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EXPONENTIAL ERGODICITY OF AN AFFINE TWO-FACTOR MODEL BASED ON THE α-ROOT PROCESS

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Abstract. We study an affine two-factor model introduced by Barczy *et al.* (2014). One component of this two-dimensional model is the so-called α-root process, which generalizes the well known CIR process. In this paper, we show that this affine two-factor model is exponentially ergodic when $\alpha \in (1,2)$.

1. Introduction

In this paper, we study a two-dimensional affine process $(Y,X) := (Y_t, X_t)_{t\geqslant 0}$ determined by the following stochastic differential equation

$$\begin{cases} \mathrm{d}Y_t = (a - bY_t)\mathrm{d}t + \sqrt[\alpha]{Y_{t-}}\mathrm{d}L_t, & t \geqslant 0, \quad Y_0 \geqslant 0 \quad \text{a.s.}, \\ \mathrm{d}X_t = (m - \theta X_t)\mathrm{d}t + \sqrt{Y_t}\mathrm{d}B_t, & t \geqslant 0, \end{cases} \tag{1.1}$$

where $a > 0, b > 0, \theta, m \in \mathbb{R}$, $\alpha \in (1,2)$, $(L_t)_{t\geqslant 0}$ is a spectrally positive α-stable Lévy process with the Lévy measure $C_\alpha z^{-1-\alpha}\mathbb{1}_{\{z>0\}}\mathrm{d}z$, with $C_\alpha := (\alpha\Gamma(-\alpha))^{-1}$, and $(B_t)_{t\geqslant 0}$ is an independent standard Brownian motion. Note that if (Y_0, X_0) is independent of $(L_t, B_t)_{t\geqslant 0}$, then the existence and uniqueness of a strong solution to the SDE (1.1) follow from [3, Theorem 2.1].

The process $(Y_t, X_t)_{t\geqslant 0}$ given by (1.1) has been introduced by Barczy *et al.* in [3]. There, it was proved that $(Y_t, X_t)_{t\geqslant 0}$ belongs to the class of regular affine processes (with state space $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$). The process Y is the so-called α-root process (sometimes referred as the stable CIR process, shorted SCIR, see [20]) and is also an affine process (with state space $\mathbb{R}_{\geqslant 0}$). It can be considered as an extension of the CIR process. The general theory of affine processes on the canonical state space $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ was initiated by Duffie *et al.* [7] and further developed in [6]. An affine process on $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ is a continuous-time Markov process taking values in $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$, whose log-characteristic function depends in an affine way on the initial state vector of the process, i.e. the log-characteristic function is linear with respect to the initial state vector. Affine processes are particularly important in financial mathematics because of their computational tractability. For example, the models of Cox *et al.* [5], Heston [12] and Vasicek [29] are all based on affine processes.

An important issue for the application of affine processes is the calibration of their parameters. This has been investigated for some well known affine models, see e.g. [26, 25, 1, 2, 4]. To study the asymptotic properties of estimators of the parameters, a comprehension of the long-time behavior of the underlying affine processes is very often required. This is one of the reasons why the stationary,

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ergodic and recurrent properties of affine processes have recently attracted many investigations, see e.g. [18, 19, 3, 20, 8, 13, 15, 14], and many others.

Concerning the two-factor model defined in (1.1), it was shown in [3] that $(Y_t, X_t)_{t\geqslant 0}$ has a stationary distribution. Using the same argument as in [17, p. 80], it can easily be seen that the stationary distribution for $(Y_t, X_t)_{t\geqslant 0}$ is actually unique. If one allows $\alpha = 2$ and replaces $(L_t)_{t\geqslant 0}$ in (1.1) by a standard Brownian motion $(W_t)_{t\geqslant 0}$ (independent of $(B_t)_{t\geqslant 0}$), then the process Y becomes the CIR process; in this case, the ergodicity of $(Y_t, X_t)_{t\geqslant 0}$ has been proved in [3]. However, the ergodicity of $(Y_t, X_t)_{t\geqslant 0}$ in the case $1 < \alpha < 2$ is still not known.

In this work we study the ergodicity problem for the two-factor model in (1.1) when $1 < \alpha < 2$. As our main result (see Theorem 6.1 below), we show that $(Y_t, X_t)_{t\geqslant 0}$ in (1.1) is exponentially ergodic if $\alpha \in (1, 2)$, complementing the results in [3]. Our approach is very close to that of [14]. The first step is to show the existence of positive transition densities of the α-root process Y_t. To achieve this, we calculate explicitly the Laplace transform of Y_t. Through a careful analysis of the decay rate of the Laplace transform of Y_t at infinity, we manage to show the positivity of the density function of Y_t using the inverse Fourier transform. In the second step, we construct a Foster-Lyapunov function for the process $(Y_t, X_t)_{t\geqslant 0}$. Using the general theory in [22, 23, 24] on the ergodicity of Markov processes, we are then able to obtain the exponential ergodicity of the process $(Y_t, X_t)_{t\geqslant 0}$ in (1.1).

Finally, we remark that the exponential ergodicity for a large class of affine processes on $\mathbb{R}_{\geqslant 0}$, including the α-root process $(Y_t)_{t\geqslant 0}$, has been derived in [20] by a coupling method. We don't know if a similar coupling argument would work for the two-dimensional affine process $(Y_t, X_t)_{t\geqslant 0}$ in (1.1).

The rest of the paper is organized as follows. In Section 2 we recall some basic facts on the process $(Y_t, X_t)_{t\geqslant 0}$. In Section 3 we derive the Laplace transform of the α-root process Y. In Section 4 we prove that the α-root process Y possesses positive transition densities. In Section 5 we construct a Foster-Lyapunov function for the process $(Y_t, X_t)_{t\geqslant 0}$. In Section 6 we show that the process $(Y_t, X_t)_{t\geqslant 0}$ is exponentially ergodic.

2. Preliminaries

In this section we recall some key facts on the affine process $(Y, X) := (Y_t, X_t)_{t\geqslant 0}$ defined by the equation (1.1), mainly due to [3].

Let $\mathbb{N}$, $\mathbb{Z}_{\geqslant 0}$, $\mathbb{R}$, $\mathbb{R}_{\geqslant 0}$ and $\mathbb{R}_{>0}$ denote the sets of positive integers, non-negative integers, real numbers, non-negative real numbers and strictly positive real numbers, respectively. Let $\mathbb{C}$ be the set of complex numbers. For $z \in \mathbb{C}\setminus\{0\}$ we denote by $\mathrm{Arg}(z)$ the principal value of its argument and by $\bar{z}$ its conjugate. We define the following subsets of $\mathbb{C}$:

$$\mathcal{U}_- := \{u \in \mathbb{C} : \mathrm{Re}\, u \leqslant 0\}, \quad \mathcal{U}_+ := \{u \in \mathbb{C} : \mathrm{Re}\, u \geqslant 0\},$$
$$\mathcal{U}_-^{\circ} := \{u \in \mathbb{C} : \mathrm{Re}\, u < 0\}, \quad \mathcal{U}_+^{\circ} := \{u \in \mathbb{C} : \mathrm{Re}\, u > 0\},$$

and

$$\mathcal{O} := \mathbb{C} \setminus \{-x : x \in \mathbb{R}_{\geqslant 0}\}.$$

For $z \in \mathbb{C}\setminus\{0\}$ let $\mathrm{Log}(z)$ be the principal value of the complex logarithm of z, i.e., $\mathrm{Log}(z) = \ln(|z|) + i\mathrm{Arg}(z)$. For $\beta \in \mathbb{R}$ define the complex power function z^β as

$$z^\beta := \exp(\beta\, \mathrm{Log}\, z), \quad z \in \mathbb{C}\setminus\{0\}. \tag{2.1}$$

By $C^2(S,\mathbb{R})$, $C_c^2(S,\mathbb{R})$ and $C_b^2(S,\mathbb{R})$ we denote the sets of $\mathbb{R}$-valued functions on S that are twice continuously differentiable, that are twice continuously differentiable with compact support and that are bounded continuous with bounded continuous first and second order partial derivatives, respectively, where the space S can be $\mathbb{R}$, $\mathbb{R}_{\geqslant 0}\times\mathbb{R}$ or $\mathbb{R}_{\geqslant 0}\times\mathbb{R}_{\geqslant 0}\times\mathbb{R}$ in this paper.

We assume that $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},\mathbb{P})$ is a filtered probability space satisfying the usual conditions, i.e., $(\Omega,\mathcal{F},\mathbb{P})$ is complete, the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$.

Let $(B_t)_{t\geqslant 0}$ be a standard $(\mathcal{F}_t)_{t\geqslant 0}$-Brownian motion and $(L_t)_{t\geqslant 0}$ be a spectrally positive $(\mathcal{F}_t)_{t\geqslant 0}$-Lévy process with the Lévy measure $C_\alpha z^{-1-\alpha}\mathbb{1}_{\{z>0\}}\mathrm{d}z$, where $1<\alpha<2$. Assume $(B_t)_{t\geqslant 0}$ and $(L_t)_{t\geqslant 0}$ are independent. Note that the characteristic function of L_1 is given by

$$\mathbb{E}\left[e^{iuL_1}\right]=\exp\left\{\int_0^\infty\left(e^{iuz}-1-iuz\right)C_\alpha z^{-1-\alpha}\mathrm{d}z\right\},\quad u\in\mathbb{R}.$$

Let $N(\mathrm{d}s,\mathrm{d}z)$ be a Poisson random measure on $\mathbb{R}^2_{>0}$ with the intensity measure $C_\alpha z^{-1-\alpha}\mathbb{1}_{\{z>0\}}\mathrm{d}s\mathrm{d}z$ and $\hat{N}(\mathrm{d}s,\mathrm{d}z)$ be its compensator. Then the Lévy-Itô representation of L takes the form

$$L_t=\gamma t+\int_0^t\int_{\{|z|<1\}}z\tilde{N}(\mathrm{d}s,\mathrm{d}z)+\int_0^t\int_{\{|z|\geqslant 1\}}zN(\mathrm{d}s,\mathrm{d}z),\quad t\geqslant 0,\tag{2.2}$$

where $\gamma:=-\mathbb{E}\left[\int_0^1\int_{\{|z|\geqslant 1\}}zN(\mathrm{d}s,\mathrm{d}z)\right]$ and $\tilde{N}(\mathrm{d}s,\mathrm{d}z):=N(\mathrm{d}s,\mathrm{d}z)-\hat{N}(\mathrm{d}s,\mathrm{d}z)$ is the compensated Poisson random measure on $\mathbb{R}^2_{>0}$ that corresponds to $N(\mathrm{d}s,\mathrm{d}z)$. We remark that $\gamma t=\int_0^t\int_{\{|z|\geqslant 1\}}z\hat{N}(\mathrm{d}s,\mathrm{d}z)$ and

$$\int_0^t\int_{\{|z|\geqslant 1\}}zN(\mathrm{d}s,\mathrm{d}z)-\gamma t,\quad t\geqslant 0,$$

is thus a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. It follows from [3, Theorem 2.1] that if (Y_0,X_0) is independent of $(L_t,B_t)_{t\geqslant 0}$, then there is a unique strong solution $(Y_t,X_t)_{t\geqslant 0}$ of the stochastic differential equation (1.1) with

$$Y_t=e^{-bt}\left(Y_0+a\int_0^t e^{bs}\mathrm{d}s+\int_0^t e^{bs}\sqrt[\alpha]{Y_{s-}}\mathrm{d}L_s\right),$$

and

$$X_t=e^{-\theta t}\left(X_0+m\int_0^t e^{\theta s}\mathrm{d}s+\int_0^t e^{\theta s}\sqrt{Y_s}\mathrm{d}B_s\right)$$

for all $t\geqslant 0$. Moreover, $(Y_t,X_t)_{t\geqslant 0}$ is a regular affine process, and the infinitesimal generator $\mathcal{A}$ of (Y,X) is given by

$$\begin{aligned}(\mathcal{A}f)(y,x)&=(a-by)\tfrac{\partial}{\partial y}f(y,x)+(m-\theta x)\tfrac{\partial}{\partial x}f(y,x)+\tfrac{1}{2}y\tfrac{\partial^2}{\partial x^2}f(y,x)\\&\quad+y\int_0^\infty\left(f(y+z,x)-f(y,x)-z\tfrac{\partial}{\partial y}f(y,x)\right)C_\alpha z^{-1-\alpha}\mathrm{d}z,\end{aligned}\tag{2.3}$$

where $(y,x)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}$ and $f\in C_c^2(\mathbb{R}_{\geqslant 0}\times\mathbb{R},\mathbb{R})$.

