - P_-v = -v$ and hence $[v|v]_\sim \geq 0$. Since $\mathcal{D}(T) \cap V_-$ is dense in M_- by Lemma 2.7, we conclude that $[v|v]_\sim \geq 0$ for $v \in M_-$. Similarly for $v \in \mathcal{D}(T) \cap V_+$ we obtain $P_+v - P_-v = v$ and thus $[v|v]_\sim \leq 0$ for all $v \in M_+$. □

Corollary 7.7 *If $\max\{r, s\} < \frac{1}{2}$, then $X_{M\pm} = X_\pm$. The operator X_- is selfadjoint and nonnegative, X_+ is selfadjoint and nonpositive.*

Proof. The assumption implies that T is strictly dichotomous. Then $M_\pm = V_\pm$ by Lemma 2.7 and hence $X_{M\pm} = X_\pm$. □

8 The Riccati equation

We keep the general assumptions of the previous section.

Lemma 8.1 *Suppose* $X_0 \in L(H_{-r}, H_{-s}^{(*)})$ *is such that its graph subspace* $U = \mathcal{R}\left(\begin{smallmatrix} I \\ X_0 \end{smallmatrix}\right)$ *is* T_0*- and* $(T_0 - \lambda)^{-1}$*-invariant. Consider the isomorphism* $\varphi : H_{-r} \to U$, $x \mapsto \left(\begin{smallmatrix} x \\ X_0 x \end{smallmatrix}\right)$. *Then*

(a) $X_0(H_{1-r}^{(*)}) \subset H_{1-s}$;

(b) $(A - BB^* X_0)x = \varphi^{-1} T_0|_U \varphi x$ *for all* $x \in H_{1-r}^{(*)}$;

(c) $A^* X_0 x + X_0 A x - X_0 BB^* X_0 x + C^* C x = 0$ *for all* $x \in H_{1-r}^{(*)}$.

Proof. First note that φ is indeed an isomorphism between H_{-r} and U since X_0 is bounded. The inverse is $\varphi^{-1} = \mathrm{pr}_1|_U$ where

$$\mathrm{pr}_1 : V_0 = H_{-r} \times H_{-s}^{(*)} \to H_{-r}$$

denotes the projection onto the first component. Recall the decomposition $T_0 = S_0 + R$ from (25) and consider the two operators $F = \varphi^{-1} T_0|_U \varphi$ and $A_0 = \mathrm{pr}_1 S_0 \varphi$, both understood as unbounded operators on H_{-r}. Since $\mathcal{D}(T_0) = \mathcal{D}(S_0) = H_{1-r}^{(*)} \times H_{1-s}$, their domains are

$$\mathcal{D}(F) = \mathcal{D}(A_0) = \left\{ x \in H_{1-r}^{(*)} \,\middle|\, X_0 x \in H_{1-s} \right\}.$$

Moreover

$$A_0 x = Ax \qquad \text{for } x \in \mathcal{D}(A_0),$$

i.e., A_0 is a restriction of A when A is considered as an operator on H_{-r} with $\mathcal{D}(A) = H_{1-r}^{(*)}$. Since φ is an isomorphism we get $\varrho(F) = \varrho(T_0|_U)$. Also $\varrho(T_0) \subset \varrho(T_0|_U)$ by the invariance of U. Therefore $i\mathbb{R} \subset \varrho(F)$. For $t \in \mathbb{R}$ we compute

$$\begin{aligned}(A_0 - F)(F - it)^{-1} &= (\mathrm{pr}_1 S_0 \varphi - \varphi^{-1} T_0 \varphi)(\varphi^{-1} T_0 \varphi - it)^{-1} \\ &= \mathrm{pr}_1 (S_0 - T_0)\varphi\varphi^{-1}(T_0 - it)^{-1}\varphi = -\mathrm{pr}_1 R (T_0 - it)^{-1}\varphi.\end{aligned}$$

From (39) in the proof of Lemma 6.1 we know that $\|(T_0 - it)^{-1}\|_{L(V_0, V_1)} \to 0$ as $t \to \infty$, and we conclude that $\|(A_0 - F)(F - it)^{-1}\| < 1$ for $t > 0$ sufficiently large. Now

$$A_0 - it = F - it + A_0 - F = \left(I + (A_0 - F)(F - it)^{-1}\right)(F - it),$$

which implies that $it \in \varrho(A_0)$. Since also $it \in \varrho(A)$ for large t and $A_0 \subset A$, it follows that in fact

$$\mathcal{D}(A_0) = \mathcal{D}(A) = H_{1-r}^{(*)}.$$

Consequently $X_0(H^{(*)}_{1-r}) \subset H_{1-s}$. Since $Fx = Ax - BB^*X_0x$ for $x \in \mathcal{D}(F) = \mathcal{D}(A_0)$, (b) is now clear. To show (c) let $x \in H^{(*)}_{1-r}$. Then $X_0x \in H_{1-s}$ and $\varphi x \in \mathcal{D}(T_0)$. By the invariance of U there exists $y \in H^{(*)}_{1-r}$ such that $T_0\varphi x = \varphi y$, i.e.,

