Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and Computational Mathematics (IMACM)

Preprint BUW-IMACM 18/32

Peng Jin, Jonas Kremer and Barbara Rüdiger

Existence of limiting distribution for affine processes

January 30, 2019

http://www.math.uni-wuppertal.de

EXISTENCE OF LIMITING DISTRIBUTION FOR AFFINE PROCESSES

PENG JIN*, JONAS KREMER, AND BARBARA RÜDIGER

ABSTRACT. In this paper, sufficient conditions are given for the existence of limiting distribution of a conservative affine process on the canonical state space $\mathbb{R}_{\geqslant 0}^m \times \mathbb{R}^n$, where m, $n \in \mathbb{Z}_{\geqslant 0}$ with $m+n > 0$. Our main theorem extends and unifies some known results for OU-type processes on $\mathbb{R}^n$ and one-dimensional CBI processes (with state space $\mathbb{R}_{\geqslant 0}$). To prove our result, we combine analytical and probabilistic techniques; in particular, the stability theory for ODEs plays an important role.

1. INTRODUCTION

Let $D := \mathbb{R}_{\geqslant 0}^m \times \mathbb{R}^n$, where $m, n \in \mathbb{Z}_{\geqslant 0}$ with $m + n > 0$. Roughly speaking, an affine process with state space D is a time-homogeneous Markov process $(X_t)_{t \geqslant 0}$ taking values in D, whose log-characteristic function depends in an affine way on the initial value of the process, that is, there exist functions ϕ, $\psi = (\psi_1, \ldots, \psi_{m+n})$ such that

$$\mathbb{E}\left[\mathrm{e}^{\langle u, X_t \rangle} \,\middle|\, X_0 = x \right] = \mathrm{e}^{\phi(t,u) + \langle \psi(t,u), x \rangle},$$

for all $u \in \mathrm{i}\mathbb{R}^{m+n}$, $t \geqslant 0$ and $x \in D$. The general theory of affine processes was initiated by Duffie, Pan and Singleton [9] and further developed by Duffie, Filipović, and Schachermayer [8]. In the seminal work of Duffie *et al.* [8], several fundamental properties of affine processes on the canonical state space D were established. In particular, the generator of D-valued affine processes is completely characterized through a set of *admissible parameters,* and the associated *generalized Riccati equations* for ϕ and ψ are introduced and studied. The results of [8] were further complemented by many subsequent developments, see, e.g., [1, 3, 4, 7, 11, 14, 16, 18].

Affine processes have found a wide range of applications in finance, mainly due to their computational tractability and modeling flexibility. Many popular models in finance, such as the models of Cox *et al.* [5], Heston [13] and Vasicek [25], are of affine type. Moreover, from the theoretical point of view, the concept of affine processes enables a unified treatment of two very important classes of continuous-time Markov processes: OU-type processes on $\mathbb{R}^n$ and CBI (continuous-state branching processes with immigration) processes on $\mathbb{R}_{\geqslant 0}^m$.

In this paper, we are concerned with the following question: when does an affine process converge in law to a limit distribution? This problem has already been dealt with in the following situations:

- Sato and Yamazato [23] provided conditions under which an OU-type process on $\mathbb{R}^n$ converges in law to a limit distribution, and they identified this type of limit distributions with the class of operator self-decomposable distributions of Urbanik [24];
- without a proof, Pinsky [22] announced the existence of a limit distribution for one-dimensional CBI processes, under a mean-reverting condition and the existence of the log-moment of the Lévy measure from the immigration mechanism. A recent proof

---

Date: January 30, 2019.

2010 *Mathematics Subject Classification.* Primary 60J25; Secondary 60G10.

Key words and phrases. affine process, limiting distribution, stationary distribution, generalized Riccati equation.

*Peng Jin is partially supported by the STU Scientific Research Foundation for Talents (No. NTF18023).

appeared in [20, Theorem 3.20 and Corollary 3.21] (see also [15, Theorem 3.16]). A stronger form of this result can be found in [17, Theorem 2.6];

- Glasserman and Kim [12] proved that affine diffusion processes on $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ introduced by Dai and Singleton [6] have limiting stationary distributions and characterized these limits;
- Barczy, Dring, Li, and Pap [2] showed stationarity of an affine two-factor model on $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$, with one component being the α-root process.

Our motivation for this paper is twofold. On the one hand, we would like to formulate a general result for affine processes with state space $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$, which unifies the above mentioned results; on the other hand, our result should also provide new results for the unsolved cases where $D = \mathbb{R}^m_{\geqslant 0}$ ($m \geqslant 2$) and $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ ($m \geqslant 1, n \geqslant 1$). As our main result (see Theorem 2.4 below), we give sufficient conditions such that an affine process X with state space $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ converges in law to a limit distribution as time goes to infinity, and we also identify this limit through its characteristic function. Using a similar argument as in [15], we will show that the limit distribution is the unique stationary distribution for X.

The rest of this paper is organized as follows. In Section 2 we recall some definitions regarding affine processes and present our main theorem, whose proof we defer to Section 4. In Section 3 we deal with the large time behavior of the function ψ and show that $\psi(t, u)$ converges exponentially fast to 0 as t goes to infinity. Finally, we prove our main theorem in Section 4.

2. Preliminaries and main result

2.1. **Notation.** Let $\mathbb{N}$, $\mathbb{Z}_{\geqslant 0}$, $\mathbb{R}$ denote the sets of positive integers, non-negative integers and real numbers, respectively. Let $\mathbb{R}^d$ be the d-dimensional ($d \geqslant 1$) Euclidean space and define

$$\mathbb{R}^d_{\geqslant 0} := \left\{x \in \mathbb{R}^d \,:\, x_i \geqslant 0,\ i = 1, \ldots, d\right\}$$

and

$$\mathbb{R}^d_{> 0} := \left\{x \in \mathbb{R}^d \,:\, x_i > 0,\ i = 1, \ldots, d\right\}.$$

For x, $y \in \mathbb{R}$, we write $x \wedge y := \min\{x, y\}$. By $\langle \cdot, \cdot \rangle$ and $\|x\|$ we denote the inner product on $\mathbb{R}^d$ and the induced Euclidean norm of a vector $x \in \mathbb{R}^d$, respectively. For a $d \times d$-matrix $A = (a_{ij})$, we write $A^\top$ for the transpose of A and define $\|A\| := (\mathrm{trace}(A^\top A))^{1/2}$. Let $\mathbb{C}^d$ be the space that consists of d-tuples of complex numbers. We define the following subsets of $\mathbb{C}^d$:

$$\mathbb{C}^d_{\leqslant 0} := \left\{u \in \mathbb{C}^d \,:\, \mathrm{Re}\, u_i \leqslant 0,\ i = 1, \ldots, d\right\}$$

and

$$\mathrm{i}\mathbb{R}^d := \left\{u \in \mathbb{C}^d \,:\, \mathrm{Re}\, u_i = 0,\ i = 1, \ldots, d\right\}.$$

The following sets of matrices are of particular importance in this work :

- $\mathbb{M}^-_d$ which stands for the set of real $d \times d$ matrices all of whose eigenvalues have strictly negative real parts. Note that $A \in \mathbb{M}^-_d$ if and only if $\|\exp\{tA\}\| \to 0$ as $t \to \infty$;
- $\mathbb{S}^+_d$ (resp. $\mathbb{S}^{++}_d$) which stands for the set of all symmetric and positive semidefinite (resp. positive definite) real $d \times d$ matrices.

If $A = (a_{ij})$ is a $d \times d$-matrix, $b = (b_1, \ldots, b_d) \in \mathbb{R}^d$ and $\mathcal{I}$, $\mathcal{J} \subset \{1, \ldots, d\}$, we write $A_{\mathcal{I}\mathcal{J}} := (a_{ij})_{i \in \mathcal{I}, j \in \mathcal{J}}$ and $b_{\mathcal{I}} := (b_i)_{i \in \mathcal{I}}$.

Let U be an open set or the closure of an open set in $\mathbb{R}^d$. We introduce the following function spaces: $C^k(U)$, $C^k_c(U)$, and $C^\infty(U)$ which denote the sets of $\mathbb{C}$-valued functions on U that are k-times continuously differentiable, that are k-times continuously differentiable with compact support, and that are smooth, respectively. The Borel σ-Algebra on U will be denoted by $\mathcal{B}(U)$.

Throughout the rest of this paper, let $D := \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$, where $m, n \in \mathbb{Z}_{\geqslant 0}$ with $m + n > 0$. Note that m or n may be 0. The set D will act as the state space of affine processes we are about to consider. The total dimension of D is denoted by $d = m + n$. We write $\mathcal{B}_b(D)$ for the Banach space of bounded real-valued Borel measurable functions f on D with norm $\|f\|_\infty := \sup_{x \in D} |f(x)|$.

For D, we write

$$I = \{1, \ldots, m\} \quad \text{and} \quad J = \{m + 1, \ldots, m + n\}$$

for the index sets of the $\mathbb{R}^m_{\geqslant 0}$-valued components and the $\mathbb{R}^n$-valued components, respectively. Define

$$\mathcal{U} := \mathbb{C}^m_{\leqslant 0} \times \mathrm{i}\mathbb{R}^n = \Big\{ u \in \mathbb{C}^d \, : \, \mathrm{Re}\, u_I \leqslant 0, \quad \mathrm{Re}\, u_J = 0 \Big\}.$$

Note that $\mathcal{U}$ is the set of all $u \in \mathbb{C}^d$, for which $x \mapsto \exp\{\langle u, x\rangle\}$ is a bounded function on D.

Further notation is introduced in the text.

2.2. **Affine processes on the canonical state space.** Affine processes on the canonical state space $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ have been systematically studied in the well-known work [8]. We remark that affine processes considered in [8] are in full generality and are allowed to have explosions or killings. In contrast to [8], in this paper we restrict ourselves to *conservative affine processes*. In terms of terminology and notation, we mainly follow, instead of [8], the paper by Keller-Ressel and Mayerhofer [16], where only the conservative case was considered.

Let us start with a time-homogeneous and conservative Markov process with state space D and semigroup (P_t) acting on $\mathcal{B}_b(D)$, that is,

$$P_t f(x) = \int_D f(\xi) p_t(x, d\xi), \quad f \in \mathcal{B}_b(D).$$

Here $p_t(x, \cdot)$ denotes the transition kernel of the Markov process. We assume that $p_0(x, \{x\}) = 1$ and $p_t(x, D) = 1$ for all $t \geqslant 0$, $x \in D$.

Let $(X, (\mathbb{P}_x)_{x \in D})$ be the canonical realization of (P_t) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0})$, where Ω is the set of all cdlg paths in D and $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$. Here $(\mathcal{F}_t)_{t \geqslant 0}$ is the filtration generated by X and $\mathcal{F} = \bigvee_{t \geqslant 0} \mathcal{F}_t$. The probability measure $\mathbb{P}_x$ on Ω represents the law of the Markov process $(X_t)_{t \geqslant 0}$ started at x, i.e., it holds that $X_0 = x$, $\mathbb{P}_x$-almost surely. The following definition is taken from [16, Definition 2.2].

Definition 2.1. The Markov process X is called *affine* with state space D, if its transition kernel $p_t(x, A) = \mathbb{P}_x(X_t \in A)$ satisfies the following:

(i) it is stochastically continuous, that is, $\lim_{s \to t} p_s(x, \cdot) = p_t(x, \cdot)$ weakly for all $t \geqslant 0,\ x \in D$, and

(ii) there exist functions $\phi : \mathbb{R}_{\geqslant 0} \times \mathcal{U} \to \mathbb{C}$ and $\psi : \mathbb{R}_{\geqslant 0} \times \mathcal{U} \to \mathbb{C}^d$ such that

$$\int_D \mathrm{e}^{\langle u, \xi \rangle} p_t(x, \mathrm{d}\xi) = \mathbb{E}_x \left[\mathrm{e}^{\langle X_t, u \rangle} \right] = \exp\{\phi(t, u) + \langle x, \psi(t, u)\rangle\} \tag{2.1}$$

for all $t \geqslant 0,\ x \in D$ and $u \in \mathcal{U}$, where $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$.

The stochastic continuity in (i) and the affine property in (ii) together imply the following regularity of the functions ϕ and ψ (see [18, Theorem 5.1]), i.e., the right-hand derivatives

$$F(u) := \frac{\partial}{\partial t} \phi(t, u) \bigg|_{t = 0+} \quad \text{and} \quad R(u) := \frac{\partial}{\partial t} \psi(t, u) \bigg|_{t = 0+} \tag{2.2}$$

exist for all $u \in \mathcal{U}$, and are continuous at $u = 0$. Moreover, according to [8, Proposition 7.4], the functions ϕ and ψ satisfy the *semi-flow property*:

$$\phi(t + s, u) = \phi(t, u) + \phi(s, \psi(t, u)) \quad \text{and} \quad \psi(t + s, u) = \psi(s, \psi(t, u)), \tag{2.3}$$

for all t, $s \geqslant 0$ with $(t+s,u) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U}$.

