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Abstract: Let X be a multi-type continuous-state branching process with immigration (CBI process) on state space $\mathbb{R}_+^d$. Denote by g_t, $t \geq 0$, the law of $X(t)$. We provide sufficient conditions under which g_t has, for each $t > 0$, a density with respect to the Lebesgue measure. Such density has, by construction, some anisotropic Besov regularity. Our approach neither relies on the use of Malliavin calculus nor on the study of corresponding Laplace transform.

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1 Introduction

Multi-type CBI processes are Markov processes with state space

$$\mathbb{R}_+^d = \{x \in \mathbb{R}^d \mid x_1, \dots, x_d \geq 0\}, \quad d \in \mathbb{N},$$

which arise as scaling limits of Galton-Watson branching processes with immigration, see, e.g., [Li06, Li11]. A remarkable feature of multi-type CBI processes is that the logarithm of their Laplace transform is an affine function of the initial state variable, i.e., multi-type CBI processes are affine processes in the sense of [DFS03, Definition 2.6]. They are also semimartingales whose characteristics can be readily deduced from their branching and immigration mechanisms. Although these processes are primarily motivated by population models, they have also found many applications in finance, especially in term-structure interest rate models and stochastic volatility models, see, e.g., [DFS03].

Let us describe these processes in more detail. According to [DFS03, Theorem 2.7] (see also [BLP15, Remark 2.5]), there exists a unique conservative Feller semigroup $(P_t)_{t\geq 0}$ acting on the

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Banach space of continuous functions vanishing at infinity with state space $\mathbb{R}_+^d$ such that its infinitesimal generator has core $C_c^2(\mathbb{R}_+^d)$ and is, for $f \in C_c^2(\mathbb{R}_+^d)$, given by

$$(Lf)(x) = \sum_{i=1}^{d} c_i x_i \frac{\partial^2 f(x)}{\partial x_i^2} + (\beta + Bx) \cdot (\nabla f)(x) + \int_{\mathbb{R}_+^d} (f(x+z) - f(x))\nu(dz) \tag{1.1}$$
$$+ \sum_{i=1}^{d} x_i \int_{\mathbb{R}_+^d} \left(f(x+z) - f(x) - \frac{\partial f(x)}{\partial x_i}(1 \wedge z_i) \right) \mu_i(dz),$$

provided that the tuple (c, β, B, ν, μ) satisfies

(i) $c = (c_1, \dots, c_d) \in \mathbb{R}_+^d$.

(ii) $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}_+^d$.

(iii) $B = (b_{ij})_{i,j \in \{1,\dots,d\}}$ is such that $b_{ij} \geq 0$ whenever $i, j \in \{1, \dots, d\}$ satisfy $i \neq j$.

(iv) ν is a Borel measure on $\mathbb{R}_+^d$ satisfying $\int_{\mathbb{R}_+^d} (1 \wedge |z|)\nu(dz) < \infty$ and $\nu(\{0\}) = 0$.

(vi) $\mu = (\mu_1, \dots, \mu_d)$, where, for each $i \in \{1, \dots, d\}$, μ_i is a Borel measure on $\mathbb{R}_+^d$ satisfying

$$\int_{\mathbb{R}_+^d} \left(|z| \wedge |z|^2 + \sum_{j \in \{1,\dots,d\} \setminus \{i\}} (1 \wedge z_j) \right) \mu_i(dz) < \infty, \quad \mu_i(\{0\}) = 0. \tag{1.2}$$

The corresponding Markov process with generator L is called a (conservative) multi-type CBI process. We call a tuple (c, β, B, ν, μ) with properties (i) – (vi) admissible. Note that this notion of admissible parameters is a special case of [DFS03, Definition 2.6], see also [BLP15, Remark 2.3] for additional comments. One of the advantages of multi-type CBI processes is their analytical tractability via Laplace transforms. More precisely, the Laplace transform of the transition semigroup P_t can be computed explicitly in terms of solutions to generalized Riccati equations. Most of the results obtained for multi-type CBI processes are based on a detailed study of these equations.

In this work we study existence of (transition) densities for multi-type CBI processes. A general expository on one-dimensional CBI processes was recently given in [CLP18], while a particular example of a two-dimensional affine process was studied in [JKR17]. Both approaches essentially rely on the study of the corresponding Riccati equations, i.e. on the Laplace transform of the transition semigroup. Results applicable for a wide class of affine processes on the state space $\mathbb{R}^n \times \mathbb{R}_+^d$ were obtained in [FMS13]. Applying their main result to the particular case of multi-type CBI processes requires that $c_1, \dots, c_d > 0$, i.e. the diffusion component is non-degenerate. Results applicable also to cases without diffusion (i.e. $c_1 = \dots = c_d = 0$) are, to the best of our knowledge, not available in arbitrary dimension. Such results should, of course, rely on the smoothing property of jumps corresponding to the branching and immigration mechanisms. We would like to mention that, similar to the diffusion case, there also exists a

Malliavin calculus for stochastic equations with jumps [BC86, Pic96, Pic97]. It is, however, much less powerful then its counterpart for diffusions.

We use some ideas developed in [FP10, DF13, Rom17], which provide a simple technique to prove existence of a density having some Besov-regularity without the use of Malliavin calculus. Their techniques were applied to Lévy driven stochastic equations with Hölder continuous coefficients [DF13], 3D Navier-Stokes equations driven by Gaussian noise [DR14], but also to the space-homogeneous Boltzmann equation [Fou15].

2 Statement of the results

2.1 The anisotropic Besov space

Due to (1.1) and the abundant choice of admissible parameters, it is reasonable to expect that the different components of a multi-type CBI processes on $\mathbb{R}_+^d$ behave very differently. Below we introduce anisotropic Besov spaces, which enable us to measure regularity for the density of each component of a CBI process separately. Similar ideas have been also applied in [FJR18] to stochastic equations driven by Lévy processes with anisotropic jumps. We call $a = (a_1, \dots, a_d)$ an anisotropy if it satisfies

$$0 < a_1, \dots, a_d < \infty \qquad \text{and} \qquad a_1 + \dots + a_d = d. \tag{2.1}$$

For $\lambda > 0$ with $\lambda / a_k \in (0,1)$, $k = 1, \dots, d$, the anisotropic Besov space $B_{1,\infty}^{\lambda,a}(\mathbb{R}^d)$ is defined as the Banach space of functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ with finite norm

$$\|f\|_{B_{1,\infty}^{\lambda,a}} := \|f\|_{L^1(\mathbb{R}^d)} + \sum_{k=1}^{d} \sup_{h \in [-1,1]} |h|^{-\lambda/a_k} \|\Delta_{he_k} f\|_{L^1(\mathbb{R}^d)}, \tag{2.2}$$

where $\Delta_h f(x) = f(x+h) - f(x)$, $h \in \mathbb{R}^d$, see [Dac03] and [Tri06] for additional details and references. Here $e_1, \dots, e_d$ denote the canonical basis vectors in $\mathbb{R}^d$. In the above definition, λ / a_k describes the smoothness in the coordinate k, its restriction to $(0,1)$ is not essential. Without this restriction we should use iterated differences in (2.2) instead (see [Tri06, Theorem 5.8.(ii)]).

2.2 Smoothing property of the noise

Let (c, β, B, ν, μ) be admissible parameters. For given $x \in \mathbb{R}^d$, let $L^x = (L^x(t))_{t \geq 0}$ be a Lévy process on $\mathbb{R}^d$ whose characteristic function $\mathbb{E}[e^{i\lambda \cdot L^x(t)}] = e^{-t\Psi_x(\lambda)}$, $\lambda \in \mathbb{R}^d$, satisfies

$$\begin{aligned} \Psi_x(\lambda) = &\sum_{j=1}^{d} 2c_j x_j \mathbb{1}_{\mathbb{R}_+}(x_j) \lambda_j^2 + \int_{\mathbb{R}_+^d} \left(1 - e^{i\lambda \cdot z}\right) \nu(dz) \\ &+ \sum_{j=1}^{d} \mathbb{1}_{\mathbb{R}_+}(x_j) x_j \int_{|z| \leq 1} \left(1 + i\lambda \cdot z - e^{i\lambda \cdot z}\right) \mu_j(dz). \end{aligned} \tag{2.3}$$

Denote by $g_t^x(dz)$ the distribution of $L^x(t)$. If this distribution has a density with respect to the Lebesgue measure, then, by abuse of notation, we denote this density also by $g_t^x(z)$. Let $(\alpha_i)_{i\in\{1,\dots,d\}} \subset (0,2]$. For $I \subset \{1,\dots,d\}$, define

$$\rho_I(x) := \min\{x_j^{1/\alpha_j} \mid j \in I\}\mathbb{1}_{\mathbb{R}_+^d}(x), \qquad \rho_\emptyset = \mathbb{1}_{\mathbb{R}_+^d}(x), \qquad \Gamma(I) = \{x \in \mathbb{R}^d \mid \rho_I(x) > 0\}. \quad (2.4)$$

The following is our main condition on the smoothing property of the noise.

(A) There exists $I \subset \{1,\dots,d\}$ and constants $(\alpha_i)_{i\in\{1,\dots,d\}} \subset (0,2]$, $C, t_0 > 0$ such that, for each $x \in \Gamma(I)$ and $t \in (0,t_0)$, the distribution g_t^x has a density with respect to the Lebesgue measure satisfying, for any $i \in \{1,\dots,d\}$ and $t \in (0,t_0)$,

$$\int_{\mathbb{R}^d} |g_t^x(z+he_i) - g_t^x(z)|dz \le \frac{C|h|}{\rho_I(x)} t^{-1/\alpha_i}, \quad h \in [-1,1]. \quad (2.5)$$

Here α_i describes the smoothness of the noise. These constants are related with an anisotropy $a = (a_i)_{i\in\{1,\dots,d\}}$ and a mean order of smoothness $\overline{\alpha}$ by

$$\frac{1}{\overline{\alpha}} = \frac{1}{d}\left(\frac{1}{\alpha_1} + \dots + \frac{1}{\alpha_d}\right), \qquad a_i = \frac{\overline{\alpha}}{\alpha_i}, \quad i \in \{1,\dots,d\}. \quad (2.6)$$

Hence larger values for α_i give higher smoothness, that is, larger values for $\overline{\alpha}$. The factor ρ_I is essential to treat the boundary behavior of multi-type CBI processes. By convention $1/0 := +\infty$, we see that (2.5) is clearly satisfied, if $\rho_I(x) = 0$, i.e. $x \notin \Gamma(I)$. In Section 6 we provide some sufficient conditions for (A). Based on these conditions, below we provide our main guiding examples.

Example 1.