$$\begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix} \begin{pmatrix} x \\ X_0x \end{pmatrix} = \begin{pmatrix} y \\ X_0y \end{pmatrix}$$

and thus

$$X_0Ax - X_0BB^*X_0x = X_0(Ax - BB^*X_0x) = X_0y = -C^*Cx - A^*X_0x.$$

□

Corollary 8.2 *If* $V_{0-} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{0-} \end{smallmatrix}\right)$ *with a bounded operator* $X_{0-} \in L(H_{-r}, H^{(*)}_{-s})$, *then* $X_{0-}(H^{(*)}_{1-r}) \subset H_{1-s}$, *the Riccati equation*

$$A^*X_{0-}x + X_{0-}Ax - X_{0-}BB^*X_{0-}x + C^*Cx = 0, \qquad x \in H^{(*)}_{1-r},$$

holds, and $A - BB^*X_{0-}$ *considered as an unbounded operator on* H_{-r} *is sectorial with spectrum* $\sigma(A - BB^*X_{0-}) \subset \mathbb{C}_-$. *In particular, it generates an exponentially stable analytic semigroup on* H_{-r}.

Proof. $A - BB^*X_{0-}$ is similar to $T_0|_{V_{0-}}$ via the isomorphism φ from the previous lemma, $\sigma(T_0|_{V_{0-}}) \subset \mathbb{C}_-$, and $T_0|_{V_{0-}}$ is sectorial by [21, Theorem 5.6]. □

Remark 8.3 If $X_{0-} \in L(H_{-r}, H^{(*)}_{-s})$ and hence $X_{0-}(H^{(*)}_{1-r}) \subset H_{1-s}$, Lemma 7.2 and 7.5 imply that $H^{(*)}_{1-r} \subset \mathcal{D}(X_{1-}) \subset \mathcal{D}(X_-)$. Since the operator $A - BB^*X_{0-}$ considered on H_{-r} has domain $H^{(*)}_{1-r}$ we find that

$$A - BB^*X_{0-} = A - BB^*X_- = A - BB^*X_{1-}.$$

Hence the Riccati equation can be written as

$$A^*X_{1-}x + X_{0-}Ax - X_{0-}BB^*X_{1-}x + C^*Cx = 0, \qquad x \in H^{(*)}_{1-r},$$

or in weak form, using $X_{0-} = X^*_{1-}$, as

$$(X_{1-}x|Ay)_{r,-r} + (Ax|X_{1-}y)_{-r,r} - (B^*X_{1-}x|B^*X_{1-}y)_U \\ + (Cx|Cy)_Y = 0, \quad x, y \in H^{(*)}_{1-r}.$$

Of course, in both Riccati equations X_{1-} may be replaced by one of its extensions X_{M-} and X_-.

Remark 8.4 For $X_{0-} \in L(H_{-r}, H_{-s}^{(*)})$ Corollary 8.2 yields that $A - BB^*X_-$ is sectorial *when considered as an operator in* H_{-r}. On the other hand, we can consider the part of $A - BB^*X_-$ in H, which we denote by $(A - BB^*X_-)|_H$. Then $(A - BB^*X_-)|_H$ is almost sectorial: First note that

$$\sigma\big((A - BB^*X_-)|_H\big) \subset \sigma(A - BB^*X_-).$$

From $A - BB^*X_- = \varphi^{-1}T_0|_{V_{0-}}\varphi$ we obtain

$$\|(A - BB^*X_- - \lambda)^{-1}\|_{L(H_{-r},H)} \le \|(T_0|_{V_{0-}} - \lambda)^{-1}\|_{L(V_{0-},V)}\|\varphi\|,$$

and (35) in conjunction with $i\mathbb{R} \subset \varrho(A - BB^*X_-)$ implies

$$\|(A - BB^*X_- - \lambda)^{-1}\|_{L(H_{-r},H)} \le \frac{c_0}{|\lambda|^{1-\max\{r,s\}}} \quad \text{for } \lambda \in i\mathbb{R} \setminus \{0\},$$

with some constant $c_0 > 0$. Moreover since $\|(T_0|_{V_{0-}} - \lambda)^{-1}\|_{L(V_0)}$ is bounded on $\mathbb{C}_+$, $\|(T_0|_{V_{0-}} - \lambda)^{-1}\|_{L(V_0,\mathcal{D}(T_0))}$ does not grow faster than $|\lambda|$ on $\mathbb{C}_+$, where $\mathcal{D}(T_0)$ is equipped with the graph norm. As the imbedding $\mathcal{D}(T_0) \hookrightarrow V$ is continuous, $\|(A - BB^*X_- - \lambda)^{-1}\|_{L(H_{-r},H)}$ does not grow faster than $|\lambda|$ on $\mathbb{C}_+$ too. The Phragmén-Lindelöf theorem then implies that

$$\|(A - BB^*X_- - \lambda)^{-1}\|_{L(H_{-r},H)} \le \frac{c_0}{|\lambda|^{1-\max\{r,s\}}} \quad \text{for } \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}$$

and hence $(A - BB^*X_-)|_H$ is almost sectorial, see [21, §5].