3. Laplace transform of the α-root process Y

In this section we study the α-root process $(Y_t)_{t\geqslant 0}$ defined by

$$\mathrm{d}Y_t = (a - bY_t)\mathrm{d}t + \sqrt[\alpha]{Y_{t-}}\mathrm{d}L_t, \quad t \geqslant 0, \quad Y_0 \geqslant 0 \quad \text{a.s.,} \tag{3.1}$$

where $a \geqslant 0$, $b > 0$, $\alpha \in (1,2)$, $(L_t)_{t\geqslant 0}$ is a spectrally positive α-stable Lévy process with the Lévy measure $C_\alpha z^{-1-\alpha}\mathbb{1}_{\{z>0\}}\mathrm{d}z$. Without any further specification, we always assume that Y_0 is independent of $(L_t)_{t\geqslant 0}$.

We remark that we have allowed $a = 0$ in (3.1), which is different as in (1.1). In this case, the SDE (3.1) turns into

$$\mathrm{d}Y_t = -bY_t\mathrm{d}t + \sqrt[\alpha]{Y_{t-}}\mathrm{d}L_t, \quad t \geqslant 0, \quad Y_0 \geqslant 0 \quad \text{a.s.,} \tag{3.2}$$

and, by [11, Theorem 6.2 and Corollary 6.3], a unique strong solution of (3.2) also exists. The α-root process Y is thus well-defined for all $a \geqslant 0$. From now on and till the end of this section, we assume temporally that $a \geqslant 0$.

The solution of the stochastic differential equation (3.1) depends obviously on its initial value Y_0. From now on, we denote by $(Y_t^y)_{t\geqslant 0}$ the α-root process starting from a constant initial value $y \in \mathbb{R}_{\geqslant 0}$, i.e., $(Y_t^y)_{t\geqslant 0}$ satisfies

$$\mathrm{d}Y_t^y = (a - bY_t^y)\mathrm{d}t + \sqrt[\alpha]{Y_{t-}^y}\mathrm{d}L_t, \quad t \geqslant 0, \quad Y_0^y = y. \tag{3.3}$$

Since the α-root process is an affine process, the corresponding characteristic functions of $(Y_t^y)_{t\geqslant 0}$ are of affine form, namely,

$$\mathbb{E}\left[e^{uY_t^y}\right] = e^{\phi(t,u)+y\psi(t,u)}, \quad u \in \mathcal{U}_-. \tag{3.4}$$

The functions ϕ and ψ in turn are given as solutions of the generalized Riccati equations

$$\begin{cases} \frac{\partial}{\partial t}\phi(t,u) = F(\psi(t,u)), & \phi(0,u) = 0, \\ \frac{\partial}{\partial t}\psi(t,u) = R(\psi(t,u)), & \psi(0,u) = u \in \mathcal{U}_-, \end{cases} \tag{3.5}$$

with

$$F(u) = au \quad \text{and} \quad R(u) = -bu + \tfrac{(-u)^\alpha}{\alpha},$$

see [3, Theorem 3.1]. An equivalent equation for ψ (see (3.6) below) was studied in [3, Theorem 3.1]. In particular, it follows from [3, Theorem 3.1] that the equation (3.6) below has a unique solution. However, the explicit form of the solution to (3.6) has not been derived in [3]. In order to study the transition densities of the α-root process, we will find the explicit form of the solution to (3.6) in the following theorem.

Proposition 3.1. *Let $a \geqslant 0$, $b > 0$. Define $v_t(\lambda) := -\psi(t, -\lambda)$, $\lambda \in \mathbb{R}_{>0}$. Then $v_t(\lambda)$ solves the differential equation*

$$\begin{cases} \frac{\partial}{\partial t}v_t(\lambda) = -bv_t(\lambda) - \frac{1}{\alpha}(v_t(\lambda))^\alpha, & t \geqslant 0, \\ v_0(\lambda) = \lambda, \end{cases} \tag{3.6}$$

where $\lambda \in \mathbb{R}_{>0}$. The unique solution to (3.6) is given by

$$v_t(\lambda) = \left(\left(\frac{1}{\alpha b} + \lambda^{(1-\alpha)}\right)e^{b(\alpha-1)t} - \frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}, \quad t \geqslant 0. \tag{3.7}$$

Moreover, the Laplace transform of Y_t^y is given by

$$\begin{aligned}\mathbb{E}\left[e^{-\lambda Y_t^y}\right] &= \exp\left\{-a\int_0^t v_s(\lambda)\mathrm{d}s - yv_t(\lambda)\right\}\\ &= \exp\left\{-a\int_0^t \left(\left(\frac{1}{\alpha b}+\lambda^{(1-\alpha)}\right)e^{b(\alpha-1)s}-\frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}\mathrm{d}s\right.\\ &\qquad\left.-y\left(\left(\frac{1}{\alpha b}+\lambda^{(1-\alpha)}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}\right\} \end{aligned} \tag{3.8}$$

for all $t \geqslant 0$ and $\lambda \in \mathbb{R}_{>0}$.

Proof. The equation (3.6) is a Bernoulli differential equation which can be transformed into a linear differential equation through a change of variables. More precisely, if we write $u_t(\lambda) := (v_t(\lambda))^{1-\alpha}$, then

$$\begin{aligned}\tfrac{\partial}{\partial t}u_t(\lambda) &= (1-\alpha)\,(v_t(\lambda))^{-\alpha}\,\tfrac{\partial}{\partial t}v_t(\lambda)\\ &= (1-\alpha)\,(v_t(\lambda))^{-\alpha}\left(-bv_t(\lambda)-\tfrac{1}{\alpha}\,(v_t(\lambda))^{\alpha}\right)\\ &= b(\alpha-1)u_t(\lambda)+\left(1-\alpha^{-1}\right)\end{aligned} \tag{3.9}$$

and $u_0(\lambda) = (v_0(\lambda))^{1-\alpha} = \lambda^{1-\alpha}$. By solving (3.9), we obtain

$$u_t(\lambda) = \left(\frac{1}{\alpha b}+\lambda 1-\alpha\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b},$$

which leads to

$$v_t(\lambda) = \left(\left(\frac{1}{\alpha b}+\lambda^{1-\alpha}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}$$

for all $t \geqslant 0$ and $\lambda \in \mathbb{R}_{>0}$. By (3.4) and (3.5) and noting that $v_t(\lambda) = -\psi(t,-\lambda)$, we get

$$\begin{aligned}\mathbb{E}\left[e^{-\lambda Y_t^y}\right] &= \exp\left\{\phi(t,-\lambda)+y\psi(t,-\lambda)\right\}\\ &= \exp\left\{a\int_0^t \psi(s,-\lambda)\mathrm{d}s - yv_t(\lambda)\right\}\\ &= \exp\left\{-a\int_0^t v_s(\lambda)\mathrm{d}s - yv_t(\lambda)\right\}\end{aligned}$$

for all $t \geqslant 0$ and $\lambda \in \mathbb{R}_{>0}$. □

Let

$$\begin{aligned}\varphi_1(t,\lambda,y) &:= \exp\left\{-y\left(\left(\frac{1}{\alpha b}+\lambda^{(1-\alpha)}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}\right\},\\ \varphi_2(t,\lambda) &:= \exp\left\{-a\int_0^t\left(\left(\frac{1}{\alpha b}+\lambda^{(1-\alpha)}\right)e^{b(\alpha-1)s}-\frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}\mathrm{d}s\right\}.\end{aligned}$$

Then

$$\mathbb{E}\left[e^{-\lambda Y_t^y}\right] = \varphi_1(t,\lambda,y)\cdot\varphi_2(t,\lambda). \tag{3.10}$$

Keeping this decomposition of the Laplace transform of Y_t^y in mind, we take a closer look at the following two special cases:

3.1. **Special case i):** $a = 0$**.** To avoid abuse of notations, we use $(Z_t^y)_{t\geqslant 0}$ to denote the strong solution of the stochastic differential equation

$$\mathrm{d}Z_t^y = -bZ_t^y\mathrm{d}t + \sqrt[\alpha]{Z_{t-}^y}\mathrm{d}L_t, \quad t \geqslant 0, \quad Z_0^y = y \geqslant 0.$$

According to (3.8), the corresponding Laplace transform of Z_t^y coincides with $\varphi_1(t, \lambda, y)$. Noting that $b > 0$, we get

$$\lim_{\lambda\to\infty} v_t(\lambda) = \left(\frac{1}{\alpha b}\left(e^{b(\alpha-1)t} - 1\right)\right)^{\frac{1}{1-\alpha}} =: d > 0 \tag{3.11}$$

for all $t > 0$. Furthermore, by dominated convergence theorem, we have

$$\begin{aligned} e^{-yd} &= \lim_{\lambda\to\infty} e^{-yv_t(\lambda)} = \lim_{\lambda\to\infty} \mathbb{E}\left[e^{-\lambda Z_t^y}\right] \\ &= \lim_{\lambda\to\infty}\left(\mathbb{E}\left[e^{-\lambda Z_t^y}\mathbb{1}_{\{Z_t^y=0\}}\right] + \mathbb{E}\left[e^{-\lambda Z_t^y}\mathbb{1}_{\{Z_t^y>0\}}\right]\right) \\ &= \mathbb{P}\left(Z_t^y = 0\right) > 0 \end{aligned} \tag{3.12}$$

for all $t > 0$ and $y \geqslant 0$.

3.2. **Special case ii):** $y = 0$**.** Consider $(Y_t^0)_{t\geqslant 0}$ that satisfies

$$\mathrm{d}Y_t^0 = (a - bY_t^0)\mathrm{d}t + \sqrt[\alpha]{Y_{t-}^0}\mathrm{d}L_t, \quad t \geqslant 0, \quad Y_0^0 = 0. \tag{3.13}$$

In view of (3.8), we easily see that the Laplace transform of Y_t^0 equals $\varphi_2(t, \lambda)$.

4. Transition densities of the α-root process Y

In this section we show that the α-root process Y has positive and continuous transition densities. Our approach is essentially based on the inverse Fourier transform.

Recall that the function $v_t(\cdot)$ given by (3.7) is defined on $\mathbb{R}_{>0}$. By considering the complex power functions, the domain of definition for $v_t(\cdot)$ can be extended to $\mathbb{C} \setminus \{0\}$. Indeed, the function

$$v_t(z) = \left(\left(\frac{1}{\alpha b} + z^{(1-\alpha)}\right)e^{b(\alpha-1)t} - \frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}, \quad z \in \mathbb{C}\setminus\{0\}, \tag{4.1}$$

is well-defined, where the complex power function is given by (2.1).

We next establish two estimates on $\int_0^t v_s(z)\mathrm{d}s$. Since the proofs are of pure analytic nature, we put them in the appendix.

Lemma 4.1. *Let $T > 1$. Then there exists a sufficiently small constant $\varepsilon_0 > 0$ such that*

$$\mathrm{Re}\left(\int_0^t v_s(z)\mathrm{d}s\right) \geqslant -C_1 + C_2|z|^{2-\alpha} \tag{4.2}$$

when $|\mathrm{Arg}(z)| \in [\pi/2 - \varepsilon_0, \pi/2 + \varepsilon_0]$ and $T^{-1} \leqslant t \leqslant T$, where $C_1, C_2 > 0$ are constants depending only on a, b, α, ε_0 and T.

Proof. See the appendix. □

Lemma 4.2. *Let ε_0 be as in the previous lemma. Then for each $t \geqslant 0$, we can find constants C_3, $C_4 > 0$, which depend only on a, b, α, ε_0 and t, such that*

$$\left| \int_0^t v_s(z) \mathrm{d}s \right| \leqslant C_3 + C_4 |z|^{2-\alpha}$$

when $\mathrm{Arg}(z) \in [\pi/2 + \varepsilon_0, \pi]$ *and* $|z| \geqslant 2$.

Proof. See the appendix. □

Now, consider the process $(Y_t^0)_{t \geqslant 0}$ given by (3.13). As shown in [9, p. 257], the function

$$\mathbb{E}\left[\exp\left(-uY_t^0\right)\right], \quad u \in \mathcal{U}_+,$$

is continuous on $\mathcal{U}_+$ and holomorphic on $\mathcal{U}_+^\circ$. On the other hand, the function $z \mapsto v_t(z)$ given in (4.1) is continuous on $\mathcal{U}_+$ and holomorphic on $\mathcal{U}_+^\circ$ for each $t \geqslant 0$. Therefore, we have

$$\mathbb{E}\left[e^{-uY_t^0}\right] = \exp\left\{-a \int_0^t v_s(u) \mathrm{d}s\right\}, \quad u \in \mathcal{U}_+. \tag{4.3}$$

Indeed, the equality (4.3) is true at least for $u \in \mathbb{R}_{>0}$ by (3.8). This and the identity theorem for holomorphic functions (see e.g. [10, Theorem III.3.2]) imply (4.3) for all $u \in \mathcal{U}_+$, since both sides of (4.3) are functions that are continuous on $\mathcal{U}_+$ and holomorphic on $\mathcal{U}_+^\circ$. In particular, the characteristic function of Y_t^0 with $t > 0$ is given by

$$\mathbb{E}\left[e^{i\xi Y_t^0}\right] = \exp\left\{-a \int_0^t v_s(i\xi) \mathrm{d}s\right\}, \quad \xi \in \mathbb{R}.$$

In the next lemma we obtain the existence of a density function for Y_t^0 when $t > 0$. Note that by [3, Theorem 1.1], we have $Y_t^0 \geqslant 0$ a.s. for each $t \geqslant 0$.