(a) Define $I_1 = \{j \in \{1,\dots,d\} \mid c_j > 0\}$ and let $I_2 := \{1,\dots,d\}\backslash I_1$. Suppose that, for each $j \in I_2$, the Lévy measure μ_j satisfies

$$\mu_j(dz) = \mathbb{1}_{\mathbb{R}_+}(z_j)\frac{dz_j}{z_j^{1+\alpha_j}} \otimes \prod_{k\neq j}\delta_0(dz_k), \quad \alpha_j \in (1,2).$$

Then (A) is satisfied for $I = \{1,\dots,d\}$ and $\alpha_j = 2\mathbb{1}_{I_1}(j) + \alpha_j\mathbb{1}_{I_2}(j)$, see Lemma 16.

(b) It is worthwhile to mention that the particular choice

$$\mu_j(dz) = \mathbb{1}_{\mathbb{R}_+^d}(z)\frac{dz}{|z|^{d+\alpha}}, \quad \alpha \in (0,2)$$

violates (1.2) for any choice of $\alpha \in (0,2)$. However, suppose that there exists $j \in \{1,\dots,d\}$ and a Lévy measure μ_j' on $\mathbb{R}_+^d$ such that

$$\mu_j(dz) = \mathbb{1}_{\mathbb{R}_+^d}(z)\mathbb{1}_{\{|z|\le 1\}}\frac{dz}{|z|^{d+\alpha}} + \mu_j'(dz), \quad \alpha \in (0,1),$$

then condition (A) is satisfied for $I = \{j\}$ and $\alpha_1 = \dots = \alpha_d = \alpha$, see Proposition 15 and Lemma 16. Nevertheless this example does not satisfy the other restrictions formulated in our main results below.

(c) Suppose that there exists a subordinator ν' on $\mathbb{R}_+^d$ such that

$$\nu(dz) = \mathbb{1}_{\mathbb{R}_+^d}(z)\mathbb{1}_{\{|z|\leq 1\}}\frac{dz}{|z|^{d+\alpha}} + \nu'(dz), \quad \alpha \in (0,1),$$

then condition (A) is satisfied for $I = \emptyset$ and $\alpha_1 = \cdots = \alpha_d = \alpha$, see Lemma 17.

It is worthwhile to mention that we may also consider more general classes branching and immigration measures which include, in particular, cases where μ_j and ν are not absolutely continuous with respect to the Lebesgue measure, see [FJR18] for additional details.

2.3 Existence of densities for multi-type CBI processes

We start with the most general case and then continue with more specific situations.

Theorem 2. *Let X be a multi-type CBI process with admissible parameters (c, β, B, ν, μ) and suppose that*

(a) Condition (A) holds for $I = \{1, \dots, d\}$ and some $\alpha_1, \dots, \alpha_d > \frac{4}{3}$.

(b) There exists $\tau \in (0,1)$ such that

$$\sum_{j=1}^{d} \int_{|z|>1} |z|^{1+\tau}\mu_j(dz) + \int_{|z|>1} |z|^{1+\tau}\nu(dz) < \infty.$$

If $X(0)$ satisfies $\mathbb{E}[|X(0)|^{1+\tau}] < \infty$, then for each $t > 0$, $X(t)$ has a density g_t on

$$\Gamma(\{1, \dots, d\}) = \{x \in \mathbb{R}_+^d \mid x_1, \dots, x_d > 0\}.$$

Moreover, $f_t(x) := \rho(x)g_t(x) \in B_{1,\infty}^{\lambda,a}(\mathbb{R}^d)$, where $\lambda > 0$ is small enough, a is defined by (2.6) and $\rho(x) = \min\{x_1^{1/\alpha_1}, \dots, x_d^{1/\alpha_d}\}\mathbb{1}_{\mathbb{R}_+^d}(x)$.

This statement is, e.g., applicable in the situation of Example 1.(a), where the smoothing property (A) is obtained from a combination of diffusion and jumps from the branching mechanism. In absence of diffusion, we can weaken the restriction on $\alpha_1, \dots, \alpha_d$ slightly.

Theorem 3. *Let X be a multi-type CBI process with admissible parameters (c, β, B, ν, μ), where $c_1 = \cdots = c_d = 0$, and suppose that*

(a) Condition (A) is satisfied for some $I \subset \{1, \dots, d\}$ and $\alpha_1, \dots, \alpha_d \in (0,2)$.

(b) There exists $\gamma_0 \in (1,2]$ and $\tau \in (0, \gamma_0 - 1)$ such that

$$\sum_{j=1}^{d} \int_{\mathbb{R}_+^d} \left(|z|^{\gamma_0}\mathbb{1}_{\{|z|\leq 1\}} + |z|^{1+\tau}\mathbb{1}_{\{|z|>1\}}\right)\mu_j(dz) + \int_{|z|>1} |z|^{1+\tau}\nu(dz) < \infty.$$

If $I = \emptyset$, then we may also take $\tau = 0$.

(c) It holds that $\alpha_1, \dots, \alpha_d > \frac{\gamma_0}{1+\gamma_0}\gamma_0$. *Moreover, for each* $j \in I$, *one has* $\alpha_j \geq 1$.

If $X(0)$ *satisfies* $\mathbb{E}[|X(0)|^{1+\tau}] < \infty$, *then for each* $t > 0$, $X(t)$ *has a density* g_t *on* $\Gamma(I)$. *Moreover,* $f_t(x) := \rho_I(x) g_t(x) \in B^{\lambda,a}_{1,\infty}(\mathbb{R}^d)$, *where* $\lambda > 0$ *is small enough,* a *is defined in* (2.6) *and* ρ_I *is given as in* (2.4).

We now make a few comments on Theorem 3.

Remark 4. *Under the above conditions,* $X(t)$ *has only a density on* $\Gamma(I)$, *i.e. the distribution may be singular on the set* $A = \{x \in \mathbb{R}^d_+ \mid x_i = 0, \quad i \in I\}$. *However, if one has* $\mathbb{P}[X(t) \in \Gamma(I)] = 1$, *then* $\mathbb{P}[X(t) \in A] = 0$ *and hence* $X(t)$ *has a density on all* $\mathbb{R}^d_+$. *Since the branching and diffusion mechanism vanishes at the boundary, one cannot avoid to study the boundary behavior of multi-type CBI processes. For results applicable to one-dimensional processes we refer to [CPGUB13], [DFM14] and [FUB14], see also the references therein. It is also possible to obtain sufficient conditions for* $\mathbb{P}[X(t) \in \Gamma(I)] = 1$ *in arbitrary dimension; this will be studied in a seperate work.*

Note that the particular choice $\gamma_0 = 2$ is always possible, in which case Theorem 3 is precisely Theorem 2. For $\gamma_0 < \frac{1+\sqrt{5}}{2}$, one has $\frac{\gamma_0^2}{1+\gamma_0} < 1$ and hence we may take $\alpha_i \in (\frac{\gamma_0^2}{1+\gamma_0}, 1)$. In this case smoothing by immigration (see Example 1.(c)) may occur, which gives the following corollary.

Corollary 5. *Let* X *be a multi-type CBI process and suppose that the same conditions as in Theorem 3 are satisfied. If* $X(0)$ *satisfies* $\mathbb{E}[|X(0)|^{1+\tau}] < \infty$, *then*

$$\mathbb{P}[X_i(t) = 0, \quad i \notin I] = 0, \quad t > 0.$$

Proof. The set $\{x \in \mathbb{R}^d_+ \mid x_i = 0, \quad i \notin I\} \subset \Gamma(I)$ has Lebesgue measure zero. Since $X(t)$ has a density on $\Gamma(I)$, the assertion is proved. $\square$

Note that this corollary is only applicable in the presence of jumps from the immigration. Indeed, if $\nu = 0$, then condition (A) can be only satisfied for $I = \{1, \dots, d\}$. Another sufficient condition for $\mathbb{P}[X_i(t) = 0, \quad i \in \{1, \dots, d\}] = 0$, $t > 0$, will be discussed in a seperate work.

Let us finally consider a particular case without diffusion where the measures $\mu_1, \dots, \mu_d$ have the specific form

$$\mu_k(dz) = \widetilde{\mu}_k(dz_k) \otimes \prod_{j \neq k} \delta_0(dz_j), \quad k \in \{1, \dots, d\}, \tag{2.7}$$

with $\widetilde{\mu}_k$ being Lévy measures on $\mathbb{R}_+$ satisfying $\widetilde{\mu}_k(\{0\}) = 0$. In this case we obtain the following analogue of our previous statements.

Theorem 6. *Let* X *be a multi-type CBI process with admissible parameters* (c, β, B, ν, μ) *and assume that* (2.7) *holds and that* $c_1 = \dots = c_d = 0$. *Moreover suppose that*

(a) Condition (A) is satisfied for some $I \subset \{1, \dots, d\}$ *and* $\alpha_1, \dots, \alpha_d \in (0, 2)$.

(b) For each $j \in \{1, \dots, d\}$ there exists $\gamma_0^j \in (1,2]$ and $\tau_j \in (0, \gamma_0^j - 1)$ such that

$$\int\limits_{\mathbb{R}_+} \left(z^{\gamma_0^j} \mathbb{1}_{\{z \leq 1\}} + z^{1+\tau_j} \mathbb{1}_{\{z > 1\}} \right) \widetilde{\mu}_j(dz) + \int\limits_{|z|>1} |z|^{1+\tau_j} \nu(dz) < \infty.$$

If $I = \emptyset$, then we may also take $\tau_1 = \dots = \tau_d = 0$.

(c) It holds that $\alpha_i > \frac{\max\{\gamma_0^1, \dots, \gamma_0^d\}}{1+\max\{\gamma_0^1, \dots, \gamma_0^d\}} \gamma_0^i$. Moreover, for each $j \in I$, one has $\alpha_j \geq 1$.

If $X(0)$ satisfies $\mathbb{E}[|X(0)|^{1+\tau}] < \infty$, then for each $t > 0$, $X(t)$ has a density g_t on $\Gamma(I)$. Moreover, $f_t(x) := \rho_I(x) g_t(x) \in B_{1,\infty}^{\lambda,a}(\mathbb{R}^d)$, where $\lambda > 0$ is small enough and a is defined in (2.6).

Note that we have not assumed anything for the drift component B. In some particular cases where B does not mix different components too much, it is possible to obtain results with less restrictions on the parameters α_i, γ_0^i, etc. It is possible, but would be awful, to formulate a general statement. It is more convenient to apply the methods of this work directly to particular models of this type.

3 Main ingredients in the proofs

3.1 Anisotropic integration by parts

Define the anisotropic Hölder-Zygmund space $C_b^{\lambda,a}(\mathbb{R}^d)$ as the Banach space of functions ϕ with finite norm

$$\|\phi\|_{C_b^{\lambda,a}} = \|\phi\|_\infty + \sum_{k=1}^{d} \sup_{h \in [-1,1]} |h|^{-\lambda/a_k} \|\Delta_{he_k} \phi\|_\infty.$$

The following is our main technical tool for the existence of a density.