Definition 2.2. We call $(a, \alpha, b, \beta, m, \mu)$ a *set of admissible parameters* for the state space D if

(i) $a \in \mathbb{S}_d^+$ and $a_{kl} = 0$ for all $k \in I$ or $l \in I$;

(ii) $\alpha = (\alpha_1, \dots, \alpha_m)$ with $\alpha_i = (\alpha_{i,kl})_{1 \leqslant k,l \leqslant d} \in \mathbb{S}_d^+$ and $\alpha_{i,kl} = 0$ if $k \in I\backslash\{i\}$ or $l \in I\backslash\{i\}$;

(iii) m is a Borel measure on $D\backslash\{0\}$ satisfying

$$\int_{D\backslash\{0\}} \left(1 \wedge \|\xi\|^2 + \sum_{i\in I} (1 \wedge \xi_i) \right) m(\mathrm{d}\xi) < \infty;$$

(iv) $\mu = (\mu_1, \dots, \mu_m)$ where every μ_i is a Borel measure on $D\backslash\{0\}$ satisfying

$$\int_{D\backslash\{0\}} \left(\|\xi\| \wedge \|\xi\|^2 + \sum_{k\in I\backslash\{i\}} \xi_k \right) \mu_i(\mathrm{d}\xi) < \infty. \tag{2.4}$$

(v) $b \in D$;

(vi) $\beta = (\beta_{ki}) \in \mathbb{R}^{d\times d}$ with $\beta_{ki} - \int_{D\backslash\{0\}} \xi_k \mu_i(\mathrm{d}\xi) \geqslant 0$ for all $i \in I$ and $k \in I \setminus \{i\}$, and $\beta_{ki} = 0$ for all $k \in I$ and $i \in J$;

We remark that our definition of admissible parameters is a special case of [8, Definition 2.6], since we require here that the parameters corresponding to killing are constant 0; moreover, the condition in (iv) is also stronger as usual, i.e., we assume that the first moment of μ_i's exists, which, by [8, Lemma 9.2], implies that the affine process under consideration is conservative. However, we should remind the reader that (2.4) is not a necessary condition for conservativeness. In fact, an example of a conservative affine process on $\mathbb{R}_{\geqslant 0}$, which violates (2.4), is provided in [21, Section 3].

We write $\psi = (\psi^I, \psi^J) \in \mathbb{C}^m \times \mathbb{C}^n$, where $\psi^I = (\psi_1, \dots, \psi_m)^\top$ and $\psi^J = (\psi_{m+1}, \dots, \psi_{m+n})^\top$. Recall that $R = (R_1, \dots, R_d)^\top : \mathcal{U} \to \mathbb{C}^d$ is given in (2.2). Define $R^I := (R_1, \dots, R_m)^\top : \mathcal{U} \to \mathbb{C}^m$. For $u \in \mathcal{U}$, we will often write $u = (v, w) \in \mathbb{C}^m_{\leqslant 0} \times \mathrm{i}\mathbb{R}^n$.

The next result is due to [8, Theorem 2.7].

Theorem 2.1. *Let $(a, \alpha, b, \beta, m, \mu)$ be a set of admissible parameters in the sense of Definition 2.2. Then there exists a (unique) conservative affine process X with state space D such that its infinitesimal generator $\mathcal{A}$ operating on a function $f \in C_c^2(D)$ is given by*

$$\begin{aligned}\mathcal{A}f(x) = &\sum_{k,l=1}^{d} \left(a_{kl} + \sum_{i=1}^{m} \alpha_{i,kl} x_i \right) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \langle b + \beta x, \nabla f(x) \rangle \\ &+ \int_{D\backslash\{0\}} \left(f(x+\xi) - f(x) - \langle \nabla_J f(x), \xi_J \rangle \mathbb{1}_{\{\|\xi\| \leqslant 1\}}(\xi) \right) m(\mathrm{d}\xi) \\ &+ \sum_{i=1}^{m} x_i \int_{D\backslash\{0\}} \left(f(x+\xi) - f(x) - \langle \nabla f(x), \xi \rangle \right) \mu_i(\mathrm{d}\xi)\end{aligned}$$

where $x \in D$, $\nabla_J := (\partial_{x_k})_{k\in J}$. Moreover, (2.1) holds for some functions $\phi(t,u)$ and $\psi(t,u)$ that are uniquely determined by the generalized Riccati differential equations: for each $u = (v,w) \in \mathbb{C}^m_{\leqslant 0} \times \mathrm{i}\mathbb{R}^n$,

$$\partial_t \phi(t,u) = F(\psi(t,u)), \quad \phi(0,u) = 0,$$

$$\partial_t \psi^I(t,u) = R^I \left(\psi^I(t,u), \mathrm{e}^{\beta_{JJ}^\top t} w \right), \quad \psi^I(0,u) = v \tag{2.5}$$

$$\psi^J(t,u) = \mathrm{e}^{\beta_{JJ}^\top t} w, \tag{2.6}$$

where

$$F(u) = \langle u, au\rangle + \langle b, u\rangle + \int_{D\setminus\{0\}} \left(\mathrm{e}^{\langle u,\xi\rangle} - 1 - \langle u_J, \xi_J\rangle \mathbb{1}_{\{\|\xi\|\leqslant 1\}}(\xi)\right) m(\mathrm{d}\xi) \tag{2.7}$$

and $R^I = (R_1, \ldots, R_m)$ *with*

$$R_i(u) = \langle u, \alpha_i u\rangle + \sum_{k=1}^{d} \beta_{ki} u_k + \int_{D\setminus\{0\}} \left(\mathrm{e}^{\langle u,\xi\rangle} - 1 - \langle u, \xi\rangle\right) \mu_i(\mathrm{d}\xi), \quad i \in I.$$

Remark 2.2. *If an affine process X with state space D and a set of admissible parameters $(a, \alpha, b, \beta, m, \mu)$ satisfy a relation as in Theorem 2.1, then we say that X is an affine process with admissible parameters $(a, \alpha, b, \beta, m, \mu)$.*

The following lemma is a consequence of the condition (iv) in Definition 2.2.

Lemma 2.3. *Let X be an affine process with state space D and admissible parameters $(a, \alpha, b, \beta, m, \mu)$. Let R and ψ be as in Theorem 2.1. For each $i \in I$ it holds that $R_i \in C^1(\mathcal{U})$ and $\psi_i \in C^1(\mathbb{R}_{\geqslant 0}\times\mathcal{U})$.*

To see that Lemma 2.3 is true, we only need to apply Lemmas 5.3 and 6.5 of [8].

2.3. **Main result.** Our main result of this paper is the following.

Theorem 2.4. *Let X be an affine process with state space $\mathbb{R}^m_{\geqslant 0}\times\mathbb{R}^n$ and admissible parameters $(a, \alpha, b, \beta, m, \mu)$ in the sense of Definition 2.2. If*

$$\beta \in \mathbb{M}_d^- \quad \text{and} \quad \int_{\{\|\xi\|>1\}} \log\|\xi\|\, m(\mathrm{d}\xi) < \infty,$$

then the law of X_t converges weakly to a limiting distribution π, which is independent of X_0 and whose characteristic function is given by

$$\int_D \mathrm{e}^{\langle u,x\rangle} \pi(\mathrm{d}x) = \exp\left\{\int_0^\infty F(\psi(s,u))\,\mathrm{d}s\right\}, \qquad u \in \mathcal{U}.$$

Moreover, the limiting distribution π is the unique stationary distribution for X.

Remark 2.5. *In virtue of the definition of admissible parameters, we can write $\beta \in \mathbb{R}^{d\times d}$ in the following way:*

$$\beta = \left(\begin{array}{c:c} \beta_{II} & 0 \\ \hdashline \beta_{JI} & \beta_{JJ} \end{array}\right), \tag{2.8}$$

where $\beta_{II} \in \mathbb{R}^{m\times m}$, $\beta_{JI} \in \mathbb{R}^{n\times m}$ and $\beta_{JJ} \in \mathbb{R}^{n\times n}$. It is easy to see that $\beta \in \mathbb{M}_d^-$ is equivalent to the fact that $\beta_{II} \in \mathbb{M}_m^-$ and $\beta_{JJ} \in \mathbb{M}_n^-$.

We now make a few comments on Theorem 2.4. To our knowledge, Theorem 2.4 seems to be the first result towards the existence of limiting distributions for affine processes on D in such a generality. It includes many previous results as special cases. In particular, it covers [12, Theorem 2.4] for affine diffusions, and partially extends [23, Theorem 4.1] for OU-type processes and [22, Corollary 2] for 1-dimensional CBI processes. However, we are not able to show $\int_{\{\|\xi\|>1\}} \log\|\xi\|\, m(\mathrm{d}\xi) < \infty$, provided that $\beta \in \mathbb{M}_d^-$ and the stationarity of X is known.

Our strategy of proving Theorem 2.4 is as follows. Clearly, to prove the weak convergence of the distribution of X_t to π, it is essential to establish the pointwise convergence of the corresponding characteristic functions, i.e.,

$$\mathbb{E}_x\left[\mathrm{e}^{\langle X_t,u\rangle}\right] = \exp\left\{\phi(t,u) + \langle x,\psi(t,u)\rangle\right\} \to \exp\left\{\int_0^\infty F(\psi(s,u))\mathrm{d}s\right\} \quad \text{as } t\to\infty.$$

We will proceed in two steps. In the first step, we prove that for each $u\in\mathcal{U}$, $\psi(t,u)$ converges to zero exponentially fast. For u in a small neighborhood of the origin, this convergence follows by a fine analysis of the generalized Riccati equations (2.5), (2.7) and an application of the linearized stability theorem for ODEs. Then, by some probabilistic arguments, we show that $\psi(t,u)$ reaches every neighborhood of the origin for large enough t. The essential observation here is the tightness of the laws of X_t, $t\geqslant 0$. This is a simple consequence of the uniform boundedness for the first moment of X_t, $t\geqslant 0$, which we show in Proposition 3.7. We thus obtain the desired convergence speed of $\psi(t,u)\to 0$ by the semi-flow property (2.3). In the second step, we show that

$$\phi(t,u) = \int_0^t F(\psi(s,u))\mathrm{d}s \to \int_0^\infty F(\psi(s,u))\mathrm{d}s \quad \text{as } t\to\infty. \tag{2.9}$$

Since $\psi(s,u)\to 0$ exponentially fast as $s\to\infty$, we will see that the convergence in (2.9) is naturally connected with the condition $\int_{\{\|\xi\|>1\}}\log\|\xi\|\, m\,(\mathrm{d}\xi)<\infty$. Finally, the stationarity of π can be derived using the semi-flow property.

3. Large time behavior of the function $\psi(t,u)$

In this section we consider an affine process X with admissible parameters (a,α,b,β,m,μ) and assume that

$$a=0,\ b=0,\ m=0. \tag{3.1}$$

In particular, we have $F\equiv 0$ as well as $\phi\equiv 0$. We will show that if $\beta\in\mathbb{M}_d^-$, then $\psi(t,u)\to 0$ exponentially fast as $t\to\infty$.

Remark 3.1. *The assumption that $a=0$, $b=0$ and $m=0$ is not essential. Indeed, Proposition 3.9, as the main result of this section, remains true if we drop Assumption (3.1). This follows from the following observation: when we study the properties of the function $\psi(t,u)$, the parameters a, b and m do not play a role.*

3.1. Uniform boundedness for the first moment of X_t, $t\geqslant 0$.

The aim we pursue in this subsection is to establish the uniform boundedness for the first moment of X_t, $t\geqslant 0$. We start with some approximations of X, which were introduced in [4].

For $K\in(1,\infty)$, let

$$\mu_{K,i}(\mathrm{d}\xi) := \mathbb{1}_{\{\|\xi\|\leqslant K\}}(\xi)\mu_i(\mathrm{d}\xi),$$

and denote by $(X_{K,t})_{t\geqslant 0}$ the affine process with admissible parameters $(a=0,\alpha,b=0,\beta,m=0,\mu_K)$, where $\mu_K=(\mu_{K,1},\ldots,\mu_{K,m})$. Then we have

$$\mathbb{E}_x\left[\mathrm{e}^{\langle X_{K,t},u\rangle}\right] = \exp\left\{\langle x,\psi_K(t,u)\rangle\right\},\quad t\geqslant 0,\ x\in D,\ u\in\mathcal{U},$$

for some function $\psi_K:\mathbb{R}_{\geqslant 0}\times\mathcal{U}\to\mathbb{C}^d$. By (2.5) and (2.6), we know that $\psi_K=(\psi_K^I,\psi^J)$, where $\psi^J(t,u)=\exp(\beta_{JJ}^\top t)w$ for $u=(v,w)\in\mathbb{C}^m_{\leqslant 0}\times\mathrm{i}\mathbb{R}^n$ and ψ_K^I satisfies the generalized Riccati equation

$$\partial_t\psi_K^I(t,u) = R_K^I\left(\psi_K^I(t,u),\mathrm{e}^{\beta_{JJ}^\top t}w\right),\quad \psi_K^I(0,u)=v\in\mathbb{C}^m_{\leqslant 0},$$

where $R_K^I = (R_{K,i}, \dots, R_{K,m})^\top$ with

$$R_{K,i}(u) = \langle u, \alpha_i u\rangle + \sum_{k=1}^d \beta_{ki} u_k + \int_{D\setminus\{0\}} \left(\mathrm{e}^{\langle u,\xi\rangle} - 1 - \langle u, \xi\rangle\right) \mu_{K,i}(\mathrm{d}\xi), \quad i \in I.$$

Lemma 3.2. *For each $t \in \mathbb{R}_{\geqslant 0}$ and $u \in \mathcal{U}$, $\psi_K(t,u)$ converges to $\psi(t,u)$ as $K \to \infty$.*

Proof. Clearly, we only need to show the pointwise convergence of ψ_K^I to ψ^I. Let $u = (v,w) \in \mathbb{C}^m_{\leqslant 0} \times \mathrm{i}\mathbb{R}^n$ and $T > 0$ be fixed.

