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Peng Jin, Jonas Kremer and Barbara Rüdiger

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MOMENTS AND ERGODICITY OF THE JUMP-DIFFUSION CIR PROCESS

PENG JIN, JONAS KREMER AND BARBARA RÜDIGER

ABSTRACT. We study the jump-diffusion CIR process, which is an extension of the Cox-Ingersoll-Ross model and whose jumps are introduced by a subordinator. We provide sufficient conditions on the Lévy measure of the subordinator under which the jump-diffusion CIR process is ergodic and exponentially ergodic, respectively. Furthermore, we characterize the existence of the κ-moment ($\kappa > 0$) of the jump-diffusion CIR process by an integrability condition on the Lévy measure of the subordinator.

1. Introduction

In the present paper, we study the jump-diffusion CIR (shorted as JCIR) process, which is an extension of the well-known Cox-Ingersoll-Ross (shorted as CIR) model introduced in [8]. The JCIR process $X = (X_t)_{t\geqslant 0}$ is defined as the unique strong solution to the stochastic differential equation (SDE)

$$\mathrm{d}X_t = (a - bX_t)\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}B_t + \mathrm{d}J_t, \quad t \geqslant 0, \quad X_0 \geqslant 0 \text{ a.s.}, \tag{1.1}$$

where $a \geqslant 0$, $b > 0$, $\sigma > 0$ are constants, $(B_t)_{t\geqslant 0}$ is a one-dimensional Brownian motion and $(J_t)_{t\geqslant 0}$ is a pure jump Lévy process with its Lévy measure ν concentrating on $(0, \infty)$ and satisfying

$$\int_0^\infty (z \wedge 1)\nu(\mathrm{d}z) < \infty. \tag{1.2}$$

We assume that X_0, $(B_t)_{t\geqslant 0}$ and $(J_t)_{t\geqslant 0}$ are independent. Note that the existence of a unique strong solution to (1.1) is guaranteed by [12, Theorem 5.1].

The importance of the the CIR model and its extensions has been demonstrated by their vast applications in mathematical finance, see, e.g., [8, 10, 11, 21], and many others. Since the CIR process is non-negative and mean-reverting, it is particularly popular in interest rates and stochastic volatility modelling. These important features are inherited by the JCIR process defined in (1.1). Moreover, compared to the CIR model, the JCIR process has included possible jumps in it, which seems to make it a more appropriate model to fit real world interest rates or volatility of asset prices. As an application of the JCIR process, Barletta and Nicolato [3] recently studied a stochastic volatility model with jumps for the sake of pricing of VIX options, where the volatility (or instantaneous variance process) of the asset price process is modelled via the JCIR process.

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An important issue for applications of the JCIR process is the estimation of its parameters. To estimate the parameters of the original CIR model, one can build some conditional least square estimators (CLSEs) based on discrete observations (see Overbeck and Rydén [29]), or some maximum likelihood estimators (MLEs) through continuous time observations (see Overbeck [28], Ben Alaya and Kebaier [5, 6]). The parameter estimation problem for the JCIR process is more complicated, since it has an additional parameter ν, which is the Lévy measure of the driving noise $(J_t)_{t\geqslant 0}$ in (1.1) and thus an infinite dimensional object. Nevertheless, based on low frequency observations, Xu [32] proposed some nonparametric estimators for ν, given that ν is absolutely continuous with respect to the Lebesgue measure. Barczy *et al.* [1] studied also the maximum likelihood estimator for the parameter b of the JCIR process.

As seen in the aforementioned works [1] and [32], to study the fine properties of the estimators, a comprehension of the moments and long-time behavior of the JCIR processes is required. In this paper we focus on these two problems and analyze their subtle dependence on the big jumps of $(J_t)_{t\geqslant 0}$. Our first main result is a characterization of the existence of moments of the JCIR process in terms of the Lévy measure ν of $(J_t)_{t\geqslant 0}$, namely, we have the following:

Theorem 1.1. *Consider the JCIR process $X=(X_t)_{t\geqslant 0}$ defined in* (1.1)*. Let $\kappa>0$ be a constant. Then the following three conditions are equivalent:*

(i) $\mathbb{E}_x[X_t^\kappa]<\infty$ *for all $x\in\mathbb{R}_{\geqslant 0}$ and $t>0$,*
(ii) $\mathbb{E}_x[X_t^\kappa]<\infty$ *for some $x\in\mathbb{R}_{\geqslant 0}$ and $t>0$,*
(iii) $\int_{\{z>1\}} z^\kappa\nu(\mathrm{d}z)<\infty$,

where the notation $\mathbb{E}_x[\cdot]$ means that the process X considered under the expectation is with the initial condition $X_0=x$.

After this paper was finished, we noticed that moments of general 1-dimensional CBI processes were recently studied in [15]. If $\kappa\geqslant 1$ and $x>0$, our Theorem 1.1 can be viewed as a special case of [15, Theorem 2.2]. However, to the authors' knowledge, the cases $0<\kappa<1$ and $\kappa\geqslant 0$ with $x=0$ can not be handled by the approach used in [15].

The second aim of this paper is to improve the results of [17] on the ergodicity of the JCIR process. For a general time-homogeneous Markov process $M=(M_t)_{t\geqslant 0}$ with state space E, let $\mathbf{P}^t(x,\cdot):=\mathbb{P}_x(M_t\in\cdot)$ denote the distribution of M_t with the initial condition $M_0=x\in E$. Following [27], we call M *ergodic* if it admits a unique invariant probability measure π such that

$$\lim_{t\to\infty}\left\|\mathbf{P}^t(x,\cdot)-\pi\right\|_{TV}=0,\quad\forall x\in E,$$

where $\|\cdot\|_{TV}$ denotes the total variation norm for signed measures. The Markov process M is called *exponentially ergodic* if it is ergodic and in addition there exists a finite-valued function B on E and a positive constant δ such that

$$\left\|\mathbf{P}^t(x,\cdot)-\pi\right\|_{TV}\leqslant B(x)e^{-\delta t},\quad\forall x\in E,\ t>0.$$

Our second main result is the following:

Theorem 1.2. *Consider the JCIR process $(X_t)_{t\geqslant 0}$ defined by* (1.1) *with parameters a,b,σ and ν, where ν is the Lévy measure of $(J_t)_{t\geqslant 0}$. Assume $a>0$. We have:*

(a) *If $\int_{\{z>1\}}\log z\nu(\mathrm{d}z)<\infty$, then X is ergodic.*

(b) *If* $\int_{\{z>1\}} z^{\kappa}\nu(\mathrm{d}z) < \infty$ *for some* $\kappa > 0$*, then* X *is exponentially ergodic.*

We remark that similar results on the ergodicity of Ornstein-Uhlenbeck type processes were derived by Masuda, see [23, Theorem 2.6]. It is also worth mentioning that Jin *et al.* [17] already found a sufficient condition for the exponential ergodicity of the JCIR process, namely, if $a > 0$, $\int_{\{z\leqslant 1\}} z\log(1/z)\nu(\mathrm{d}z) < \infty$ and $\int_{\{z>1\}} z\nu(\mathrm{d}z) < \infty$. It is seen from part (b) of our Theorem 1.2 that these conditions can be significantly relaxed.

Our method to prove the (exponential) ergodicity of the JCIR process in question is based on the general theory of Meyn and Tweedie [25, 27] for ergodicity of Markov processes. As the first step, using a decomposition of its characteristic function, we show existence of positive transition densities of the JCIR process (see Proposition 3.1), which improves a similar result in [17]. In the second step, we construct some Foster-Lyapunov functions for the JCIR process which enable us to prove the asserted (exponential) ergodicity by using the results in [25, 26, 27]. For the construction of the Foster-Lyapunov functions we will use some ideas from [23].

The remainder of the article is organized as follows. In Section 2 we first introduce some notation and recall some basic facts on the JCIR process, then we establish an estimate for the moments of Bessel distributed random variables, which is crucial to Theorem 1.1. In Section 3 we will show that the JCIR process possesses positive transition densities. In Section 4 we will prove Theorem 1.1. Sections 5 and 6 are devoted to the proof of Theorem 1.2.

2. Preliminaries and notation

2.1. **Notation.** Let $\mathbb{N}$, $\mathbb{Z}_{\geqslant 0}$, $\mathbb{R}$, $\mathbb{R}_{\geqslant 0}$ and $\mathbb{R}_{>0}$ denote the sets of positive integers, non-negative integers, real numbers, non-negative real numbers and strictly positive real numbers, respectively. Let $\mathbb{C}$ be the set of complex numbers. We define the following subset of $\mathbb{C}$:

$$\mathcal{U} := \{u \in \mathbb{C} \,:\, \operatorname{Re} u \leqslant 0\}\,.$$

We denote the Borel σ-algebra on $\mathbb{R}_{\geqslant 0}$ simply by $\mathcal{B}(\mathbb{R}_{\geqslant 0})$.

By $C^2(\mathbb{R}_{\geqslant 0}, \mathbb{R})$, and $C^2_c(\mathbb{R}_{\geqslant 0}, \mathbb{R})$ we denote the sets of $\mathbb{R}$-valued functions on $\mathbb{R}_{\geqslant 0}$ that are twice continuously differentiable, and that are twice continuously differentiable with compact support, respectively. For $a, b \in \mathbb{R}$, we denote by $a \wedge b$ and $a \vee b$ the minimum and maximum of a and b, respectively.

We assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions, i.e., $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$.