Lemma 7. *Let $a = (a_1, \dots, a_d)$ be an anisotropy in the sense of (2.1) and $\lambda, \eta > 0$ be such that $(\lambda + \eta)/a_k \in (0,1)$ holds for all $k = 1, \dots, d$. Suppose that q is a finite measure over $\mathbb{R}^d$ and there exists $A > 0$ such that, for all $\phi \in C_b^{\eta,a}(\mathbb{R}^d)$ and all $k = 1, \dots, d$,*

$$\left| \int\limits_{\mathbb{R}^d} (\phi(x + he_k) - \phi(x)) q(dx) \right| \leq A \|\phi\|_{C_b^{\eta,a}} |h|^{(\lambda+\eta)/a_k}, \quad \forall h \in [-1,1]. \tag{3.1}$$

Then q has a density g with respect to the Lebesgue measure such that

$$\|g\|_{B_{1,\infty}^{\lambda,a}} \leq q(\mathbb{R}^d) + 3dA(2d)^{\eta/\lambda} \left(1 + \frac{\lambda}{\eta} \right)^{1+\frac{\eta}{\lambda}}.$$

A proof of this Lemma is given in [FJR18]. The isotropic case, i.e. $a_1 = \dots = a_d = 1$, was first given in [DF13, DR14, Fou15]. Note that the restriction $(\lambda + \eta)/a_k \in (0,1)$, $k = 1, \dots, d$, is not essential since we may always replace $\lambda, \eta > 0$ by some smaller values which satisfy this condition and (3.1).

3.2 Multi-type CBI processes as strong solutions to stochastic equations

Our proof relies on the representation of multi-type CBI processes as solutions to a stochastic differential equation which is described below. Let (c, β, B, ν, μ) be a tuple of admissible parameters and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space rich enough to support following objects

(i) A d-dimensional Brownian motion $W = (W(t))_{t \geq 0}$.

(ii) Poisson random measures $N_1, \dots, N_d$ on $\mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+$ with compensators

$$\widehat{N}_j(du, dz, dr) = du\mu_j(dz)dr, \quad j \in \{1, \dots, d\}.$$

(iii) A Poisson random measure N_ν on $\mathbb{R}_+ \times \mathbb{R}_+^d$ with compensator $\widehat{N}_\nu(ds, dz) = ds\nu(dz)$.

The objects $W, N_\nu, N_1, \dots, N_d$ are supposed to be mutually independent. Denote by $\widetilde{N}_j = N_j - \widehat{N}_j$, $j \in \{1, \dots, d\}$, and $\widetilde{N}_\nu = N_\nu - \widehat{N}_\nu$ the corresponding compensated Poisson random measures. Let $X(0)$ be a random variable independent of the noise $W, N_\nu, N_1, \dots, N_d$. Then

$$\begin{aligned} X(t) = X(0) &+ \int_0^t \left(\beta + \widetilde{B}X(s)\right) ds + \sum_{k=1}^d \sqrt{2c_k} e_k \int_0^t \sqrt{X_k(s)} dW_k(s) + \int_0^t \int_{\mathbb{R}_+^d} z N_\nu(ds, dz) \qquad (3.2) \\ &+ \sum_{j=1}^d \int_0^t \int_{|z| \leq 1} \int_{\mathbb{R}_+} z \mathbb{1}_{\{r \leq X_j(s-)\}} \widetilde{N}_j(ds, dz, dr) + \sum_{j=1}^d \int_0^t \int_{|z| > 1} \int_{\mathbb{R}_+} z \mathbb{1}_{\{r \leq X_j(s-)\}} N_j(ds, dz, dr) \end{aligned}$$

has a pathwise unique strong solution, see [BLP15]. Here $\widetilde{B} = (\widetilde{b}_{ij})_{i,j \in \{1, \dots, d\}}$ is obtained by changing the compensator of the jump operator involving $(\mu_1, \dots, \mu_d)$. It is given by

$$\widetilde{b}_{ij} = b_{ij} + \mathbb{1}_{\{i \neq j\}} \int_{|z| \leq 1} z_i \mu_j(dz) - \mathbb{1}_{\{i = j\}} \mu_i(\{|z| > 1\}), \quad i, j \in \{1, \dots, d\}.$$

Note that $\widetilde{B}$ is well-defined and has non-negative off-diagonal entries. An application of the Itô-formula shows that X solves the martingale problem with generator (1.1), i.e. is a multi-type CBI process with admissible parameters (c, β, B, ν, μ).

3.3 Structure of the work

This work is organized as follows. In Section 4 we provide a general statement on the existence of densities for solutions to stochastic equations with Hölder continuous coefficients driven by a Brownian motion and a Poisson random measure. Our main results for multi-type CBI processes are then deduced in Section 5 from the results obtained in Section 4. Section 6 is devoted to the discussion of sufficient conditions for (A), while particular examples illustrating how our main results from Section 2 can be applied are discussed in Section 7. Some technical estimates for stochastic integrals with respect to Poisson random measures are collected in the appendix.

4 A general criterion for existence of a density

4.1 Description of the model

In this section we prove a general statement applicable to a wide class of stochastic equations driven by Brownian motions and Poisson random measures. Such equations should, in particular, include (3.2). Motivated by multi-type CBI processes we consider unbounded coefficients and treat the case of compensated small jumps, jumps of finite variation and big jumps separately.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with the usual conditions, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets and $(\mathcal{F}_t)_{t\geq 0}$ is a right-continuous filtration over $\mathcal{F}$. Suppose that the stochastic basis is rich enough to support the following objects

(i) A d-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $W = (W(t))_{t\geq 0}$.

(ii) An $(\mathcal{F}_t)_{t\geq 0}$-Poisson random measure N with compensator $\widehat{N}(du, dz) = dum(dz)$ on $\mathbb{R}_+ \times E$, where m is a σ-finite measure on some Polish space E.

Both terms are supposed to be independent. Denote by $\widetilde{N} = N - \widehat{N}$ the corresponding compensated Poisson random measure. Let $X(0)$ be an $\mathcal{F}_0$-measurable random variable independent of W and N. Consider an $(\mathcal{F}_t)_{t\geq 0}$-adapted cádlág-process $X = (X(t))_{t\geq 0}$ satisfying

$$X(t) = X(0) + \int_0^t b(X(u))du + \int_0^t \sigma(X(t))dW(t) + \int_0^t \int_{E_0} \sigma^0(X(u-), z)\widetilde{N}(du, dz) \tag{4.1}$$
$$+ \int_0^t \int_{E_1} \sigma^1(X(u-), z)N(du, dz) + \int_0^t \int_{E_2} \sigma^2(X(u-), z)N(du, dz),$$

where $E = E_0 \cup E_1 \cup E_2$ and E_0, E_1, E_2 are disjoint sets with $m(E_2) < \infty$. We suppose that $b, \sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\sigma^0, \sigma^1, \sigma^2 : \mathbb{R}^d \times E \longrightarrow \mathbb{R}^d$ are measurable, and satisfy

$$\sup_{|x|\leq R} \left(|b(x)| + |\sigma(x)| + \int_{E_0} |\sigma^0(x, z)|^2 m(dz) + \int_{E_1} |\sigma^1(x, z)| m(dz) \right) < \infty, \quad R > 0.$$

This implies, in particular, that the corresponding stochastic integrals in (4.1) are well-defined. Here E_0 corresponds to small (compensated) jumps, E_1 to jumps of finite variation and E_2 to big jumps.

Remark 8. *(i) One typically absorbs the finite variation terms into the definition of σ^0, σ^2, i.e., one has $E^1 = \emptyset$ and $\sigma^1 = 0$. However, having applications in mind it is reasonable to treat this cases differently.*

(ii) At first one may think that (3.2) *is more general, since it contains different independent Poisson random measures. However, since the particular form of E is not specified, we also cover this case as it is shown in Section 5.*

(iii) It is straightforward to extend all results obtained below to time-dependent coefficients.

4.2 Hölder regularity in time

Motivated by (3.2), we suppose that the coefficients of (4.1) are Hölder continuous and not necessarily bounded. Since an unbounded function f might be Hölder continuous with exponent $\gamma \in (0,1]$ without being Hölder continuous with exponent $\gamma' \in (0,\gamma)$, we have to keep track of the Hölder continuity for each component separately, see also Section 7 for particular examples. Below we suppose that the following conditions are satisfied:

(B1) For each $i \in \{1,\dots,d\}$, there exist $J_i(b) \subset \{1,\dots,d\}$, $\theta_i(b) \in [0,1]$ and $C > 0$ such that

$$|b_i(x) - b_i(y)| \leq C \sum_{j \in J_i(b)} |x_j - y_j|^{\theta_i(b)}.$$

(B2) For each $i \in \{1,\dots,d\}$, there exist $J_i(\sigma^0), J_i(\sigma^1), J_i(\sigma^2) \subset \{1,\dots,d\}$, $\theta_i(\sigma^0), \theta_i(\sigma^1), \theta_i(\sigma^2) \in [0,1]$, $\gamma_i(\sigma^0) \in (1,2]$, $\gamma_i(\sigma^1) \in (0,1]$, $\gamma_i(\sigma^2) \in (0, \gamma_i(\sigma^0)]$ and $C > 0$ such that

$$\int_{E_k} |\sigma_i^k(x,z) - \sigma_i^k(y,z)|^{\gamma_i(\sigma^k)} m(dz) \leq C \sum_{j \in J_i(\sigma^k)} |x_j - y_j|^{\theta_i(\sigma^k)\gamma_i(\sigma^k)}, \quad k \in \{0,1,2\}.$$

(B3) For each $i \in \{1,\dots,d\}$, there exists $J_i(\sigma) \subset \{1,\dots,d\}$ and $\theta_i(\sigma) \in [0,1]$ such that

$$|\sigma_{ik}(x) - \sigma_{ik}(y)| \leq \sum_{j \in J_i(\sigma)} |x_j - y_j|^{\theta_i(\sigma)}, \quad k \in \{1,\dots,d\}.$$

Thus $(\theta_i(b), \theta_i(\sigma), \theta_i(\sigma^0), \theta_i(\sigma^1), \theta_i(\sigma^2))$, $i \in \{1,\dots,d\}$, describe the Hölder exponents for the coefficients with respect to the space variables while the coupling of different components is described by the sets $J_i(b), J_i(\sigma), J_i(\sigma^0), J_i(\sigma^1), J_i(\sigma^2)$, $i \in \{1,\dots,d\}$. These sets are motivated by the particular form of (3.2). Define

$$\gamma_i = \max\{\mathbb{1}_{\sigma_i \neq 0} 2, \mathbb{1}_{\sigma_i^0 \neq 0}\gamma_i(\sigma^0), \mathbb{1}_{\sigma_i^1 \neq 0}\gamma_i(\sigma^1), \mathbb{1}_{\sigma_i^2 \neq 0}\gamma_i(\sigma^2)\},$$

where $\sigma_i = (\sigma_{i1},\dots,\sigma_{id})$, and similarly let

$$\gamma_{*,i} = \min\{\mathbb{1}_{\sigma_i \neq 0} 2, \mathbb{1}_{\sigma_i^0 \neq 0}\gamma_i(\sigma^0), \mathbb{1}_{\sigma_i^1 \neq 0}\gamma_i(\sigma^1), \mathbb{1}_{\sigma_i^2 \neq 0}\gamma_i(\sigma^2)\}.$$

We start with an estimate on time Hölder regularity for processes X given as in (4.1).