By the Riccati equations for ψ^I and ψ_K^I, we get

$$\psi^I(t,u) = v + \int_0^t R^I\left(\psi^I(s,u), \mathrm{e}^{\beta_{JJ}^\top s} w\right) \mathrm{d}s, \quad t \geqslant 0, \tag{3.2}$$

and

$$\psi_K^I(t,u) = v + \int_0^t R_K^I\left(\psi_K^I(s,u), \mathrm{e}^{\beta_{JJ}^\top s} w\right) \mathrm{d}s, \quad t \geqslant 0. \tag{3.3}$$

In view of the formula (6.16) in the proof of [8, Propostion 6.1], we have

$$\begin{aligned}\sup_{t\in[0,T]} \left\|\psi_K^I(t,u)\right\|^2 &\leqslant \sup_{t\in[0,T]} \left(\|v\|^2 + c_1 \int_0^t \left(1 + \left\|\mathrm{e}^{\beta_{JJ}^\top s} w\right\|^2\right) \mathrm{d}s\right) \\ &\qquad \times \exp\left\{c_1 \int_0^t \left(1 + \left\|\mathrm{e}^{\beta_{JJ}^\top s} w\right\|^2\right) \mathrm{d}s\right\} \\ &\leqslant \left(\|v\|^2 + c_1 \int_0^T \left(1 + \left\|\mathrm{e}^{\beta_{JJ}^\top s} w\right\|^2\right) \mathrm{d}s\right) \\ &\qquad \times \exp\left\{c_1 \int_0^T \left(1 + \left\|\mathrm{e}^{\beta_{JJ}^\top s} w\right\|^2\right) \mathrm{d}s\right\},\end{aligned} \tag{3.4}$$

for some positive constant c_1. Moreover, by checking carefully the proof of [8, Propostion 6.1] and noting that $\mu_{K,i} \leqslant \mu_i$, we can actually choose c_1 in such a way that it depends only on the parameters α, β, μ. So c_1 is independent of K. Similarly, the same inequality holds for ψ^I:

$$\begin{aligned}\sup_{t\in[0,T]} \left\|\psi^I(t,u)\right\|^2 &\leqslant \left(\|v\|^2 + c_1 \int_0^T \left(1 + \left\|\mathrm{e}^{\beta_{JJ}^\top s} w\right\|^2\right) \mathrm{d}s\right) \\ &\qquad \times \exp\left\{c_1 \int_0^T \left(1 + \left\|\mathrm{e}^{\beta_{JJ}^\top s} w\right\|^2\right) \mathrm{d}s\right\}.\end{aligned}$$

According to Lemma 2.3, the mapping $u \mapsto R^I(u) : \mathcal{U} \to \mathbb{C}^m$ is locally Lipschitz continuous. Therefore, for each $L > 0$, there exists a constant $c_2 = c_2(L) > 0$ such that

$$\|R_i(u_1) - R_i(u_2)\| \leqslant c_2 \|u_1 - u_2\|, \quad \text{for all } i \in I \text{ and } \|u_1\|, \|u_2\| \leqslant L. \tag{3.5}$$

In addition, it is easy to see that for $u \in \mathcal{U}$,

$$\begin{aligned}\|R_i(u) - R_{K,i}(u)\| &= \left|\int_{\{\|\xi\|>K\}} \left(\mathrm{e}^{\langle u,\xi\rangle} - 1 - \langle u,\xi\rangle\right) \mu_i(\mathrm{d}\xi)\right| \\ &\leqslant \int_{\{\|\xi\|>K\}} 2\mu_i(\mathrm{d}\xi) + \|u\| \int_{\{\|\xi\|>K\}} \|\xi\|\, \mu_i(\mathrm{d}\xi) \\ &\leqslant \varepsilon_K (1 + \|u\|),\end{aligned} \tag{3.6}$$

where $\varepsilon_K := \sum_{i=1}^m \int_{\{\|\xi\|>K\}} (2 + \|\xi\|)\, \mu_i(\mathrm{d}\xi)$. Note that $\varepsilon_K \to 0$ as $K \to \infty$ by dominated convergence.

Let

$$g_K(t) := \left\|\psi^I(t,u) - \psi^I_K(t,u)\right\|, \quad t \in [0,T].$$

By (3.2) and (3.3), we have

$$\begin{aligned} g_K(t) &\leqslant \left\| \int_0^t R^I\left(\psi^I(s,u), \mathrm{e}^{\beta_{JJ}^\top s} w\right) \mathrm{d}s - \int_0^t R^I_K\left(\psi^I_K(s,u), \mathrm{e}^{\beta_{JJ}^\top s} w\right) \mathrm{d}s \right\| \\ &\leqslant \sum_{i=1}^m \int_0^t \left\| R_i\left(\psi^I(s,u), \mathrm{e}^{\beta_{JJ}^\top s} w\right) - R_i\left(\psi^I_K(s,u), \mathrm{e}^{\beta_{JJ}^\top s} w\right)\right\| \mathrm{d}s \\ &\quad + \sum_{i=1}^m \int_0^t \left\| R_i\left(\psi^I_K(s,u), \mathrm{e}^{\beta_{JJ}^\top s} w\right) - R_{K,i}\left(\psi^I_K(s,u), \mathrm{e}^{\beta_{JJ}^\top s} w\right)\right\| \mathrm{d}s. \end{aligned} \tag{3.7}$$

In virtue of (3.4), there exists a constant $c_3 = c_3(T) > 0$ such that

$$\sup_{K\in[1,\infty)} \sup_{s\in[0,T]} \left\|\psi^I_K(s,u)\right\| \leqslant c_3 < \infty,$$

which implies

$$\sup_{K\in[1,\infty)} \sup_{s\in[0,T]} \left\|\left(\psi^I_K(s,u), \mathrm{e}^{\beta_{JJ}^\top s} w\right)\right\| \leqslant c_4 < \infty. \tag{3.8}$$

So, for $0 < s \leqslant T$, we get

$$\left\| R_i\left(\psi^I(s,u), \mathrm{e}^{\beta_{JJ}^\top s} w\right) - R_i\left(\psi^I_K(s,u), \mathrm{e}^{\beta_{JJ}^\top s} w\right)\right\| \leqslant c_5 \left\|\psi^I(s,u) - \psi^I_K(s,u)\right\| \tag{3.9}$$

from (3.5), and obtain

$$\left\| R_i\left(\psi^I_K(s,u), \mathrm{e}^{\beta_{JJ}^\top s} w\right) - R_{K,i}\left(\psi^I_K(s,u), \mathrm{e}^{\beta_{JJ}^\top s} w\right)\right\| \leqslant \varepsilon_K (1 + c_6) \tag{3.10}$$

from (3.6) and (3.8). Here, c_5, $c_6 > 0$ are constants not depending on K.

Combining (3.7), (3.9) and (3.10) yields, for $t \in [0,T]$,

$$\begin{aligned} g_K(t) &\leqslant c_5 m \int_0^t \left\|\psi^I(s,u) - \psi^I_K(s,u)\right\| \mathrm{d}s + m\varepsilon_K (1 + c_6) t \\ &= c_5 m \int_0^t g_K(s)\mathrm{d}s + m\varepsilon_K (1 + c_6) t. \end{aligned}$$

Gronwall's inequality implies

$$\begin{aligned} g_K(t) &\leqslant m\varepsilon_K (1 + c_6) t + m^2 \varepsilon_K (1 + c_6) c_5 \int_0^t s\mathrm{e}^{c_5 m(t-s)} \mathrm{d}s \\ &\leqslant m\varepsilon_K (1 + c_6) \left(T + c_5 m T^2 \mathrm{e}^{c_5 m T}\right), \qquad t \in [0,T]. \end{aligned}$$

Since $\varepsilon_K \to 0$ as $K \to \infty$, we see that $g_K(t) \to 0$ and thus

$$\psi^I_K(t,u) \to \psi^I(t,u), \quad \text{for all } t \in [0,T].$$

□

For $K \in (1,\infty)$, the generator $\mathcal{A}_K$ of $(X_{K,t})_{t\geqslant 0}$ is given by

$$\begin{aligned} \mathcal{A}_K f(x) &= \sum_{k,l=1}^d \left(\sum_{i=1}^m \alpha_{i,kl} x_i\right) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \langle \beta x, \nabla f(x)\rangle \\ &\quad + \sum_{i=1}^m x_i \int_{D\setminus\{0\}} \left(f(x+\xi) - f(x) - \langle \nabla f(x), \xi\rangle\right) \mu_{K,i}(\mathrm{d}\xi), \end{aligned}$$

defined for every $f \in C^2_c(D)$.

To avoid the complication of discussing the domain of definition for the generator $\mathcal{A}_K$, we introduce the operator $\mathcal{A}_K^\sharp$, which was also used in [8].

Definition 3.1. If $f \in C^2(D)$ is such that for all $x \in D$,

$$\sum_{i=1}^m \int_{D\setminus\{0\}} |f(x+\xi) - f(x) - \langle \nabla f(x), \xi\rangle| \, \mu_{K,i}(\mathrm{d}\xi) < \infty,$$

then we say that $\mathcal{A}_K^\sharp f$ is well-defined and let

$$\mathcal{A}_K^\sharp f(x) := \sum_{k,l=1}^d \left(\sum_{i=1}^m \alpha_{i,kl} x_i\right) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \langle \beta x, \nabla f(x)\rangle$$
$$+ \sum_{i=1}^m x_i \int_{D\setminus\{0\}} \left(f(x+\xi) - f(x) - \langle \nabla f(x), \xi\rangle\right) \mu_{K,i}(\mathrm{d}\xi)$$

for $x \in D$.

It is easy to see that if $f \in C^2(D)$ has bounded first and second order derivatives, then $\mathcal{A}_K^\sharp f$ is well-defined.

Recall that the matrix β can be written as in (2.8). We define the following matrices

$$M_1 := \int_0^\infty \mathrm{e}^{t\beta_{II}^\top} \mathrm{e}^{t\beta_{II}} \mathrm{d}t \quad \text{and} \quad M_2 := \int_0^\infty \mathrm{e}^{t\beta_{JJ}^\top} \mathrm{e}^{t\beta_{JJ}} \mathrm{d}t.$$

Since $\beta_{II} \in \mathbb{M}_m^-$ and $\beta_{JJ} \in \mathbb{M}_n^-$, the matrices M_1 and M_2 are well-defined. Moreover, we have that $M_1 \in \mathbb{S}_m^{++}$ and $M_2 \in \mathbb{S}_n^{++}$. In the following we will often write $x = (y,z) \in \mathbb{R}_{\geqslant 0}^m \times \mathbb{R}^n$ for $x \in D$. For y_1, $y_2 \in \mathbb{R}_{\geqslant 0}^m$ and z_1, $z_2 \in \mathbb{R}^n$, we define

$$\langle y_1, y_2\rangle_I := \int_0^\infty \langle \mathrm{e}^{t\beta_{II}} y_1, \mathrm{e}^{t\beta_{II}} y_2\rangle \mathrm{d}t \quad \text{and} \quad \langle z_1, z_2\rangle_J := \int_0^\infty \langle \mathrm{e}^{t\beta_{JJ}} z_1, \mathrm{e}^{t\beta_{JJ}} z_2\rangle \mathrm{d}t.$$

It is easily verified that $\langle\cdot,\cdot\rangle_I$ and $\langle\cdot,\cdot\rangle_J$ define inner products on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Moreover, we have that

$$\langle y_1, y_2\rangle_I = y_2^\top M_1 y_1 = \langle y_1, M_1 y_2\rangle \quad \text{and} \quad \langle z_1, z_2\rangle_J = z_2^\top M_2 z_1 = \langle z_1, M_2 z_2\rangle.$$

The norms on $\mathbb{R}^m$ and $\mathbb{R}^n$ induced by the scalar products $\langle\cdot,\cdot\rangle_I$ and $\langle\cdot,\cdot\rangle_J$ are denoted by

$$\|y\|_I := \sqrt{\langle y, y\rangle_I} \quad \text{and} \quad \|z\|_J := \sqrt{\langle z, z\rangle_J},$$

respectively.

In the following lemma we construct a Lyapunov function V for $(X_{K,t})_{t\geqslant 0}$. Note that the definition of V does not depend on K.

Lemma 3.3. *Assume $m \geqslant 1$ and $n \geqslant 1$. Suppose that $\beta \in \mathbb{M}_d^-$. Let $V \in C^2(D,\mathbb{R})$ be such that $V > 0$ on D and*

$$V(x) = \left(\langle y, y\rangle_I + \varepsilon \langle z, z\rangle_J\right)^{1/2}, \quad \text{whenever } x = (y,z) \in \mathbb{R}_{\geqslant 0}^m \times \mathbb{R}^n \text{ with } \|x\| > 2.$$

Here $\varepsilon > 0$ is some small enough constant. Then $\mathcal{A}_K^\sharp V$ is well-defined and V is a Lyapunov function for $(X_{K,t})_{t\geqslant 0}$, that is, there exist positive constants c and C such that

$$\mathcal{A}_K^\sharp V(x) \leqslant -cV(x) + C, \quad \text{for all } x \in D.$$

Moreover, the constants c and C can be chosen to be independent of K.