2.2. **The JCIR process.** Let $(B_t)_{t\geqslant 0}$ be a standard $(\mathcal{F}_t)_{t\geqslant 0}$-Brownian motion and $(J_t)_{t\geqslant 0}$ be a 1-dimensional $(\mathcal{F}_t)_{t\geqslant 0}$-Lévy process whose characteristic function is given by

$$\mathbb{E}\left[e^{uJ_t}\right] = \exp\left\{t\int_0^{\infty} \left(e^{uz} - 1\right)\nu(\mathrm{d}z)\right\}, \quad (t, u) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U},$$

where ν satisfies (1.2). We assume that $(B_t)_{t\geqslant 0}$ and $(J_t)_{t\geqslant 0}$ are independent. The Lévy-Itô representation of $(J_t)_{t\geqslant 0}$ takes the form

$$J_t = \int_0^t \int_0^\infty z N(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0, \tag{2.1}$$

where $N(\mathrm{d}t, \mathrm{d}z) = \sum_{s\leqslant t} \delta_{(s,\Delta J_s)}(\mathrm{d}t, \mathrm{d}z)$ is a Poisson random measure on $\mathbb{R}_{\geqslant 0}$, where $\Delta J_s := J_s - J_{s-}$, $s > 0$, $\Delta J_0 := 0$, and $\delta_{(s,x)}$ denotes the Dirac measure concentrated at $(s,x) \in \mathbb{R}^2_{\geqslant 0}$.

It follows from [12, Theorem 5.1] that if X_0 is independent of $(B_t)_{t\geqslant 0}$ and $(J_t)_{t\geqslant 0}$, then there is a unique strong solution $(X_t)_{t\geqslant 0}$ to the SDE (1.1). Since the diffusion coefficient in the SDE (1.1) is degenerate at zero and only positive jumps are possible, the JCIR process $(X_t)_{t\geqslant 0}$ stays non-negative if $X_0 \geqslant 0$. This fact can be shown rigorously with the help of comparison theorems for SDEs, for more details we refer to [12]. Using Itô's formula, it is easy to see that

$$X_t = e^{-bt}\left(X_0 + a\int_0^t e^{bs}\mathrm{d}s + \sigma\int_0^t e^{bs}\sqrt{X_s}\mathrm{d}B_s + \int_0^t e^{bs}\mathrm{d}J_s\right), \quad t \geqslant 0.$$

Moreover, the JCIR process $(X_t)_{t\geqslant 0}$ is a *regular affine process*, and the infinitesimal generator $\mathcal{A}$ of X is given by

$$\left(\mathcal{A}f\right)(x) = (a - bx)\frac{\partial f(x)}{\partial x} + \frac{1}{2}\sigma^2 x \frac{\partial^2 f(x)}{\partial x^2} + \int_0^\infty \left(f(x+z) - f(x)\right)\nu(\mathrm{d}z), \tag{2.2}$$

where $x \in \mathbb{R}_{\geqslant 0}$ and $f \in C^2_c(\mathbb{R}_{\geqslant 0}, \mathbb{R})$. If we write

$$\begin{aligned} \left(\mathcal{D}f\right)(x) &= (a - bx)\frac{\partial f(x)}{\partial x} + \frac{1}{2}\sigma^2 x \frac{\partial^2 f(x)}{\partial x^2}, \\ \left(\mathcal{J}f\right)(x) &= \int_0^\infty \left(f(x+z) - f(x)\right)\nu(\mathrm{d}z), \end{aligned}$$

where $x \in \mathbb{R}_{\geqslant 0}$ and $f \in C^2_c(\mathbb{R}_{\geqslant 0}, \mathbb{R})$, we see that $\mathcal{A}f = \mathcal{D}f + \mathcal{J}f$.

Remark 2.1. *Let $a, b \in \mathbb{R}_{>0}$. If $\int_{\{z>1\}} \log z \nu(\mathrm{d}z) < \infty$, then it follows from* [20, Theorem 3.16] *that the JCIR process converges in law to a limit distribution π. Moreover, as shown in* [18, p.80]*, the limit distribution π is also the unique invariant distribution of the JCIR process.*

Finally, we introduce some notation. Note that the strong solution $(X_t)_{t\geqslant 0}$ of the SDE (1.1) obviously depends on its initial value X_0. From now on, we denote by $(X^x_t)_{t\geqslant 0}$ the JCIR process starting from a constant initial value $x \in \mathbb{R}_{\geqslant 0}$, i.e., $(X^x_t)_{t\geqslant 0}$ satisfies

$$\mathrm{d}X^x_t = (a - bX^x_t)\mathrm{d}t + \sigma\sqrt{X^x_t}\mathrm{d}B_t + \mathrm{d}J_t, \quad t \geqslant 0, \quad X^x_0 = x \in \mathbb{R}_{\geqslant 0}. \tag{2.3}$$

2.3. **Bessel distribution.** Suppose α and β are positive constants. We call a probability measure $m_{\alpha,\beta}$ on $(\mathbb{R}_{\geqslant 0}, \mathcal{B}(\mathbb{R}_{\geqslant 0}))$ a Bessel distribution with parameters α and β if

$$m_{\alpha,\beta}(\mathrm{d}x) := e^{-\alpha}\delta_0(\mathrm{d}x) + \beta e^{-\alpha-\beta x}\sqrt{\alpha(\beta x)^{-1}} I_1\left(2\sqrt{\alpha\beta x}\right)\mathrm{d}x, \quad x \in \mathbb{R}_{\geqslant 0}, \tag{2.4}$$

where δ_0 denotes the Dirac measure at the origin and I_1 is the modified Bessel function of the first kind, namely,

$$I_1(r) = \frac{r}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}r^2\right)^k}{k!(k+1)!}, \quad r \in \mathbb{R}. \tag{2.5}$$

Let $\widehat{m}_{\alpha,\beta}(u) := \int_{\mathbb{R}_{\geqslant 0}} \exp\{ux\} m_{\alpha,\beta}(\mathrm{d}x)$ for $u \in \mathcal{U}$ denote the characteristic function of the Bessel distribution $m_{\alpha,\beta}$. It follows from [17, p.291] that

$$\widehat{m}_{\alpha,\beta}(u) = \exp\left\{\frac{\alpha u}{\beta - u}\right\}, \quad u \in \mathcal{U}.$$

To study the moments of the JCIR process, the lemma below plays a substantial role.

Lemma 2.2. *Let $\kappa > 0$ and $\delta > 0$ be positive constants. Then*

(i) *there exists a positive constant $C_1 = C_1(\kappa)$ such that for all $\alpha > 0$ and $\beta > 0$,*

$$\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x) \leqslant C_1 \frac{1+\alpha^\kappa}{\beta^\kappa}.$$

(ii) *there exists a positive constant $C_2 = C_2(\kappa, \delta)$ such that for all $\alpha \geqslant \delta$ and $\beta > 0$,*

$$\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x) \geqslant C_2 \frac{\alpha^\kappa}{\beta^\kappa}.$$

Proof. (i) If $0 < \kappa \leqslant 1$, then we can use Jensen's inequality to obtain

$$\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x) \leqslant \left(\int_{\mathbb{R}_{\geqslant 0}} x m_{\alpha,\beta}(\mathrm{d}x)\right)^\kappa = \left(\frac{\alpha}{\beta}\right)^\kappa, \tag{2.6}$$

where the last identity holds because of

$$\int_{\mathbb{R}_{\geqslant 0}} x m_{\alpha,\beta}(\mathrm{d}x) = \frac{\partial}{\partial u}\widehat{m}_{\alpha,\beta}(u)\Big|_{u=0} = \frac{\alpha}{\beta}.$$

For $\kappa = n \in \mathbb{N}$ with $n \geqslant 2$, by (2.4) and (2.5), we have for all $\alpha, \beta > 0$,

$$\begin{aligned}
\int_{\mathbb{R}_{\geqslant 0}} x^n m_{\alpha,\beta}(\mathrm{d}x) &= \int_{\mathbb{R}_{\geqslant 0}} x^n \left(e^{-\alpha}\delta_0(\mathrm{d}x) + \beta e^{-\alpha-\beta x}\sqrt{\alpha(\beta x)^{-1}} I_1\left(2\sqrt{\alpha\beta x}\right)\mathrm{d}x\right) \\
&= e^{-\alpha}\sum_{k=0}^{\infty}\frac{(\alpha\beta)^{k+1}}{k!(k+1)!}\int_0^\infty x^{n+k}e^{-\beta x}\mathrm{d}x \\
&= \frac{e^{-\alpha}}{\beta^n}\sum_{k=0}^{\infty}\frac{\alpha^{k+1}(n+k)!}{k!(k+1)!} \\
&= \frac{e^{-\alpha}}{\beta^n}\sum_{k=0}^{n-2}\frac{\alpha^{k+1}(n+k)!}{k!(k+1)!} \\
&\quad + \frac{e^{-\alpha}\alpha^n}{\beta^n}\sum_{k=n-1}^{\infty}\frac{\alpha^{k+1-n}}{(k+1-n)!}\cdot\frac{(k+1)\cdots(k+n)}{(k+2-n)\cdots(k+1)}.
\end{aligned} \tag{2.7}$$

Since

$$\lim_{k\to\infty}\frac{(k+1)\cdots(k+n)}{(k+2-n)\cdots(k+1)} = 1,$$

it follows from (2.7) that

$$\begin{aligned}\int_{\mathbb{R}_{\geqslant 0}} x^n m_{\alpha,\beta}(\mathrm{d}x) &\leqslant c_1 \frac{e^{-\alpha}}{\beta^n}\left(\alpha+\alpha^2+\cdots+\alpha^{n-1}+\alpha^n\sum_{m=0}^{\infty}\frac{\alpha^m}{m!}\right)\\ &\leqslant c_2\left(\frac{1}{\beta^n}+\frac{\alpha^n}{\beta^n}\right), \quad \text{for all } \alpha,\beta>0, \qquad (2.8)\end{aligned}$$

where c_1 and c_2 are positive constants depending on n.