Lemma 4.3. *Assume $a > 0$ and $b > 0$. Then for each $t > 0$, Y_t^0 possesses a density function $f_{Y_t^0}$ given by*

$$f_{Y_t^0}(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \exp\left\{-a \int_0^t v_s(-i\xi) \mathrm{d}s\right\} \mathrm{d}\xi, \quad x \geqslant 0. \tag{4.4}$$

Moreover, the function $f_{Y_t^0}(x)$ is jointly continuous in $(t, x) \in (0, \infty) \times \mathbb{R}_{\geqslant 0}$, and $f_{Y_t^0}(\cdot) \in C^\infty(\mathbb{R}_{\geqslant 0})$ for each $t > 0$.

Proof. Let $T > 1$ be fixed. By Lemma 4.1, there exist constants c_1, $c_2 > 0$ such that

$$\left| \exp\left\{-a \int_0^t v_s(-i\xi) \mathrm{d}s\right\} \right| = \exp\left\{\mathrm{Re}\left(-a \int_0^t v_s(-i\xi) \mathrm{d}s\right)\right\} \leqslant c_1 e^{-c_2 |\xi|^{2-\alpha}} \tag{4.5}$$

for all $\xi \in \mathbb{R}$ and $t \in [1/T, T]$, which implies that $\xi \mapsto \exp\{-a \int_0^t v_s(-i\xi) \mathrm{d}s\}$ is integrable on $\mathbb{R}$. Therefore, by the inversion formula of Fourier transform, Y_t^0 has a density $f_{Y_t^0}$ given by (4.4). The joint continuity of the density $f_{Y_t^0}(x)$ in (t, x) follows from (4.5), (4.4) and dominated convergence theorem. The smoothness property of $f_{Y_t^0}(\cdot)$ is a consequence of (4.5) and [27, Proposition 28.1]. □

We remark that for each $t > 0$, the function $f_{Y_t^0}(x)$ given in (4.4) is actually well-defined also for $x < 0$, although $f_{Y_t^0}(x) \equiv 0$ for $x \leqslant 0$, which is due to the fact that $Y_t^0 \geqslant 0$ a.s.. Next, we would like to know if $f_{Y_t^0}(x) > 0$ when $x > 0$. The next lemma partly answers this question.

Lemma 4.4. *For each $t > 0$, the density function $f_{Y_t^0}(\cdot)$ of Y_t^0 is almost everywhere positive on $\mathbb{R}_{\geqslant 0}$.*

Proof. Basically, the idea of the proof is as follows. We will show the following:

Claim. *The function*

$$x \mapsto f_{Y_t^0}(x), \quad x \in \mathbb{R}_{>0},$$

can be extended to a holomorphic function on $\mathcal{U}_+^\circ$.

If this claim is true, then the set $A_n := \{x > 1/n : f_{Y_t^0}(x) = 0\}$ with $n \in \mathbb{N}$ must be discrete, that is, for each $x \in A_n$, one can find a neighbourhood of x whose intersection with A_n equals x; otherwise the identity theorem for holomorphic functions implies that $f_{Y_t^0}(x) \equiv 0$ for $x > 0$. As a consequence, A_n is countable, which implies that $A := \cup_{n\in\mathbb{N}} A_n$ is also countable and thus has Lebesgue measure 0.

Let $x > 0$ be fixed. We will complete the proof of the above claim in several steps.

"*Step 1*": We derive a simpler representation for $f_{Y_t^0}(x)$. We have

$$\begin{aligned} f_{Y_t^0}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi} \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{0}^{\infty} e^{-ix\xi} \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\mathrm{d}\xi \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{0} e^{-ix\xi} \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{ix\xi} \exp\left\{-a\int_0^t v_s(i\xi)\mathrm{d}s\right\}\mathrm{d}\xi \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{0} e^{-ix\xi} \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\mathrm{d}\xi. \end{aligned} \tag{4.6}$$

For $\xi < 0$, we have

$$\begin{aligned} \overline{v_s(-i\xi)} &= \left(\left(\tfrac{1}{\alpha b} + \overline{(-i\xi)^{1-\alpha}}\right) e^{b(\alpha-1)s} - \tfrac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}} \\ &= \left(\left(\tfrac{1}{\alpha b} + (i\xi)^{1-\alpha}\right) e^{b(\alpha-1)s} - \tfrac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}} = v_s(i\xi), \end{aligned}$$

which implies

$$\overline{e^{-ix\xi}\exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}} = e^{ix\xi}\exp\left\{-a\int_0^t v_s(i\xi)\mathrm{d}s\right\}. \tag{4.7}$$

By (4.6) and (4.7), we get

$$f_{Y_t^0}(x) = \mathrm{Re}\left(\frac{1}{\pi}\int_{-\infty}^{0} e^{-ix\xi}\exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\mathrm{d}\xi\right). \tag{4.8}$$

For simplicity, let

$$I := \frac{1}{\pi}\int_{-\infty}^{0} e^{-ix\xi}\exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\mathrm{d}\xi. \tag{4.9}$$

"*Step 2*": We calculate I by contour integration. By a change of variables $z := -i\xi$, we get

$$I = \frac{-i}{\pi}\int_0^{i\infty} e^{xz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z$$

$$= \lim_{K\to\infty}\frac{-i}{\pi}\int_{iK^{-1}}^{iK} e^{xz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z. \tag{4.10}$$

Define two paths $\Gamma_{1,K}$ and $\Gamma_{2,K}$ by

$$\Gamma_{1,K}(\vartheta) := Ke^{i\vartheta}, \quad \vartheta\in\left[\frac{\pi}{2},\pi\right] \quad \text{and} \quad \Gamma_{2,K}(\vartheta) := K^{-1}e^{i\vartheta}, \quad \vartheta\in\left[\frac{\pi}{2},\pi\right].$$

According to (4.1), we see that the function

$$z\mapsto e^{yz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}, \quad z\in\mathcal{O}_1 := \left\{\rho\exp(i\vartheta) : \rho>0, \vartheta\in\left[\frac{\pi}{2},\pi\right]\right\},$$

can be extended to a holomorphic function on $\mathcal{O}_2 := \{\rho\exp(i\vartheta) : \rho > 0, \vartheta\in(0,3\pi/2)\}$. Therefore, we have

(4.11)

$$\int_{iK^{-1}}^{iK} e^{xz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z$$
$$= \int_{-K^{-1}}^{-K} e^{xz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z - \int_{\Gamma_{1,K}} e^{xz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z$$
$$+ \int_{\Gamma_{2,K}} e^{xz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z.$$

Since $\lim_{z\to 0} e^{xz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\} = 1$, it follows that

$$\lim_{K\to\infty}\int_{\Gamma_{2,K}} e^{xz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z\Big) = 0. \tag{4.12}$$

To estimate the second term on the right-hand side of (4.11), we divide the path $\Gamma_{1,K}$ into two parts, namely

$$\Gamma_{11,K}(\vartheta) := Ke^{i\vartheta}, \quad \vartheta\in\left[\frac{\pi}{2},\frac{\pi}{2}+\varepsilon_0\right] \quad \text{and} \quad \Gamma_{12,K}(\vartheta) =: Ke^{i\vartheta}, \quad \vartheta\in\left[\frac{\pi}{2}+\varepsilon_0,\pi\right],$$

with $\varepsilon_0 > 0$ being the constant appearing in Lemmas 4.1 and 4.2. Then

$$\int_{\Gamma_{1,K}} e^{xz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z$$
$$= \int_{\Gamma_{11,K}} e^{xz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z + \int_{\Gamma_{12,K}} e^{xz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z$$
$$:= II_1(K) + II_2(K).$$

If we can show that $\lim_{K\to\infty} II_1(K) = 0$ and $\lim_{K\to\infty} II_2(K) = 0$, then it follows from (4.10), (4.11) and (4.12) that

$$I = \frac{-i}{\pi}\int_0^{-\infty} e^{xz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z. \tag{4.13}$$

"*Step 3*": We show that $\lim_{K\to\infty} II_1(K) = 0$. If $\vartheta \in [\pi/2, \pi/2+\varepsilon_0]$, then

$$\left| e^{xKe^{i\vartheta}} \right| = e^{\mathrm{Re}\left(xKe^{i\vartheta}\right)} = e^{xK\cos(\vartheta)} \leqslant 1.$$

By Lemma 4.1, we get

$$\begin{aligned} |II_1(K)| &= \left| \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\varepsilon_0} iKe^{i\vartheta} e^{xKe^{i\vartheta}} e^{-a\int_0^t v_s\left(Ke^{i\vartheta}\right)\mathrm{d}s} \mathrm{d}\vartheta \right| \\ &\leqslant K \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\varepsilon_0} \left| e^{-a\int_0^t v_s\left(Ke^{i\vartheta}\right)\mathrm{d}s} \right| \mathrm{d}\vartheta \leqslant K\varepsilon_0 e^{aC_1 - aC_2K^{2-\alpha}}, \end{aligned} \tag{4.14}$$

which implies

$$\lim_{K\to\infty} |II_1(K)| \leqslant \lim_{K\to\infty} K\varepsilon_0 e^{aC_1-aC_2K^{2-\alpha}} = 0.$$

"*Step 4*": We show that $\lim_{K\to\infty} II_2(K) = 0$. In case $\vartheta \in [\pi/2+\varepsilon_0, \pi]$, then

$$\left| e^{xKe^{i\vartheta}} \right| = e^{\mathrm{Re}\left(xKe^{i\vartheta}\right)} = e^{xK\cos(\vartheta)} \leqslant e^{xK\cos\left(\frac{\pi}{2}+\varepsilon_0\right)} = e^{-xK\sin(\varepsilon_0)}. \tag{4.15}$$

So

$$\begin{aligned} |II_2(K)| &= \left| \int_{\frac{\pi}{2}+\varepsilon_0}^{\pi} iKe^{i\vartheta} e^{xKe^{i\vartheta}} \exp\left\{ -a \int_0^t v_s\left(Ke^{i\vartheta}\right)\mathrm{d}s \right\} \mathrm{d}\vartheta \right| \\ &\leqslant K \int_{\frac{\pi}{2}+\varepsilon_0}^{\pi} \left| e^{xKe^{i\vartheta}} \right| \left| \exp\left\{ -a \int_0^t v_s\left(Ke^{i\vartheta}\right)\mathrm{d}s \right\} \right| \mathrm{d}\vartheta \\ &\leqslant Ke^{-xK\sin(\varepsilon_0)} \int_{\frac{\pi}{2}+\varepsilon_0}^{\pi} \exp\left\{ a \left| \int_0^t v_s\left(Ke^{i\vartheta}\right)\mathrm{d}s \right| \right\} \mathrm{d}\vartheta. \end{aligned}$$

By Lemma 4.2, we get

$$\lim_{K\to\infty} |II_2(K)| \leqslant \lim_{K\to\infty} K\left(\tfrac{\pi}{2}-\varepsilon_0\right) e^{-xK\sin(\varepsilon_0)} e^{aC_3} e^{aC_4K^{2-\alpha}} = 0.$$

"*Step 5*": By (4.8), (4.9) and (4.13), we get

$$\begin{aligned} f_{Y_t^0}(x) &= \mathrm{Re}\left(\frac{-i}{\pi} \int_0^{-\infty} e^{xz} \exp\left\{ -a \int_0^t v_s(z)\mathrm{d}s \right\} \mathrm{d}z \right) \\ &= \mathrm{Re}\left(\frac{i}{\pi} \int_0^{\infty} e^{-xz} \exp\left\{ -a \int_0^t v_s(-z)\mathrm{d}s \right\} \mathrm{d}z \right) \\ &= -\mathrm{Im}\left(\frac{1}{\pi} \int_0^{\infty} e^{-xz} \exp\left\{ -a \int_0^t v_s(-z)\mathrm{d}s \right\} \mathrm{d}z \right) \\ &= \frac{1}{\pi} \int_0^{\infty} e^{-xz} \left\{ -\mathrm{Im}\left(\exp\left\{ -a \int_0^t v_s(-z)\mathrm{d}s \right\} \right) \right\} \mathrm{d}z. \end{aligned}$$