Now suppose in addition that $\max\{r, s\} < \frac{1}{2}$ and that $X_- \in L(H)$, e.g. as a consequence of Theorem 6.11. Then

$$(A - BB^*X_-)|_H = \varphi|_H^{-1}T|_{V_-}\varphi|_H$$

where $\varphi|_H : H \to V_-$, $x \mapsto \binom{x}{X_-x}$ is an isomorphism. Since T is bisectorial by Theorem 5.9, $T|_{V_-}$ is sectorial by [21, Theorem 5.6], and hence $(A-BB^*X_-)|_H$ is sectorial too.

References

[1] H. Bart, I. Gohberg, M. A. Kaashoek. *Minimal factorization of matrix and operator functions*, volume 1 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, 1979.

[2] J. M. Berezanskiĭ. *Expansions in eigenfunctions of selfadjoint operators.* Translated from the Russian by R. Bolstein, J. M. Danskin, J. Rovnyak and L. Shulman. Translations of Mathematical Monographs, Vol. 17. American Mathematical Society, Providence, R.I., 1968.

[3] S. Bittanti, A. J. Laub, J. C. Willems, editors. *The Riccati equation.* Communications and Control Engineering Series. Springer-Verlag, Berlin, 1991.

[4] P. Bubák, C. V. M. van der Mee, A. C. M. Ran. *Approximation of solutions of Riccati equations.* SIAM J. Control Optim., **44(4)** (2005), 1419–1435.

[5] F. M. Callier, L. Dumortier, J. Winkin. *On the nonnegative self-adjoint solutions of the operator Riccati equation for infinite-dimensional systems.* Integral Equations Operator Theory, **22(2)** (1995), 162–195.

[6] R. F. Curtain, H. J. Zwart. *An Introduction to Infinite Dimensional Linear Systems Theory.* Springer, New York, 1995.

[7] K.-J. Engel, R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations.* Springer, New York, 2000.

[8] M. Haase. *The functional calculus for sectorial operators*, volume 169 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, 2006.

[9] P. Jonas, H. Langer. *Some questions in the perturbation theory of J-nonnegative operators in Kreĭn spaces.* Math. Nachr., **114** (1983), 205–226.

[10] P. Jonas, C. Trunk. *On a class of analytic operator functions and their linearizations.* Math. Nachr., **243** (2002), 92–133.

[11] S. G. Kreĭn. *Linear Differential Equations in Banach Space.* Amer. Math. Soc., Providence, 1971.

[12] C. R. Kuiper, H. J. Zwart. *Connections between the algebraic Riccati equation and the Hamiltonian for Riesz-spectral systems.* J. Math. Systems Estim. Control, **6(4)** (1996), 1–48.

[13] P. Lancaster, L. Rodman. *Algebraic Riccati Equations.* Oxford University Press, Oxford, 1995.

[14] H. Langer, A. C. M. Ran, B. A. van de Rotten. *Invariant subspaces of infinite dimensional Hamiltonians and solutions of the corresponding Riccati equations.* In *Linear Operators and Matrices*, volume 130 of Oper. Theory Adv. Appl., pages 235–254. Birkhäuser, Basel, 2002.

[15] M. R. Opmeer, R. F. Curtain. *New Riccati equations for well-posed linear systems.* Systems Control Lett., **52(5)** (2004), 339–347.

[16] A. J. Pritchard, D. Salamon. *The linear quadratic control problem for infinite-dimensional systems with unbounded input and output operators.* SIAM J. Control Optim., **25(1)** (1987), 121–144.

[17] O. Staffans. *Well-posed linear systems*, volume 103 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2005.

[18] C. Tretter, C. Wyss. *Dichotomous Hamiltonians with unbounded entries and solutions of Riccati equations.* J. Evol. Equ., **14(1)** (2014), 121–153.

[19] M. Tucsnak, G. Weiss. *Observation and control for operator semigroups.* Birkhäuser Advanced Texts. Birkhäuser Verlag, Basel, 2009.

[20] M. Weiss, G. Weiss. *Optimal control of stable weakly regular linear systems.* Math. Control Signals Systems, **10(4)** (1997), 287–330.

[21] M. Winklmeier, C. Wyss. *On the Spectral Decomposition of Dichotomous and Bisectorial Operators.* Integral Equations Operator Theory, **82(1)** (2015), 119–150.

[22] C. Wyss. *Hamiltonians with Riesz bases of generalised eigenvectors and Riccati equations.* Indiana Univ. Math. J., **60** (2011), 1723–1766.

[23] C. Wyss, B. Jacob, H. J. Zwart. *Hamiltonians and Riccati equations for linear systems with unbounded control and observation operators.* SIAM J. Control Optim., **50** (2012), 1518–1547.