For the remaining possible κ, namely, $\kappa > 1$ and $\kappa \notin \mathbb{N}$, we can find $n \in \mathbb{N}$ and $\varepsilon \in (0,1]$ such that $2\kappa = n + \varepsilon$. By (2.6), (2.8) and Hölder's inequality, we get for all $\alpha, \beta > 0$,

$$\begin{aligned}\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x) &\leqslant \left(\int_{\mathbb{R}_{\geqslant 0}} x^n m_{\alpha,\beta}(\mathrm{d}x)\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}_{\geqslant 0}} x^\varepsilon m_{\alpha,\beta}(\mathrm{d}x)\right)^{\frac{1}{2}}\\ &\leqslant c_3\left(\frac{1+\alpha^n}{\beta^n}\right)^{\frac{1}{2}}\left(\frac{\alpha}{\beta}\right)^{\frac{\varepsilon}{2}} \leqslant c_4\frac{\alpha^{\varepsilon/2}+\alpha^{(n+\varepsilon)/2}}{\beta^{(n+\varepsilon)/2}} \leqslant c_5\frac{1+\alpha^\kappa}{\beta^\kappa},\end{aligned}$$

where c_3, c_4 and c_5 are positive constants depending on κ.

(ii) If $\kappa \geqslant 1$, using again Jensen's inequality, we obtain for all $\alpha, \beta > 0$,

$$\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x) \geqslant \left(\int_{\mathbb{R}_{\geqslant 0}} x m_{\alpha,\beta}(\mathrm{d}x)\right)^\kappa = \left(\frac{\alpha}{\beta}\right)^\kappa.$$

Suppose now $0 < \kappa < 1$ and let $\theta := 1-\kappa \in (0,1)$. Consider a random variable $\eta > 0$ such that

$$\eta \sim \left(1-e^{-\alpha}\right)^{-1}\left(m_{\alpha,\beta}(\mathrm{d}x) - e^{-\alpha}\delta_0(\mathrm{d}x)\right). \qquad (2.9)$$

Then for $u \geqslant 0$, we have

$$\begin{aligned}\mathbb{E}\left[e^{-u\eta}\right] &= \left(1-e^{-\alpha}\right)^{-1}\left(\widehat{m}_{\alpha,\beta}(-u) - e^{-\alpha}\right)\\ &= \left(1-e^{-\alpha}\right)^{-1}\left(\exp\left\{\frac{-\alpha u}{\beta+u}\right\} - \exp\left\{-\alpha\right\}\right).\end{aligned}$$

Since, by the Fubini's theorem,

$$\begin{aligned}\int_0^\infty \frac{\partial}{\partial u}\mathbb{E}\left[e^{-u\eta}\right]u^{\theta-1}\mathrm{d}u &= -\int_0^\infty \mathbb{E}\left[Ye^{-u\eta}\right]u^{\theta-1}\mathrm{d}u\\ &= -\mathbb{E}\left[\int_0^\infty \eta e^{-u\eta}u^{\theta-1}\mathrm{d}u\right] = -\mathbb{E}\left[\Gamma(\theta)\eta^{1-\theta}\right],\end{aligned}$$

it follows that

$$\begin{aligned}\mathbb{E}\left[\eta^\kappa\right] &= \frac{-1}{\Gamma(\theta)}\int_0^\infty \frac{\partial}{\partial u}\mathbb{E}\left[e^{-u\eta}\right]u^{\theta-1}\mathrm{d}u\\ &= \frac{\alpha\beta}{\Gamma(\theta)\left(1-e^{-\alpha}\right)}\int_0^\infty \exp\left\{\frac{-\alpha u}{\beta+u}\right\}\frac{u^{\theta-1}}{(\beta+u)^2}\mathrm{d}u. \qquad (2.10)\end{aligned}$$

By (2.9) and (2.10), we see that

$$\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x) = \frac{\alpha\beta}{\Gamma(\theta)}\int_0^\infty \exp\left\{\frac{-\alpha u}{\beta+u}\right\}\frac{u^{\theta-1}}{(\beta+u)^2}\mathrm{d}u, \quad u \in \mathbb{R}_{\geqslant 0}.$$

By a change of variables $w := \alpha u/\beta$, we get

$$\begin{aligned}\int_{\mathbb{R}_{\geqslant 0}} x^{\kappa} m_{\alpha,\beta}(\mathrm{d}x) &= \frac{\alpha\beta}{\Gamma(\theta)} \int_0^\infty \exp\left\{-\alpha + \frac{\alpha\beta}{\beta + \frac{\beta w}{\alpha}}\right\} \frac{\left(\frac{\beta w}{\alpha}\right)^{-\kappa}}{\left(\beta + \frac{\beta w}{\alpha}\right)^2} \frac{\beta}{\alpha} \mathrm{d}w \\ &= \frac{1}{\Gamma(\theta)} \left(\frac{\alpha}{\beta}\right)^{\kappa} \int_0^\infty \exp\left\{\frac{-\alpha w}{\alpha + w}\right\} \frac{w^{-\kappa}}{(1 + w/\alpha)^2} \mathrm{d}w \\ &=: \frac{1}{\Gamma(\theta)} \left(\frac{\alpha}{\beta}\right)^{\kappa} I(\alpha). \qquad (2.11)\end{aligned}$$

By Fatou's lemma,

$$\begin{aligned}\liminf_{\alpha\to\infty} I(\alpha) &\geqslant \int_0^\infty \liminf_{\alpha\to\infty} \exp\left\{\frac{-\alpha w}{\alpha + w}\right\} \frac{w^{-\kappa}}{(1 + w/\alpha)^2} \mathrm{d}w \\ &= \int_0^\infty \exp\{-w\} w^{-\kappa} \mathrm{d}w = \Gamma(1-\kappa) > 0.\end{aligned}$$

On the other hand, the function $(0, \infty) \ni \alpha \mapsto I(\alpha)$ is positive and continuous. So we can find a positive constant c_6 depending on κ and δ such that $I(\alpha) \geq c_6$ for all $\alpha \in [\delta, \infty)$, which, together with (2.11), implies the assertion. □

3. Positivity of the transition densities of the JCIR process

The aim of this section is to prove that the JCIR process X has positive transition densities. Our approach is similar to that in [16, Proposition 4.5] and is based on the representation of the law of X_t^x as the convolution of two probability measures, one of which is the distribution of the normal CIR process. Before we prove the positivity of the transition densities, we recall the characteristic function of X_t^x and a decomposition of it, established in [17].

Recall that $(X_t^x)_{t\geqslant 0}$ is given in (2.3). Assume $a \in \mathbb{R}_{\geqslant 0}$ and $b, \sigma \in \mathbb{R}_{>0}$. Following [17], the characteristic function of X_t^x has the form

(3.1)

$$\begin{aligned}\mathbb{E}\left[e^{uX_t^x}\right] &= \left(1 - \tfrac{\sigma^2 u}{2b}\left(1 - e^{-bt}\right)\right)^{-\frac{2a}{\sigma^2}} \cdot \exp\{x\psi(t,u)\} \\ &\quad \cdot \exp\left\{\int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right) \nu(\mathrm{d}z)\mathrm{d}s\right\}, \quad (t,u) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U},\end{aligned}$$

where the function $\psi(t, u)$ is given by

$$\psi(t,u) = \frac{ue^{-bt}}{1 - \frac{\sigma^2 u}{2b}\left(1 - e^{-bt}\right)}. \qquad (3.2)$$

As mentioned in [17], the product of the first two terms on the right-hand side of (3.1) is the characteristic function of the CIR process. More precisely, consider the unique strong solution $(Y_t^x)_{t\geqslant 0}$ of the following stochastic differential equation (1.1)

$$\mathrm{d}Y_t^x = (a - bY_t^x)\mathrm{d}t + \sqrt{Y_t^x}\mathrm{d}B_t, \quad t \geqslant 0, \quad Y_0^x = x \in \mathbb{R}_{\geqslant 0} \text{ a.s.}. \qquad (3.3)$$

where $a \in \mathbb{R}_{\geqslant 0}$, and $b, \sigma \in \mathbb{R}_{>0}$. So $(Y_t^x)_{t\geqslant 0}$ is the CIR process starting from x. Note that (3.3) is a special case of (2.3) with $J_t \equiv 0$ (corresponding to $\nu = 0$). By (3.1), we obtain

$$\mathbb{E}\left[e^{uY_t^x}\right] = \left(1 - \tfrac{\sigma^2 u}{2b}\left(1 - e^{-bt}\right)\right)^{-\frac{2a}{\sigma^2}} \exp\left\{\frac{xue^{-bt}}{1-\frac{\sigma^2 u}{2b}(1-e^{-bt})}\right\} \tag{3.4}$$

for all $t \geqslant 0$ and $u \in \mathcal{U}$.

We now turn to the third term on the right-hand side of (3.1). Let $Z := (Z_t)_{t\geqslant 0}$ be the unique strong solution of the stochastic differential equation

$$\mathrm{d}Z_t = -bZ_t\mathrm{d}t + \sigma\sqrt{Z_t}\mathrm{d}B_t + \mathrm{d}J_t, \quad t \geqslant 0, \quad Z_0 = 0 \text{ a.s.}, \tag{3.5}$$

where $\sigma \in \mathbb{R}_{>0}$. It is easy to see that (3.5) is also a special case of (2.3) with $a = x = 0$. Again by (3.1), we have

$$\mathbb{E}\left[e^{uZ_t}\right] = \exp\left\{\int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right)\nu(\mathrm{d}z)\mathrm{d}s\right\}, \quad (t,u) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U}. \tag{3.6}$$

It follows from (3.1), (3.4) and (3.6) that

$$\mathbb{E}\left[e^{uX_t^x}\right] = \mathbb{E}\left[e^{uY_t^x}\right]\mathbb{E}\left[e^{uZ_t}\right]$$

for all $t \geqslant 0$ and $u \in \mathcal{U}$. Let $\mu_{Y_t^x}$ and μ_{Z_t} be the probability laws of Y_t^x and Z_t induced on $(\mathbb{R}_{\geqslant 0}, \mathcal{B}(\mathbb{R}_{\geqslant 0}))$, respectively. Then the probability law $\mu_{X_t^x}$ of X_t^x is given by

$$\mu_{X_t^x} = \mu_{Y_t^y} * \mu_{Z_t}, \tag{3.7}$$

where $*$ denotes the convolution of two measures.