Proof. For $x_1=(y_1,z_1)\in\mathbb{R}^m_{\geqslant 0}\times\mathbb{R}^n$ and $x_2=(y_2,z_2)\in\mathbb{R}^m_{\geqslant 0}\times\mathbb{R}^n$, we define

$$\langle x_1,x_2\rangle_\beta := \langle y_1,z_1\rangle_I+\varepsilon\langle y_2,z_2\rangle_J,$$

where $\varepsilon>0$ is a small constant to be determined later. Set $\tilde V(x):=(\langle x,x\rangle_\beta)^{1/2}$, $x\in D$. Then $\widetilde V$ is smooth on $\{x\in D:\|x\|>1\}$. By the extension lemma for smooth functions (see [19, Lemma 2.26]), we can easily find a function $V\in C^\infty(D,\mathbb{R})$ such that $V>0$ on D and $V(x)=\widetilde V(x)=(\langle x,x\rangle_\beta)^{1/2}$ for $\|x\|>2$. So for all $x=(y,z)\in\mathbb{R}^m_{\geqslant 0}\times\mathbb{R}^n$ with $\|x\|>2$, we have

$$\nabla V(y,z)=V(y,z)^{-1}\begin{pmatrix} M_1y\\ \varepsilon M_2 z\end{pmatrix} \tag{3.11}$$

and

$$\nabla^2 V(y,z)=\begin{pmatrix} \frac{M_1}{V(y,z)}-\frac{(M_1y)(M_1y)^\top}{V(y,z)^3} & \frac{-\varepsilon(M_1y)(M_2z)^\top}{V(y,z)^3}\\ \frac{-\varepsilon(M_1y)(M_2z)^\top}{V(y,z)^3} & \frac{\varepsilon M_2}{V(y,z)}-\frac{\varepsilon^2(M_2z)(M_2z)^\top}{V(y,z)^3}\end{pmatrix}. \tag{3.12}$$

We write $\mathcal{A}^\sharp_K V=\mathcal{D}V+\mathcal{J}_K V$, where

$$\mathcal{D}V(x):=\sum_{k,l=1}^d\langle\alpha_{I,kl},x_I\rangle\frac{\partial^2V(x)}{\partial x_k\partial x_l}+\langle\beta x,\nabla V(x)\rangle, \tag{3.13}$$

$$\mathcal{J}_KV(x):=\sum_{i=1}^m x_i\int_{D\setminus\{0\}}\left(V\left(x+\xi\right)-V\left(x\right)-\langle\nabla V(x),\xi\rangle\right)\mu_{K,i}\left(\mathrm{d}\xi\right). \tag{3.14}$$

We now estimate $\mathcal{D}V(x)$ and $\mathcal{J}_KV(x)$ separately. Let us first consider $\mathcal{D}V(x)$. We may further split $\mathcal{D}V(x)$ into the drift part and the diffusion part.

Drift. Recall that $\beta_{IJ}=0$. Consider $x=(y,z)$ with $\|x\|>2$. It follows from (3.11) that

$$\begin{aligned}\langle\beta x,\nabla V(x)\rangle&=\langle\begin{pmatrix}\beta_{II}y\\ \beta_{JI}y+\beta_{JJ}z\end{pmatrix},\begin{pmatrix}V(y,z)^{-1}M_1y\\ V(y,z)^{-1}\varepsilon M_2z\end{pmatrix}\rangle\\ &=V(y,z)^{-1}\left(\langle\beta_{II}y,M_1y\rangle+\langle\beta_{JI}y,\varepsilon M_2z\rangle+\langle\beta_{JJ}z,\varepsilon M_2z\rangle\right).\end{aligned}$$

The first and the third inner product on the right-hand side may be estimated similarly. Namely, we have

$$V(y,z)^{-1}\langle\beta_{II}y,M_1y\rangle=\frac12V(y,z)^{-1}y^\top\left(M_1\beta_{II}+\beta_{II}^\top M_1\right)y.$$

The definition of M_1 implies

$$\begin{aligned}M_1\beta_{II}+\beta_{II}^\top M_1&=\int_0^\infty\left(\mathrm{e}^{t\beta_{II}^\top}\mathrm{e}^{t\beta_{II}}\beta_{II}+\beta_{II}^\top\mathrm{e}^{t\beta_{II}^\top}\mathrm{e}^{t\beta_{II}}\right)\mathrm{d}t\\ &=\int_0^\infty\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{t\beta_{II}^\top}\mathrm{e}^{t\beta_{II}}\right)\mathrm{d}t\\ &=\mathrm{e}^{t\beta_{II}^\top}\mathrm{e}^{t\beta_{II}}\Big|_{t=0}^\infty\\ &=-I_m,\end{aligned}$$

where I_m denotes the $m\times m$ identity matrix. Hence

$$V(y,z)^{-1}\langle\beta_{II}y,M_1y\rangle=-\frac12V(y,z)^{-1}y^\top y.$$

Since all norms on $\mathbb{R}^m$ are equivalent, we have

$$-y^\top y\leqslant -c_1y^\top M_1y=-c_1\langle y,y\rangle_I\leqslant -c_1\|y\|_I^2,$$

for some positive constant c_1 that is independent of K. So

$$V(y,z)^{-1}\langle \beta_{II}y, M_1 y\rangle \leqslant -c_1\|y\|_I^2 V(y,z)^{-1}. \tag{3.15}$$

In the very same way we obtain

$$V(y,z)^{-1}\langle \beta_{JJ}z, \varepsilon M_2 z\rangle \leqslant -c_2\varepsilon\|z\|_J^2 V(y,z)^{-1}, \tag{3.16}$$

for some constant $c_2 > 0$. To estimate the remaining term, we can use Cauchy Schwarz inequality to obtain

$$\begin{aligned}\left|V(y,z)^{-1}\langle \beta_{JI}y, \varepsilon M_2 z\rangle\right| &\leqslant \varepsilon V(y,z)^{-1}\,\|\beta_{JI}y\|\,\|M_2 z\| \\ &\leqslant c_3\varepsilon V(y,z)^{-1}\,\|y\|\,\|z\|,\end{aligned}$$

for some constant $c_3 > 0$. Using the fact that all norms on $\mathbb{R}^d$ are equivalent, we get

$$\begin{aligned}\left|V(y,z)^{-1}\langle \beta_{JI}y, \varepsilon M_2 z\rangle\right| &\leqslant \varepsilon c_4 V(y,z)^{-1}\|y\|_I\|z\|_J \\ &= c_4\frac{\sqrt{\varepsilon}\sqrt{\langle y,y\rangle_I}\sqrt{\varepsilon\langle z,z\rangle_J}}{\sqrt{\langle y,y\rangle_I + \varepsilon\langle z,z\rangle_J}} \\ &\leqslant c_4\sqrt{\varepsilon}\|y\|_I. \end{aligned} \tag{3.17}$$

Combining (3.15), (3.16) and (3.17), we obtain

$$\begin{aligned}\langle \beta x, \nabla V(x)\rangle &\leqslant -c_1\|y\|_I^2 V(y,z)^{-1} - \varepsilon c_2\|z\|_J^2 V(y,z)^{-1} + c_4\sqrt{\varepsilon}\|y\|_I \\ &\leqslant -c_5\left(\langle y,y\rangle_I + \varepsilon\langle z,z\rangle_J\right)V(y,z)^{-1} + c_4\sqrt{\varepsilon}\|y\|_I \\ &\leqslant -c_5 V(y,z) + c_4\sqrt{\varepsilon}V(y,z),\end{aligned}$$

where $c_5 := c_1 \wedge c_2 > 0$. Since c_4 and c_5 depend only on β but not on ε, by choosing $\varepsilon = \varepsilon_0 > 0$ sufficiently small, we get

$$\langle \beta x, \nabla V(x)\rangle \leqslant -c_6 V(x), \quad x \in D \quad \text{with } \|x\| > 2. \tag{3.18}$$

From now on we take $\varepsilon = \varepsilon_0$ as fixed. In particular, the upcoming constants $c_7 - c_{11}$ may depend on ε.

Diffusion. By (3.12), we have

$$\left|\frac{\partial^2 V(x)}{\partial x_k \partial x_l}\right| \leqslant \frac{c_7}{V(x)}, \quad \text{for all } \|x\| > 2,\ k,l \in \{1,\ldots,d\}, \tag{3.19}$$

where $c_7 > 0$ is a constant. This implies

$$\sup_{x\in D}\left|x_i\frac{\partial^2 V(x)}{\partial x_k \partial x_l}\right| < \infty, \quad \text{for all } i \in I \text{ and } k,l \in \{1,\ldots,d\}.$$

We conclude that

$$\left|\sum_{k,l=1}^{d}\left(\sum_{i\in I}\alpha_{i,kl}x_i\right)\frac{\partial^2 V(x)}{\partial x_k \partial x_l}\right| \leqslant c_8, \quad \text{for all } x \in D, \tag{3.20}$$

where $c_8 > 0$ is a constant.

Turning to the jump part $\mathcal{J}_K$, we define for $i \in I$ and $k \in \mathbb{N}$,

$$\mathcal{J}_{k,i,*}V(x) := x_i\int_{\{0<\|\xi\|<k\}}\left(V(x+\xi) - V(x) - \langle \nabla V(x), \xi\rangle\right)\mu_{K,i}(\mathrm{d}\xi),$$

and

$$\mathcal{J}^*_{k,i}V(x) := x_i\int_{\{\|\xi\|\geqslant k\}}\left(V(x+\xi) - V(x) - \langle \nabla V(x), \xi\rangle\right)\mu_{K,i}(\mathrm{d}\xi).$$

So $\mathcal{J}_K V(x) = \sum_{i\in I}(\mathcal{J}_{k,i,*}V(x) + \mathcal{J}^*_{k,i}V(x))$.

Big jumps. By the mean value theorem, we get

$$\begin{aligned}
\left|\mathcal{J}^*_{k,i}V(x)\right| &\leqslant \|x_i\| \int_{\{\|\xi\|\geqslant k\}} \left(\|\nabla V\|_\infty \|\xi\| + \|\nabla V(x)\|\,\|\xi\|\right)\mu_i(\mathrm{d}\xi) \\
&\leqslant 2\|x\|\,\|\nabla V\|_\infty \int_{\{\|\xi\|\geqslant k\}} \|\xi\|\,\mu_i(\mathrm{d}\xi) \\
&\leqslant c_9(1+V(x)) \int_{\{\|\xi\|\geqslant k\}} \|\xi\|\,\mu_i(\mathrm{d}\xi) < \infty,
\end{aligned} \tag{3.21}$$

where we used that $\|\nabla V\|_\infty = \sup_{x\in D}\|\nabla V(x)\| < \infty$, as a consequence of (3.11). Hence, by dominated convergence, we can find large enough $k = k_0 > 0$ such that

$$\left|\mathcal{J}^*_{k_0,i}V(x)\right| \leqslant \frac{1}{2}c_6(1+V(x)), \quad x\in D.$$

Small jumps. To estimate the small jump part, we apply (3.19) and the mean value theorem, yielding for $\|x\| > 3k_0$,

$$\begin{aligned}
\left|\mathcal{J}_{k_0,i,*}V(x)\right| &\leqslant \left| x_i \int_{\{0<\|\xi\|<k_0\}} \left(\int_0^1 \langle \nabla V(x+r\xi) - \nabla V(x), \xi\rangle\right) \mathrm{d}r\mu_{K,i}(\mathrm{d}\xi)\right| \\
&\leqslant \|x_i\| \sup_{\widetilde{x}\in B_{k_0}(x)} \left\|\nabla^2 V(\widetilde{x})\right\| \int_{\{0<\|\xi\|<k_0\}} \|\xi\|^2\,\mu_i(\mathrm{d}\xi) \\
&\leqslant c_7\|x\| \sup_{\widetilde{x}\in B_{k_0}(x)} \frac{1}{V(\tilde{x})} \int_{\{0<\|\xi\|<k_0\}} \|\xi\|^2\,\mu_i(\mathrm{d}\xi) \\
&\leqslant c_{10}\frac{\|x\|}{\|x\|-k_0} \leqslant 2c_{10} < \infty,
\end{aligned} \tag{3.22}$$

with some positive constant c_{10} not depending on K. Here $B_{k_0}(x)$ denotes the ball with center x and radius k_0. Note that $\mathcal{J}_{k_0,i,*}V(x)$ is continuous in $x\in D$. Hence, we conclude that

$$|\mathcal{J}_K V(x)| \leqslant \frac{1}{2}c_6 V(x) + c_{11}, \quad x\in D.$$

Combining the latter inequality with (3.18) and (3.20), we obtain the desired result, namely,

$$\mathcal{A}^\sharp_K V(x) = \mathcal{D}V(x) + \mathcal{J}_K V(x) \leqslant -\frac{1}{2}c_6 V(x) + c_{12}, \quad x\in D.$$

□

Remark 3.4. *For the function V defined in the last lemma, we can easily find positive constants c_1, c_2, c_3, c_4 such that for all $x\in D$,*

$$V(x) \leqslant c_1\|x\| + c_2 \quad \text{and} \quad \|x\| \leqslant c_3 V(x) + c_4. \tag{3.23}$$

Proposition 3.5. *Assume $m\geqslant 1$ and $n\geqslant 1$. Suppose that $\beta\in\mathbb{M}^-_d$. Let c, C and V be the same as in Lemma 3.3. Then*

$$\mathbb{E}_x\left[V(X_{K,t})\right] \leqslant \mathrm{e}^{-ct}V(x) + c^{-1}C \quad \text{for all } K\geqslant 1,\ x\in D \text{ and } t\in\mathbb{R}_{\geqslant 0}. \tag{3.24}$$

Proof. Let $x\in D$, $K\geqslant 1$ and $T>0$ be fixed. The proof is divided into three steps.

Step 1: We show that

$$\sup_{t\in[0,T]} \mathbb{E}_x\left[\|X_{K,t}\|^2\right] < \infty. \tag{3.25}$$

Since $\mu_{K,i}$ has compact support, it follows that $\int_{\{\|\xi\|>1\}}\|\xi\|^k\mu_{K,i}(\mathrm{d}\xi)<\infty$ for all $k\in\mathbb{N}$. By [8, Lemmas 5.3 and 6.5], we know that $\psi_K\in C^2(\mathbb{R}_+\times\mathcal{U})$. Moreover, by [8, Theorem 2.16], we have

$$\mathbb{E}_x\left[\|X_{K,t}\|^2\right]=-\sum_{l=1}^{d}\left(\langle x,\partial^2_{\lambda_l}\psi_K(t,\mathrm{i}\lambda)|_{\lambda=0}\rangle+\langle x,\partial_{\lambda_l}\psi_K(t,\mathrm{i}\lambda)|_{\lambda=0}\rangle^2\right),$$

where the right-hand side is a continuous function in $t\in[0,T]$. So (3.25) follows.

Step 2: We show that

$$\sup_{t\in[0,T]}\mathbb{E}_x\left[V\left(X_{K,t}\right)\right]<\infty. \tag{3.26}$$

In fact, (3.26) follows from (3.23) and (3.25).