Let $x_0 > 0$ be fixed. By Lemma 4.2, for $z \in \mathbb{R}_{\geqslant 0}$ and $x \in \mathbb{C}$ with $\mathrm{Re}(x) \geqslant x_0$, we have

$$\left| ze^{-xz}\mathrm{Im}\left(\exp\left\{ -a \int_0^t v_s(-z)\mathrm{d}s \right\} \right) \right| \leqslant ze^{-\mathrm{Re}(xz)} \left| \exp\left\{ -a \int_0^t v_s(-z)\mathrm{d}s \right\} \right|$$

$$\leqslant ze^{-x_0 z}\left|\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\right|$$

$$\leqslant ze^{-x_0 z}\exp\left\{aC_3 + aC_4|z|^{2-\alpha}\right\}, \tag{4.16}$$

where the right-hand side of (4.16) is an integrable function (with the variable z) on $\mathbb{R}_{\geqslant 0}$. By Lebesgue differential theorem, we see that the function

$$x \mapsto \frac{1}{\pi}\int_0^\infty e^{-xz}\left\{-\mathrm{Im}\left(\exp\left\{-a\int_0^t v_s(-z)\mathrm{d}s\right\}\right)\right\}\mathrm{d}z, \quad x \in \mathcal{U}_+^{\mathrm{o}},$$

is holomorphic, which means that $x \mapsto f_{Y_t^0}(x)$ has a holomorphic extension on $\mathcal{U}_+^{\mathrm{o}}$. This completes the proof. □

With the help of the previous lemma, we are now able to prove the main result of this section. Recall that the process $(Y_t^y)_{t\geqslant 0}$ is given by (3.3).

Proposition 4.5. *Assume $a > 0$ and $b > 0$. Then for each $y \geqslant 0$ and $t > 0$, Y_t^y possesses a density function $f_{Y_t^y}$ given by*

$$f_{Y_t^y}(x) := \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ix\xi}\exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s - yv_t(-i\xi)\right\}\mathrm{d}\xi, \quad x \geqslant 0, \tag{4.17}$$

where $f_{Y_t^y}(\cdot) \in C^\infty(\mathbb{R}_{\geqslant 0})$ and $f_{Y_t^y}(x) > 0$ for all $x > 0$. Moreover, the function $f_{Y_t^y}(x)$ is jointly continuous in $(t, y, x) \in (0,\infty)\times\mathbb{R}_{\geqslant 0}\times\mathbb{R}_{\geqslant 0}$.

Proof. In view of (3.8) and (3.10), we have

$$\mathbb{E}\left[e^{i\xi Y_t^y}\right] = \mathbb{E}\left[e^{i\xi Y_t^0}\right]\cdot\mathbb{E}\left[e^{i\xi Z_t^y}\right] = \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s - yv_t(-i\xi)\right\}, \tag{4.18}$$

where $\xi \in \mathbb{R}$. It follows from (4.5) that

$$\left|\mathbb{E}\left[e^{i\xi Y_t^y}\right]\right| \leqslant \left|\mathbb{E}\left[e^{i\xi Y_t^0}\right]\right| \leq c_1 e^{-c_2|\xi|^{2-\alpha}}$$

for all $\xi \in \mathbb{R}$ and $t \in [1/T, T]$, where $T > 1$ and c_1, $c_2 > 0$ are constants depending on T. It follows that for $t > 0$, Y_t^y has a density $f_{Y_t^y}$ given by (4.17). Proceeding in the same way as in Lemma 4.3, we obtain the desired continuity and smoothness properties of $f_{Y_t^y}$.

We next show that if $t > 0$, then $f_{Y_t^y}(x) > 0$ for all $x > 0$. According to (4.18), we see that the law of Y_t^y, denoted by $\mu_{Y_t^y}$, is the convolution of the laws of Z_t^y and Y_t^0, which we denote by $\mu_{Z_t^y}$ and $\mu_{Y_t^0}$, respectively. So $\mu_{Y_t^y} = \mu_{Z_t^y} * \mu_{Y_t^0}$. From this we deduce that for all $x > 0$,

$$f_{Y_t^y}(x) = \int_{\mathbb{R}_{\geqslant 0}} f_{Y_t^0}(x-z)\mu_{Z_t^y}(\mathrm{d}z)$$

$$= \int_{(0,\infty)} f_{Y_t^0}(x-z)\mu_{Z_t^y}(\mathrm{d}z) + f_{Y_t^0}(x)\mu_{Z_t^y}(\{0\}). \tag{4.19}$$

By Lemma 4.4, the density function $f_{Y_t^0}(x)$ of Y_t^0 is strictly positive for almost all $x > 0$. In the following we consider a fixed $x > 0$ and distinguish between two cases.

"*Case 1*": $f_{Y_t^0}(x) > 0$. It follows from (4.19) that

$$f_{Y_t^y}(x) \geqslant f_{Y_t^0}(x)\mu_{Z_t^y}(\{0\}) > 0, \tag{4.20}$$

where we used the fact that $\mu_{Z_t^y}(\{0\}) = \mathbb{P}(Z_t^y = 0) > 0$, as shown in (3.12).

"*Case 2*": $f_{Y_t^0}(x) = 0$. Then $x \in A_n$ for a large enough n, where the set A_n is the same as in the proof of Lemma 4.4. Since A_n is discrete, we can find a small enough $\delta > 0$ such that

$$f_{Y_t^0}(x - z) > 0, \tag{4.21}$$

for all $z \in (0, \delta]$. We next show that $\mu_{Z_t^y}((0, \delta]) > 0$. By (3.11), (3.12) and L'Hospital's Rule, we get

$$\begin{aligned}
&\lim_{\lambda\to\infty} \left(\mathbb{E}\left[e^{-\lambda(Z_t^y - \delta)}\right] - \mathbb{E}\left[e^{-\lambda(Z_t^y-\delta)} \mathbb{1}_{\{Z_t^y = 0\}}\right] \right) \\
&\qquad = \lim_{\lambda\to\infty} e^{\lambda\delta} \left(\mathbb{E}\left[e^{-\lambda Z_t^y}\right] - \mathbb{P}(Z_t^y = 0) \right) \\
&\qquad = \lim_{\lambda\to\infty} e^{\lambda\delta} \left(e^{-yv_t(\lambda)} - e^{-yd} \right) \\
&\qquad = \lim_{\lambda\to\infty} \delta^{-1} e^{\lambda\delta} y e^{-yv_t(\lambda)} \left(v_t(\lambda)\right)^{\alpha} e^{b(\alpha-1)t} \lambda^{-\alpha} = \infty.
\end{aligned} \tag{4.22}$$

Suppose that $\mathbb{P}(Z_t^y \in (0, \delta]) = 0$. Then we can use dominated convergence theorem to get

$$\begin{aligned}
&\lim_{\lambda\to\infty} \left(\mathbb{E}\left[e^{-\lambda(Z_t^y - \delta)}\right] - \mathbb{E}\left[e^{-\lambda(Z_t^y-\delta)} \mathbb{1}_{\{Z_t^y = 0\}}\right] \right) \\
&\quad = \lim_{\lambda\to\infty} \left(\mathbb{E}\left[e^{-\lambda(Z_t^y - \delta)} \mathbb{1}_{\{0 < Z_t^y \leqslant \delta\}}\right] + \mathbb{E}\left[e^{-\lambda(Z_t^y-\delta)} \mathbb{1}_{\{Z_t^y > \delta\}}\right] \right) = 0,
\end{aligned}$$

which contradicts (4.22). Consequently, the assumption that $\mathbb{P}(Z_t^y \in (0, \delta]) = 0$ is not true and we thus get $\mathbb{P}(Z_t^y \in (0, \delta]) > 0$. Now, by (4.19) and (4.21), we get

$$f_{Y_t^y}(x) \geqslant \int_{(0,\delta]} f_{Y_t^0}(x - z) \mu_{Z_t^y}(\mathrm{d}z) > 0. \tag{4.23}$$

Summarizing the above two cases, we have $f_{Y_t^y}(x) > 0$ for all $x > 0$. This completes the proof. ☐

5. A Foster-Lyapunov function for (Y, X)

We now turn back to the two-dimensional affine process $(Y, X) = (Y_t, X_t)_{t \geqslant 0}$ defined in (1.1). Our aim of this section is to construct a Foster-Lyapunov function for (Y, X).

For a functional $\Phi(Y, X)$ based on the process (Y, X), we use $\mathbb{E}_{(y,x)}[\Phi(Y, X)]$ to indicate that the process (Y, X) considered under the expectation is with the initial condition $(Y_0, X_0) = (y, x)$, where $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$ is constant. The notation $\mathbb{P}_{(y,x)}(\Phi(Y, X) \in \cdot)$ is similarly defined.

Lemma 5.1. *Let $h \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $h(x) \geqslant 1$ for all $x \in \mathbb{R}$ and $h(x) = |x|$ whenever $|x| \geqslant 2$. Define*

$$V(y, x) := \beta y + h(x), \quad y \geqslant 0,\ x \in \mathbb{R},$$

where $\beta > 0$ is a constant. If β is sufficiently large, then V is a Foster-Lyapunov function for (Y, X), that is, there exist constants $c, M > 0$ such that

$$\mathbb{E}_{(y,x)}[V(Y_t, X_t)] \leqslant e^{-ct} V(y, x) + \tfrac{M}{c} \tag{5.1}$$

for all $(y,x)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}$ *and* $t\geqslant 0$.

Proof. Define $g(t,y,x):=\exp\{ct\}V(y,x)$, where $c>0$ is a constant to be determined later. It is easy to see that $g\in C^2(\mathbb{R}_{\geqslant 0}\times\mathbb{R}_{\geqslant 0}\times\mathbb{R},\mathbb{R})$. We define the functions g_1', g_2', g_3' and $g_{3,3}''$ by

$$g_1'(t,y,x):=\tfrac{\partial}{\partial t}g(t,y,x)=ce^{ct}V(y,x),\quad g_2'(t,y,x):=\tfrac{\partial}{\partial y}g(t,y,x)=\beta e^{ct},$$
$$g_3'(t,y,x):=\tfrac{\partial}{\partial x}g(t,y,x)=e^{ct}\tfrac{\partial}{\partial x}h(x),\quad g_{3,3}''(t,y,x):=\tfrac{\partial^2}{\partial x^2}g(t,y,x)=e^{ct}\tfrac{\partial^2}{\partial x^2}h(x).$$

If the process $(Y_t,X_t)_{t\geqslant 0}$ starts from (y,x), i.e., $(Y_0,X_0)=(y,x)$, then we can use the Lévy-Itô decomposition of $(L_t)_{t\geqslant 0}$ in (2.2) to obtain that for each $t\geqslant 0$,

$$\begin{cases} Y_t=y+\int_0^t\gamma\sqrt[\alpha]{Y_s}\mathrm{d}s+\int_0^t(a-bY_s)\mathrm{d}s\\ \qquad+\int_0^t\int_{\{|z|<1\}}z\sqrt[\alpha]{Y_{s-}}\tilde{N}(\mathrm{d}s,\mathrm{d}z)+\int_0^t\int_{\{|z|\geqslant 1\}}z\sqrt[\alpha]{Y_{s-}}N(\mathrm{d}s,\mathrm{d}z),\\ X_t=x+\int_0^t(m-\theta X_s)\mathrm{d}s+\int_0^t\sqrt{Y_s}\mathrm{d}B_s,\end{cases}\tag{5.2}$$

where γ, $N(\mathrm{d}s,\mathrm{d}z)$ and $\tilde{N}(\mathrm{d}s,\mathrm{d}z)$ are as in (2.2). By (5.2) and applying Itô's formula for g (see [28, Theorem 94]), we obtain that for each $t\geqslant 0$,