Proposition 3.1. *Assume $a > 0$. For each $x \in \mathbb{R}_{\geqslant 0}$ and $t \in \mathbb{R}_{>0}$, the random variable X_t^x possesses a density function $f_{X_t^x}(y)$, $y \geqslant 0$ with respect to the Lebesgue measure. Moreover, the density function $f_{X_t^x}(y)$ is strictly positive for all $y \in \mathbb{R}_{>0}$.*

Proof. According to [17, Lemma 1], X_t^x possesses a density function given by

$$f_{X_t^x}(y) = \int_{\mathbb{R}_{\geqslant 0}} f_{Y_t^x}(y-z)\mu_{Z_t}(\mathrm{d}z), \quad y \geqslant 0,$$

where $f_{Y_t^x}(y)$, $y \in \mathbb{R}$ denotes the density function of Y_t^x, $t > 0$. Since $(Y_t^x)_{t\geqslant 0}$ is the CIR process, as well-known, we have $f_{Y_t^x}(y) > 0$ for $y > 0$ and $f_{Y_t^x}(y) \equiv 0$ for $y < 0$ (see, e.g., Cox *et al.* [8, Formula (18)] or Jeanblanc *et al.* [14, Proposition 6.3.2.1] in case $x > 0$ and Ikeda and Watanabe [13, p.222] in case $x = 0$). It remains to prove the strict positivity of $f_{X_t^x}(y)$ for all $y \in \mathbb{R}_{>0}$.

Let $t > 0$ and $y > 0$ be fixed. It follows that

$$f_{X_t^x}(y) \geqslant \int_{[0,\delta]} f_{Y_t^x}(y-z)\mu_{Z_t}(\mathrm{d}z),$$

where $\delta > 0$ is small enough with $\delta < y$. Since $f_{Y_t^x}(y-z) > 0$ for all $z \in [0,\delta]$, it is enough to check that $\mu_{Z_t}([0,\delta]) > 0$. If $\mathbb{P}(Z_t = 0) > 0$, then we are done. So we now suppose

$$\mathbb{P}(Z_t = 0) = 0. \tag{3.8}$$

Let

$$\Delta_t(u) = \int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right) \nu(\mathrm{d}z)\mathrm{d}s, \quad u \in \mathcal{U},$$

where ψ is given in (3.2). By (3.8), we conclude

$$\begin{aligned}
&\mathbb{E}\left[e^{u(Z_t-\delta)}\right] - \mathbb{E}\left[e^{u(Z_t-\delta)}\mathbb{1}_{\{Z_t=0\}}\right] \\
&\qquad = e^{-u\delta}\left(\mathbb{E}\left[e^{uZ_t}\right] - \mathbb{E}\left[e^{uZ_t}\mathbb{1}_{\{Z_t=0\}}\right]\right) \\
&\qquad = e^{-u\delta}\left(e^{\Delta_t(u)} - \mathbb{P}\left(Z_t=0\right)\right) \\
&\qquad = e^{-u\delta/2}e^{\Delta_t(u)-u\delta/2}. \qquad (3.9)
\end{aligned}$$

For all $u \in (-\infty,-1]$ and $s \in [0,t]$, we have

$$\begin{aligned}
\frac{\partial}{\partial u}\left(e^{z\psi(s,u)} - 1\right) &= \frac{ze^{-bs}}{\left(1 - \frac{\sigma^2 u}{2b}\left(1-e^{-bs}\right)\right)^2} \exp\left\{\frac{zue^{-bs}}{1 - \frac{\sigma^2 u}{2b}\left(1-e^{-bs}\right)}\right\} \\
&\leqslant ze^{-bs}\mathbb{1}_{\{z\leqslant 1\}} + ze^{-bs}e^{-c_1 z}\mathbb{1}_{\{z>1\}} \leqslant c_2 e^{-bs}(z\wedge 1), \qquad (3.10)
\end{aligned}$$

for some positive constants c_1 and c_2. By the differentiation lemma [4, Lemma 16.2], we see that $\Delta_t(u)$ is differentiable at $u \in (-\infty,-1]$ and

$$\frac{\partial}{\partial u}\left(\Delta_t(u)\right) = \int_0^t \int_0^\infty \frac{\partial}{\partial u}\left(e^{z\psi(s,u)} - 1\right)\nu(\mathrm{d}z)\mathrm{d}s, \quad u \in (-\infty,-1]. \qquad (3.11)$$

Note that $\partial/(\partial u)(\exp\{z\psi(s,u)\} - 1) > 0$ for $z > 0$, $u \in (-\infty,-1]$ and $s \in [0,t]$. Therefore, $\Delta_t(u)$ is strictly increasing in u on $(-\infty,-1]$. Moreover, we have

$$\lim_{u\to-\infty} \frac{\partial}{\partial u}\left(e^{z\psi(s,u)} - 1\right) = \exp\left\{\frac{-2bz}{\sigma^2\left(e^{bs}-1\right)}\right\} \lim_{u\to-\infty} \frac{ze^{-bs}}{\left(1 - \frac{\sigma^2 u}{2b}\left(1-e^{-bs}\right)\right)^2} = 0.$$

By (3.10), (3.11) and the Lebesgue dominated convergence theorem, $\partial/(\partial u)\Delta_t(u) \to 0$ as $u \to -\infty$. So $\partial/(\partial u)(\Delta_t(u) - u\delta/2) \to -\delta/2$ as $u \to -\infty$, which implies that $\Delta_t(u) - u\delta/2$ is monotone in u for sufficiently small u and thus

$$\lim_{u\to-\infty} e^{-u\delta/2}e^{\Delta_t(u)-u\delta/2} = \infty. \qquad (3.12)$$

It follows from (3.9) and (3.12) that

$$\lim_{u\to-\infty}\left(\mathbb{E}\left[e^{u(Z_t-\delta)}\right] - \mathbb{E}\left[e^{u(Z_t-\delta)}\mathbb{1}_{\{Z_t=0\}}\right]\right) = \infty.$$

Now, we must have $\mathbb{P}(Z_t \in (0,\delta]) > 0$, otherwise

$$\begin{aligned}
&\lim_{u\to-\infty}\left(\mathbb{E}\left[e^{u(Z_t-\delta)}\right] - \mathbb{E}\left[e^{u(Z_t-\delta)}\mathbb{1}_{\{Z_t=0\}}\right]\right) \\
&\qquad = \lim_{u\to-\infty}\left(\mathbb{E}\left[e^{u(Z_t-\delta)}\mathbb{1}_{\{0<Z_t\leqslant\delta\}}\right] + \mathbb{E}\left[e^{u(Z_t-\delta)}\mathbb{1}_{\{Z_t>\delta\}}\right]\right) = 0.
\end{aligned}$$

This completes the proof. □

4. Moments of the JCIR process

In this section we prove Theorem 1.1. Our approach is essentially motivated by the proof of [30, Theorem 25.3].

Proof of Theorem 1.1. "(iii)⇒(i)": Let $\kappa > 0$ be a constant. Suppose that $\int_{\{z>1\}} z^\kappa \nu(\mathrm{d}z) < \infty$. Let $x \in \mathbb{R}_{\geqslant 0}$ and $t > 0$ be arbitrary. Note that for all $(t,u) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U}$,

$$
\begin{aligned}
\mathbb{E}\left[e^{uZ_t}\right] &= \exp\left\{\int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right)\nu(\mathrm{d}z)\mathrm{d}s\right\} \\
&= \exp\left\{\int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right)\nu_1(\mathrm{d}z)\mathrm{d}s\right\} \\
&\quad \cdot \exp\left\{\int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right)\nu_2(\mathrm{d}z)\mathrm{d}s\right\},
\end{aligned} \tag{4.1}
$$

where $\nu_1(\mathrm{d}z) := \mathbb{1}_{\{z\leqslant 1\}}\nu(\mathrm{d}z)$ and $\nu_2(\mathrm{d}z) := \mathbb{1}_{\{z>1\}}\nu(\mathrm{d}z)$. Similarly to (3.5), for $i = 1, 2$, we define $(Z_t^i)_{t\geqslant 0}$ as the unique strong solution of

$$
\mathrm{d}Z_t^i = -bZ_t^i\mathrm{d}t + \sigma\sqrt{Z_t^i}\mathrm{d}B_t + \mathrm{d}J_t^i, \quad t \geqslant 0, \quad Z_0^i = 0 \text{ a.s.},
$$

where $(J_t^i)_{t\geqslant 0}$ is a subordinator of pure jump-type with Lévy measure ν_i. By (3.6), we have

$$
\mathbb{E}\left[e^{uZ_t^i}\right] = \exp\left\{\int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right)\nu_i(\mathrm{d}z)\mathrm{d}s\right\}, \quad i = 1, 2, \ (t,u) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U}. \tag{4.2}
$$

It follows from (4.1) and (4.2) that

$$
\mu_{Z_t} = \mu_{Z_t^1} * \mu_{Z_t^2}. \tag{4.3}
$$

Let $f(y) := (|y| \vee 1)^\kappa$, $y \in \mathbb{R}$. Then f is locally bounded and submultiplicative by [30, Proposition 25.4], i.e., there exists a constant $c_1 > 0$ such that $f(y_1 + y_2) \leqslant c_1 f(y_1) f(y_2)$ for all $y_1, y_2 \in \mathbb{R}$. Further, it is easy to see that for any constant $c > 0$, there exists a constant $c_2 > 0$ such that $f(y) \leqslant c_2 \exp\{c|y|\}$, $y \in \mathbb{R}$. By (3.7) and (4.3), we get

$$
\begin{aligned}
\mathbb{E}\left[f\left(X_t^x\right)\right] &\leqslant c_1^2 \mathbb{E}\left[f\left(Y_t^x\right)\right] \mathbb{E}\left[f\left(Z_t^1\right)\right] \mathbb{E}\left[f\left(Z_t^2\right)\right] \\
&\leqslant c_1^2 c_2 \mathbb{E}\left[f\left(Y_t^x\right)\right] \mathbb{E}\left[e^{cZ_t^1}\right] \mathbb{E}\left[f\left(Z_t^2\right)\right].
\end{aligned} \tag{4.4}
$$

By [5, Proposition 3], we have $\mathbb{E}[f(Y_t^x)] < \infty$. The finiteness of the exponential moments of Z_t^1, i.e., $\mathbb{E}[\exp\{cZ_t^1\}] < \infty$, follows by [19, Theorem 2.14 (b)], since $(J_t^1)_{t\geqslant 0}$ has only small jumps.