Step 3: We show that (3.24) is true. It follows from [8, Theorem 2.12] and [8, Lemma 10.1] that

$$f\left(X_{K,t}\right)-f\left(X_{K,0}\right)-\int_0^t\mathcal{A}_Kf\left(X_{K,s}\right)\mathrm{d}s,\quad t\in\mathbb{R}_{\geqslant0}, \tag{3.27}$$

is a $\mathbb{P}_x$-martingale for every $f\in C_c^2(D)$. Note that V belongs to $C^2(D)$ but does not have compact support. Let $\varphi\in C_c^\infty(\mathbb{R}_{\geqslant0})$ be such that $\mathbb{1}_{[0,1]}\leqslant\varphi\leqslant\mathbb{1}_{[0,2]}$, and define $(\varphi_j)_{j\geqslant1}\subset C_c^\infty(D)$ by $\varphi_j(y):=\varphi(\|y\|^2/j^2)$. Then

$$\varphi_j(y)=1\quad\text{for }\|y\|\leqslant j\quad\text{and}\quad\varphi_j(y)=0\quad\text{for }\|y\|>\sqrt{2}j,$$

and $\varphi_j\to1$ as $j\to\infty$. For $j\in\mathbb{N}$, we then define

$$V_j(y):=V(y)\varphi_j(y),\quad y\in D.$$

So $V_j\in C_c^2(D)$. In view of (3.27) and [10, Chap.4, Lemma 3.2], it follows that

$$\mathrm{e}^{ct}V_j\left(X_{K,t}\right)-V_j\left(X_{K,0}\right)-\int_0^t\mathrm{e}^{cs}\mathcal{A}_KV_j\left(X_{K,s}\right)\mathrm{d}s-\int_0^tc\mathrm{e}^{cs}V_j\left(X_{K,s}\right)\mathrm{d}s,\quad t\in\mathbb{R}_{\geqslant0},$$

is a $\mathbb{P}_x$-martingale, and hence

$$\mathrm{e}^{ct}\mathbb{E}_x\left[V_j\left(X_{K,t}\right)\right]-V_j\left(x\right)=\mathbb{E}_x\left[\int_0^t\mathrm{e}^{cs}\left(\mathcal{A}_KV_j\left(X_{K,s}\right)+cV_j\left(X_{K,s}\right)\right)\mathrm{d}s\right].$$

Now, a simple calculation shows

$$\|\nabla\varphi_j(y)\|\leqslant\frac{2\|y\|}{j^2}\|\varphi'\|_\infty\leqslant\frac{2c_1\|y\|}{j^2},$$

for some constant $c_1>0$. Therefore, by (3.23), we get

$$\begin{aligned}\|\nabla V_j(y)\|&=\mathbb{1}_{\{\|y\|\leqslant\sqrt{2}j\}}\|\varphi_j(y)\nabla V(y)+V(y)\nabla\varphi_j(y)\|\\&\leqslant\mathbb{1}_{\{\|y\|\leqslant\sqrt{2}j\}}\left(\|\nabla V\|_\infty+c_2\left(1+\|y\|\right)\frac{2c_1\|y\|}{j^2}\right)\\&\leqslant c_3\frac{(1+j)\,j}{j^2},\end{aligned}\tag{3.28}$$

where c_2 and c_3 are positive constants. A similar calculation yields that there exists a constant $c_4>0$ such that

$$\left\|\nabla^2\varphi_j(y)\right\|\leqslant c_4\frac{\|y\|^2+j^2}{j^4}.$$

So

$$\|\nabla^2V_j(y)\|\leqslant\mathbb{1}_{\{\|y\|\leqslant\sqrt{2}j\}}\left(\|\nabla^2V\|_\infty+2\|\nabla V\|_\infty\|\nabla\varphi_j(y)\|+\|V(y)\|\|\nabla^2\varphi_j(y)\|\right)$$

$$\leqslant \mathbb{1}_{\{\|y\|\leqslant\sqrt{2}j\}}\left(c_5+\frac{c_6\|y\|}{j^2}+c_7(1+\|y\|)\frac{\|y\|^2+j^2}{j^4}\right)$$

$$\leqslant c_8\frac{1+j+j^2}{j^2}, \tag{3.29}$$

where c_5, c_6, c_7, $c_8>0$ are constants. Define $\mathcal{D}V_j$ and $\mathcal{J}_KV_j$ similarly as in (3.13) and (3.14), respectively. It holds obviously that

$$|\mathcal{D}V_j(y)|\leqslant c_9\|y\|\left(\|\nabla V_j\|_\infty+\|\nabla^2V_j\|_\infty\right),\quad y\in D.$$

Similarly as in (3.21) and (3.22), we have that for all $y\in D$,

$$|\mathcal{J}_KV_j(y)|\leqslant c_{10}\|y\|\sum_{i=1}^m\Big(\|\nabla V_j\|_\infty\int_{\{\|\xi\|\geqslant1\}}\|\xi\|\,\mu_i(\mathrm{d}\xi)$$
$$+\|\nabla^2V_j\|_\infty\int_{\{0<\|\xi\|<1\}}\|\xi\|^2\,\mu_i(\mathrm{d}\xi)\Big).$$

Using (3.28), (3.29) and the above estimates for $\mathcal{D}V_j$ and $\mathcal{J}_KV_j$, we obtain

$$|\mathcal{A}_KV_j(y)|\leqslant c_{11}(1+\|y\|),\quad y\in D, \tag{3.30}$$

where $c_{11}>0$ is a constant not depending on j. The dominated convergence theorem implies $\lim_{j\to\infty}\mathcal{A}_KV_j(y)=\mathcal{A}_K^\sharp V(y)$ for all $y\in D$. By (3.26), (3.30) and again dominated convergence, it follows that

$$\mathrm{e}^{ct}\mathbb{E}_x\left[V\left(X_{K,t}\right)\right]-V\left(x\right)=\mathbb{E}_x\left[\int_0^t\mathrm{e}^{cs}\left(\mathcal{A}_K^\sharp V\left(X_{K,s}\right)+cV\left(X_{K,s}\right)\right)\mathrm{d}s\right].$$

Applying Lemma 3.3 yields

$$\mathrm{e}^{ct}\mathbb{E}_x\left[V\left(X_{K,t}\right)\right]-V\left(x\right)\leqslant\mathbb{E}_x\left[\int_0^t\mathrm{e}^{cs}C\mathrm{d}s\right]\leqslant c^{-1}C\mathrm{e}^{ct},$$

which implies

$$\mathbb{E}_x\left[V\left(X_{K,t}\right)\right]\leqslant\mathrm{e}^{-ct}V(x)+c^{-1},\quad \text{for} t\in[0,T].$$

Since $x\in D$, $K\geqslant1$ and $T>0$ are arbitrary, the assertion follows. □

Arguing similarly as in Lemma 3.3 and Proposition 3.5, we obtain also an analog result for the case where $m\geqslant1$ and $n=0$.

Proposition 3.6. *Assume $m\geqslant1$ and $n=0$. Suppose that $\beta\in\mathbb{M}_d^-$. Let $V\in C^2(D,\mathbb{R})$ be such that $V>0$ on D and*

$$V(x)=\langle x,x\rangle_I^{1/2},\quad \textit{whenever } \|x\|>2.$$

Then $\mathcal{A}_K^\sharp V$ is well-defined and there exist positive constants c and C, independent of K, such that

$$\mathcal{A}_K^\sharp V(x)\leqslant-cV(x)+C,\quad \forall x\in D.$$

Moreover, for all $K\geqslant1$, $t\geqslant0$ and $x\in D$, it holds that

$$\mathbb{E}_x\left[V\left(X_{K,t}\right)\right]\leqslant\mathrm{e}^{-ct}V(x)+c^{-1}C.$$

We are now ready to prove the uniform boundedness for the first moment of X_t, $t\geqslant0$.

Proposition 3.7. *Let X be an affine process satisfying* (3.1)*. Suppose that $\beta\in\mathbb{M}_d^-$. Then*

$$\sup_{t\geqslant0}\mathbb{E}_x\left[\|X_t\|\right]<\infty\quad \textit{for all } x\in D. \tag{3.31}$$

Proof. If $m = 0$ and $n \geqslant 1$, then $(X_t)_{t\geqslant 0}$ degenerates to a deterministic motion governed by the vector field $x \mapsto \beta x$. In this case we have

$$X_t = \mathrm{e}^{\beta t} X_0,$$

so (3.31) follows from the assumption that $\beta \in \mathbb{M}_d^-$.

For the case where $m \geqslant 1$, by Propositions 3.5 and 3.6, we have

$$\mathbb{E}_x \left[V\left(X_{K,t}\right)\right] \leqslant \mathrm{e}^{-ct} V(x) + c^{-1} C, \quad \text{for all } K \geqslant 1, x \in D \text{ and } t \in \mathbb{R}_{\geqslant 0}, \tag{3.32}$$

where c, $C > 0$ are constants not depending on K.

Let $x \in D$ be fixed and assume without loss of generality that $X_0 = x$ a.s. In view of Lemma 3.2 and Skorokhod's representation theorem (see, e.g., [10, Chap.3, Theorem 1.8]), there exist some probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ on which $(\widetilde{X}_{K,t})_{K\geqslant 1}$ and $\widetilde{X}_t$ are defined such that $\widetilde{X}_{K,t}$ and $\widetilde{X}_t$ have the same distributions as $X_{K,t}$ and X_t, respectively, and $\widetilde{X}_{K,t} \to \widetilde{X}_t$ $\widetilde{\mathbb{P}}$-almost surely as $K \to \infty$. Hence $V(\widetilde{X}_{K,t}) \to V(\widetilde{X}_t)$ $\widetilde{\mathbb{P}}$-almost surely as $K \to \infty$. By (3.32) and Fatou's lemma, we have

$$\begin{aligned} \mathbb{E}_x \left[V\left(X_t\right)\right] = \widetilde{\mathbb{E}}\left[V\left(\widetilde{X}_t\right)\right] &\leqslant \liminf_{K\to\infty} \widetilde{\mathbb{E}}\left[V\left(\widetilde{X}_{K,t}\right)\right] \\ &= \liminf_{K\to\infty} \mathbb{E}_x \left[V\left(X_{K,t}\right)\right] \\ &\leqslant \mathrm{e}^{-ct} V\left(x\right) + c^{-1} C \end{aligned}$$

for all $t \geqslant 0$. By (3.23), the assertion follows. □

3.2. **Exponential convergence of $\psi(t,u)$ to zero.** In this subsection we study the convergence speed of $\psi(t,u) \to 0$ as $t \to \infty$.

Lemma 3.8. *Suppose that $\beta \in \mathbb{M}_d^-$. There exist $\delta > 0$ and constants $C_1, C_2 > 0$ such that for all $u \in \mathcal{U}$ with $\|u\| < \delta$,*

$$\|\psi\left(t,u\right)\| \leqslant C_1 \exp\left\{-C_2 t\right\}, \quad t \geqslant 0. \tag{3.33}$$

Proof. For $u \in \mathcal{U}$, we can write $u = (v,w) \in \mathbb{C}^m_{\leqslant 0} \times \mathrm{i}\mathbb{R}^n$ and further $v = x + \mathrm{i}y$ and $w = \mathrm{i}z$, where $x \in \mathbb{R}^m_{\leqslant 0}$, $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$. Therefore,

$$\psi(t,u) = \psi\left(t,v,w\right) = \begin{pmatrix} \psi^I\left(t, x+\mathrm{i}y, \mathrm{i}z\right) \\ \mathrm{i}\mathrm{e}^{\beta_{JJ}^\top t} z \end{pmatrix}.$$

For $x \in \mathbb{R}^m_{\leqslant 0}$, $y \in \mathbb{R}^m$, and $z \in \mathbb{R}^n$, we define

$$\widetilde{\psi}\left(t,x,y,z\right) := \begin{pmatrix} \operatorname{Re}\psi^I\left(t,x+\mathrm{i}y,\mathrm{i}z\right) \\ \operatorname{Im}\psi^I\left(t,x+\mathrm{i}y,\mathrm{i}z\right) \\ \mathrm{e}^{\beta_{JJ}^\top t} z \end{pmatrix} = \begin{pmatrix} \vartheta \\ \eta \\ \zeta \end{pmatrix}, \quad t \geqslant 0.$$

Recall that $\psi^I(t,u)$ satisfies the Riccati equation

$$\partial_t \psi^I(t,v,w) = R^I\left(\psi^I(t,v,w), \mathrm{e}^{\beta_{JJ}^\top t} w\right), \qquad \psi^I(0,v,w) = v.$$