Lemma 9. *Suppose that (B1) – (B3) are satisfied, fix $i \in \{1,\dots,d\}$ and let $\eta \in (0, \gamma_{*,i}]$. Then, there exists a constant $C > 0$ such that, for all $0 \leq s \leq t \leq s+1$ and any X as in (4.1), one has*

$$\mathbb{E}[|X_i(t) - X_i(s)|^\eta] \leq C(t-s)^{\frac{\eta}{\gamma_i}} M_i(t,\eta),$$

where the constant C is independent of X, and

$$M_i(t,\eta) = \sum_{k=1}^{d} \sup_{u\in[s,t]} \mathbb{E}[|\sigma_{ik}(X(u))|^2]^{\eta/2} + \sup_{u\in[0,t]} \begin{cases} \mathbb{E}[|b_i(X(u))|^\eta], & \eta \geq 1 \\ \mathbb{E}[|b_i(X(u))|]^\eta, & \eta \in (0,1) \end{cases}$$

$$+ \sup_{u\in[0,t]} \mathbb{E}\left[\int_{E_0} |\sigma_i^0(X(u),z)|^{\gamma_i(\sigma^0)} m(dz)\right]^{\eta/\gamma_i(\sigma^0)}$$

$$+ \sup_{u\in[0,t]} \mathbb{E}\left[\int_{E_1} |\sigma_i^1(X(u),z)|^{\gamma_i(\sigma^1)} m(dz)\right]^{\eta/\gamma_i(\sigma^1)}$$

$$+ \sup_{u\in[0,t]} \mathbb{E}\left[\int_{E_2} |\sigma_i^2(X(u),z)|^{\gamma_i(\sigma^2)} m(dz)\right]^{\eta/\gamma_i(\sigma^2)}.$$

Proof. Observe that

$$\mathbb{E}[|X_i(t) - X_i(s)|^\eta] \leq C\mathbb{E}\left[\left|\int_s^t b_i(X(u))du\right|^\eta\right] + C\sum_{k=1}^{d} \mathbb{E}\left[\left|\int_s^t \sigma_{ik}(X(u))dW_k(u)\right|^\eta\right]$$

$$+ C\mathbb{E}\left[\left|\int_s^t \int_{E_0} \sigma_i^0(X(u-),z)\widetilde{N}(du,dz)\right|^\eta\right]$$

$$+ C\mathbb{E}\left[\left|\int_s^t \int_{E_1} \sigma_i^1(X(u-),z)N(du,dz)\right|^\eta\right] + C\mathbb{E}\left[\left|\int_s^t \int_{E_2} \sigma_i^2(X(u-),z)N(du,dz)\right|^\eta\right].$$

The first term is, for $\eta \geq 1$, estimated by the Hölder inequality

$$\mathbb{E}\left[\left|\int_s^t b_i(X(u))du\right|^\eta\right] \leq C(t-s)^\eta \sup_{u\in[0,t]} \mathbb{E}\left[|b_i(X(u))|^\eta\right]$$

and for $\eta \in (0,1)$ we get

$$\mathbb{E}\left[\left|\int_s^t b_i(X(u))du\right|^\eta\right] \leq \mathbb{E}\left[\left|\int_s^t b_i(X(u))du\right|\right]^\eta \leq (t-s)^\eta \sup_{u\in[0,t]} \mathbb{E}\left[|b_i(X(u))|\right]^\eta.$$

For the stochastic integral with respect to the Brownian motion we obtain from the BDG-inequality

$$\mathbb{E}\left[\left|\int_s^t \sigma_{ik}(X(u))dW_k(u)\right|^\eta\right] \leq \mathbb{E}\left[\left|\int_s^t |\sigma_{ik}(X(u))|^2 du\right|^{\eta/2}\right]$$

$$\leq (t-s)^{\eta/2} \sup_{u\in[s,t]} \mathbb{E}[|\sigma_{ik}(X(u))|^2]^{\eta/2}.$$

For the second stochastic integral we get by Lemma 19.(a) immediately

$$\mathbb{E}\left[\left|\int_s^t\int_{E_0}\sigma_i^0(X(u-),z)\widetilde{N}(du,dz)\right|^\eta\right]\leq C(t-s)^{\eta/\gamma_i(\sigma^0)}\sup_{u\in[0,t]}\mathbb{E}\left[\int_{E_0}|\sigma_i^0(X(u),z)|^{\gamma_i(\sigma^0)}m(dz)\right]^{\eta/\gamma_i(\sigma^0)}.$$

For the third stochastic integral we apply Lemma 19.(b) so that

$$\mathbb{E}\left[\left|\int_s^t\int_{E_1}\sigma_i^1(X(u-),z)N(du,dz)\right|^\eta\right]\leq C(t-s)^{\eta/\gamma_i(\sigma^1)}\sup_{u\in[0,t]}\mathbb{E}\left[\int_{E_1}|\sigma_i^1(X(u),z)|^{\gamma_i(\sigma^1)}m(dz)\right]^{\eta/\gamma_i(\sigma^1)}.$$

For the last stochastic integral we obtain by Lemma 19.(b) and $m(E_2)<\infty$

$$\mathbb{E}\left[\left|\int_s^t\int_{E_2}\sigma_i^2(X(u-),z)N(du,dz)\right|^\eta\right]\leq C(t-s)^{\eta/\gamma_i(\sigma^2)}\sup_{u\in[0,t]}\mathbb{E}\left[\int_{E_2}|\sigma_i^2(X(u),z)|^{\gamma_i(\sigma^2)}m(dz)\right]^{\eta/\gamma_i(\sigma^2)}.$$

The assertion now follows from $t-s\leq 1$ and since $\gamma_i(\sigma^0),\gamma_i(\sigma^1),\gamma_i(\sigma^2),2\leq\gamma_i$. $\square$

4.3 The approximation

For coefficients $b,\sigma,\sigma^0,\sigma^1,\sigma^2$ satisfying (B1) – (B3), $i\in\{1,\dots,d\}$ and $k\in\{0,1,2\}$, define

$$\kappa(\sigma_i^k)=\begin{cases}\frac{1}{\gamma_i(\sigma^k)}+\frac{\theta_i(\sigma^k)}{\gamma_i^*}, & \sigma_i^k \text{ is not constant}\\ +\infty, & \sigma_i^k \text{ is constant}\end{cases},$$

$$\kappa(\sigma_i)=\begin{cases}\frac{1}{2}+\frac{\theta_i(\sigma)}{\gamma_i^*}, & \sigma_i \text{ is not constant}\\ +\infty, & \sigma_i \text{ is constant}\end{cases},$$

$$\kappa(b_i)=\begin{cases}1+\frac{\theta_i(b)}{\gamma_i^*}, & b_i \text{ is not constant}\\ +\infty, & b_i \text{ is constant}\end{cases},$$

where $\sigma_i=(\sigma_{i1},\dots,\sigma_{1d})$ and $\gamma_i^*=\max\{\gamma_j\mid j\in J_i(b)\cup J_i(\sigma)\cup J_i(\sigma^k),\ k\in\{0,1,2\}\}$. Define

$$\kappa_i=\min\left\{\kappa(\sigma_i),\kappa(\sigma_i^0),\kappa(\sigma_i^1),\kappa(\sigma_i^2),\kappa(b_i)\right\}.$$

Hence, in cases where some of the coefficients $b_i,\sigma_i,\sigma_i^0,\sigma_i^1,\sigma_i^2$ are constant, we omit the corresponding terms in the definition of κ_i and set $J_i(b)=\emptyset$ or $J_i(\sigma)=\emptyset$ or $J_i(\sigma_i^k)=\emptyset$, respectively. The following is the main estimate for this section.

Proposition 10. *Suppose that (B1) – (B3) are satisfied and fix $i\in\{1,\dots,d\}$. Moreover, suppose that, for each $j\in\{1,\dots,d\}$,*

$$\max\{\mathbb{1}_{J_i(b)}(j)\theta_i(b),\mathbb{1}_{J_i(\sigma)}(j)2\theta_i(\sigma),\mathbb{1}_{J_i(\sigma^k)}(j)\theta_i(\sigma^k)\gamma_i(\sigma^k),\}\leq\gamma_{*,j},\quad k\in\{0,1,2\}. \tag{4.2}$$

Let X be as in (4.1) and define, for $t > 0$ and $\varepsilon \in (0, 1 \wedge t]$, the approximation $X_i^\varepsilon(t) = U_i^\varepsilon(t) + V_i^\varepsilon(t)$, where

$$U_i^\varepsilon(t) = X_i(t-\varepsilon) + \varepsilon b_i(X(t-\varepsilon)) + \int_{t-\varepsilon}^{t} \int_{E_2} \sigma_i^2(X(t-\varepsilon), z) N(du, dz), \tag{4.3}$$

$$\begin{aligned} V_i^\varepsilon(t) &= \sum_{k=1}^{d} \sigma_{ik}(X(t-\varepsilon))(W_k(t) - W_k(t-\varepsilon)) \\ &\quad + \int_{t-\varepsilon}^{t} \int_{E_0} \sigma_i^0(X(t-\varepsilon), z) \widetilde{N}(du, dz) + \int_{t-\varepsilon}^{t} \int_{E_1} \sigma_i^1(X(t-\varepsilon), z) N(du, dz). \end{aligned} \tag{4.4}$$