We next show that $\mathbb{E}[f(Z_t^2)] < \infty$. Note that $(J_t^2)_{t\geqslant 0}$ has only big jumps. By [17, Lemma 2], we know that Z_t^2 is compound Poisson distributed, namely, we can find a probability measure ρ_t on $\mathbb{R}_{\geqslant 0}$ such that

$$
\mathbb{E}\left[e^{uZ_t^2}\right] = e^{\lambda_t(\widehat{\rho}_t(u) - 1)}, \quad (t,u) \in \mathbb{R}_{>0} \times \mathcal{U},
$$

where $\lambda_t > 0$ and $\widehat{\rho}_t$ denotes the characteristic function of the measure ρ_t. More precisely, according to [17, see p.292], we have

$$\rho_t = \lambda_t^{-1} \int_0^t \int_{\{z>1\}} m_{\alpha(z,s),\beta(z,s)} \nu(\mathrm{d}z)\mathrm{d}s,$$

where $m_{\alpha(z,s),\beta(z,s)}$ is a Bessel distribution with parameters $\alpha(z,s)$ and $\beta(z,s)$ given by

$$\alpha(z,s) := \frac{2bz}{\sigma^2\left(e^{bs}-1\right)} \quad \text{and} \quad \beta(z,s) := \frac{2be^{bs}}{\sigma^2\left(e^{bs}-1\right)},$$

and

$$\lambda_t = \int_0^t \int_{\{z>1\}} \left(1 - e^{-\alpha(z,s)}\right) \nu(\mathrm{d}z)\mathrm{d}s < \infty.$$

By the Fubini's theorem, we obtain

$$\int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t(\mathrm{d}y) = \lambda_t^{-1} \int_0^t \int_{\{z>1\}} \left(\int_{\mathbb{R}_{\geqslant 0}} f(y) m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y) \right) \nu(\mathrm{d}z)\mathrm{d}s. \tag{4.5}$$

By Lemma 2.2, we have

$$\begin{aligned} \int_{\mathbb{R}_{\geqslant 0}} f(y) m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y) &\leqslant \int_{\mathbb{R}_{\geqslant 0}} (1+y^\kappa) m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y) \\ &\leqslant 1 + C_1 \frac{1+\alpha(z,s)^\kappa}{\beta(z,s)^\kappa} \leqslant 1 + C_1\sigma^{2\kappa}(2b)^{-\kappa}(1-e^{-bs})^\kappa + C_1 e^{-\kappa bs} z^\kappa. \end{aligned} \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$\int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t(\mathrm{d}y) < \infty. \tag{4.7}$$

Moreover, using (4.7) together with the submultiplicativity of f, we get

$$\begin{aligned} \int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t^{*n}(\mathrm{d}y) &= \int_{\mathbb{R}_{\geqslant 0}} \cdots \int_{\mathbb{R}_{\geqslant 0}} f(y_1+\cdots+y_n)\rho_t(\mathrm{d}y_1)\cdots\rho_t(\mathrm{d}y_n) \\ &\leqslant c_1^n \left(\int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t(\mathrm{d}y) \right)^n < \infty, \end{aligned} \tag{4.8}$$

which implies

$$\mathbb{E}\left[f\left(Z_t^2\right)\right] = \int_{\mathbb{R}_{\geqslant 0}} f(y)\mu_{Z_t^2}(\mathrm{d}y) = e^{-\lambda_t} \sum_{n=0}^\infty \frac{\lambda_t^n}{n!} \int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t^{*n}(\mathrm{d}y) < \infty. \tag{4.9}$$

By (4.4) and (4.9), we obtain $\mathbb{E}\left[f\left(X_t^x\right)\right] < \infty$. It follows easily that $\mathbb{E}\left[(X_t^x)^\kappa\right] < \infty$.

"(i)⇒(ii)": It is clear.

"(ii)⇒(iii)": Suppose now that $\mathbb{E}\left[(X_t^x)^\kappa\right] < \infty$ for some $x \in \mathbb{R}_{\geqslant 0}$ and $t > 0$. By (3.7), we obtain

$$\mathbb{E}\left[(X_t^x)^\kappa\right] = \int_{\mathbb{R}_{\geqslant 0}} \int_{\mathbb{R}_{\geqslant 0}} (y+z)^\kappa \mu_{Y_t^x}(\mathrm{d}y)\mu_{Z_t}(\mathrm{d}z) < \infty.$$

So $\int_{\mathbb{R}_{\geqslant 0}} (y+z)^\kappa \mu_{Z_t}(\mathrm{d}z) < \infty$ for some $y \in \mathbb{R}_{\geqslant 0}$, which implies

$$\mathbb{E}\left[Z_t^\kappa\right] = \int_{\mathbb{R}_{\geqslant 0}} z^\kappa \mu_{Z_t}(\mathrm{d}z) \leqslant \int_{\mathbb{R}_{\geqslant 0}} (y+z)^\kappa \mu_{Z_t}(\mathrm{d}z) < \infty. \tag{4.10}$$

Similarly, we can use (4.10) and (4.3) to conclude that $(Z_t^2)_{t\geqslant 0}$ has finite moment of order κ. Let the function f be as above. Then $\mathbb{E}\left[f\left(Z_t^2\right)\right] \leqslant 1+\mathbb{E}\left[\left(Z_t^2\right)^\kappa\right] < \infty$. Since now all the summands in the last identity of (4.9) are finite, the summand corresponding to $n=1$ is also finite and thus

$$\int_{\mathbb{R}_{\geqslant 0}} y^\kappa \rho_t(\mathrm{d}y) \leqslant \int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t(\mathrm{d}y) < \infty.$$

By the Fubini's theorem, we obtain

$$\int_{\mathbb{R}_{\geqslant 0}} y^\kappa \rho_t(\mathrm{d}y) = \lambda_t^{-1}\int_0^t \int_{\{z>1\}} \left(\int_{\mathbb{R}_{\geqslant 0}} y^\kappa m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y)\right)\nu(\mathrm{d}z)\mathrm{d}s < \infty. \tag{4.11}$$

Noting that for all $s\in[0,t]$ and $z>1$,

$$\alpha(z,s) = \frac{2bz}{\sigma^2\left(e^{bs}-1\right)} \geqslant \frac{2b}{\sigma^2\left(e^{bt}-1\right)}.$$

By Lemma 2.2, we can find a constant $c_3 = c_3(t) > 0$ such that

$$\int_{\mathbb{R}_{\geqslant 0}} y^\kappa m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y) \geqslant c_3\left(\frac{\alpha(z,s)}{\beta(z,s)}\right)^\kappa = c_3 z^\kappa e^{-\kappa bs}, \quad s\in[0,t],\ z>1. \tag{4.12}$$

It follows from (4.11) and (4.12) that $\int_{\{z>1\}} z^\kappa \nu(\mathrm{d}z) < \infty$. □

Remark 4.1. *In Theorem 1.1 we have given a complete characterization of the existence of fractional moments for the JCIR process. For an explicit formula of integral moments of general CBI processes, the reader is referred to Barzy et al.* [2].

Based on the proof of Theorem 1.1 we get the following corollary.

Corollary 4.2. *Let $\kappa>0$ be a constant. Suppose $\int_{\{z>1\}} z^\kappa\nu(\mathrm{d}z) < \infty$. Then, for all $x\in\mathbb{R}_{\geqslant 0}$ and $T>0$,*

$$\sup_{t\in[0,T]} \mathbb{E}_x\left[X_t^\kappa\right] < \infty.$$

Proof. Let f, Z_t^1 and Z_t^2 be as in the proof of Theorem 1.1. Note that $|y|^\kappa \leqslant f(y) \leq |y|^\kappa + 1$ for all $y\in\mathbb{R}$. Since $\sup_{t\in\mathbb{R}_{\geqslant 0}} \mathbb{E}[(Y_t^x)^\kappa] < \infty$ due to [5, Proposition 3], by (4.4), it suffices to check that

$$\sup_{t\in[0,T]} \mathbb{E}\left[e^{cZ_t^1}\right] < \infty \quad \text{and} \quad \sup_{t\in[0,T]} \mathbb{E}\left[\left(Z_t^2\right)^\kappa\right] < \infty, \quad T>0,$$

where $c>0$ is a constant to be chosen. It follows from [19, Theorem 2.14 (b)] that

$$\mathbb{E}\left[e^{cZ_t^1}\right] = \exp\left\{\int_0^t \int_0^1 \left(e^{z\psi(s,c)}-1\right)\nu_1(\mathrm{d}z)\mathrm{d}s\right\} < \infty, \quad c\in\mathbb{R},$$

where ψ is given in (3.2). Now, we choose $c>0$ sufficiently small such that $\psi(s,c)\geqslant 0$ for all $s\in\mathbb{R}_{\geqslant 0}$. Hence, $\sup_{t\in[0,T]}\mathbb{E}[\exp\{cZ_t^1\}] \leqslant \mathbb{E}[\exp\{cZ_T^1\}] < \infty$. We next show that $\sup_{t\in[0,T]}\mathbb{E}\left[\left(Z_t^2\right)^\kappa\right] < \infty$. By (4.5), (4.8) and (4.9), we have for all $t\in[0,T]$,

$$\mathbb{E}\left[f\left(Z_t^2\right)\right] \leqslant \exp\left\{-\lambda_t + c_1\lambda_t\int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t(\mathrm{d}y)\right\}$$

$$
\begin{aligned}
&= \exp\left\{-\lambda_t + c_1 \int_0^t \int_{\{z>1\}} \left(\int_{\mathbb{R}_{\geqslant 0}} f(y) m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y)\right) \nu(\mathrm{d}z)\mathrm{d}s\right\} \\
&\leqslant \exp\left\{c_1 \int_0^T \int_{\{z>1\}} \left(\int_{\mathbb{R}_{\geqslant 0}} f(y) m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y)\right) \nu(\mathrm{d}z)\mathrm{d}s\right\} \\
&= \exp\left\{c_1 \lambda_T \int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_T(\mathrm{d}y)\right\}. \qquad (4.13)
\end{aligned}
$$