$$\begin{aligned}
&g(t,Y_t,X_t)-g(0,Y_0,X_0)\\
&=\int_0^t g_1'(s,Y_s,X_s)\mathrm{d}s+\int_0^t g_2'(s,Y_s,X_s)\gamma\sqrt[\alpha]{Y_s}\mathrm{d}s\\
&\quad+\int_0^t g_2'(s,Y_s,X_s)(a-bY_s)\mathrm{d}s+\int_0^t g_3'(s,Y_s,X_s)(m-\theta X_s)\mathrm{d}s\\
&\quad+\frac12\int_0^t g_{3,3}''(s,Y_s,X_s)Y_s\mathrm{d}s+\int_0^t g_3'(s,Y_s,X_s)\sqrt{Y_s}\mathrm{d}B_s\\
&\quad+\int_0^t\int_{\{|z|<1\}}\Big(g(s,Y_{s-}+z\sqrt[\alpha]{Y_{s-}},X_{s-})-g(s,Y_{s-},X_{s-})\Big)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\\
&\quad+\int_0^t\int_{\{|z|\geqslant 1\}}\Big(g(s,Y_{s-}+z\sqrt[\alpha]{Y_{s-}},X_{s-})-g(s,Y_{s-},X_{s-})\Big)N(\mathrm{d}s,\mathrm{d}z)\\
&\quad+\int_0^t\int_{\{|z|<1\}}\Big(g(s,Y_s+z\sqrt[\alpha]{Y_s},X_s)\\
&\qquad\qquad-g(s,Y_s,X_s)-z\sqrt[\alpha]{Y_s}g_2'(s,Y_s,X_s)\Big)C_\alpha z^{-1-\alpha}\mathrm{d}s\mathrm{d}z
\end{aligned}$$

$$=\int_0^t(\mathcal{L}g)(s,Y_s,X_s)\mathrm{d}s+\int_0^t g_1'(s,Y_s,X_s)\mathrm{d}s+M_t(g),\tag{5.3}$$

where

$$\begin{aligned}
M_t(g):=&\int_0^t g_3'(s,Y_s,X_s)\sqrt{Y_s}\mathrm{d}B_s\\
&+\int_0^t\int_{\{|z|<1\}}\Big(g(s,Y_{s-}+z\sqrt[\alpha]{Y_{s-}},X_{s-})-g(s,Y_{s-},X_{s-})\Big)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\\
&+\int_0^t\int_{\{|z|\geqslant 1\}}\Big(g(s,Y_{s-}+z\sqrt[\alpha]{Y_{s-}},X_{s-})-g(s,Y_{s-},X_{s-})\Big)N(\mathrm{d}s,\mathrm{d}z)\\
&-\int_0^t\int_{\{|z|\geqslant 1\}}\Big(g(s,Y_s+z\sqrt[\alpha]{Y_s},X_s)-g(s,Y_s,X_s)\Big)\hat{N}(\mathrm{d}s,\mathrm{d}z)
\end{aligned}$$

and $\mathcal{L}g$ is defined by

$$
\begin{aligned}
(\mathcal{L}g)(t,y,x) := (a-by)g_2'(t,y,x) + (m-\theta x)g_3'(t,y,x) + \frac{1}{2}yg_{3,3}''(t,y,x) \\
+ \int_{\{|z|<1\}} \left(g(t,y+z\sqrt[\alpha]{y},x) - g(t,y,x) - z\sqrt[\alpha]{y}g_2'(t,y,x)\right) C_\alpha z^{-1-\alpha}\mathrm{d}z \\
+ \int_{\{|z|\geqslant 1\}} \left(g(t,y+z\sqrt[\alpha]{y},x) - g(t,y,x)\right) C_\alpha z^{-1-\alpha}\mathrm{d}z + \gamma\sqrt[\alpha]{y}g_2'(t,y,x)
\end{aligned}
$$

for $(t,y,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \times \mathbb{R}$. By a change of variable $\tilde{z} := z\sqrt[\alpha]{y}$ and an easy computation, we see that $\mathcal{L}g = \mathcal{A}g$, where $\mathcal{A}$ is given in (2.3). As a result, it follows from (5.3) that for each $t \geqslant 0$,

$$
\begin{aligned}
& g(t,Y_t,X_t) - g(0,Y_0,X_0) \\
& \quad = \int_0^t (\mathcal{A}g)(s,Y_s,X_s)\mathrm{d}s + \int_0^t g_1'(s,Y_s,X_s)\mathrm{d}s + M_t(g).
\end{aligned}
\tag{5.4}
$$

The rest of the proof is divided into three steps:

"*Step 1*": We show that $(M_t(g))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$, where $(\mathcal{F}_t)_{t\geqslant 0}$ is the same as in Sect. 2. To achieve this, we can use the same argument as in [3]. Define

$$
\begin{aligned}
M_t^1(g) &:= \int_0^t g_3'(s,Y_s,X_s)\sqrt{Y_s}\mathrm{d}B_s, \\
M_t^2(g) &:= \int_0^t \int_{\{|z|<1\}} \Big(g(s,Y_{s-}+z\sqrt[\alpha]{Y_{s-}},X_{s-}) - g(s,Y_{s-},X_{s-})\Big)\tilde{N}(\mathrm{d}s,\mathrm{d}z), \\
&\quad + \int_0^t \int_{\{|z|\geqslant 1\}} \Big(g(s,Y_{s-}+z\sqrt[\alpha]{Y_{s-}},X_{s-}) - g(s,Y_{s-},X_{s-})\Big) N(\mathrm{d}s,\mathrm{d}z) \\
&\quad - \int_0^t \int_{\{|z|\geqslant 1\}} \Big(g(s,Y_s+z\sqrt[\alpha]{Y_s},X_s) - g(s,Y_s,X_s)\Big)\hat{N}(\mathrm{d}s,\mathrm{d}z),
\end{aligned}
$$

where $t \geqslant 0$. By noting that g_2' and g_3' are both bounded, we can proceed in the same way as in [3, Theorem 2.1] to prove that $(M_t^1(g))_{t\geqslant 0}$,

$$
\begin{aligned}
M_t^{3,n}(g) := \int_0^t \int_{\{|z|<1\}} \bigg(& g(s,Y_{s-}\wedge n + z\sqrt[\alpha]{Y_{s-}\wedge n},X_{s-}) \\
& - g(s,Y_{s-}\wedge n,X_{s-})\bigg)\tilde{N}(\mathrm{d}s,\mathrm{d}z), \quad t \geqslant 0, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
M_t^{4,n}(g) := \int_0^t \int_{\{|z|\geqslant 1\}} \bigg(& g(s,Y_{s-}\wedge n + z\sqrt[\alpha]{Y_{s-}\wedge n},X_{s-}) \\
& - g(s,Y_{s-}\wedge n,X_{s-})\bigg)N(\mathrm{d}s,\mathrm{d}z) \\
- \int_0^t \int_{\{|z|\geqslant 1\}} \bigg(& (g(s,Y_s\wedge n + z\sqrt[\alpha]{Y_s\wedge n},X_s) \\
& - g(s,Y_s\wedge n,X_s)\bigg)\hat{N}(\mathrm{d}s,\mathrm{d}z), \quad t \geqslant 0,
\end{aligned}
$$

are all martingales with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$, where $n\in\mathbb{N}$ is arbitrary. We omit the details here. For each $n\in\mathbb{N}$, define

$$\eta_t^n(g) := M_t^2(g) - M_t^{3,n}(g) - M_t^{4,n}(g), \quad t\geqslant 0. \tag{5.5}$$

Noting that $g(s,y+z,x)-g(s,y,x)=\beta z\exp(ct)$, we get

$$\begin{aligned}\eta_t^n(g) &= \int_0^t\int_{\{|z|<1\}} \mathbb{1}_{\{Y_{s-}>n\}} e^{cs}\beta z\sqrt[\alpha]{Y_{s-}}\tilde{N}(\mathrm{d}s,\mathrm{d}z)\\ &\quad + \int_0^t\int_{\{|z|\geqslant 1\}} \mathbb{1}_{\{Y_{s-}>n\}} e^{cs}\beta z\sqrt[\alpha]{Y_{s-}}N(\mathrm{d}s,\mathrm{d}z)\\ &\quad - \int_0^t\int_{\{|z|\geqslant 1\}} \mathbb{1}_{\{Y_{s-}>n\}} e^{cs}\beta z\sqrt[\alpha]{Y_{s-}}\hat{N}(\mathrm{d}s,\mathrm{d}z)\\ &= \int_0^t \mathbb{1}_{\{Y_{s-}>n\}} e^{cs}\beta\sqrt[\alpha]{Y_{s-}}\mathrm{d}L_s, \qquad t\geqslant 0,\end{aligned}$$

where we used the Lévy-Itô decomposition in (2.2) to get the second equality. It follows from [20, Remark A.8] that for each $t\geqslant 0$, there exist some constant $c_1>0$ such that

$$\mathbb{E}_{(y,x)}\left[\sup_{s\in[0,t]}|\eta_s^n(g)|\right] \leqslant c_1\mathbb{E}_{(y,x)}\left[\left(\int_0^t \mathbb{1}_{\{Y_s>n\}}Y_s\mathrm{d}s\right)^{\frac{1}{\alpha}}\right].$$

Since $\mathbb{E}_{(y,x)}[Y_s]\leqslant c_2(1+y\exp(-bs/\alpha))$ for all $s\in[0,t]$ by [20, Proposition 2.8], where $c_2>0$ is some constant, it follows that $\mathbb{E}_{(y,x)}\left[\int_0^t Y_s\mathrm{d}s\right]<\infty$ and further $\mathbb{E}_{(y,x)}\left[\left(\int_0^t Y_s\mathrm{d}s\right)^{1/\alpha}\right]<\infty$. Therefore, by the dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\mathbb{E}_{(y,x)}\left[\sup_{s\in[0,t]}|\eta_s^n(g)|\right]\leqslant c_1\lim_{n\to\infty}\mathbb{E}_{(y,x)}\left[\left(\int_0^t \mathbb{1}_{\{Y_s>n\}}Y_s\mathrm{d}s\right)^{\frac{1}{\alpha}}\right]=0. \tag{5.6}$$

As shown in the proof of [3, Theorem 2.1], the martingale property of $(M_t^2(g))_{t\geqslant 0}$ now follows from (5.5), (5.6) and the fact that both $(M_t^{3,n}(g))_{t\geqslant 0}$ and $(M_t^{4,n}(g))_{t\geqslant 0}$ are martingales. It is clear that $(M_t(g))_{t\geqslant 0}=(M_t^1(g)+M_t^2(g))_{t\geqslant 0}$ is also a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

"*Step 2*": We determine the constant $c>0$ and find another constant $M>0$ such that

$$(\mathcal{A}V)(y,x)\leqslant -cV(y,x)+M \tag{5.7}$$

for all $(y,x)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}$, where $\mathcal{A}$ is given by (2.3). For the function V, we have $V\in C^2(\mathbb{R}_{\geqslant 0}\times\mathbb{R},\mathbb{R})$,

$$\tfrac{\partial}{\partial y}V(y,x)=\beta,\quad \tfrac{\partial}{\partial x}V(y,x)=\tfrac{\partial}{\partial x}h(x)=\begin{cases}\frac{x}{|x|}, & \text{if } |x|>2\\ h'(x), & \text{if } |x|\leqslant 2,\end{cases}$$

and

$$\tfrac{\partial^2}{\partial x^2}V(y,x)=\tfrac{\partial^2}{\partial x^2}h(x):=\begin{cases}0, & \text{if } |x|>2,\\ h''(x), & \text{if } |x|\leqslant 2,\end{cases}$$

where h' and h'' denote the first and second order derivatives of the function h, respectively. So

$$\begin{aligned}(\mathcal{A}V)(y,x) &= (a-by)\beta + (m-\theta x)\tfrac{\partial}{\partial x}h(x) + \tfrac{1}{2}y\tfrac{\partial^2}{\partial x^2}h(x)\\ &\quad + y\int_0^\infty \left(\beta(y+z) + h(x) - \beta y - h(x) - z\beta\right) C_\alpha z^{-1-\alpha}\mathrm{d}z\\ &= (a-by)\beta + (m-\theta x)\tfrac{\partial}{\partial x}h(x) + \tfrac{1}{2}y\tfrac{\partial^2}{\partial x^2}h(x).\end{aligned}$$

By choosing $\beta > 0$ large enough, we obtain that for all $(y,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$,

$$\begin{aligned}(\mathcal{A}V)(y,x) &= a\beta - \tfrac{by\beta}{2} - \theta x\tfrac{\partial}{\partial x}h(x) + \left(-\tfrac{b\beta}{2} + \tfrac{1}{2}\tfrac{\partial^2}{\partial x^2}h(x)\right)y + m\tfrac{\partial}{\partial x}h(x)\\ &\leqslant a\beta - \tfrac{by\beta}{2} - \theta\left(h(x)\mathbb{1}_{\{x>2\}} + h(x)\mathbb{1}_{\{x<-2\}}\right) + 0 + c_3\\ &\leqslant a\beta - \tfrac{by\beta}{2} - \theta\left(h(x)\mathbb{1}_{\{|x|>2\}} + h(x)\mathbb{1}_{\{|x|\leqslant 2\}}\right) + c_4\\ &= a\beta - \tfrac{by\beta}{2} - \theta h(x) + c_4 = -\tfrac{b\beta}{2}y - \theta h(x) + c_5, \qquad (5.8)\end{aligned}$$

where we used the boundedness of $|h'|$, $|h''|$ and $|h|\mathbb{1}_{\{|x|\leqslant 2\}}$ to get the first and second inequality. Here c_3, c_4 and c_5 are some positive constants. Now, we see that (5.7) holds with $c := \min(b/2, \theta)$ and $M := c_5$.