Then, for any $0 < \eta \le 1 \wedge \gamma_{*,i}$,

$$\mathbb{E}[|X_i(t) - X_i^\varepsilon(t)|^\eta] \le C \varepsilon^{\eta \kappa_i} H_i(t, \eta), \quad t > 0, \quad \varepsilon \in (0, 1 \wedge t],$$

where the constant $C > 0$ is independent of ε, t and X, and

$$\begin{aligned} H_i(t,\eta) = & \sum_{j \in J_i(\sigma^0)} M_j(t, \theta_i(\sigma^0)\gamma_i(\sigma^0))^{\eta/\gamma_i(\sigma^0)} + \sum_{j \in J_i(\sigma^1)} M_j(t, \theta_i(\sigma^1)\gamma_i(\sigma^1))^{\eta/\gamma_i(\sigma^1)} \\ & + \sum_{j \in J_i(\sigma^2)} M_j(t, \theta_i(\sigma^2)\gamma_i(\sigma^2))^{\eta/\gamma_i(\sigma^2)} + \sum_{j \in J_i(b)} M_j(t, \theta_i(b))^\eta + \sum_{j \in J_i(\sigma)} M_j(t, 2\theta_i(\sigma))^{\eta/2}. \end{aligned}$$

Proof. Fix $t > 0$, $\varepsilon \in (0, 1 \wedge t]$ and let $\eta \in (0, 1 \wedge \gamma_{*,i}]$. Then

$$\begin{aligned} \mathbb{E}[|X_i(t) - X_i^\varepsilon(t)|^\eta] &\le R_0 + R_1 + R_2 + R_3 + R_4, \\ R_0 &= \mathbb{E}\left[\left| \int_{t-\varepsilon}^{t} \int_{E_0} \left(\sigma_i^0(X(u-), z) - \sigma_i^0(X(t-\varepsilon), z) \right) \widetilde{N}(du, dz) \right|^\eta \right], \\ R_1 &= \mathbb{E}\left[\left| \int_{t-\varepsilon}^{t} \int_{E_1} \left(\sigma_i^1(X(u-), z) - \sigma_i^1(X(t-\varepsilon), z) \right) N(du, dz) \right|^\eta \right], \\ R_2 &= \mathbb{E}\left[\left| \int_{t-\varepsilon}^{t} \int_{E_2} \left(\sigma_i^2(X(u-), z) - \sigma_i^2(X(t-\varepsilon), z) \right) N(du, dz) \right|^\eta \right], \\ R_3 &= \mathbb{E}\left[\left| \int_{t-\varepsilon}^{t} (b_i(X(u)) - b_i(X(t-\varepsilon))) du \right|^\eta \right], \\ R_4 &= \sum_{k=1}^{d} \mathbb{E}\left[\left| \int_{t-\varepsilon}^{t} \left(\sigma_{ik}(X(u)) - \sigma_{ik}(X(t-\varepsilon)) \right) dW_k(u) \right|^\eta \right]. \end{aligned}$$

For R_0 we first apply Lemma 19, then condition (B2) and finally Lemma 9 to obtain

$$
\begin{aligned}
R_0 &\leq C\varepsilon^{\eta/\gamma_i(\sigma^0)} \sup_{u\in(t-\varepsilon,t]} \mathbb{E}\left[\int\limits_{E_0} \left|\sigma_i^0(X(u-),z) - \sigma_i^0(X(t-\varepsilon),z)\right|^{\gamma_i(\sigma^0)} m(dz)\right]^{\eta/\gamma_i(\sigma^0)} \\
&\leq C\varepsilon^{\eta/\gamma_i(\sigma^0)} \sum_{j\in J_i(\sigma^0)} \sup_{u\in(t-\varepsilon,t]} \mathbb{E}\left[|X_j(u-) - X_j(t-\varepsilon)|^{\theta_i(\sigma^0)\gamma_i(\sigma^0)}\right]^{\eta/\gamma_i(\sigma^0)} \\
&\leq C\varepsilon^{\eta/\gamma_i(\sigma^0)} \sum_{j\in J_i(\sigma^0)} \varepsilon^{\eta\theta_i(\sigma^0)/\gamma_j} M_j(t,\theta_i(\sigma^0)\gamma_i(\sigma^0))^{\eta/\gamma_i(\sigma^0)}.
\end{aligned}
$$

For R_1 we apply Lemma 19.(b), use assumption (B2) and proceed as before to deduce

$$
R_1 \leq \varepsilon^{\eta/\gamma_i(\sigma^1)} \sum_{j\in J^i(\sigma^1)} \varepsilon^{\eta\theta_i(\sigma^1)/\gamma_j} M_j(t,\theta_i(\sigma^1)\gamma_i(\sigma^1))^{\eta/\gamma_i(\sigma^1)}.
$$

For R_2 we apply Lemma 19.(b) and proceed as before to deduce

$$
R_2 \leq \varepsilon^{\eta/\gamma_i(\sigma^2)} \sum_{j\in J^i(\sigma^2)} \varepsilon^{\eta\theta_i(\sigma^2)/\gamma_j} M_j(t,\theta_i(\sigma^2)\gamma_i(\sigma^2))^{\eta/\gamma_i(\sigma^2)}.
$$

For R_3 we apply (B1) and Lemma 9 so that

$$
\begin{aligned}
R_3 &\leq C\mathbb{E}\left[\sum_{j\in J_i(b)} \int\limits_{t-\varepsilon}^{t} |X_j(u-) - X_j(t-\varepsilon)|^{\theta_i(b)} du\right]^{\eta} \\
&\leq C\varepsilon^{\eta} \sum_{j\in J_i(b)} \sup_{u\in(t-\varepsilon,t]} \mathbb{E}[|X_j(u-) - X_j(t-\varepsilon)|^{\theta_i(b)}]^{\eta} \\
&\leq C\varepsilon^{\eta} \sum_{j\in J_i(b)} \varepsilon^{\eta\theta_i(b)/\gamma_j} M_j(t,\theta_i(b))^{\eta}.
\end{aligned}
$$

For the last term we obtain from the BDG-inequality, (B3) and Lemma 9

$$
\begin{aligned}
R_4 &\leq C\varepsilon^{\eta/2} \sup_{u\in(t-\varepsilon,t]} \mathbb{E}\left[|\sigma_{ik}(X(u)) - \sigma_{ik}(X(t-\varepsilon))|^2\right]^{\eta/2} \\
&\leq C\varepsilon^{\eta/2} \sum_{j\in J_i(\sigma)} \sup_{u\in(t-\varepsilon,t]} \mathbb{E}\left[|X_j(u) - X_j(t-\varepsilon)|^{2\theta_i(\sigma)}\right]^{\eta/2} \\
&\leq C\varepsilon^{\eta/2} \sum_{j\in J_i(\sigma)} \varepsilon^{\eta\theta_i(\sigma)/\gamma_j} M_j(t,2\theta_i(\sigma))^{\eta/2}.
\end{aligned}
$$

Collecting all estimates and using the definition of $H_i(t,\eta)$ and κ_i gives the assertion. $\square$

4.4 Main estimate

In this section we prove our main estimate used to prove existence of densities. For each $t \in (0,1]$ and $x \in \mathbb{R}^d$ define

$$L^x(t) := \sigma(x)W(t) + \int_0^t \int_{E_0} \sigma^0(x,z)\widetilde{N}(du,dz) + \int_0^t \int_{E_1} \sigma^1(x,z)N(du,dz). \tag{4.5}$$

The following condition guarantees that the noise part has some smoothing property. As usual write $1/0 := +\infty$.

(B4) There exist $\rho : \mathbb{R}^d \longrightarrow [0,\infty)$ and $(\alpha_i)_{i\in\{1,\dots,d\}} \subset (0,2]$ such that $L^x(t)$ has, for each $t \in (0,1]$ and $x \in \Gamma := \{y \in \mathbb{R}^d \mid \rho(y) > 0\}$, a density g_t^x with respect to the Lebesgue measure and, for all $i \in \{1,\dots,d\}$,

$$\limsup_{t\to 0} t^{1/\alpha_i} \int_{\mathbb{R}^d} |g_t^x(z+e_ih) - g_t^x(z)|\, dz \le \frac{|h|}{\rho(x)}, \quad h \in [-1,1]. \tag{4.6}$$

Note that, for $x \notin \Gamma$, the right-hand side of (4.6) equals to $+\infty$ in which case nothing has to be verified. The following is our main estimate for this section.

Proposition 11. *Assume that (B1) – (B4) are satisfied and suppose that* (4.2) *holds for all* $i \in \{1,\dots,d\}$. *Let* $(X(t))_{t\ge 0}$ *be as in* (4.1) *with the additional properties*

(i) There exists $\tau > 0$ *such that, for each* $i \in \{1,\dots,d\}$, *one has*

$$G_{j,i}(t) < \infty, \qquad j \in J_i(b) \cup J_i(\sigma) \cup J_i(\sigma^0) \cup J_i(\sigma^1) \cup J_i(\sigma^2),$$

where $\zeta_i = \max\{1, 2\theta_i(\sigma), \theta_i(\sigma^0)\gamma_i(\sigma^0), \theta_i(\sigma^1)\gamma_i(\sigma^1), \theta_i(\sigma^2)\gamma_i(\sigma^2)\}$ *and*

$$\begin{aligned} G_{j,i}(t) = &\sum_{k=1}^d \sup_{u\in[0,t]} \mathbb{E}[|\sigma_{jk}(X(u))|^2] + \sup_{u\in[0,t]} \mathbb{E}\left[\int_{E_0} |\sigma_j^0(X(u),z)|^{\gamma_i(\sigma^0)} m(dz)\right] \\ &+ \sup_{u\in[0,t]} \mathbb{E}\left[\int_{E_1} |\sigma_j^1(X(u),z)|^{\gamma_i(\sigma^1)} m(dz)\right] + \sup_{u\in[0,t]} \mathbb{E}\left[\int_{E_2} |\sigma_j^2(X(u),z)|^{\gamma_i(\sigma^2)} m(dz)\right] \\ &+ \sup_{u\in[0,t]} \mathbb{E}\left[|b_j(X(u))|^{\zeta_i}\right] + \sup_{u\in[0,t]} \mathbb{E}[\rho(X(u))^{1+\tau}], \end{aligned}$$

(ii) There exists $\delta > 0$ *such that, for any* $t > 0$ *and* $\varepsilon \in (0, 1\wedge t]$,

$$\mathbb{E}[|\rho(X(t)) - \rho(X(t-\varepsilon))|] \le C\varepsilon^\delta, \tag{4.7}$$

where $C = C_t > 0$ *is independent of* ε *and locally bounded in* t.