So

$$\begin{aligned} \partial_t \widetilde{\psi}(t,x,y,z) &= \begin{pmatrix} \partial_t \operatorname{Re}\psi^I\left(t,x+\mathrm{i}y,\mathrm{i}z\right) \\ \partial_t \operatorname{Im}\psi^I\left(t,x+\mathrm{i}y,\mathrm{i}z\right) \\ \partial_t \mathrm{e}^{\beta_{JJ}^\top t} z \end{pmatrix} \\ &= \begin{pmatrix} \operatorname{Re} R^I\left(\psi^I\left(t,x+\mathrm{i}y,\mathrm{i}z\right), \mathrm{i}\mathrm{e}^{\beta_{JJ}^\top t} z\right) \\ \operatorname{Im} R^I\left(\psi^I\left(t,x+\mathrm{i}y,\mathrm{i}z\right), \mathrm{i}\mathrm{e}^{\beta_{JJ}^\top t} z\right) \\ \beta_{JJ}^\top \mathrm{e}^{\beta_{JJ}^\top t} z \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} \operatorname{Re} R^I \left(\operatorname{Re} \psi^I (t, x+\mathrm{i}y, \mathrm{i}z) + \mathrm{i} \operatorname{Im} \psi^I (t, x+\mathrm{i}y, \mathrm{i}z), \mathrm{i}\mathrm{e}^{\beta_{JJ}^\top t} z \right) \\ \operatorname{Im} R^I \left(\operatorname{Re} \psi^I (t, x+\mathrm{i}y, \mathrm{i}z) + \mathrm{i} \operatorname{Im} \psi^I (t, x+\mathrm{i}y, \mathrm{i}z), \mathrm{i}\mathrm{e}^{\beta_{JJ}^\top t} z \right) \\ \beta_{JJ}^\top \mathrm{e}^{\beta_{JJ}^\top t} z \end{pmatrix} \\
&= \begin{pmatrix} \operatorname{Re} R^I (\vartheta + \mathrm{i}\eta, \mathrm{i}\zeta) \\ \operatorname{Im} R^I (\vartheta + \mathrm{i}\eta, \mathrm{i}\zeta) \\ \beta_{JJ}^\top \zeta \end{pmatrix} \\
&=: \widetilde{R} (\vartheta, \eta, \zeta),
\end{aligned}
$$

where the map $\mathbb{R}^m_{\leqslant 0} \times \mathbb{R}^m \times \mathbb{R}^n \ni (\vartheta, \eta, \zeta) \mapsto \widetilde{R} (\vartheta, \eta, \zeta)$ is C^1 by [8, Lemma 5.3]. Hence $\widetilde{\psi}(t, x, y, z)$ solves the equation

$$
\partial_t \widetilde{\psi}(t, x, y, z) = \widetilde{R} \left(\widetilde{\psi}(t, x, y, z) \right), \quad t \geqslant 0, \quad \psi(0, x, y, z) = (x, y, z). \tag{3.34}
$$

Similarly to [8, p.1011, (6.7)], we have, for $u = (x + \mathrm{i}y, \mathrm{i}z)$,

$$
\begin{aligned}
\operatorname{Re} R_i (x + \mathrm{i}y, \mathrm{i}z) = {} & \alpha_{i,ii} x_i^2 - \langle \alpha_i \operatorname{Im} u, \operatorname{Im} u \rangle + \sum_{k=1}^m \beta_{ki} x_k \\
& + \int_{D \setminus \{0\}} \left(\mathrm{e}^{\langle \xi_I, x \rangle} \cos \langle \operatorname{Im} u, \xi \rangle - 1 - \langle \xi_I, x \rangle \right) \mu_i (\mathrm{d}\xi)
\end{aligned} \tag{3.35}
$$

and

$$
\begin{aligned}
\operatorname{Im} R_i (x + \mathrm{i}y, \mathrm{i}z) = {} & 2\alpha_{i,ii} x_i y_i + \langle \beta_{Ii}, y \rangle + \langle \beta_{Ji}, z \rangle \\
& + \int_{D \setminus \{0\}} \left(\mathrm{e}^{\langle \xi_I, x \rangle} \sin \langle \operatorname{Im} u, \xi \rangle - \langle \operatorname{Im} u, \xi \rangle \right) \mu_i (\mathrm{d}\xi).
\end{aligned} \tag{3.36}
$$

Since $\widetilde{R} : \mathbb{R}^m_{\leqslant 0} \times \mathbb{R}^{m+n} \to \mathbb{R}^{2m+n}$ is C^1, so

$$
\begin{aligned}
& \left\| \widetilde{R} (\vartheta, \eta, \zeta) - D\widetilde{R}(\mathbf{0}) (\vartheta, \eta, \zeta)^\top \right\| \\
&\quad = \left\| \widetilde{R} (\vartheta, \eta, \zeta) - \widetilde{R}(\mathbf{0}) - D\widetilde{R}(\mathbf{0}) (\vartheta, \eta, \zeta)^\top \right\| \\
&\quad = \left\| \int_0^1 D\widetilde{R} (r (\vartheta, \eta, \zeta)) (\vartheta, \eta, \zeta)^\top \mathrm{d}r - \int_0^1 D\widetilde{R}(\mathbf{0}) (\vartheta, \eta, \zeta)^\top \mathrm{d}r \right\| \\
&\quad \leqslant \sup_{0 \leqslant r \leqslant 1} \left\| D\widetilde{R} (r (\vartheta, \eta, \zeta)) - D\widetilde{R}(\mathbf{0}) \right\| \cdot \left\| (\vartheta, \eta, \zeta)^\top \right\| \\
&\quad = o \left(\left\| (\vartheta, \eta, \zeta)^\top \right\| \right)
\end{aligned} \tag{3.37}
$$

holds. Here, $D\widetilde{R}(\vartheta, \eta, \zeta)$ denotes the Jacobian, i.e., the matrix consisting of all first-order partial derivatives of the vector-valued function $(\vartheta, \eta, \zeta) \mapsto \widetilde{R}(\vartheta, \eta, \zeta)$. According to (3.35) and (3.36), we see that $D\widetilde{R}(\mathbf{0})$ is a matrix taking the form

$$
D\widetilde{R}(\mathbf{0}) = \left(\begin{array}{cc:c} \beta_{II}^\top & 0 & 0 \\ 0 & \beta_{II}^\top & * \\ \hdashline 0 & 0 & \beta_{JJ}^\top \end{array} \right)
$$

where $*$ is a $(m \times n)$-matrix. By the Riccati equation (3.34) for $\widetilde{\psi}$, we can write

$$
\partial_t \widetilde{\psi} (t, x, y, z) = D\widetilde{R}(\mathbf{0}) \widetilde{\psi} (t, x, y, z) + \left(\widetilde{R} \left(\widetilde{\psi}(t, x, y, z) \right) - D\widetilde{R}(\mathbf{0}) \widetilde{\psi} (t, x, y, z) \right).
$$

From (3.37) it follows that

$$\lim_{\|(\vartheta,\eta,\zeta)\|\to 0} \frac{\left\| \widetilde{R}(\vartheta,\eta,\zeta) - D\widetilde{R}(\mathbf{0})(\vartheta,\eta,\zeta)^{\top} \right\|}{\|(\vartheta,\eta,\zeta)\|} = 0.$$

By assumption, we know that $\beta_{II} \in \mathbb{M}_m^-$ and $\beta_{JJ} \in \mathbb{M}_n^-$, which ensures $D\widetilde{R}(\mathbf{0}) \in \mathbb{M}_{2m+n}^-$. Now, an application of the linearized stability theorem (see, e.g., [26, VII. Stability Theorem, p.311]) yields that $\widetilde{\psi}$ is asymptotically stable at $\mathbf{0}$. Moreover, as shown in the proof of [26, VII. Stability Theorem, p.311], we can find constants δ, c_1, $c_2 > 0$ such that

$$\left\| \widetilde{\psi}(t,x,y,z) \right\| \leqslant c_1 \mathrm{e}^{-c_2 t}, \quad \forall\, t \geqslant 0,\ (x,y,z) \in B_\delta(0) \cap \mathbb{R}^m_{\leqslant 0} \times \mathbb{R}^{m+n},$$

where $B_\delta(0)$ denotes the ball with center 0 and radius δ. By the definition of $\widetilde{\psi}$, the latter inequality implies that (3.33) is true. The lemma is proved. □

Next, we extend the estimate in Lemma 3.8 to all $u \in \mathcal{U}$.

Proposition 3.9. *Let X be an affine process satisfying* (3.1)*. Suppose that $\beta \in \mathbb{M}_d^-$. Then for every $u \in \mathcal{U}$, there exist positive constants c_1, c_2, which depend on u, such that*

$$\|\psi(t,u)\| \leqslant c_1 \exp\{-c_2 t\}, \quad t \geqslant 0.$$

Proof. Our proof is inspired by the proof of [12, Theorem 2.4]. By Proposition 3.7, we have $\sup_{t\in\mathbb{R}_{\geqslant 0}} \mathbb{E}_x[\|X_t\|] < \infty$ for all $x \in D$. Then for $M > 0$,

$$\mathbb{P}_x(\|X_t\| > M) \leqslant \frac{\mathbb{E}_x[\|X_t\|]}{M} \leqslant \frac{\sup_{t\geqslant 0}\mathbb{E}_x[\|X_t\|]}{M},$$

which implies

$$\sup_{t\geqslant 0} \mathbb{P}_x(\|X_t\| > M) \to 0 \quad \text{as } M \to \infty.$$

We see that under $\mathbb{P}_x$, the sequence $\{X_t,\, t \geqslant 0\}$ is tight. Consider an arbitrary subsequence $\{X_{t'}\}$. Then it contains a further subsequence $\{X_{t''}\}$ converging in law to some limiting random vector, say X^a. Since $X_{t''}$ converges weakly to X^a as $t'' \to \infty$, Lévy's continuity theorem implies that the characteristic function of $X_{t''}$ converges pointwise to that of X^a, namely,

$$\lim_{t''\to\infty} \mathbb{E}_x[\exp\{\langle u, X_{t''}\rangle\}] = \mathbb{E}[\exp\{\langle u, X^a\rangle\}], \quad \text{for all } u \in \mathcal{U}.$$

We know by Proposition 3.8 that the original sequence $\{X_t\}$ satisfies

$$\lim_{t\to\infty} \mathbb{E}_x[\exp\{\langle u, X_t\rangle\}] = \lim_{t\to\infty} \exp\{\langle x, \psi(t,u)\rangle\} = 1$$

for all $u \in \mathcal{U}$ with $\|u\| < \delta$. As a consequence, we get

$$\mathbb{E}[\exp\{\langle u, X^a\rangle\}] = 1, \quad \text{for all } u \in \mathcal{U} \quad \text{with} \quad \|u\| < \delta. \tag{3.38}$$

We claim that $X^a = 0$ almost surely. To prove this, we consider an arbitrary $z \in \mathbb{R}^d$ with $z \neq 0$. Then there exists an $u_0 \in$
$mathbbR^d$ with $\|u_0\| < \delta$ such that $0 < \langle u_0, z\rangle < \pi/6$, and hence $0 < \cos(\langle u_0, z\rangle) < 1$. Continuity of cosinus implies that there exists an $\varepsilon > 0$ such that $0 \notin B_\varepsilon(z) := \{y \in \mathbb{R}^d : \|y - z\| < \varepsilon\}$ and $0 < \cos(\langle u_0, y\rangle) < 1$ for all $y \in B_\varepsilon(z)$. Suppose that $\mathbb{P}(X^a \in B_\varepsilon(z)) > 0$. It follows that

$$\mathbb{E}\left[\cos(\langle u_0, X^a\rangle)\, \mathbb{1}_{\{X^a \in B_\varepsilon(z)\}}\right] < \mathbb{P}(X^a \in B_\varepsilon(z)),$$

which in turn implies

$$\begin{aligned} \operatorname{Re}\mathbb{E}[\exp\{\mathrm{i}\langle u_0, X^a\rangle\}] &= \mathbb{E}[\cos(\langle u_0, X^a\rangle)] \\ &\leqslant \mathbb{E}\left[\cos(\langle u_0, X^a\rangle)\, \mathbb{1}_{\{X^a \in B_\varepsilon(z)\}}\right] \\ &\quad + \mathbb{E}\left[\cos(\langle u_0, X^a\rangle)\, \mathbb{1}_{\{X^a \notin B_\varepsilon(z)\}}\right] \end{aligned}$$

$$< \mathbb{P}\left(X^a \in B_\varepsilon(z)\right) + \mathbb{P}\left(X^a \notin B_\varepsilon(z)\right)$$
$$= 1,$$

a contradiction to (3.38). We conclude that $\mathbb{P}(X^a \in B_\varepsilon(z)) = 0$. Since $z \neq 0$ is arbitrary, X^a must be 0 almost surely. Now we have shown that every subsequence of $\{X_t\}$ contains a further subsequence converging weakly to δ_0, so the original sequence $\{X_t\}$ must converge to δ_0 weakly. In view of this, we now denote X^a by X_∞ which is 0 almost surely. We have thus shown that for all $x \in D$ and $u \in \mathcal{U}$,

$$\exp\{\langle x, \psi(t,u)\rangle\} = \mathbb{E}_x\left[\exp\{\langle u, X_t\rangle\}\right] \to 1 \quad \text{as} \quad t \to \infty. \tag{3.39}$$

From the above convergence of $\exp\{\langle x, \psi(t,u)\rangle\}$ to 1, we infer that for each $i = 1, \ldots, d$,

$$\operatorname{Re}\psi_i(t,u) \to 0 \quad \text{as } t \to \infty. \tag{3.40}$$

Moreover, we must have $\sup_{t\in[0,\infty)} |\psi_i(t,u)| \leqslant C$ for some constant $C = C(u) < \infty$, otherwise, by continuity, $\operatorname{Im}\psi_i(t,u)$ hits the set $\{2k\pi + \pi/2 : \ k \in \mathbb{Z}\}$ infinitely many times as $t \to \infty$, so $\sin(\operatorname{Im}\psi_i(t,u)) = 1$ infinitely often, contradicting the fact that $\exp\{\langle x, \psi(t,u)\rangle\} \to 1$ for all $x \in D$.

Let $z,\, z' \in \mathbb{C}$ be two different accumulation points of $\{\psi_1(t,u),\ t \geqslant 0\}$ as $t \to \infty$, that is, we can find sequences $t_n, t'_n \to \infty$ such that $\psi_1(t_n,u) \to z$ and $\psi_1(t'_n,u) \to z'$. Using once again the convergence in (3.39), we obtain that $z = \mathrm{i}2\pi k_1$ and $z' = \mathrm{i}2\pi k_2$ for some $k_1, k_2 \in \mathbb{Z}$. By (3.40) and a similar argument as in the last paragraph, $\psi_1(t,u)$ is not allowed to fluctuate between z and z', showing that $z = z'$. So $z = \mathrm{i}2\pi k_1$ is the only accumulation point of $\{\psi_1(t,u),\ t \geqslant 0\}$, and $\psi_1(t,u) \to z = \mathrm{i}2\pi k_1$ as $t \to \infty$. Moreover, we must have $k_1 = 0$, otherwise for some $x \in D$ we get $\exp\{x_1 2\pi \mathrm{i} k_1\} \neq 1$, which is impossible due to (3.39). We conclude that

$$\psi_1(t,u) \to 0 \quad \text{as } t \to \infty \text{ for all } u \in \mathcal{U}.$$

In the same way it follows that $\psi_i(t,u) \to 0$ as $t \to 0$ for all $i = 2, \ldots, d$ and $u \in \mathcal{U}$.