It follows from (4.7) and (4.13) that

$$
\sup_{t\in[0,T]} \mathbb{E}\left[\left(Z_t^2\right)^{\kappa}\right] \leqslant \sup_{t\in[0,T]} \mathbb{E}\left[f\left(Z_t^2\right)\right] \leqslant \exp\left\{c_1\lambda_T \int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_T(\mathrm{d}y)\right\} < \infty.
$$

This completes the proof. □

5. Ergodicity of the JCIR process

In this section we prove the ergodicity of the JCIR process X provided that

$$
\int_{\{z>1\}} \log z \nu(\mathrm{d}z) < \infty. \qquad (5.1)
$$

Our approach is based on the general theory of Meyn and Tweedie [27] for the ergodicity of Markov processes. The essential step is to find a Foster-Lyapunov function in the sense of [27, condition (CD2)]. In view of (5.1), we choose the Foster-Lyapunov function to be $V(x) = \log(1+x)$, $x \in \mathbb{R}_{\geqslant 0}$. We first show that this function V is in the domain of the *extended generator* (see [27, pp. 521-522] for a definition) of X.

Lemma 5.1. *Suppose* (5.1) *is true. Let* $V(x) := \log(1+x)$, $x \in \mathbb{R}_{\geqslant 0}$. *Then for all* $t > 0$ *and* $x \in \mathbb{R}_{\geqslant 0}$, *we have* $\mathbb{E}_x\left[\int_0^t |\mathcal{A}V(X_s)|\,\mathrm{d}s\right] < \infty$ *and*

$$
\mathbb{E}_x\left[V(X_t)\right] = V(x) + \mathbb{E}_x\left[\int_0^t \mathcal{A}V(X_s)\,\mathrm{d}s\right], \qquad (5.2)
$$

where $\mathcal{A}$ *is given in* (2.2). *In other words,* V *is in the domain of the extended generator of* X.

Proof. It is easy to see that $V \in C^2(\mathbb{R}_{\geqslant 0}, \mathbb{R})$ and

$$
V'(x) := \frac{\partial}{\partial x} V(x) = (1+x)^{-1} \quad \text{and} \quad V''(x) := \frac{\partial^2}{\partial x^2} V(x) = -(1+x)^{-2}.
$$

Let $x \in \mathbb{R}_{\geqslant 0}$ be fixed and assume that $X_0 = x$ almost surely. In view of the Lévy-Itô decomposition of $(J_t)_{t\geqslant 0}$ in (2.1), we have

$$
X_t = x + \int_0^t (a - bX_s)\mathrm{d}s + \sigma \int_0^t \sqrt{X_s}\mathrm{d}B_s + \int_0^t \int_0^\infty zN(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0,
$$

where $N(\mathrm{d}s, \mathrm{d}z)$ is defined in (2.1). By Itô's formula, we obtain

$$
\begin{aligned}
V(X_t) - V(X_0) = &\int_0^t (a - bX_s) V'(X_s)\,\mathrm{d}s + \frac{\sigma^2}{2} \int_0^t X_s V''(X_s)\,\mathrm{d}s \\
&+ \sigma \int_0^t \sqrt{X_s} V'(X_s)\,\mathrm{d}B_s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_0^\infty \left(V\left(X_{s-}+z\right)-V\left(X_{s-}\right)\right)N(\mathrm{d}s,\mathrm{d}z)\\
&=\int_0^t \left(a-bX_s\right)V'\left(X_s\right)\mathrm{d}s+\frac{\sigma^2}{2}\int_0^t X_sV''\left(X_s\right)\mathrm{d}s\\
&+\int_0^t\int_0^\infty \left(V\left(X_{s-}+z\right)-V\left(X_{s-}\right)\right)\nu(\mathrm{d}z)\mathrm{d}s\\
&+\sigma\int_0^t \sqrt{X_s}V'\left(X_s\right)\mathrm{d}B_s\\
&+\int_0^t\int_0^\infty \left(V\left(X_{s-}+z\right)-V\left(X_{s-}\right)\right)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\\
&=\int_0^t \left(\mathcal{A}V\right)\left(X_s\right)\mathrm{d}s+M_t(V),\quad t\geqslant 0,
\end{aligned}\tag{5.3}
$$

where $\widetilde{N}(\mathrm{d}s,\mathrm{d}z):=N(\mathrm{d}s,\mathrm{d}z)-\nu(\mathrm{d}z)\mathrm{d}s$ and

$$
\begin{aligned}
M_t(V):=&\ \sigma\int_0^t \sqrt{X_s}V'\left(X_s\right)\mathrm{d}B_s\\
&+\int_0^t\int_{\{z\leqslant 1\}} \left(V\left(X_{s-}+z\right)-V\left(X_{s-}\right)\right)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\\
&+\int_0^t\int_{\{z>1\}} \left(V\left(X_{s-}+z\right)-V\left(X_{s-}\right)\right)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\\
=&\ D_t+J_{*,t}+J_t^*.
\end{aligned}
$$

Clearly, if $(M_t(V))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$, by taking the expectation of both sides of (5.3), we see that condition (5.2) holds.

We start to prove that $(M_t(V))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Since

$$
\mathbb{E}_x\left[(D_t)^2\right]=\sigma^2\int_0^t\mathbb{E}_x\left[X_s\left(1+X_s\right)^{-2}\right]\mathrm{d}s\leqslant\sigma^2\int_0^t\mathbb{E}_x\left[\left(1+X_s\right)^{-1}\right]\mathrm{d}s\leqslant t\sigma^2<\infty,
$$

it follows that $(D_t)_{t\geqslant 0}$ is a square-integrable martingale. Note that

$$
|V(y+z)-V(y)|\leqslant z\sup_{y\in\mathbb{R}_{\geqslant 0}}|V'(y)|\leqslant z,\quad y,z\in\mathbb{R}_{\geqslant 0}.\tag{5.4}
$$

Therefore,

$$
\mathbb{E}_x\left[\int_0^t\int_{\{z\leqslant 1\}}\left(V\left(X_{s-}+z\right)-V\left(X_{s-}\right)\right)^2\nu(\mathrm{d}z)\mathrm{d}s\right]\leqslant t\int_{\{z\leqslant 1\}}z^2\nu(\mathrm{d}z)<\infty,
$$

which implies that $(J_{*,t})_{t\geqslant 0}$ is also a square-integrable martingale by [13, pp. 62, 63]. If $y\in\mathbb{R}_{\geqslant 0}$ and $z>1$, then

$$
|V(y+z)-V(y)|=\log\left(1+\frac{z}{1+y}\right)\leqslant\log(1+z)\leqslant\log(2)+\log(z).\tag{5.5}
$$

So

$$
\mathbb{E}_x\left[\int_0^t\int_{\{z>1\}}\left|V(X_{s-}+z)-V(X_{s-})\right|\nu(\mathrm{d}z)\mathrm{d}s\right]
$$

$$\leqslant t \int_{\{z>1\}} (\log(2) + \log(z))\,\nu(\mathrm{d}z)$$
$$= t\log(2)\nu(\{z>1\}) + t\int_{\{z>1\}} \log(z)\nu(\mathrm{d}z) < \infty, \quad t \geqslant 0,$$

and hence, by [13, Lemma 3.1 and p. 62], $(J_t^*)_{t\geqslant 0}$ is a martingale. Consequently, $(M_t(V))_{t\geqslant 0} = (D_t + J_{*,t} + J_t^*)_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

Next, we show that $\mathbb{E}_x\left[\int_0^t |\mathcal{A}V(X_s)|\,\mathrm{d}s\right] < \infty$ for all $t \geqslant 0$. By the decomposition of $\mathcal{A}$ into a *diffusion part* $\mathcal{D}$ and a *jump part* $\mathcal{J}$ as introduced in Section 2, we can write $\mathcal{A}V = \mathcal{D}V + \mathcal{J}V$. Concerning the diffusion part $\mathcal{D}V$, it is easy to see that

$$\sup_{y\in\mathbb{R}_{\geqslant 0}} |(\mathcal{D}V)(y)| = \sup_{y\in\mathbb{R}_{\geqslant 0}} \left|(a-by)(1+y)^{-1} - \frac{\sigma^2}{2}y(1+y)^{-2}\right| < \infty. \tag{5.6}$$

For the jump part $\mathcal{J}V$, we decompose it further as $\mathcal{J}V = \mathcal{J}_*V + \mathcal{J}^*V$, where

$$(\mathcal{J}_*V)(y) = \int_{\{z\leqslant 1\}} (V(y+z) - V(y))\,\nu(\mathrm{d}z), \tag{5.7}$$

$$(\mathcal{J}^*V)(y) = \int_{\{z>1\}} (V(y+z) - V(y))\,\nu(\mathrm{d}z). \tag{5.8}$$

By (5.4), we have

$$|(\mathcal{J}_*V)(y)| \leqslant \int_{\{z\leqslant 1\}} z\nu(\mathrm{d}z) < \infty, \quad y \in \mathbb{R}_{\geqslant 0}. \tag{5.9}$$

Concerning $\mathcal{J}^*$, it follows from (5.5) that

$$|(\mathcal{J}^*V)(y)| \leqslant \log(2)\nu(\{z>1\}) + \int_{\{z>1\}} \log z\nu(\mathrm{d}z) < \infty, \quad y \in \mathbb{R}_{\geqslant 0}. \tag{5.10}$$

Combining (5.6), (5.9) and (5.10) yields that $|\mathcal{A}V|$ is bounded on $\mathbb{R}_{\geqslant 0}$, which implies $\mathbb{E}_x\left[\int_0^t |\mathcal{A}V(X_s)|\,\mathrm{d}s\right] < \infty$ for all $t \geqslant 0$. □

We are ready to prove the ergodicity of the JCIR process $(X_t)_{t\geqslant 0}$ under (5.1).