Let $a = (a_i)_{i\in\{1,\dots,d\}}$ be an anisotropy and $\eta \in (0,1)$ with

$$\left(1+\frac{1}{\tau}\right)\frac{\eta}{a_i} \leq 1 \wedge \gamma_{*,i}, \quad i \in \{1,\dots,d\}. \tag{4.8}$$

Then there exists a constant $C = C_{t,\eta} > 0$ (locally bounded in t) and $\varepsilon_0 \in (0, 1\wedge t)$ such that, for any $\varepsilon \in (0,\varepsilon_0)$, $h \in [-1,1]$, $\phi \in C_b^{\eta,a}(\mathbb{R}^d)$ and $i \in \{1,\dots,d\}$,

$$|\mathbb{E}\left[\rho(X(t))\Delta_{he_i}\phi(X(t))\right]| \leq C\|\phi\|_{C_b^{\eta,a}}\left(|h|^{\eta/a_i}\varepsilon^{\delta} + |h|\varepsilon^{-1/\alpha_i} + \max_{j\in\{1,\dots,d\}}\varepsilon^{\eta\kappa_j/a_j}\right).$$

Proof. For $\varepsilon \in (0,1\wedge t)$ let $X^\varepsilon(t)$ be the approximation from Proposition 10. Then

$$\begin{aligned}
|\mathbb{E}\left[\rho(X(t))\Delta_{he_i}\phi(X(t))\right]| &\leq R_1 + R_2 + R_3,\\
R_1 &= |\mathbb{E}\left[\Delta_{he_i}\phi(X(t))\left(\rho(X(t)) - \rho(X(t-\varepsilon))\right)\right]|,\\
R_2 &= \mathbb{E}\left[|\Delta_{he_i}\phi(X(t)) - \Delta_{he_i}\phi(X^\varepsilon(t))|\rho(X(t-\varepsilon))\right],\\
R_3 &= |\mathbb{E}\left[\rho(X(t-\varepsilon))\Delta_{he_i}\phi(X^\varepsilon(t))\right]|.
\end{aligned}$$

For the first term we can use (4.7) to obtain

$$R_1 \leq \|\phi\|_{C_b^{\eta,a}}|h|^{\eta/a_i}\mathbb{E}[|\rho(X(t)) - \rho(X(t-\varepsilon))|] \leq C\|\phi\|_{C_b^{\eta,a}}|h|^{\eta/a_i}\varepsilon^{\delta}.$$

For R_2, the Hölder inequality with $\frac{1}{1+\tau} + \frac{1}{1+\frac{1}{\tau}} = 1$ implies

$$\begin{aligned}
R_2 &\leq C\|\phi\|_{C_b^{\eta,a}}\max_{j\in\{1,\dots,d\}}\mathbb{E}\left[\rho(X(t-\varepsilon))|X_j(t) - X_j^\varepsilon(t)|^{\eta/a_j}\right]\\
&\leq C\|\phi\|_{C_b^{\eta,a}}\sup_{u\in[0,t]}\mathbb{E}\left[\rho(X(u))^{1+\tau}\right]^{1/(1+\tau)}\max_{j\in\{1,\dots,d\}}\mathbb{E}\left[|X_j(t) - X_j^\varepsilon(t)|^{\left(1+\frac{1}{\tau}\right)\frac{\eta}{a_j}}\right]^{\frac{\tau}{1+\tau}}\\
&\leq C\|\phi\|_{C_b^{\eta,a}}\sup_{u\in[0,t]}\mathbb{E}\left[\rho(X(u))^{1+\tau}\right]^{1/(1+\tau)}\max_{j\in\{1,\dots,d\}}\varepsilon^{\eta\kappa_j/a_j},
\end{aligned}$$

where in the last inequality we have used (4.8) and $G_{j,i}(t) < \infty$ so that Lemma 9 is applicable. Let us turn to R_3. Let g_t^x be the density given by (B4) and write $X^\varepsilon(t) = U^\varepsilon(t) + V^\varepsilon(t)$, where $U^\varepsilon(t)$ and $V^\varepsilon(t)$ are given by (4.3) and (4.4). By (B4) there exists $\varepsilon_0 > 0$ small enough such that for any $\varepsilon \in (0,\varepsilon_0)$,

$$\begin{aligned}
R_3 &= \left|\mathbb{E}\left[\int_{\mathbb{R}^d}\rho(X(t-\varepsilon))(\Delta_{he_i}\phi)(U^\varepsilon(t)+z)g_\varepsilon^{X(t-\varepsilon)}(z)dz\right]\right|\\
&= \left|\mathbb{E}\left[\int_{\mathbb{R}^d}\rho(X(t-\varepsilon))\phi(U^\varepsilon(t)+z)(\Delta_{-he_i}g_\varepsilon^{X(t-\varepsilon)})(z)dz\right]\right| \leq C\|\phi\|_{C_b^{\eta,a}}|h|\varepsilon^{-1/\alpha_i},
\end{aligned}$$

where we have used (4.6). Summing up the estimates for R_0, R_1, R_2, R_3 yields the assertion. □

Remark 12. *Note that for bounded coefficients $b, \sigma, \sigma^0, \sigma^1, \sigma^2$ the restriction $G_{j,i}(t) < \infty$ is automatically satisfied. More generally, in many cases it suffices to show that X has finite second moments. For the particular case of multi-type CBI processes even less is sufficient, see Section 2.*

4.5 Existence of the density

The following is the main result on the existence of densities for (4.1).

Theorem 13. *Assume that (B1) – (B4) are satisfied and suppose that* (4.2) *holds for all* $i \in \{1, \dots, d\}$ *and,*

$$\kappa_i \alpha_i > 1, \quad \forall i \in \{1, \dots, d\}. \tag{4.9}$$

Let $(X(t))_{t\geq 0}$ *be as in* (4.1) *with the properties (i) and (ii) from Proposition 11. Define an anisotropy* $a = (a_i)_{i\in\{1,\dots,d\}}$ *and a mean order of smoothness* $\overline{\alpha}$ *as in* (2.6)*. Then there exists* $\lambda \in (0,1)$ *such that the finite measure* q_t *given by*

$$q_t(A) = \mathbb{E}[\rho(X(t))\mathbb{1}_A(X(t))], \quad \forall A \subset \mathbb{R}^d \quad \textit{Borel} ,$$

has, for every $t > 0$*, a density* $g_t \in B^{\lambda,a}_{1,\infty}(\mathbb{R}^d)$ *with respect to the Lebesgue measure and*

$$\|g_t\|_{B^{\lambda,a}_{1,\infty}} \leq q_t(\mathbb{R}^d) + h(t)(1\wedge t)^{-1/\alpha^{\min}},$$

where $h : [0,\infty) \longrightarrow (0,\infty)$ *is locally bounded in* t *and* $\alpha^{\min} = \min\{\alpha_1, \dots, \alpha_d\}$.

Proof. Let $t > 0$ be fixed. It suffices to show that Lemma 7 is applicable to q_t. Using (4.9) we obtain $\frac{\kappa_j}{a_j} > 1/\overline{\alpha}$ for all $j \in \{1, \dots, d\}$ and hence $\frac{a_j}{\kappa_j}\frac{1}{a_i} < \frac{\overline{\alpha}}{a_i} = \alpha_i$ for all $i, j \in \{1, \dots, d\}$. Hence we find $\eta \in (0,1)$ and $c_1, \dots, c_d > 0$ such that, for all $i, j \in \{1, \dots, d\}$,

$$0 < \left(1 + \frac{1}{\tau}\right)\frac{\eta}{a_i} < 1 \wedge \gamma_{*,i}, \qquad \frac{a_j}{\kappa_j}\frac{1}{a_i} < c_i < \alpha_i\left(1 - \frac{\eta}{a_i}\right).$$

Define

$$\lambda = \min_{i,j\in\{1,\dots,d\}}\left\{c_i\delta a_i,\ a_i - \eta - \frac{a_i c_i}{\alpha_i},\ \eta\left(c_i a_i \frac{\kappa_j}{a_j} - 1\right)\right\} > 0.$$

Let $\phi \in C_b^{\eta,a}(\mathbb{R}^d)$. By Proposition 11 we obtain, for $h \in [-1,1]$, $\varepsilon = |h|^{c_i}(1\wedge t)$ and $i \in \{1, \dots, d\}$,

$$\begin{aligned}
|\mathbb{E}\left[\rho(X(t))\Delta_{he_i}\phi(X(t))\right]| &\leq C\|\phi\|_{C_b^{\eta,a}}\left(|h|^{\eta/a_i}\varepsilon^{\delta} + |h|\varepsilon^{-1/\alpha_i} + \max_{j\in\{1,\dots,d\}}\varepsilon^{\eta\kappa_j/a_j}\right)\\
&\leq \frac{C\|\phi\|_{C_b^{\eta,a}}}{(1\wedge t)^{1/\alpha_i}}\left(|h|^{\eta/a_i + c_i\delta} + |h|^{1-c_i/\alpha_i} + \max_{j\in\{1,\dots,d\}}|h|^{c_i\eta\kappa_j/a_j}\right)\\
&= \frac{C\|\phi\|_{C_b^{\eta,a}}}{(1\wedge t)^{1/\alpha_i}}|h|^{\eta/a_i}\left(|h|^{c_i\delta} + |h|^{1-\eta/a_i - c_i/\alpha_i} + \max_{j\in\{1,\dots,d\}}|h|^{c_i\eta\kappa_j/a_j - \eta/a_i}\right)\\
&\leq \frac{C\|\phi\|_{C_b^{\eta,a}}}{(1\wedge t)^{1/\alpha_i}}|h|^{(\eta+\lambda)/a_i}.
\end{aligned}$$

The assertion now follows from Lemma 7. □

By inspection of the proof, we obtain the following extension.

Remark 14. *Estimate* (4.7) *can be replaced by the integrability condition*

$$\sup_{t\in[0,T]}\mathbb{E}[\rho(X(t))^{-1}] < \infty, \quad \forall T > 0.$$

In such a case $X(t)$ *has, for* $t > 0$*, a density on* $\mathbb{R}^d$ *(not only on* Γ*).*

5 Application to multi-type CBI processes

5.1 Proof of Theorem 2

Our aim is to show that Theorem 13 is applicable. Let us first show that (3.2) is a particular case of (4.1). Indeed, letting $b(x) := \beta + \widetilde{B}x$ and $\sigma(x) = \mathrm{diag}(\sqrt{2c_1 x_1}, \dots, \sqrt{2c_d x_d})$ we see that the first two terms have the desired form. Concerning the jumps let $E = \mathbb{R}_+^d \times \mathbb{R}_+ \times \{1, \dots, d+1\}$ and set $E_0 = \{z \in \mathbb{R}_+^d \mid |z| \leq 1\} \times \mathbb{R}_+ \times \{1, \dots, d\}$, $E_2 = \{z \in \mathbb{R}_+^d \mid |z| > 1\} \times \mathbb{R}_+ \times \{1, \dots, d\}$, $E_1 = \mathbb{R}_+^d \times \mathbb{R}_+ \times \{d+1\}$. Define the corresponding intensity measure $m(d\xi)$, where $\xi = (z, r, k) \in E$, by

$$m(d\xi) = \sum_{j=1}^{d} \mu_j(dz) dr \delta_j(dk) + \nu(dz)\delta_0(dr)\delta_{d+1}(dk).$$