Finally, we prove that the convergence of $\psi(t,u)$ to zero as $t \to \infty$ is exponentially fast. Since $\psi(t,u)$ converges to 0 as $t \to \infty$, there exists a $t_0 > 0$ such that $\|\psi(t_0,u)\| < \delta$. Combining Lemma 3.8 with the semi-flow property of ψ, we conclude that

$$\|\psi(t+t_0,u)\| = \|\psi(t,\psi(t_0,u))\| \leqslant c_1 \mathrm{e}^{-c_2 t}, \quad t \geqslant 0,$$

for some positive constants c_1 and c_2. Hence,

$$\|\psi(t,u)\| \leqslant c_3 \mathrm{e}^{-c_2 t}, \quad t \geqslant t_0.$$

Since $\sup_{t\in[0,t_0]} \|\psi(t,u)\| < c_4$, where $c_4 > 0$ is a constant, it follows that

$$\|\psi(t,u)\| \leqslant c_5 \mathrm{e}^{-c_2 t}, \quad t \geqslant 0,$$

with another constant $c_5 > 0$. This completes our proof. □

4. Proof of the main result

In this section we will prove Theorem 2.4.

Let X be an affine process with state space D and admissible parameters $(a, \alpha, b, \beta, m, \mu)$. Recall that $F(u)$ is given by (2.7). We start with the following lemma.

Lemma 4.1. *Suppose $\beta \in \mathbb{M}_d^-$ and $\int_{\{\|\xi\|>1\}} \log\|\xi\|\, m(\mathrm{d}\xi) < \infty$. Then*

$$\int_0^\infty |F(\psi(s,u))|\, \mathrm{d}s < \infty \quad \textit{for all } u \in \mathcal{U}.$$

Proof. Let $u \in \mathcal{U}$ be fixed. By Remark 3.1 and Proposition 3.9, we can find constants c_1, $c_2 > 0$ depending on u such that

$$\|\psi(s,u)\| \leqslant c_1 \mathrm{e}^{-c_2 s}, \quad s \geqslant 0. \tag{4.1}$$

It is clear that finiteness of $\int_0^\infty |F(\psi(s,u))| \,\mathrm{d}s$ depends only on the jump part of F. We define

$$\begin{aligned}\mathcal{I}(u) &= \int_0^\infty \int_{\{0<\|\xi\|\leqslant 1\}} \left| \mathrm{e}^{\langle \xi, \psi(s,u)\rangle} - 1 - \langle \psi^J(s,u), \xi_J\rangle \right| m(\mathrm{d}\xi)\,\mathrm{d}s \\ &\quad + \int_0^\infty \int_{\{\|\xi\|>1\}} \left| \mathrm{e}^{\langle \xi, \psi(s,u)\rangle} - 1 \right| m(\mathrm{d}\xi)\,\mathrm{d}s \\ &=: \mathcal{I}_*(u) + \mathcal{I}^*(u).\end{aligned}$$

With the latter fact in mind, we start with the big jumps. We can apply Fubini's theorem to get

$$\mathcal{I}^*(u) = \int_{\{\|\xi\|>1\}} \int_0^\infty \left| \mathrm{e}^{\langle \xi, \psi(s,u)\rangle} - 1 \right| \mathrm{d}s m(\mathrm{d}\xi).$$

Let us define $I_1(\xi) := \int_0^\infty |\exp\{\langle \psi(s,u), \xi\rangle\} - 1| \,\mathrm{d}s$. For $\|\xi\| > 1$, by a change of variables $t := \exp\{-c_2 s\} \|\xi\|$, we get $\mathrm{d}s = -c_2^{-1} t^{-1} \mathrm{d}t$, and hence

$$\begin{aligned} I_1(\xi) &= -\frac{1}{c_2} \int_{\|\xi\|}^0 \frac{1}{t} \left| \mathrm{e}^{\langle \xi, \psi(s^{-1}(t),u)\rangle} - 1 \right| \mathrm{d}t \\ &= \frac{1}{c_2} \int_0^{\|\xi\|} \frac{1}{t} \left| \mathrm{e}^{\langle \xi, \psi(s^{-1}(t),u)\rangle} - 1 \right| \mathrm{d}t \\ &\leqslant \frac{1}{c_2} \int_0^1 \frac{1}{t} \left| \mathrm{e}^{\langle \xi, \psi(s^{-1}(t),u)\rangle} - 1 \right| \mathrm{d}t + \frac{1}{c_2} \int_1^{\|\xi\|} \frac{2}{t} \mathrm{d}t \\ &=: I_2(\xi) + I_3(\xi). \end{aligned}$$

Note that

$$\begin{aligned} \left| \mathrm{e}^{\langle \xi, \psi(s^{-1}(t),u)\rangle} - 1 \right| &= \left| \int_0^1 \mathrm{e}^{r\langle \xi, \psi(s^{-1}(t),u)\rangle} \langle \xi, \psi(s^{-1}(t), u)\rangle \mathrm{d}r \right| \\ &\leqslant \left| \langle \xi, \psi(s^{-1}(t), u)\rangle \right|. \end{aligned}$$

Using (4.1), we obtain

$$\begin{aligned} I_2(\xi) &\leqslant \frac{1}{c_2} \int_0^1 \frac{1}{t} \left| \langle \psi(s^{-1}(t), u), \xi\rangle \right| \mathrm{d}t \\ &\leqslant \frac{1}{c_2} \int_0^1 \frac{1}{t} \left\| \psi(s^{-1}(t), u) \right\| \|\xi\| \,\mathrm{d}t \\ &\leqslant \frac{1}{c_2} \int_0^1 \frac{c_1}{t} \mathrm{e}^{-c_2 s^{-1}(t)} \|\xi\| \,\mathrm{d}t. \end{aligned}$$

Since $s^{-1}(t) = \log(t\|\xi\|^{-1})(-c_2)^{-1}$, it follows that

$$I_2(\xi) \leqslant \frac{1}{c_2} \int_0^1 c_1 \mathrm{d}t = \frac{c_1}{c_2}.$$

On the other hand, it is easy to see that

$$I_3(\xi) \leqslant \frac{2}{c_2} \log \|\xi\|,$$

Having established the latter inequalities, we conclude that

$$\begin{aligned}|\mathcal{I}^*(u)| &\leqslant \int_{\{\|\xi\|>1\}} (I_2(\xi)+I_3(\xi))\, m(\mathrm{d}\xi)\\ &\leqslant \int_{\{\|\xi\|>1\}} \left(\frac{c_1}{c_2}+\frac{2}{c_2}\log\|\xi\|\right) m(\mathrm{d}\xi)\\ &= \frac{c_1}{c_2} m(\{\|\xi\|>1\}) + \frac{2}{c_2}\int_{\{\|\xi\|>1\}} \log\|\xi\|\, m(\mathrm{d}\xi).\end{aligned}$$

Because the Lévy measure $m(\mathrm{d}\xi)$ integrates $\mathbb{1}_{\{\|\xi\|>1\}}\log\|\xi\|$ by assumption, we see that

$$\mathcal{I}^*(u) < \infty. \tag{4.2}$$

We now turn to $\mathcal{I}_*(\xi)$. We can write

$$\begin{aligned}&\mathrm{e}^{\langle \xi,\psi(s,u)\rangle} - 1 - \langle \psi^J(s,u), \xi_J\rangle\\ &= \int_0^1 \mathrm{e}^{r\langle \xi,\psi(s,u)\rangle}\langle \psi(s,u), \xi\rangle \mathrm{d}r - \langle \psi^J(s,u), \xi_J\rangle\\ &= \int_0^1 \mathrm{e}^{r\langle \xi,\psi(s,u)\rangle}\langle \psi^I(s,u), \xi_I\rangle \mathrm{d}r + \int_0^1 \left(\mathrm{e}^{r\langle \xi,\psi(s,u)\rangle}-1\right)\langle \psi^J(s,u), \xi_J\rangle \mathrm{d}r\\ &= \int_0^1 \mathrm{e}^{r\langle \xi,\psi(s,u)\rangle}\langle \psi^I(s,u), \xi_I\rangle \mathrm{d}r\\ &\quad + \int_0^1\int_0^1 \mathrm{e}^{rr'\langle \xi,\psi(s,u)\rangle} r\langle \xi,\psi(s,u)\rangle\langle \psi^J(s,u), \xi_J\rangle \mathrm{d}r\mathrm{d}r'.\end{aligned}$$

Noting (4.1) and $\mathrm{Re}\,(\langle \xi,\psi(s,u)\rangle)\leqslant 0$, we deduce that for $\|\xi\|\leqslant 1$ and $s\geqslant 0$,

$$\begin{aligned}\left|\mathrm{e}^{\langle \xi,\psi(s,u)\rangle} - 1 - \langle \psi^J(s,u), \xi_J\rangle\right| &\leqslant \left\|\psi^I(s,u)\right\|\|\xi_I\| + \left\|\psi(s,u)\right\|\|\xi\|\left\|\psi^J(s,u)\right\|\|\xi_J\|\\ &\leqslant (c_1+c_1^2)\mathrm{e}^{-c_2 s}\left(\|\xi_I\| + (\|\xi_I\|+\|\xi_J\|)\|\xi_J\|\right)\\ &\leqslant (c_1+c_1^2)\mathrm{e}^{-c_2 s}\left(2\|\xi_I\| + \|\xi_J\|^2\right).\end{aligned} \tag{4.3}$$

So

$$\mathcal{I}_*(u) \leqslant (c_1+c_1^2)\int_0^\infty \mathrm{e}^{-c_2 s}\mathrm{d}s \int_{\{0<\|\xi\|\leqslant 1\}} \left(2\|\xi_I\| + \|\xi_J\|^2\right) m(\mathrm{d}\xi) < \infty,$$

where the finiteness of the integral on the right-hand side follows by Definition 2.2 (iii). Since (4.2) holds, it follows that

$$\int_0^\infty |F(\psi(s,u))|\,\mathrm{d}s \leqslant \mathcal{I}(u) = \mathcal{I}_*(u) + \mathcal{I}^*(u) < \infty.$$

The lemma is proved. □

We are now ready to prove our main result.

Proof of Theorem 2.4. Recall that the characteristic function of X_t is given by

$$\mathbb{E}_x\left[\mathrm{e}^{\langle u,X_t\rangle}\right] = \exp\left\{\phi(t,u) + \langle x,\psi(t,u)\rangle\right\}, \quad (t,u)\in\mathbb{R}_{\geqslant 0}\times\mathcal{U}.$$

Using Remark 3.1, Theorem 3.9 and Lemma 4.1, we have that $\psi(t,u)\to 0$ and

$$\phi(t,u) = \int_0^t F(\psi(s,u))\,\mathrm{d}s \to \int_0^\infty F(\psi(s,u))\,\mathrm{d}s, \quad \text{as } t\to\infty.$$

We now verify that $\int_0^\infty F(\psi(s,u))\,\mathrm{d}s$ is continuous at $u=0$. It is easy to see that that $\int_0^T F(\psi(s,u))\,\mathrm{d}s$ is continuous at $u=0$. It suffices to show that the convergence $\lim_{T\to\infty}\int_0^T F(\psi(s,u))\,\mathrm{d}s =$

$\int_0^\infty F(\psi(s,u))\,\mathrm{d}s$ is uniform for u in a small neighborhood of 0. By (3.33), there exist $\delta>0$ and constants c_1, $c_2>0$ such that for all $B_\delta(0)\cap\mathcal{U}$,

$$\|\psi(t,u)\|\leqslant c_1\exp\{-c_2t\},\quad t\geqslant 0.$$

Define

$$\begin{aligned}\mathcal{I}_T(u)&=\int_T^\infty\int_{\{0<\|\xi\|\leqslant 1\}}\left|\mathrm{e}^{\langle\xi,\psi(s,u)\rangle}-1-\langle\psi^J(s,u),\xi_J\rangle\right|m(\mathrm{d}\xi)\,\mathrm{d}s\\&+\int_T^\infty\int_{\{1<\|\xi\|\leqslant K\}}\left|\mathrm{e}^{\langle\xi,\psi(s,u)\rangle}-1\right|m(\mathrm{d}\xi)\,\mathrm{d}s\\&+\int_T^\infty\int_{\{\|\xi\|>K\}}\left|\mathrm{e}^{\langle\xi,\psi(s,u)\rangle}-1\right|m(\mathrm{d}\xi)\,\mathrm{d}s\\&=:\mathcal{I}_{*,T}(u)+\mathcal{I}_T^*(u)+\mathcal{I}_T^{**}(u),\end{aligned}$$

where $K>0$. Let $\varepsilon>0$ be arbitrary. By Fubini's theorem,

$$\mathcal{I}_T^{**}(u)=\int_{\{\|\xi\|>K\}}\int_T^\infty\left|\mathrm{e}^{\langle\xi,\psi(s,u)\rangle}-1\right|\mathrm{d}s m(\mathrm{d}\xi).$$

Set $I_1(\xi):=\int_T^\infty|\exp\{\langle\psi(s,u),\xi\rangle\}-1|\,\mathrm{d}s$. As in the proof of Lemma 4.1, we introduce a change of variables $t:=\exp\{-c_2(s-T)\}\|\xi\|$ and obtain for $\|\xi\|>1$,

$$\begin{aligned}I_1(\xi)&=\frac{1}{c_2}\int_0^{\|\xi\|}\frac{1}{t}\left|\mathrm{e}^{\langle\xi,\psi(s^{-1}(t),u)\rangle}-1\right|\mathrm{d}t\\&\leqslant\frac{1}{c_2}\int_0^1\frac{1}{t}\left|\mathrm{e}^{\langle\xi,\psi(s^{-1}(t),u)\rangle}-1\right|\mathrm{d}t+\frac{1}{c_2}\int_1^{\|\xi\|}\frac{2}{t}\mathrm{d}t\\&\leqslant\frac{1}{c_2}\int_0^1\frac{c_1}{t}\mathrm{e}^{-c_2s^{-1}(t)}\|\xi\|\,\mathrm{d}t+\frac{2}{c_2}\log\|\xi\|\\&\leqslant\frac{1}{c_2}\int_0^1c_1e^{-c_2T}\mathrm{d}t+\frac{2}{c_2}\log\|\xi\|.\end{aligned}\tag{4.4}$$

So

$$\begin{aligned}\mathcal{I}_T^{**}(u)&\leqslant\int_{\{\|\xi\|>K\}}\left(\frac{c_1}{c_2}e^{-c_2T}+\frac{2}{c_2}\log\|\xi\|\right)m(\mathrm{d}\xi)\\&\leqslant\frac{c_1}{c_2}m(\{\|\xi\|>K\})+\frac{2}{c_2}\int_{\{\|\xi\|>K\}}\log\|\xi\|\,m(\mathrm{d}\xi).\end{aligned}$$

We now choose $K>0$ large enough such that $\mathcal{I}_T^{**}(u)<\varepsilon/3$.