"*Step* 3": We prove (5.1). By (5.4), (5.7) and the martingale property of $(M_t(g))_{t\geqslant 0}$, we obtain

$$\begin{aligned}&e^{ct}\mathbb{E}_{(y,x)}\left[V(Y_t,X_t)\right] - V(y,x)\\ &\quad= \mathbb{E}_{(y,x)}\left[g(t,Y_t,X_t)\right] - \mathbb{E}_{(y,x)}\left[g(0,Y_0,X_0)\right]\\ &\quad= \mathbb{E}_{(y,x)}\left[\int_0^t \left(e^{cs}(\mathcal{A}V)(Y_s,X_s) + ce^{cs}V(Y_s,X_s)\right)\mathrm{d}s\right]\\ &\quad\leqslant \mathbb{E}_{(y,x)}\left[\int_0^t \left(e^{cs}(-cV(Y_s,X_s)+M) + ce^{cs}V(Y_s,X_s)\right)\mathrm{d}s\right]\\ &\quad= \mathbb{E}_{(y,x)}\left[\int_0^t Me^{cs}\mathrm{d}s\right] \leqslant \frac{M}{c}e^{ct}\end{aligned}$$

for all $(y,x) \in \mathbb{R}_{\geqslant 0}\times\mathbb{R}$ and $t \geqslant 0$, which implies (5.1). This completes the proof. □

Remark 5.2. *To see the existence of a function $h \in C^\infty(\mathbb{R},\mathbb{R})$ that fulfills the conditions of Lemma 5.1, we can proceed in the following way: let $\rho \in C^\infty(\mathbb{R},\mathbb{R})$ be such that $\rho(x) = 1$ for $x \geqslant 2$, $\rho(x) = 0$ for $x \leqslant 1$ and $0 \leqslant \rho(x) \leqslant 1$ for $1 \leqslant x \leqslant 2$. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) := \int_0^x \rho(r)\mathrm{d}r$, $x \in \mathbb{R}$. Then*

$$F(x) = \begin{cases} 0, & x \leqslant 1,\\ \in [0,1], & 1 < x \leqslant 2,\\ x - 2 + \int_1^2 \rho(r)\mathrm{d}r, & x > 2.\end{cases}$$

We now define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) := F(|x|) + 2 - F(2)$, $x \in \mathbb{R}$. Then h satisfies the conditions required in Lemma 5.1.

6. Exponential ergodicity of (Y,X)

In this section we prove our main result, namely, the exponential ergodicity of the affine two factor model $(Y,X) = (Y_t,X_t)_{t\geqslant 0}$.

Let $\|\cdot\|_{TV}$ denote the total variation norm for signed measures on $\mathbb{R}_{\geqslant 0}\times\mathbb{R}$, namely,

$$\|\mu\|_{TV} := \sup\{|\mu(A)|\},$$

where μ is a signed measure on $\mathbb{R}_{\geqslant 0}\times\mathbb{R}$ and the above supremum is running for all Borel sets A in $\mathbb{R}_{\geqslant 0}\times\mathbb{R}$.

Let $\mathbf{P}^t(y,x,\cdot) := \mathbb{P}_{(y,x)}((Y_t,X_t)\in\cdot)$ denote the distribution of $(Y_t,X_t)_{t\geqslant 0}$ with the initial condition $(Y_0,X_0)=(y_0,x_0)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}$.

By [3, Theorem 3.1] and the argument in [17, p.80], there exists a unique invariant probability measure π for the two dimensional process $(Y_t,X_t)_{t\geqslant 0}$. Roughly speaking, if for each $(y,x)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}$, the convergence of the distribution $\mathbf{P}^t(y,x,\cdot)$ to π as $t\to\infty$ is exponentially fast with respect to the total variation norm, then we say that the process $(Y_t,X_t)_{t\geqslant 0}$ is exponentially ergodic.

The main result of this paper is the following:

Theorem 6.1. *Consider the two-dimensional affine process $(Y,X)=(Y_t,X_t)_{t\geqslant 0}$ defined by (1.1) with parameters $\alpha\in(1,2)$, $a>0$, $b>0$, $m\in\mathbb{R}$ and $\theta>0$. Then $(Y_t,X_t)_{t\geqslant 0}$ is exponentially ergodic, that is, there exist constants $\delta\in(0,1)$ and $B\in(0,\infty)$ such that*

$$\|\mathbf{P}^t(y,x,\cdot)-\pi\|_{TV}\leqslant B\,(V(y,x)+1)\,e^{-\delta t} \tag{6.1}$$

for all $t\geqslant 0$ and $(y,x)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}$.

Proof. We basically follow the proof of [15, Theorem 6.3]. The essential idea is to use the so called Foster-Lyapunov criteria developed in [24] for the geometric ergodicity of Markov chains.

We first consider the skeleton chain $(Y_n,X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$, which is a Markov chain on the state space $\mathbb{R}_{\geqslant 0}\times\mathbb{R}$ with transition kernel $\mathbf{P}^n(y,x,\cdot)$. It is easy to see that the measure π is also an invariant probability measure for the chain $(Y_n,X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$.

Let the function V be the same as in Lemma 5.1 and the constant $\beta>0$ there be sufficiently large. The Markov property together with Lemma 5.1 implies that

$$\begin{aligned}&\mathbb{E}\big[V(Y_{n+1},X_{n+1})\,|\,(Y_0,X_0),(Y_1,X_1),\ldots,(Y_n,X_n)\big]\\&\qquad=\int_{\mathbb{R}_{\geqslant 0}}\int_{\mathbb{R}}V(y,x)\mathbf{P}^1(Y_n,X_n,\mathrm{d}y\mathrm{d}x)\leqslant e^{-c}V(Y_n,X_n)+\frac{M}{c},\end{aligned}$$

where c and M are the positive constants in Lemma 5.1. If we set $V_0:=V$ and $V_n:=V(Y_n,X_n)$, $n\in\mathbb{N}$, then

$$\mathbb{E}[V_1]\leqslant e^{-c}V_0(Y_0,X_0)+\frac{M}{c}$$

and, for all $n\in\mathbb{N}$,

$$\mathbb{E}\left[V_{n+1}\,|\,(Y_0,X_0),(Y_1,X_1),\ldots(Y_n,X_n)\right]\leqslant e^{-c}V_n+\frac{M}{c}.$$

In order to apply [22, Theorem 6.3] for the chain $(Y_n,X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$, it remains to verify the following conditions:

(a) the Lebesgue measure λ on $\mathbb{R}_{\geqslant 0}\times\mathbb{R}$ is an irreducibility measure for the chain $(Y_n,X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$;
(b) the chain $(Y_n,X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ is aperiodic (the definition of aperiodicity can be found in [21, p.114]);
(c) all compact sets of the state space $\mathbb{R}_{\geqslant 0}\times\mathbb{R}$ are petite (see [23, p.500] for a definition).

We now proceed to prove (a)-(c).

In order to prove (a), we will use the same argument as in [3, Theorem 4.1]. It is enough to check that for each $(y_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$, the measure $\mathbf{P}^1(y_0, x_0, \cdot)$ is absolutely continuous with respect to the Lebesgue measure with a density function $p_1(y, x|y_0, x_0)$ that is strictly positive for almost all $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$. Indeed, let A be a Borel set of $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$ with $\lambda(A) > 0$. Then

$$\mathbb{P}_{(y_0,x_0)}\left(\tau_A < \infty\right) \geqslant \mathbf{P}^1\left(y_0, x_0, A\right) = \iint_A p_0(y, x|y_0, x_0)\mathrm{d}y\mathrm{d}x > 0$$

for all $(y_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$, where the stopping time τ_A is defined by $\tau_A := \inf\{n \geqslant 0 : (Y_n, X_n) \in A\}$.

Next, we prove the existence of the density $p_1(y, x|y_0, x_0)$ with the required property. Recall that

$$Y_1 = e^{-b}\left(y_0 + a\int_0^1 e^{bs}\mathrm{d}s + \int_0^1 e^{bs}\sqrt[\alpha]{Y_{s-}}\mathrm{d}L_s\right),$$

and

$$X_1 = e^{-\theta}\left(x_0 + m\int_0^1 e^{\theta s}\mathrm{d}s + \int_0^1 e^{\theta s}\sqrt{Y_s}\mathrm{d}B_s\right),$$

provided that $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$. For $(\bar{y}, \bar{x}) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$, we have

$$\begin{aligned}\mathbb{P}_{(y_0,x_0)}\left(Y_1 < \bar{y}, X_1 < \bar{x}\right) &= \mathbb{E}_{(y_0,x_0)}\left[\mathbb{P}_{(y_0,x_0)}\left(Y_1 < \bar{y}, X_1 < \bar{x} \mid Y_1\right)\right] \\ &= \mathbb{E}_{(y_0,x_0)}\left[\mathbb{E}_{(y_0,x_0)}\left[\mathbb{1}_{\{Y_1<\bar{y}\}}\mathbb{1}_{\{X_1<\bar{x}\}} \,\middle|\, Y_1\right]\right] \\ &= \mathbb{E}_{(y_0,x_0)}\left[\mathbb{1}_{\{Y_1<\bar{y}\}}\mathbb{E}_{(y_0,x_0)}\left[\mathbb{1}_{\{X_1<\bar{x}\}} \,\middle|\, Y_1\right]\right]. \qquad (6.2)\end{aligned}$$

Note that $(Y_t)_{t\geqslant 0}$ and the Brownian motion $(B_t)_{t\geqslant 0}$ are independent, since $(L_t)_{t\geqslant 0}$ and $(B_t)_{t\geqslant 0}$ are independent and $(Y_t)_{t\geqslant 0}$ is a strong solution. Therefore, the conditional distribution of X_1 given $(Y_t)_{t\in[0,1]}$ is a normal distribution with mean $x_0\exp(-\theta) + m\left(1 - \exp(-\theta)\right)/\theta$ and variance $\exp\left(-2\theta\right)\int_0^1 Y_s \exp\left(2\theta s\right)\mathrm{d}s$. Hence, we get that for $\bar{x} \in \mathbb{R}$,

$$\begin{aligned}&\mathbb{E}_{(y_0,x_0)}\left[\mathbb{1}_{\{X_1<\bar{x}\}} \,\middle|\, Y_1\right] \\ &= \mathbb{E}_{(y_0,x_0)}\left[\mathbb{E}_{(y_0,x_0)}\left[\mathbb{1}_{\{X_1<\bar{x}\}} \,\middle|\, (Y_t)_{0\leqslant t\leqslant 1}\right] \,\middle|\, Y_1\right] \\ &= \mathbb{E}_{(y_0,x_0)}\left[\int_{-\infty}^{\bar{x}} \varrho\left(r - e^{-\theta}x_0 - \tfrac{m}{\theta}\left(1 - e^{-\theta}\right); e^{-2\theta}\int_0^1 e^{2\theta s}Y_s\mathrm{d}s\right)\mathrm{d}r \,\middle|\, Y_1\right], \qquad (6.3)\end{aligned}$$

where $\varrho(r; \sigma^2)$ is the density of the normal distribution with variance $\sigma^2 > 0$, i.e.,

$$\varrho(r; \sigma^2) := \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{r^2}{2\sigma^2}}, \quad r \in \mathbb{R}.$$

Note that the assumption $a > 0$ ensures that

$$\mathbb{P}_{(y_0,x_0)}\left(\int_0^1 e^{2\theta s}Y_s\mathrm{d}s > 0\right) = 1.$$