Finally choose $\sigma_i^1(x, \xi) = z_i$ and

$$\sigma_i^0(x, \xi) = z_i \mathbb{1}_{\{r \leq x_k\}} \mathbb{1}_{\mathbb{R}_+}(x_k) \mathbb{1}_{\{1,\dots,d\}}(k), \qquad \sigma_i^2(x, \xi) = z_i \mathbb{1}_{\{r \leq x_k\}} \mathbb{1}_{\mathbb{R}_+}(x_k) \mathbb{1}_{\{1,\dots,d\}}(k).$$

Then it is not difficult to see that (3.2) is equivalent in law to (4.1) with paramters defined above. It is easily seen from the Itô formula that both equations pose the same martingale problem. Hence they describe the same law, which is sufficient for our purposes. Let us show that conditions (B1) – (B4) are satisfied. Indeed (B1) is satisfied for $J_i(b) = \{1, \dots, d\}$ and $\theta_i(b) = 1$. Concerning condition (B2) we see that

$$\begin{aligned}
\int_{E_0} |\sigma_i^0(x,\xi) - \sigma_i^0(y,\xi)|^2 m(d\xi) &= \sum_{k=1}^{d} \int_{|z|\leq 1} \int_0^\infty |\mathbb{1}_{\{r \leq x_k\}} \mathbb{1}_{\mathbb{R}_+}(x_k) - \mathbb{1}_{\{r \leq y_k\}} \mathbb{1}_{\mathbb{R}_+}(y_k)| z_i^2 m(dz, dr, \{k\}) \\
&= \sum_{k=1}^{d} \int_{|z|\leq 1} z_i^2 \mu_k(dz) |x_k - y_k| \\
&\leq \max_{j \in \{1,\dots,d\}} \int_{|z|\leq 1} |z|^2 \mu_j(dz) \sum_{k=1}^{d} |x_k - y_k|
\end{aligned}$$

and hence we may choose $J_i(\sigma^0) = \{1, \dots, d\}$, $\theta_i(\sigma^0) = \frac{1}{2}$ and $\gamma_i(\sigma^0) = 2$. For the integral against σ^1 we obtain $\int_{E_1} |\sigma_i^1(x,\xi) - \sigma_i^1(y,\xi)| m(d\xi) = 0$, i.e. $J_i(\sigma^1) = \emptyset$, $\theta_i(\sigma^1) = 1$ and $\gamma_i(\sigma^1) = 1$. In the same way we show that

$$\int_{E_2} |\sigma_i^2(x,\xi) - \sigma_i^2(y,\xi)|^{1+\tau} m(d\xi) \leq \max_{j \in \{1,\dots,d\}} \int_{|z|>1} |z|^{1+\tau} \mu_j(dz) \sum_{k=1}^{d} |x_k - y_k|,$$

i.e. $J_i(\sigma^2) = \{1, \dots, d\}$, $\theta_i(\sigma^2) = \frac{1}{1+\tau}$ and $\gamma_i(\sigma^2) = 1 + \tau$. This shows that (B2) is satisfied. Condition (B3) is clearly satisfied with $J_i(\sigma) = \{i\}$ and $\theta_i(\sigma) = \frac{1}{2}$. For the noise part (4.5) appearing in condition (B4) we obtain

$$\begin{aligned} L_i^x(t) &= \mathbb{1}_{\mathbb{R}_+}(x_i)\sqrt{2c_i x_i}B_i(t) + \int_0^t \int_{E_0} z_i \mathbb{1}_{\{r\le x_k\}}\mathbb{1}_{\mathbb{R}_+}(x_k)\widetilde{N}(du,d\xi) + \int_0^t\int_{E_1} z_i N(du,d\xi) \\ &= \mathbb{1}_{\mathbb{R}_+}(x_i)\sqrt{2c_i x_i}B_i(t) + \sum_{j=1}^d \int_0^t \int_{|z|\le 1}\int_{\mathbb{R}_+} z_i \mathbb{1}_{\{r\le x_j\}}\mathbb{1}_{\mathbb{R}_+}(x_j)\widetilde{N}_j(du,dz,dr) + \int_0^t\int_{\mathbb{R}_+^d} z_i N_\nu(du,dz), \end{aligned}$$

where $N_\nu, N_1, \dots, N_d$ are given as in (3.2) and the second equality holds in law. Hence L_i^x given by (4.5) is precisely (2.3). In particular, (2.5) is precisely (B4) with $\rho(x) = \min\{x_1,\dots,x_d\}\mathbb{1}_{\mathbb{R}_+^d}(x)$. Observe that $G_{ji}(t)$ satisfies

$$G_{ji}(t) \le C\left(1 + \sup_{u\in[0,t]}\mathbb{E}[|X(u)|] + \sup_{u\in[0,t]}\mathbb{E}[|X(u)|^{1+\tau}]\right) \le C\left(1+\sup_{u\in[0,t]}\mathbb{E}[|X(u)|^{1+\tau}]\right),$$

i.e. it suffices to show that the right-hand side is finite. However, in view of assumption (b) from Theorem 2, this property can be classically shown by Gronwall. Note that, by $\gamma_i(\sigma^1) = 1$, one has $\gamma_{*,i} = 1$ and hence (4.2) is satisfied. Next observe that $\gamma_i = 2$ and hence (4.7) follows from

$$\mathbb{E}[|\rho(X(t)) - \rho(X(t-\varepsilon))|] \le \sum_{j=1}^d \mathbb{E}[|X_j(t) - X_j(t-\varepsilon)|^{1/\alpha_j}] \le C\sum_{j=1}^d \varepsilon^{\frac{1}{2\alpha_j}} \le C\varepsilon^{1/4}, \tag{5.1}$$

where we have used Lemma 9 which is applicable due to $\gamma_{*,j} = 1 \ge \frac{3}{4} > 1/\alpha_j$. Finally, we have $\kappa_i = \frac{3}{4}$ and hence (4.9) is equivalent to $\alpha_i > \frac{4}{3}$, which proves the assertion.

5.2 Proof of Theorem 3

We proceed similarly to the previous case. Namely, observe that (3.2), with $c_1 = \dots = c_d = 0$, is equivalent in law to (4.1) for the particular choice $\sigma(x) = 0$ and $b, E, E_0, E_1, E_2, m, \sigma^0, \sigma^1, \sigma^2$ the same as in the proof of Theorem 2. Conditions (B1) – (B3) are satisfied for $J_i(\sigma) = J_i(\sigma^1) = \emptyset$, $J_i(\sigma^0) = J_i(\sigma^2) = J_i(b) = \{1,\dots,d\}$, $\theta_i(b) = 1$, $\theta_i(\sigma^0) = \frac{1}{\gamma_0}$, $\theta_i(\sigma^1) = 1$, $\theta_i(\sigma^2) = \frac{1}{1+\tau}$, $\theta_i(\sigma) = 1$, $\gamma_i(\sigma^0) = \gamma_0$, $\gamma_i(\sigma^1) = 1$ and $\gamma_i(\sigma^2) = 1+\tau$. The noise part (4.5) appearing in condition (B4) is precisely (2.3), i.e. (B4) follows from condition (A) with $\rho(x) = \rho_I(x)$. Estimating G_{ji} as before, we see that, for $I = \emptyset$ and hence $\rho_\emptyset = 1$, we may take $\tau = 0$. Condition (4.2) can be shown as in the proof of Theorem 2. For (4.7) we obtain

$$\mathbb{E}[|\rho_I(X(t)) - \rho_I(X(t-\varepsilon))|] \le \sum_{j\in I}\mathbb{E}[|X_j(t) - X_j(t-\varepsilon)|^{1/\alpha_j}].$$

Since, for $j \in I$, we have $\alpha_j \ge 1 = \gamma_{*,j}$, we may proceed exactly as in (5.1). Finally, we have $\gamma_i = \gamma_i^* = \gamma_0$ and hence $\kappa_i = \frac{1}{\gamma_0}\left(1 + \frac{1}{\gamma_0}\right)$. Thus (4.9) is equivalent to $\alpha_i > \frac{\gamma_0}{1+\gamma_0}\gamma_0$, which proves the assertion.

5.3 Proof of Theorem 6

We proceed similarly to the previous cases. Namely, (3.2) is equivalent in law to (4.1) for the same choice as in the proof of Theorem 3. A simple computation shows that conditions (B1) – (B3) are satisfied for $J_i(\sigma) = J_i(\sigma^1) = \emptyset$, $J_i(\sigma^0) = J_i(\sigma^2) = \{i\}$, $J_i(b) = \{1, \dots, d\}$, $\theta_i(b) = 1$, $\theta_i(\sigma^0) = \frac{1}{\gamma_0^i}$, $\theta_i(\sigma^1) = 1$, $\theta_i(\sigma^2) = \frac{1}{1+\tau_i}$, $\theta_i(\sigma) = 1$, $\gamma_i(\sigma^0) = \gamma_0^i$, $\gamma_i(\sigma^1) = 1$ and $\gamma_i(\sigma^2) = 1 + \tau_i$. The noise part (4.5) appearing in condition (B4) is precisely (2.3), i.e. (B4) follows from condition (A) with $\rho(x) = \rho_I(x)$. The function G_{ji} can be estimated exactly as before (here we need that $J_i(\sigma^0) = J_i(\sigma^2) = \{i\}$). Using $\gamma_{*,j} = 1$ we see that (4.2) is satisfied. Condition (4.7) can be shown in the same way as in the proof of Theorem 3. Finally, we have $\gamma_i = \gamma_0^i$, thus $\gamma_i^* = \max\{\gamma_0^1, \dots, \gamma_0^i\} =: \gamma^*$, $\kappa_i = \frac{1}{\gamma_0^i}\left(1 + \frac{1}{\gamma^*}\right)$. Hence (4.9) is equivalent to $\alpha_i > \frac{\gamma^*}{1+\gamma^*}\gamma_0^i$, which proves the assertion.

6 On the smoothing property (A)

The following is due to [DF13, Lemma 3.3].

Proposition 15. *Let Z be a Lévy process with Lévy measure m and symbol*

$$\Psi_m(\xi) = \int\limits_{\mathbb{R}^d} \left(1 + i\xi \cdot z \mathbb{1}_{\{|z| \le 1\}} - e^{i\xi \cdot z}\right) m(dz).$$

Suppose that there exist $\alpha \in (0, 2]$ and $c, C > 0$ with

$$c|\xi|^\alpha \le \mathrm{Re}(\Psi_m(\xi)) \le C|\xi|^\alpha, \quad \forall \xi \in \mathbb{R}^d, \quad |\xi| \gg 1.$$

Then for each $t > 0$, $Z(t)$ has a smooth density f_t and there exists a constant $C > 0$ such that

$$\|\nabla f_t\|_{L^1(\mathbb{R}^d)} \le C t^{-1/\alpha}, \quad t > 0.$$

Below we provide two sufficient conditions for (A). Our first result is a more general version of Example 1.(a).