Proof of Theorem 1.2 (a). In view of [27, Theorem 5.1], to prove the ergodicity of the JCIR process $(X_t)_{t\geqslant 0}$, it is enough to check that

(i) $(X_t)_{t\geqslant 0}$ is a non-explosive (Borel) right process (see, e.g., [31, p.38] or [22, p.67] for a definition of a (Borel) right process);
(ii) all compact sets of the state space $\mathbb{R}_{\geqslant 0}$ are petite for some skeleton chain (see [26, p.500] for a definition);
(iii) there exist positive constants c, M such that

$$(\mathcal{A}V)(x) \leqslant -c + M\mathbb{1}_K(x), \quad x \in \mathbb{R}_{\geqslant 0}, \tag{5.11}$$

for some compact subset $K \subset \mathbb{R}_{\geqslant 0}$, where $V(x) = \log(1+x)$, $x \in \mathbb{R}_{\geqslant 0}$.

We proceed to prove (i)-(iii).

In view of [22, Corollary 4.1.4], $(X_t)_{t\geqslant 0}$ is a right process, since it possesses the Feller property as an affine process (see [9, Theorem 2.7]).

According to Proposition 3.1, we can proceed in the very same way as in Jin *et al.* [17, Theorem 1] to see that for each $n \in \mathbb{Z}_{\geqslant 0}$ the δ-skeleton chain $X_{n\delta}$, $\delta > 0$ being a constant, is irreducible with respect to the Lebesgue measure on $\mathbb{R}_{\geqslant 0}$. Since $(X_t)_{t\geqslant 0}$ has the Feller property, the claim (ii) now follows from [24, Proposition 6.2.8].

Finally, we prove (iii). As shown in the proof of Lemma 5.1, $|\mathcal{A}V|$ is bounded on $\mathbb{R}_{\geqslant 0}$. Therefore, to get (5.11), it suffices to show that $\lim_{x\to\infty}\mathcal{A}V(x)$ exists and is negative. As before, we write $\mathcal{A}V = \mathcal{D}V + \mathcal{J}V$. It is easy to see that

$$\lim_{x\to\infty}(\mathcal{D}V)(x) = \lim_{x\to\infty}\left[(a-bx)(1+x)^{-1} - \frac{\sigma^2}{2}x(1+x)^{-2}\right] = -b.$$

Next, we consider the jump part $\mathcal{J}V$. Note that

$$V(x+z) - V(x) = \log\left(1 + \frac{z}{1+x}\right) \longrightarrow 0 \quad \text{as } x \to \infty.$$

On the other hand, by (5.4) and (5.5), we have

$$|V(x+z) - V(x)| \leqslant z\mathbb{1}_{\{z\leqslant 1\}} + [\log(2) + \log(z)]\,\mathbb{1}_{\{z>1\}},$$

where the function on the right-hand side is integrable with respect to ν. By the dominated convergence theorem, we obtain $\lim_{x\to\infty}(\mathcal{J}V)(x) = 0$. This completes the proof. □

Remark 5.2. *According to the discussion after* [7, Proposition 2.5]*, a direct but important consequence of our ergodic result is the following: under the assumptions of Theorem* 1.2 (a)*, for all Borel measurable functions* $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ *with* $\int_{\mathbb{R}_{\geqslant 0}} |f(x)|\pi(\mathrm{d}x) < \infty$*, it holds*

$$\mathbb{P}\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X_s)\mathrm{d}s = \int_{\mathbb{R}_{\geqslant 0}} f(x)\pi(\mathrm{d}x)\right) = 1. \tag{5.12}$$

The convergence (5.12) *may be very useful for parameter estimation of the JCIR process.*

6. Exponential ergodicity of the JCIR process

Our aim of this section is to show that the JCIR process X is exponentially ergodic if

$$\int_{\{z>1\}} z^{\kappa}\nu(dz) < \infty \quad \text{for some } \kappa > 0. \tag{6.1}$$

As in previous works (see, e.g., [17] and [16]) the following proposition will play an essential role in proving exponential ergodicity of the JCIR process X, provided that (6.1) holds.

Proposition 6.1. *Suppose* (6.1) *is true. Let* $V \in C^2(\mathbb{R}_{\geqslant 0}, \mathbb{R})$ *be nonnegative and such that* $V(x) = x^{\kappa \wedge 1}$ *for* $x \geqslant 1$. *Then there exist positive constants* c, M *such that*

$$\mathbb{E}_x \left[V(X_t) \right] \leqslant e^{-ct} V(x) + \frac{M}{c} \tag{6.2}$$

for all $(t, x) \in \mathbb{R}^2_{\geqslant 0}$.

Proof. If $\kappa \geqslant 1$, then it follows from (6.1) that $\int_{\{z>1\}} z\nu(dz) < \infty$, which, together with [17, Lemma 3], implies

$$\mathbb{E}_x \left[X_t \right] \leqslant x e^{-bt} + M_1, \quad t > 0, x \geqslant 0,$$

for some constant $0 < M_1 < \infty$. In this case, we have

$$\begin{aligned}
\mathbb{E}_x \left[V(X_t) \right] &= \mathbb{E}_x \left[V(X_t) \mathbb{1}_{\{X_t > 1\}} \right] + \mathbb{E}_x \left[V(X_t) \mathbb{1}_{\{X_t \leqslant 1\}} \right] \\
&\leqslant \mathbb{E}_x \left[X_t \right] + \sup_{y \in [0,1]} |V(y)| \\
&\leqslant x e^{-bt} + M_1 + \sup_{y \in [0,1]} |V(y)| \\
&\leqslant \left(V(x) + 1 \right) e^{-bt} + M_1 + \sup_{y \in [0,1]} |V(y)| \\
&\leqslant V(x) e^{-bt} + M_2,
\end{aligned}$$

where $M_2 := 1 + M_1 + \sup_{y \in [0,1]} |V(y)| < \infty$ is a constant. Hence (6.2) is true when $\kappa \geqslant 1$. So in the following we assume $0 < \kappa < 1$.

Define $g(t, x) := \exp(ct) V(x)$, where $c \in \mathbb{R}_{>0}$ is a constant to be determined later. Then,

$$\begin{aligned}
g'_t(t, x) &:= \frac{\partial}{\partial t} g(t, x) = c e^{ct} V(x), \\
g'_x(t, x) &:= \frac{\partial}{\partial x} g(t, x) = \begin{cases} \kappa e^{ct} x^{\kappa - 1}, & x > 1, \\ e^{ct} V'(x), & x \in [0, 1], \end{cases} \\
g''_x(t, x) &:= \frac{\partial^2}{\partial x^2} g(t, x) = \begin{cases} \kappa(\kappa - 1) e^{ct} x^{\kappa - 2}, & x > 1, \\ e^{ct} V''(x), & x \in [0, 1]. \end{cases}
\end{aligned}$$

Applying Itô's formula for $g(t, X_t)$, we obtain

$$g(t, X_t) - g(0, X_0) = \int_0^t (\mathcal{L} g)(s, X_s) \mathrm{d}s + \int_0^t g'_s(s, X_s) \mathrm{d}s + M_t(g), \quad t \geqslant 0, \tag{6.3}$$

where the operator $\mathcal{L}$ is given by $(\mathcal{L} g)(s, X_s) = \exp\{cs\} (\mathcal{A} V)(X_s)$ with $\mathcal{A}$ as in (2.2) and

$$\begin{aligned}
M_t(g) &:= \sigma \int_0^t \sqrt{X_s} g'_x(s, X_s) \mathrm{d}B_s + \int_0^t \int_0^\infty \left(g(s, X_{s-} + z) - g(s, X_{s-}) \right) \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&= G_t(g) + J_t(g), \quad \text{for all } t \geqslant 0.
\end{aligned}$$

We will complete the proof in three steps.