For $\mathcal{I}_T^*(u)$, by (4.4), we have

$$\begin{aligned}I_1(\xi)&=\frac{1}{c_2}\int_0^{\|\xi\|}\frac{1}{t}\left|\mathrm{e}^{\langle\xi,\psi(s^{-1}(t),u)\rangle}-1\right|\mathrm{d}t\\&\leqslant\frac{1}{c_2}\int_0^{\|\xi\|}\frac{c_1}{t}\mathrm{e}^{-c_2s^{-1}(t)}\|\xi\|\,\mathrm{d}t\\&\leqslant\frac{1}{c_2}\int_0^{\|\xi\|}c_1e^{-c_2T}\mathrm{d}t\\&\leqslant\frac{c_1}{c_2}e^{-c_2T}\|\xi\|,\end{aligned}$$

which imples

$$\mathcal{I}_T^*(u)\leqslant\int_{\{1<\|\xi\|\leq K\}}\left(\frac{c_1}{c_2}e^{-c_2T}\|\xi\|\right)m(\mathrm{d}\xi)$$

$$\leqslant \frac{c_1}{c_2} e^{-c_2 T} \int_{\{1<\|\xi\|\leq K\}} \|\xi\| \, m\,(\mathrm{d}\xi) \to 0, \quad \text{as } T \to \infty.$$

So we find $T_1 > 0$ such that for $T > T_1$, $\mathcal{I}_T^*(u) < \varepsilon/3$. It follows from (4.3) that

$$\mathcal{I}_{*,T}(u) \leqslant (c_1 + c_1^2) \int_T^\infty \mathrm{e}^{-c_2 s} \mathrm{d}s \int_{\{0<\|\xi\|\leqslant 1\}} \left(2\|\xi_I\| + \|\xi_J\|^2\right) m\,(\mathrm{d}\xi) \to 0, \quad \text{as } T \to \infty.$$

Hence there exists $T_2 > T_1$ such that for $T > T_2$, $\mathcal{I}_{*,T}(u) < \varepsilon/3$. Finally, we get for $T > T_2$,

$$\int_T^\infty |F(\psi(s,u))| \, \mathrm{d}s \leqslant \mathcal{I}_{*,T}(u) + \mathcal{I}_T^*(u) + \mathcal{I}_T^{**}(u) < \varepsilon.$$

Moreover, the particular choice of above K, T_1, T_2 do not depend on $u \in B_\delta(0) \cap \mathcal{U}$. We thus obtain the desired uniform convergence and further the continuity of $\int_0^\infty F(\psi(s,u))\,\mathrm{d}s$ at $u = 0$.

By Lévy's continuity theorem, the limiting distribution of X_t exists and we denote it by π. The limiting distribution π has characteristic function

$$\int_D \mathrm{e}^{\langle u,x\rangle} \pi(\mathrm{d}x) = \exp\left\{\int_0^\infty F(\psi(s,u))\,\mathrm{d}s\right\}.$$

We now verify that π is the unique stationary distribution. We start with the stationarity. Suppose that X_0 is distributed according to π. Then, for any $u \in \mathcal{U}$,

$$\begin{aligned}
\mathbb{E}_\pi[\exp\{\langle u, X_t\rangle\}] &= \int_D \exp\{\phi(t,u) + \langle x, \psi(t,u)\rangle\} \pi(\mathrm{d}x) \\
&= \mathrm{e}^{\phi(t,u)} \int_D \exp\{\langle x, \psi(t,u)\rangle\} \pi(\mathrm{d}x) \\
&= \mathrm{e}^{\phi(t,u)} \int_D \mathrm{e}^{\langle x,\eta\rangle} \pi(\mathrm{d}x),
\end{aligned}$$

where we substituted $\eta := \psi(t,u)$ in the last equality. Note that the integral on the right-hand side of the last equality is the characteristic function of the limit distribution π. Therefore, using the semi-flow property of ψ in (2.3), we have

$$\begin{aligned}
\mathbb{E}_\pi[\exp\{\langle u, X_t\rangle\}] &= \mathrm{e}^{\phi(t,u)} \exp\left\{\int_0^\infty F(\psi(s,\eta))\,\mathrm{d}s\right\} \\
&= \mathrm{e}^{\phi(t,u)} \exp\left\{\int_0^\infty F(\psi(s,\psi(t,u)))\,\mathrm{d}s\right\} \\
&= \mathrm{e}^{\phi(t,u)} \exp\left\{\int_0^\infty F(\psi(t+s,u))\,\mathrm{d}s\right\} \\
&= \mathrm{e}^{\phi(t,u)} \exp\left\{\int_t^\infty F(\psi(s,u))\,\mathrm{d}s\right\}.
\end{aligned}$$

So, by the generalized Riccati equation (2.5) for ϕ,

$$\mathbb{E}_\pi[\exp\{\langle u, X_t\rangle\}] = \exp\left\{\int_0^\infty F(\psi(s,u))\,\mathrm{d}s\right\} = \int_D \mathrm{e}^{\langle x,u\rangle} \pi(\mathrm{d}x).$$

Hence π is a stationary distribution for X.

Finally, we prove the uniqueness of stationary distributions for X. We proceed as in [15, p.80]. Suppose that there exists another stationary distribution π'. Let X_0 be distributed according to π'. Recall that for all $u \in \mathcal{U}$, $\psi(t,u) \to 0$ as $t \to \infty$ in virtue of Theorem 3.9 and, by Lemma 4.1, $\phi(t,u) \to \int_0^\infty F(\psi(t,u))\,\mathrm{d}s$ as $t \to \infty$. Hence, by dominated convergence,

$$\int_D \mathrm{e}^{\langle x,u\rangle} \pi'(\mathrm{d}x) = \lim_{t\to\infty} \mathbb{E}_{\pi'}[\exp\{\langle u, X_t\rangle\}]$$

$$= \lim_{t\to\infty} \int_D \exp\left\{\phi(t,u) + \langle x, \psi(t,u)\rangle\right\} \pi'(\mathrm{d}x)$$
$$= \int_D \exp\left\{\int_0^\infty F\left(\psi(s,u)\right) \mathrm{d}s\right\} \pi'(\mathrm{d}x)$$
$$= \exp\left\{\int_0^\infty F\left(\psi(s,u)\right) \mathrm{d}s\right\} = \int_D \mathrm{e}^{\langle x,u\rangle} \pi(\mathrm{d}x).$$

So $\pi = \pi'$. □

Acknowledgements. We would like to thank Martin Friesen for several helpful discussions.

References

1. Leif B. G. Andersen and Vladimir V. Piterbarg, *Moment explosions in stochastic volatility models*, Finance Stoch. **11** (2007), no. 1, 29–50. MR 2284011
2. Mátyás Barczy, Leif Döring, Zenghu Li, and Gyula Pap, *Stationarity and ergodicity for an affine two-factor model*, Adv. in Appl. Probab. **46** (2014), no. 3, 878–898. MR 3254346
3. Mátyás Barczy, Zenghu Li, and Gyula Pap, *Stochastic differential equation with jumps for multi-type continuous state and continuous time branching processes with immigration*, ALEA Lat. Am. J. Probab. Math. Stat. **12** (2015), no. 1, 129–169. MR 3340375
4. ______, *Moment formulas for multitype continuous state and continuous time branching process with immigration*, J. Theoret. Probab. **29** (2016), no. 3, 958–995. MR 3540486
5. John C. Cox, Jonathan E. Ingersoll, Jr., and Stephen A. Ross, *A theory of the term structure of interest rates*, Econometrica **53** (1985), no. 2, 385–407.
6. Qiang Dai and Kenneth J. Singleton, *Specification analysis of affine term structure models*, The Journal of Finance **55** (2000), no. 5, 1943–1978.
7. D. A. Dawson and Zenghu Li, *Skew convolution semigroups and affine Markov processes*, Ann. Probab. **34** (2006), no. 3, 1103–1142. MR 2243880
8. D. Duffie, D. Filipović, and W. Schachermayer, *Affine processes and applications in finance*, Ann. Appl. Probab. **13** (2003), no. 3, 984–1053.
9. Darrell Duffie, Jun Pan, and Kenneth Singleton, *Transform analysis and asset pricing for affine jump-diffusions*, Econometrica **68** (2000), no. 6, 1343–1376. MR 1793362
10. Stewart N. Ethier and Thomas G. Kurtz, *Markov processes: Characterization and convergence*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1986. MR 838085 (88a:60130)
11. Damir Filipović and Eberhard Mayerhofer, *Affine diffusion processes: theory and applications*, Advanced financial modelling, Radon Ser. Comput. Appl. Math., vol. 8, Walter de Gruyter, Berlin, 2009, pp. 125–164. MR 2648460
12. Paul Glasserman and Kyoung-Kuk Kim, *Moment explosions and stationary distributions in affine diffusion models*, Math. Finance **20** (2010), no. 1, 1–33. MR 2599675
13. Steven L. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Review of Financial Studies (1993), 6:327?343.
14. Rudra P. Jena, Kyoung-Kuk Kim, and Hao Xing, *Long-term and blow-up behaviors of exponential moments in multi-dimensional affine diffusions*, Stochastic Process. Appl. **122** (2012), no. 8, 2961–2993. MR 2931348
15. Martin Keller-Ressel, *Moment explosions and long-term behavior of affine stochastic volatility models*, Math. Finance **21** (2011), no. 1, 73–98. MR 2779872 (2012e:91126)
16. Martin Keller-Ressel and Eberhard Mayerhofer, *Exponential moments of affine processes*, Ann. Appl. Probab. **25** (2015), no. 2, 714–752. MR 3313754
17. Martin Keller-Ressel and Aleksandar Mijatović, *On the limit distributions of continuous-state branching processes with immigration*, Stochastic Process. Appl. **122** (2012), no. 6, 2329–2345. MR 2922631
18. Martin Keller-Ressel, Walter Schachermayer, and Josef Teichmann, *Affine processes are regular*, Probab. Theory Related Fields **151** (2011), no. 3-4, 591–611. MR 2851694 (2012k:60219)
19. John M. Lee, *Introduction to smooth manifolds*, second ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013. MR 2954043
20. Zenghu Li, *Measure-valued branching Markov processes*, Probability and its Applications (New York), Springer, Heidelberg, 2011. MR 2760602

21. Eberhard Mayerhofer, Johannes Muhle-Karbe, and Alexander G. Smirnov, *A characterization of the martingale property of exponentially affine processes*, Stochastic Process. Appl. **121** (2011), no. 3, 568–582. MR 2763096
22. Mark A. Pinsky, *Limit theorems for continuous state branching processes with immigration*, Bull. Amer. Math. Soc. **78** (1972), 242–244. MR 0295450
23. Ken-iti Sato and Makoto Yamazato, *Operator-self-decomposable distributions as limit distributions of processes of Ornstein-Uhlenbeck type*, Stochastic Process. Appl. **17** (1984), no. 1, 73–100. MR 738769
24. K. Urbanik, *Self-decomposable probability distributions on* R^m, Zastos. Mat. **10** (1969), 91–97. MR 0245068
25. Oldrich Vasicek, *An equilibrium characterization of the term structure [reprint of J. Financ. Econ.* **5** *(1977), no. 2, 177–188]*, Financial risk measurement and management, Internat. Lib. Crit. Writ. Econ., vol. 267, Edward Elgar, Cheltenham, 2012, pp. 724–735. MR 3235239
26. Wolfgang Walter, *Ordinary differential equations*, Graduate Texts in Mathematics, vol. 182, Springer-Verlag, New York, 1998, Translated from the sixth German (1996) edition by Russell Thompson, Readings in Mathematics. MR 1629775

(Peng Jin) Department of Mathematics, Shantou University, Shantou, Guangdong 515063, China
E-mail address: pjin@stu.edu.cn

(Jonas Kremer) Fakultät für Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, 42119 Wuppertal, Germany
E-mail address: jkremer@uni-wuppertal.de

(Barbara Rüdiger) Fakultät für Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, 42119 Wuppertal, Germany
E-mail address: ruediger@uni-wuppertal.de