By [16, Theorem 6.3] and considering the conditional distribution of $\int_0^1 e^{2\theta s}Y_s\mathrm{d}s$ given Y_1, we can find a probability kernel $K_{(y_0,x_0)}(\cdot, \cdot)$ from $\mathbb{R}_{\geqslant 0}$ to $\mathbb{R}_{\geqslant 0}$ such that

$$\mathbb{P}_{(y_0,x_0)}\left(\int_0^1 e^{2\theta s}Y_s\mathrm{d}s \in \cdot \,\middle|\, Y_1\right) = K_{(y_0,x_0)}(Y_1, \cdot)$$

and

$$K_{(y_0,x_0)}(z,\mathbb{R}_{>0})=1,\quad \text{for all } z>0. \tag{6.4}$$

So

$$\begin{aligned}
&\mathbb{E}_{(y_0,x_0)}\left[\int_{-\infty}^{\bar{x}}\varrho\left(r-e^{-\theta}x_0-\tfrac{m}{\theta}\left(1-e^{-\theta}\right);e^{-2\theta}\int_0^1 e^{2\theta s}Y_s\mathrm{d}s\right)\mathrm{d}r\,\middle|\,Y_1\right]\\
&\quad=\int_0^{\infty}\left(\int_{-\infty}^{\bar{x}}\varrho\left(r-e^{-\theta}x_0-\tfrac{m}{\theta}\left(1-e^{-\theta}\right);e^{-2\theta}w\right)\mathrm{d}r\right)K_{(y_0,x_0)}(Y_1,\mathrm{d}w)\\
&\quad=\int_{-\infty}^{\bar{x}}\left(\int_0^{\infty}\varrho\left(r-e^{-\theta}x_0-\tfrac{m}{\theta}\left(1-e^{-\theta}\right);e^{-2\theta}w\right)K_{(y_0,x_0)}(Y_1,\mathrm{d}w)\right)\mathrm{d}r. \qquad (6.5)
\end{aligned}$$

It follows from (6.2), (6.3) and (6.5) that for all $(\bar{y},\bar{x})\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}$,

$$\begin{aligned}
\mathbb{P}_{(y_0,x_0)}\left(Y_1<\bar{y},X_1<\bar{x}\right)=\int_0^{\bar{y}}\int_{-\infty}^{\bar{x}}\Bigg(\int_0^{\infty}&\varrho\left(r-e^{-\theta}x_0-\tfrac{m}{\theta}\left(1-e^{-\theta}\right);e^{-2\theta}w\right)\\
&\cdot K_{(y_0,x_0)}(z,\mathrm{d}w)\Bigg)f_{Y_1^{y_0}}(z)\mathrm{d}r\mathrm{d}z, \qquad (6.6)
\end{aligned}$$

where $f_{Y_1^{y_0}}$ is given in (4.17). Define

$$p_1(y,x|y_0,x_0):=f_{Y_1^{y_0}}(y)\int_0^{\infty}\varrho\left(x-e^{-\theta}x_0-\tfrac{m}{\theta}\left(1-e^{-\theta}\right);e^{-2\theta}w\right)K_{(y_0,x_0)}(y,\mathrm{d}w).$$

By (6.4) and the fact that $f_{Y_1^{y_0}}(y)$ is strictly positive for all $y>0$ (see Theorem 4.5), for each $(y_0,x_0)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}$, the density $p_1(y,x|y_0,x_0)$ is strictly positive for almost all $(y,x)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}$. Moreover, by (6.6), we have

$$\mathbb{P}_{(y_0,x_0)}\left(Y_1<\bar{y},X_1<\bar{x}\right)=\int_0^{\bar{y}}\int_{-\infty}^{\bar{x}}p_1(y,x|y_0,x_0)\mathrm{d}y\mathrm{d}x$$

for all $(\bar{y},\bar{x})\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}$. So $p_1(\cdot,\cdot|y_0,x_0)$ is the density function of (Y_t,X_t) given that $(Y_0,X_0)=(y_0,x_0)$.

To prove (b), i.e., the aperiodicity of the skeleton chain $(Y_n,X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$, we use a contradiction argument. Suppose that the period l of the chain $(Y_n,X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ is greater than 1 (see [21, p.114] for a definition of the period of a Markov chain). Then we can find disjoint Borel sets $A_1,A_2,\cdots,A_l$ such that

$$\lambda(A_i)>0,\ i=1,\cdots,l,\quad \cup_{i=1}^{l}A_i=\mathbb{R}_{\geqslant 0}\times\mathbb{R}, \tag{6.7}$$

$$\mathbf{P}^1(y_0,x_0,A_{i+1})=1 \tag{6.8}$$

for all $(y_0,x_0)\in A_i,\ i=1,\cdots,l-1$, and

$$\mathbf{P}^1(y_0,x_0,A_1)=1$$

for all $(y_0,x_0)\in A_l$. By (6.8), we have

$$\iint_{(A_2)^c}p_1(y,x|y_0,x_0)\mathrm{d}y\mathrm{d}x=0,\quad (y_0,x_0)\in A_1,$$

and further

$$\iint_{A_1}p_1(y,x|y_0,x_0)\mathrm{d}y\mathrm{d}x=0,\quad (y_0,x_0)\in A_1.$$

However, since for each $(y_0,x_0)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}$, the density $p_1(y,x|y_0,x_0)$ is strictly positive for almost all $(y,x)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}$, we must have $\lambda(A_1)=0$, which contradicts

(6.7). Therefore, the assumption that $l \geqslant 2$ is not true. So we have $l=1$.

In view of [22, Theorem 3.4 (ii)], to prove (c), it is enough to check the Feller property of the skeleton chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$. By [6, Theorem 2.7], the two-dimensional process $(Y_t, X_t)_{t\geqslant 0}$, as an affine process, possesses the Feller property. So the skeleton chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ has also the Feller property.

Now, we can apply [22, Theorem 6.3] and thus find constants $\delta \in (0,1)$, $B \in (0,\infty)$ such that

$$\|\mathbf{P}^n(y,x,\cdot)-\pi\|_{TV} \leqslant B\,(V(y,x)+1)\,e^{-\delta n} \tag{6.9}$$

for all $n \in \mathbb{Z}_{\geqslant 0}$, $(y,x)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}$. For the remainder of the proof, i.e., to extend the inequality (6.9) to all $t \geqslant 0$, we can interpolate in the same way as in [24, p.536], and we omit the details. This completes the proof. □

Appendix

Proof of Lemma 4.1. We will complete the proof in three steps.

"*Step 1*": Consider $\rho \geqslant 2$ and $\vartheta \in [\pi/2-\varepsilon, \pi/2+\varepsilon]$, where $\varepsilon > 0$ is a small constant whose exact value will be determined later. We introduce a change of variables

$$z := \left(\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{(1-\alpha)}\right)e^{b(\alpha-1)s}-\frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}$$

and define $\Gamma_0 : [0,t]\to\mathbb{C}$ by

$$\Gamma_0(s) := \left(\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{(1-\alpha)}\right)e^{b(\alpha-1)s}-\frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}, \quad s\in[0,t].$$

Then we get

$$\begin{aligned}\int_0^t v_s\left(\rho e^{i\vartheta}\right)\mathrm{d}s &= \int_0^t \left(\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{1-\alpha}\right)e^{b(\alpha-1)s}-\frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}\mathrm{d}s\\ &= -\frac{1}{b}\int_{\Gamma_0}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z.\end{aligned} \tag{6.10}$$

Next, we derive a lower bound for $\mathrm{Re}\big(\int_0^t v_s\left(\rho e^{i\vartheta}\right)\mathrm{d}s\big)$.

Let Γ_0^* be the range of Γ_0. Since $\Gamma_0^*\subset\mathcal{O}$ and $z\mapsto\left(1+z^{\alpha-1}/(\alpha b)\right)^{-1}$ is analytic in $\mathcal{O}$, we have

(6.11)

$$\int_{\Gamma_0}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z = \int_{\rho e^{i\vartheta}}^{\left(\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{(1-\alpha)}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z.$$

Here and after, the notation

$$\int_{w_1}^{w_2}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z$$

means the integral $\int_{\Gamma_{[w_1,w_2]}} \left(1 + z^{\alpha-1}/(\alpha b)\right)^{-1} \mathrm{d}z$, where $\Gamma_{[w_1,w_2]}$ is the directed segment joining w_1 and w_2 and is defined by

$$\Gamma_{[w_1,w_2]} : [0,1] \to \mathbb{C} \quad \text{with} \quad \Gamma_{[w_1,w_2]}(r) := (1-r)w_1 + rw_2, \quad r \in [0,1].$$

By (6.10), (6.11) and the holomorphicity of $z \mapsto \left(1 + z^{\alpha-1}/(\alpha b)\right)^{-1}$ on $\mathcal{O}$, we obtain

$$\begin{aligned}\int_0^t v_s\left(\rho e^{i\vartheta}\right) \mathrm{d}s &= \frac{1}{b}\int_{e^{i\vartheta}}^{\rho e^{i\vartheta}} \left(1 + \frac{z^{\alpha-1}}{\alpha b}\right)^{-1} \mathrm{d}z \\ &\quad + \frac{1}{b}\int_{\left(\left(\frac{1}{\alpha b} + (\rho e^{i\vartheta})^{(1-\alpha)}\right)e^{b(\alpha-1)t} - \frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}}^{e^{i\vartheta}} \left(1 + \frac{z^{\alpha-1}}{\alpha b}\right)^{-1} \mathrm{d}z. \end{aligned} \tag{6.12}$$

Since the second term on the right-hand of (6.12) is continuous in $(t, \rho, \vartheta) \in [1/T, T] \times [2, \infty) \times [\pi/2 - \varepsilon, \pi/2 + \varepsilon]$ and converges to

$$\frac{1}{b}\int_{\left(\left(e^{b(\alpha-1)t} - 1\right)\frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}}^{e^{i\vartheta}} \left(1 + \frac{z^{\alpha-1}}{\alpha b}\right)^{-1} \mathrm{d}z$$

(uniformly in $(t, \vartheta) \in [1/T, T] \times [\pi/2 - \varepsilon, \pi/2 + \varepsilon]$) as $\rho \to \infty$, it must be bounded, i.e., we have

$$\left| \frac{1}{b}\int_{\left(\left(e^{b(\alpha-1)t} - 1\right)\frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}}}^{e^{i\vartheta}} \left(1 + \frac{z^{\alpha-1}}{\alpha b}\right)^{-1} \mathrm{d}z \right| \leqslant c_3 \tag{6.13}$$

for all $t \in [1/T, T]$, $\vartheta \in [\pi/2 - \varepsilon, \pi/2 + \varepsilon]$ and $\rho \geqslant 2$, where $c_3 = c_3(\varepsilon, T) > 0$ is some constant.

Now, define $\Gamma_\vartheta : [0,1] \to \mathbb{C}$ by

$$\Gamma_\vartheta(r) := (1-r)e^{i\vartheta} + r\rho e^{i\vartheta}, \quad r \in [0,1],$$

and let Γ_ϑ^* be the range of Γ_ϑ. We can calculate the real part of the first integral appearing on the right-hand side of (6.12) by

$$\begin{aligned}&\mathrm{Re}\left(\int_{e^{i\vartheta}}^{\rho e^{i\vartheta}} \left(1 + \frac{z^{\alpha-1}}{\alpha b}\right)^{-1} \mathrm{d}z\right) \\ &\quad = \mathrm{Re}\left(\int_{\Gamma_\vartheta} \left(1 + \frac{z^{\alpha-1}}{\alpha b}\right)^{-1} \mathrm{d}z\right) \\ &\quad = \mathrm{Re}\left(\int_0^1 \left(1 + \frac{(\Gamma_v(r))^{\alpha-1}}{\alpha b}\right)^{-1} \partial_r \Gamma_\vartheta(r) \mathrm{d}r\right) \\ &\quad = \mathrm{Re}\left(\int_0^1 \frac{(\rho-1)e^{i\vartheta}}{1 + (\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}} \mathrm{d}r\right) \\ &\quad = \int_0^1 \left|\frac{(\rho-1)e^{i\vartheta}}{1 + (\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}}\right| \cos\left(\mathrm{Arg}\left(\frac{(\rho-1)e^{i\vartheta}}{1 + (\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}}\right)\right) \mathrm{d}r. \end{aligned} \tag{6.14}$$

For $r \in [0,1]$, we have

$$\begin{aligned}\mathrm{Arg}\left(1 + (\Gamma_\vartheta(0))^{\alpha-1}(\alpha b)^{-1}\right) &\leqslant \mathrm{Arg}\left(1 + (\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}\right) \\ &\leqslant \mathrm{Arg}\left(1 + (\Gamma_\vartheta(1))^{\alpha-1}(\alpha b)^{-1}\right). \end{aligned} \tag{6.15}$$

Define δ_ϑ by

$$\begin{aligned}\delta_\vartheta &:= (\alpha-1)\vartheta - \operatorname{Arg}\left(1+(\Gamma_\vartheta(0))^{\alpha-1}(\alpha b)^{-1}\right)\\ &= (\alpha-1)\vartheta - \operatorname{Arg}\left(1+e^{i(\alpha-1)\vartheta}(\alpha b)^{-1}\right) \in (0,(\alpha-1)\vartheta).\end{aligned} \tag{6.16}$$