"*Step 1*": We check that $(M_t(g))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. First, note that

$$G_t(g) := \sigma \int_0^t \frac{\partial}{\partial x} g(s, X_s)\sqrt{X_s}\mathrm{d}B_s, \quad t \geqslant 0,$$

is a square-integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Indeed, for each $t \geqslant 0$, we have

$$\begin{aligned} &\mathbb{E}_x\left[\left(\sigma \int_0^t \sqrt{X_s} g'_x(s, X_s)\mathrm{d}B_s\right)^2\right] \\ &= \sigma^2 \int_0^t e^{2cs}\mathbb{E}\left[\mathbb{1}_{\{X_s\leqslant 1\}} X_s V'(X_s)\right]\mathrm{d}s + \sigma^2\kappa^2 \int_0^t e^{2cs}\mathbb{E}\left[\mathbb{1}_{\{X_s>1\}} X_s^{2\kappa-1}\right]\mathrm{d}s. \end{aligned} \tag{6.4}$$

Clearly, we have $|\mathbb{1}_{\{X_s\leqslant 1\}} X_s V'(X_s)| \leqslant \sup_{y\in[0,1]} |V'(y)| < \infty$, which implies that the first integral on the right-hand side of (6.4) is finite. Since $|\mathbb{1}_{\{X_s>1\}} X_s^{2\kappa-1}| \leqslant |X_s|^\kappa$, by (6.1) and Proposition 4.2, we see that the second integral on the right-hand side of (6.4) is finite as well. Hence, $(G_t(g))_{t\geqslant 0}$ is a square-integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

Next, we prove that $J_t(g)$, $t \geqslant 0$, is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. We define

$$\begin{aligned} J_{*,t}(V) &:= \int_0^t \int_{\{z\leqslant 1\}} e^{cs}\left(V(X_{s-}+z) - V(X_{s-})\right)\widetilde{N}(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0, \\ J_t^*(V) &:= \int_0^t \int_{\{z>1\}} e^{cs}\left(V(X_{s-}+z) - V(X_{s-})\right)\widetilde{N}(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0. \end{aligned}$$

So $J_t(g) = J_{*,t}(V) + J_t^*(V)$ for $t \geqslant 0$. In what follows, we establish some elementary inequalities for V. For $y \geqslant 1$, we have

$$\begin{aligned} \mathbb{1}_{\{z\leqslant 1\}}(z)|V(y+z) - V(y)| &= \mathbb{1}_{\{z\leqslant 1\}}(z)\left((y+z)^\kappa - y^\kappa\right) \\ &= \mathbb{1}_{\{z\leqslant 1\}}(z) y^\kappa \left(\left(1+\frac{z}{y}\right)^\kappa - 1\right) \\ &\leqslant \mathbb{1}_{\{z\leqslant 1\}}(z)\kappa y^{\kappa-1} z \leqslant \mathbb{1}_{\{z\leqslant 1\}}(z) z, \end{aligned} \tag{6.5}$$

where we used Bernoulli's inequality to obtain the first inequality in (6.5). Moreover, it is easy to see that for $y \geqslant 1$,

$$\mathbb{1}_{\{z>1\}}(z)|V(y+z) - V(y)| \leqslant \mathbb{1}_{\{z>1\}}(z)\left(y^\kappa + z^\kappa - y^\kappa\right) \leqslant \mathbb{1}_{\{z>1\}}(z) z^\kappa. \tag{6.6}$$

For $y \in [0, 1]$, using the mean value theorem, we get

$$\mathbb{1}_{\{z\leqslant 1\}}(z)|V(y+z) - V(y)| \leqslant z \sup_{y\in[0,2]} |V'(y)| \leqslant c_1 z, \tag{6.7}$$

for some constant $c_1 > 0$. Finally, for $y \in [0, 1]$, again by Bernoulli's inequality, we have

$$\begin{aligned} \mathbb{1}_{\{z>1\}}(z)|V(y+z) - V(y)| &\leqslant \mathbb{1}_{\{z>1\}}(z)\left((y+z)^\kappa + |V(y)|\right) \\ &\leqslant \mathbb{1}_{\{z>1\}}(z)\left(z^\kappa\left(1+\kappa\frac{y}{z}\right) + |V(y)|\right) \\ &\leqslant \mathbb{1}_{\{z>1\}}(z)\left(z^\kappa + 1 + |V(y)|\right) \\ &\leqslant \mathbb{1}_{\{z>1\}}(z)\left(z^\kappa + c_2\right), \end{aligned} \tag{6.8}$$

where $c_2 := 1 + \sup_{y\in[0,1]} |V(y)| < \infty$ is a positive constant. Now, from (6.5) and (6.7), we deduce that

$$\mathbb{E}_x\left[\int_0^t \int_{\{z\leqslant 1\}} e^{cs}|V(X_{s-}+z)-V(X_{s-})|\nu(\mathrm{d}z)\mathrm{d}s\right]$$
$$\leqslant (1+c_1)\int_0^t e^{cs}\mathrm{d}s\int_{\{z\leqslant 1\}} z\nu(\mathrm{d}z) < \infty, \quad t\geqslant 0.$$

It follows from [13, p.62 and Lemma 3.1] that $(J_{*,t}(V))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Using (6.6) and (6.8), we obtain

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^t \int_{\{z>1\}} e^{cs}|V(X_{s-}+z)-V(X_{s-})|\nu(\mathrm{d}z)\mathrm{d}s\right]\\
&\quad\leqslant \int_0^t \int_{\{z>1\}} e^{cs}\left(z^\kappa + c_2\right)\nu(\mathrm{d}s)\mathrm{d}s\\
&\quad= \int_0^t e^{cs}\mathrm{d}s\left(\int_{\{z>1\}} z^\kappa\nu(\mathrm{d}z) + c_2\nu(\{z>1\})\right) < \infty, \quad t\geqslant 0.
\end{aligned}$$

As a consequence, we see that $(J_t^*(V))_{t\geqslant 0}$ is also a martingale. Clearly, $(M_t(g))_{t\geqslant 0} = (G_t(g)+J_t(g))_{t\geqslant 0}$ is now a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

"*Step 2*": We determine the constant $c \in \mathbb{R}_{>0}$ and find another positive constant $M < \infty$ such that

$$(\mathcal{A}V)(y) = (\mathcal{D}V)(y) + (\mathcal{J}V)(y) \leqslant -cV(y) + M, \quad y\in\mathbb{R}_{\geqslant 0}. \tag{6.9}$$

Consider the jump part $\mathcal{J}V = \mathcal{J}_*V + \mathcal{J}^*V$, where $\mathcal{J}_*V$ and $\mathcal{J}^*V$ are defined by (5.7) and (5.8), respectively. For all $x\in\mathbb{R}_{\geqslant 0}$, using (6.5) and (6.7), we obtain

$$(\mathcal{J}_*V)(y) = \int_{\{z\leqslant 1\}} |V(y+z)-V(y)|\nu(\mathrm{d}z) \leqslant (1+c_1)\int_{\{z\leqslant 1\}} z\nu(\mathrm{d}z) < \infty.$$

For $\mathcal{J}^*V$, we can use (6.6) and (6.8) to obtain that for all $y\in\mathbb{R}_{\geqslant 0}$,

$$\begin{aligned}
(\mathcal{J}^*V)(x) &= \int_{\{z>1\}} |V(y+z)-V(y)|\nu(\mathrm{d}z)\\
&\leqslant \int_{\{z>1\}} z^\kappa\nu(\mathrm{d}z) + c_2\nu(\{z>1\}) < \infty.
\end{aligned}$$

Next, we estimate $\mathcal{D}V$. Since,

$$V'(x) = \kappa x^{\kappa-1} \quad \text{and} \quad V''(x) = \kappa(\kappa-1)x^{\kappa-2} \quad \text{for } x\geqslant 1,$$

we see that

$$\begin{aligned}
(\mathcal{D}V)(x) &= (a-bx)V'(x) + \frac{\sigma^2 x}{2}V''(x)\\
&= -b\kappa x^\kappa + \kappa x^{\kappa-1}\left(a + \frac{\sigma^2(\kappa-1)}{2}\right) \leqslant -b\kappa x^\kappa + c_3
\end{aligned}$$

for all $x\geqslant 1$. Here $c_3 < \infty$ is a positive constant. After all we get that for all $x\geqslant 1$,

$$(\mathcal{A}V)(x) \leqslant -b\kappa V(x) + c_4$$

where $c_4 < \infty$ is a positive constant. By noting that $V \in C^2(\mathbb{R}_{\geqslant 0}, \mathbb{R})$, we see that

$$\sup_{y\in[0,1]} |V(y)| < \infty \quad \text{and} \quad \sup_{y\in[0,1]} |(\mathcal{A}V)(y)| < \infty.$$

Consequently, (6.9) holds for all $x \geqslant 0$.

"*Step 3*": We prove (6.2). Note that $(\mathcal{L}g)(s,x) = \exp\{cs\}(\mathcal{A}V)(x)$. By (6.3), (6.9) and the martingale property of $(M_t(g))_{t\geqslant 0}$, we obtain that for all $(x,t) \in \mathbb{R}^2_{\geqslant 0}$,

$$\begin{aligned} e^{ct}\mathbb{E}_x\left[V(X_t)\right] - V(x) &= \mathbb{E}_x\left[g(t,X_t) - g(0,X_0)\right] \\ &= \mathbb{E}_x\left[\int_0^t \left(e^{cs}(\mathcal{A}V)(X_s) + ce^{cs}V(X_s)\right)\mathrm{d}s\right] \\ &\leqslant \mathbb{E}_x\left[\int_0^t \left(e^{cs}\left(-cV(X_s) + M\right) + ce^{cs}V(X_s)\right)\mathrm{d}s\right] \\ &= \mathbb{E}_x\left[\int_0^t e^{cs}M\mathrm{d}s\right] \leqslant \frac{M}{c}e^{ct}. \end{aligned}$$

So (6.2) is true. With this our proof is complete. □

Based on Proposition 6.1, we are now ready to prove Theorem 1.2 (b).

Proof of Theorem 1.2 (b). In view of Proposition 3.1 and Proposition 6.1, to obtain the exponential ergodicity of X, we can follow almost the very same lines as in the proof of [17, Theorem 1]. We remark that the *strong aperiodicity* condition used in the proof of [17, Theorem 1] can be safely replaced by the *aperiodicity* condition (the definition of aperiodicity can be found in [24, p.114]), due to [25, Theorem 6.3]. Moreover, the *aperiodicity* of the skeleton chain can be obtained by following the same arguments as in part (b) of the proof of [16, Theorem 6.1]. This completes the proof. □

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(Peng Jin) Fakultät für Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, 42119 Wuppertal, Germany
E-mail address, Peng Jin: jin@uni-wuppertal.de

(Jonas Kremer) Fakultät für Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, 42119 Wuppertal, Germany
E-mail address, Jonas Kremer: j.kremer@uni-wuppertal.de

(Barbara Rüdiger) Fakultät für Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, 42119 Wuppertal, Germany
E-mail address, Barbara Rüdiger: ruediger@uni-wuppertal.de