It is easy to see that

$$\operatorname{Arg}\left(1+(\Gamma_\vartheta(1))^{\alpha-1}(\alpha b)^{-1}\right) < (\alpha-1)\vartheta. \tag{6.17}$$

By (6.15), (6.16) and (6.17), we get

$$\operatorname{Arg}\left(1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}\right) \in [(\alpha-1)\vartheta-\delta_\vartheta,(\alpha-1)\vartheta), \quad r\in[0,1].$$

As a result,

$$\operatorname{Arg}\left(\frac{(\rho-1)e^{i\vartheta}}{1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}}\right) \in ((2-\alpha)\vartheta,(2-\alpha)\vartheta+\delta_\vartheta], \quad r\in[0,1]. \tag{6.18}$$

Note that $0<\delta_{\pi/2}<(\alpha-1)\pi/2$ by (6.16). Since δ_ϑ is continuous in ϑ, we see that

$$0 < \lim_{\vartheta\to\frac{\pi}{2}}\{(2-\alpha)\vartheta+\delta_\vartheta\} = (2-\alpha)\frac{\pi}{2}+\delta_{\frac{\pi}{2}} < \frac{\pi}{2}.$$

Set

$$c_4 := \frac{\pi}{2} - \left((2-\alpha)\frac{\pi}{2}+\delta_{\frac{\pi}{2}}\right) \in \left(0,\frac{\pi}{2}\right).$$

Now, we choose the constant $\varepsilon_0>0$ small enough such that

$$0<(2-\alpha)\vartheta<(2-\alpha)\vartheta+\delta_\vartheta \le \frac{\pi}{2}-\frac{c_4}{2} \tag{6.19}$$

for all $\vartheta\in[\pi/2-\varepsilon_0,\pi/2+\varepsilon_0]$. It follows from (6.18) and (6.19) that for all $\vartheta\in[\pi/2-\varepsilon_0,\pi/2+\varepsilon_0]$ and $r\in[0,1]$,

$$\cos\left(\operatorname{Arg}\left(\frac{(\rho-1)e^{i\vartheta}}{1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}}\right)\right) \geqslant \cos\left(\frac{\pi}{2}-\frac{c_4}{2}\right) =: c_5 > 0. \tag{6.20}$$

In view of (6.14) and (6.20), we get

$$\begin{aligned}&\operatorname{Re}\left(\int_{e^{i\vartheta}}^{\rho e^{i\vartheta}}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z\right)\\ &\geqslant \cos\left(\frac{\pi}{2}-\frac{c_4}{2}\right)\int_0^1\left|\frac{(\rho-1)e^{i\vartheta}}{1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}}\right|\mathrm{d}r\\ &= c_5\int_0^1\frac{\rho-1}{\left|1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}\right|}\mathrm{d}r \geqslant c_5\int_0^1\frac{\rho-1}{1+\left|(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}\right|}\mathrm{d}r\\ &= c_5\int_0^1\frac{\rho-1}{1+(1-r+r\rho)^{\alpha-1}(\alpha b)^{-1}}\mathrm{d}r = c_5\int_0^{\rho-1}\frac{1}{1+(1+r)^{\alpha-1}(\alpha b)^{-1}}\mathrm{d}r\\ &\geqslant \frac{c_5}{1+(\alpha b)^{-1}}\int_0^{\rho-1}\frac{1}{(1+r)^{\alpha-1}}\mathrm{d}r = c_5\alpha b(1+\alpha b)^{-1}(2-\alpha)^{-1}\left(\rho^{2-\alpha}-1\right).\end{aligned} \tag{6.21}$$

Combining (6.12), (6.13) and (6.21) yields

(6.22)
$$\operatorname{Re}\left(\int_0^t v_s\left(\rho e^{i\vartheta}\right)\mathrm{d}s\right) \geqslant c_6\rho^{2-\alpha} - c_7, \quad \rho \geqslant 2,\ \vartheta \in \left[\frac{\pi}{2}-\varepsilon_0, \frac{\pi}{2}+\varepsilon_0\right],\ t \in [1/T, T],$$

where c_6, $c_7 > 0$ are constants that depend only on a, b, α, ε_0 and T.

"*Step 2*": The case with $\rho \geqslant 2$ and $\vartheta \in [-\pi/2-\varepsilon_0, -\pi/2+\varepsilon_0]$ can be similarly treated, and we thus get

$$\operatorname{Re}\left(\int_0^t v_s\left(\rho e^{i\vartheta}\right)\mathrm{d}s\right) \geqslant c_8\rho^{2-\alpha} - c_9 \tag{6.23}$$

for all $\rho \geqslant 2$, $\vartheta \in [-\pi/2-\varepsilon_0, -\pi/2+\varepsilon_0]$ and $t \in [1/T, T]$, where $c_8,\ c_9 > 0$ are constants depending only on a, b, α, ε_0 and T.

"*Step 3*": Since $\int_0^t v_s\left(\rho e^{i\vartheta}\right)\mathrm{d}s$ is continuous in (t, ρ, ϑ), we can find a constant $c_{10} > 0$ such that

$$\operatorname{Re}\left(\int_0^t v_s\left(\rho e^{i\vartheta}\right)\mathrm{d}s\right) \geqslant -c_{10} \tag{6.24}$$

for all $0 \leqslant \rho \leqslant 2$, $\vartheta \in [-\pi/2-\varepsilon_0, -\pi/2+\varepsilon_0] \cup [\pi/2-\varepsilon_0, \pi/2+\varepsilon_0]$ and $t \in [1/T, T]$. The estimate (4.2) now follows from (6.22), (6.23) and (6.24). □

Proof of Lemma 4.2. Let $\rho \geq 2$ and $\vartheta \in [\pi/2+\varepsilon_0, \pi]$. Our aim is to show

$$\left|\int_0^t v_s(\rho e^{i\vartheta})\mathrm{d}s\right| \leqslant C_3 + C_4\rho^{2-\alpha} \tag{6.25}$$

for some constants C_3, $C_4 > 0$ that depend only on a, b, α, ε_0 and t. Using the change of variables

$$z := \left(\frac{1}{\alpha b} + \left(\rho e^{i\vartheta}\right)^{(1-\alpha)}\right) e^{b(\alpha-1)s} - \frac{1}{\alpha b},$$

we get

$$\int_0^t v_s(\rho e^{i\vartheta})\mathrm{d}s = \int_0^t \left(\left(\frac{1}{\alpha b} + \left(\rho e^{i\vartheta}\right)^{1-\alpha}\right) e^{b(\alpha-1)s} - \frac{1}{\alpha b}\right)^{\frac{1}{1-\alpha}} \mathrm{d}s$$
$$= \frac{1}{b(\alpha-1)} \int_{(\rho e^{i\vartheta})^{1-\alpha}}^{\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{1-\alpha}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}} z^{\frac{1}{1-\alpha}} \left(z + \frac{1}{\alpha b}\right)^{-1} \mathrm{d}z. \tag{6.26}$$

Since $\vartheta \in [\pi/2+\varepsilon_0, \pi]$, we have $(1-\alpha)\vartheta \in [(1-\alpha)\pi, (1-\alpha)(\pi/2+\varepsilon_0)]$, which implies

$$|\sin((1-\alpha)\vartheta)| \geqslant \min\{\sin((\alpha-1)\pi), \sin((\alpha-1)(\pi/2+\varepsilon_0))\} =: c_1 > 0. \tag{6.27}$$

Note that $z \mapsto z^{1/(1-\alpha)}(z + 1/(\alpha b))^{-1}$ is holomorphic on $\mathcal{O}$. So we have

$$\int_{(\rho e^{i\vartheta})^{1-\alpha}}^{(\frac{1}{\alpha b}+(\rho e^{i\vartheta})^{1-\alpha})e^{b(\alpha-1)t}-\frac{1}{\alpha b}} z^{\frac{1}{1-\alpha}} \left(z + \frac{1}{\alpha b}\right)^{-1} \mathrm{d}z$$
$$= \int_{(\rho e^{i\vartheta})^{1-\alpha}}^{(\rho e^{i\vartheta})^{1-\alpha}+2} z^{\frac{1}{1-\alpha}} \left(z + \frac{1}{\alpha b}\right)^{-1} \mathrm{d}z$$

$$+\int_{(\rho e^{i\vartheta})^{1-\alpha}+2}^{\left(\frac{1}{\alpha b}+(\rho e^{i\vartheta})^{1-\alpha}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}} z^{\frac{1}{1-\alpha}}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z. \tag{6.28}$$

Since

$$\lim_{\rho\to\infty}\int_{(\rho e^{i\vartheta})^{1-\alpha}+2}^{\left(\frac{1}{\alpha b}+(\rho e^{i\vartheta})^{1-\alpha}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}} z^{\frac{1}{1-\alpha}}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z = \int_{2}^{\frac{1}{\alpha b}\left(e^{b(\alpha-1)t}-1\right)} z^{\frac{1}{1-\alpha}}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z,$$

where the convergence is uniform in $\vartheta\in[\pi/2+\varepsilon_0,\pi]$, we can find a constant $c_2>0$ such that

$$\left|\int_{(\rho e^{i\vartheta})^{1-\alpha}+2}^{\left(\frac{1}{\alpha b}+(\rho e^{i\vartheta})^{1-\alpha}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}} z^{\frac{1}{1-\alpha}}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z\right|\leqslant c_2 \tag{6.29}$$

for all $\rho\geqslant 2$ and $\vartheta\in[\pi/2+\varepsilon_0,\pi]$.

We now proceed to estimate the first term on the right-hand side of (6.28). Define

$$\Gamma_{\vartheta,\rho}(r):=\left(\rho e^{i\vartheta}\right)^{1-\alpha}+r,\quad r\in[0,2].$$

By (6.27), we have

$$|\rho^{1-\alpha}e^{(1-\alpha)i\vartheta}+r|\geqslant\rho^{1-\alpha}|\sin((1-\alpha)\vartheta)|\geqslant c_1\rho^{1-\alpha}, \tag{6.30}$$

where $r\in[0,2]$ and $\vartheta\in[\pi/2+\varepsilon_0,\pi]$. If $r\in[2\rho^{1-\alpha},2]$, then

$$|\rho^{1-\alpha}e^{(1-\alpha)i\vartheta}+r|\geqslant r-\rho^{1-\alpha}\geqslant\frac{r}{2}. \tag{6.31}$$

It follows from (6.30) and (6.31) that for $\rho\geqslant 2$ and $\vartheta\in[\pi/2+\varepsilon_0,\pi]$,

$$\begin{aligned}
&\left|\int_{(\rho e^{i\vartheta})^{1-\alpha}}^{(\rho e^{i\vartheta})^{1-\alpha}+2} z^{\frac{1}{1-\alpha}}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z\right|\\
&\quad=\left|\int_0^2\left(\Gamma_{\vartheta,\rho}(r)\right)^{\frac{1}{1-\alpha}}\left(\Gamma_{\vartheta,\rho}(r)+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}r\right|\\
&\quad\leqslant c_3\int_0^2|\Gamma_{\vartheta,\rho}(r)|^{\frac{1}{1-\alpha}}\mathrm{d}r=c_3\int_0^2\left|\rho^{1-\alpha}e^{(1-\alpha)i\vartheta}+r\right|^{\frac{1}{1-\alpha}}\mathrm{d}r\\
&\quad=c_3\int_0^{2\rho^{1-\alpha}}\left|\rho^{1-\alpha}e^{(1-\alpha)i\vartheta}+r\right|^{\frac{1}{1-\alpha}}\mathrm{d}r\\
&\qquad+c_3\int_{2\rho^{1-\alpha}}^2\left|\rho^{1-\alpha}e^{(1-\alpha)i\vartheta}+r\right|^{\frac{1}{1-\alpha}}\mathrm{d}r\\
&\quad\leqslant c_3\int_0^{2\rho^{1-\alpha}}\left(c_1\rho^{1-\alpha}\right)^{\frac{1}{1-\alpha}}\mathrm{d}r+c_3 2^{1/(\alpha-1)}\int_{2\rho^{1-\alpha}}^2 r^{\frac{1}{1-\alpha}}\mathrm{d}r\\
&\quad=2c_3c_1^{1/(1-\alpha)}\rho^{2-\alpha}+c_3 2^{1/(\alpha-1)}\,\frac{\alpha-1}{\alpha-2}r^{\frac{2-\alpha}{1-\alpha}}\Big|_{r=2\rho^{1-\alpha}}^2\leqslant c_4\rho^{2-\alpha}+c_5,
\end{aligned} \tag{6.32}$$

where c_3, c_4, $c_5>0$ are some constants. Combining (6.26), (6.28), (6.29) and (6.32) yields (6.25). □

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