Lemma 16. *Define $I_1 = \{j \in \{1, \dots, d\} \mid c_j > 0\}$ and let $I_2 := \{1, \dots, d\} \backslash I_1$. Suppose that, for each $j \in I_2$, there exists a Lévy measure $\widetilde{\mu}_j$ on $\mathbb{R}_+$ with $\widetilde{\mu}_j(\{0\}) = 0$ and another Lévy measure μ' on $\mathbb{R}_+^d$ with $\mu_j'(\{0\}) = 0$ satisfying (1.2) such that*

$$\mu_j(dz) = \widetilde{\mu}_j(dz_j) \otimes \prod_{k \ne j} \delta_0(dz_k) + \mu_j'(dz).$$

Moreover, assume that there exists $\alpha_j \in (0, 2)$ and constants $c, C > 0$ with

$$c|\lambda|^{\alpha_j} \le \int\limits_{|z| \le 1} (1 - \cos(\lambda \cdot z))\, \widetilde{\mu}_j(dz) \le c|\lambda|^{\alpha_j}, \quad \lambda \in \mathbb{R}, \quad |\lambda| \gg 1.$$

Then (A) is satisfied for $I = \{1, \dots, d\}$ and $\alpha_j = 2\mathbb{1}_{I_1}(j) + \alpha_j \mathbb{1}_{I_2}(j)$.

Proof. Fix $x \in \mathbb{R}^d_+$ such that $x_1, \dots, x_d > 0$. Write $L^x(t) = L^x_1(t) + L^x_2(t)$ where L^x_1, L^x_2 are independent Lévy processes with symbols

$$\Psi^1_x(\lambda) = \sum_{j\in I_1} 2c_j x_j \lambda_j^2 + \sum_{j\in I_2} x_j \int\limits_{(0,1)} \left(1 + i\lambda_j \cdot z - e^{i\lambda_j \cdot z}\right) \widetilde{\mu}_j(dz),$$

$$\Psi^2_x(\lambda) = \sum_{j\in I_2} x_j \int\limits_{|z|\le 1} \left(1 + i\lambda \cdot z - e^{i\lambda\cdot z}\right) \mu'_j(dz)$$

$$+ \sum_{j\in I_1} x_j \int\limits_{|z|\le 1} \left(1 + i\lambda \cdot z - e^{-\lambda\cdot z}\right) \mu_j(dz) + \int\limits_{\mathbb{R}^d_+} \left(1 - e^{i\lambda\cdot z}\right) \nu(dz).$$

Then $g^x_t = f^1_t * f^2_t$, where f^j_t is the infinite divisible distribution of L^x_i, $i \in \{1,2\}$. Observe that, for $\lambda \in \mathbb{R}^d$ sufficiently large and $\delta = \min\{\alpha_j \mid j \in I_2\} \wedge 2$,

$$\mathrm{Re}(\Psi^1_x(\lambda)) \ge C \min\{x_1, \dots, x_d\} |\lambda|^\delta.$$

Hence f^1_t has a smooth density, and thus also g^x_t has a smooth density. Let $(B_j)_{j\in I_1}$ be a collection of independent one-dimensional Brownian motions and let $(Z_j)_{j\in I_2}$ be a collection of independent one-dimensional Lévy processes with symbols

$$\Psi_{Z_j}(\lambda) = \int\limits_{(0,1)} \left(1 + i\lambda \cdot z - e^{i\lambda\cdot z}\right) \widetilde{\mu}_j(dz), \quad \lambda \in \mathbb{R}, \quad j \in I_2.$$

All these processes are supposed to be mutually independent. Then L^x_1 satisfies in law

$$L^x_1(t) = \sum_{j\in I_1} e_j B_j(2c_j x_j t) + \sum_{j\in I_2} e_j Z_j(x_j t)$$

and hence $f^1_t(z) = \prod_{j\in I_1} h_{2c_j x_j t}(z_j) \cdot \prod_{j\in I_2} \widetilde{h}^j_{x_j t}(z_j)$, where $h_t(z)$ is the gaussian density of $B_j(t)$ and $\widetilde{h}^j_t(z)$ is the smooth density of $Z_j(t)$. By Proposition 15 we obtain

$$\int\limits_{\mathbb{R}} \left| \frac{\partial h_t(z)}{\partial z} \right| dz \le C t^{-1/2}, \qquad \int\limits_{\mathbb{R}} \left| \frac{\partial \widetilde{h}^j_t(z)}{\partial z} \right| dz \le C t^{-1/\alpha_j}, \quad t > 0.$$

Thus we obtain, for $j \in I_1$,

$$\int\limits_{\mathbb{R}^d} \left| \frac{\partial f^1_t(z)}{\partial z_j} \right| dz \le \frac{C}{\sqrt{x_j}} t^{-1/2} \le \frac{C}{\rho(x)} t^{-1/2}, \quad t > 0,$$

and similarly, for $j \in I_2$,

$$\int\limits_{\mathbb{R}^d} \left| \frac{\partial f^1_t(z)}{\partial z_j} \right| dz \le \frac{C}{x_j^{1/\alpha_j}} t^{-1/\alpha_j} \le \frac{C}{\rho(x)} t^{-1/\alpha_j}, \quad t > 0.$$

The assertion follows from

$$\int_{\mathbb{R}^d} |g_t^x(z+he_j) - g_t^x(z)|\, dz \le |h| \int_{\mathbb{R}^d} \left| \frac{\partial g_t^x(z)}{\partial z_j} \right| dz \le |h| \int_{\mathbb{R}^d} \left| \frac{\partial f_t^1(z)}{\partial z_j} \right| dz, \quad j \in \{1, \dots, d\}.$$

□

It is also possible to obtain the smoothing property (A) from the jump measure of the immigration mechanism. Our second result is a more general version of Example 1.(c).

Lemma 17. *Suppose that there exists $\alpha \in (0,1)$ and constants $c, C > 0$ such that*

$$c|\lambda|^\alpha \le \int_{\mathbb{R}_+^d} (1 - \cos(\lambda \cdot z))\, \nu(dz) \le C|\lambda|^\alpha, \quad |\lambda| \gg 1.$$

Then (A) is satisfied for $\alpha = \alpha_1 = \cdots = \alpha_d$ and $I = \emptyset$.

Proof. Write $L^x = L_1^x + L_2^x$ where L_j^x are Lévy processes with symbols

$$\Psi_x^1(\lambda) = \int_{\mathbb{R}_+^d} \left(1 - e^{i\lambda\cdot z}\right) \nu(dz),$$

$$\Psi_x^2(\lambda) = \sum_{j=1}^d 2c_j x_j \mathbb{1}_{\mathbb{R}_+}(x_j)\lambda_j^2 + \sum_{j=1}^d x_j \mathbb{1}_{\mathbb{R}_+}(x_j) \int_{|z|\le 1} \left(1 + i\lambda\cdot z - e^{i\lambda\cdot z}\right) \mu_j(dz).$$

Then $g_t^x = f_t^1 * f_t^2$, where f_t^j is the distribution of L_j^x, $j \in \{1,2\}$. Using Proposition 15 we see that

$$\int_{\mathbb{R}^d} |\nabla g_t^x(z)| dz \le \int_{\mathbb{R}^d} |\nabla f_t^1(z)| dz \le Ct^{-1/\alpha}, \quad t \to 0.$$

This proves the assertion. □

7 Some examples

In this section we provide some simple examples showing how our main results from Section 2 can be applied. Let (c, β, B, μ, ν) be admissible parameters with $\nu = 0$ and suppose that there exist $\alpha_1, \dots, \alpha_d \in (1,2)$ such that

$$\mu_k(dz) = \frac{dz_k}{z_k^{1+\alpha_k}} \otimes \prod_{j\ne k} \delta_0(dz_j) + \mu_k'(dz), \quad k \in \{1, \dots, d\},$$

where μ_k' are Lévy measures on $\mathbb{R}_+^d$ satisfying $\mu_k'(\{0\}) = 0$ and (1.2). Then we obtain the following:

(a) Theorem 2 is applicable, provided $\alpha_1, \dots, \alpha_d > \frac{4}{3}$ and μ_k' integrates $\mathbb{1}_{\{|z|>1\}}|z|^{1+\tau}$, for some $\tau \in (0,1)$ and all $k \in \{1, \dots, d\}$.

(b) If $c_1 = \cdots = c_d = 0$, then Theorem 3 is applicable, provided

$$\min\{\alpha_1, \dots, \alpha_d\} > \frac{\max\{\alpha_1, \dots, \alpha_d\}^2}{1 + \max\{\alpha_1, \dots, \alpha_d\}}, \tag{7.1}$$

and μ_k' integrates $\mathbb{1}_{\{|z|>1\}}|z|^{1+\tau}$, for some $\tau \in (0,1)$ and all $k \in \{1, \dots, d\}$. Note that (7.1) is weaker than $\max\{\alpha_1, \dots, \alpha_d\} > \frac{4}{3}$.

(c) Suppose that $c_1 = \cdots = c_d = 0$ and $\mu_k' = 0$. Then Theorem 6 is applicable. Note that the corresponding multi-type CBI process can also be obtained as the pathwise unique strong solution to the Lévy driven stochastic equation

$$X_i(t) = X_i(0) + \int_0^t \left(\beta_i + \sum_{j=1}^d b_{ij} X_j(s) \right) ds + \int_0^t X_i(s-)^{1/\alpha_i} dZ_i(s),$$

where $Z_1, \dots, Z_d$ are independent one-dimensional Lévy processes with symbols

$$\Psi_k(\xi) = \int_0^\infty \left(1 + i\xi z - e^{i\xi z} \right) \frac{dz}{z^{1+\alpha_k}}, \quad \xi \in \mathbb{R}, \quad k \in \{1, \dots, d\}.$$

We remark that the above statements in (a) - (c) also hold for $\nu \neq 0$, provided $\int_{|z|>1} |z|^{1+\tau} \nu(dz) < \infty$, for some $\tau \in (0,1)$. Below we provide one example, where existence of a density is deduced from the smoothing property of the immigration mechanism.

Example 18. *Let (c, β, B, μ, ν) be admissible parameters with $c_1 = \cdots = c_d = 0$, $\mu_1, \dots, \mu_d$ are such that, for some $\gamma_0 \in \left(1, \frac{1+\sqrt{5}}{2}\right)$,*

$$\int_{\mathbb{R}_+^d} \left(|z|^{\gamma_0} \mathbb{1}_{\{|z| \leq 1\}} + |z| \mathbb{1}_{\{|z|>1\}} \right) \mu_k(dz) < \infty, \quad k \in \{1, \dots, d\},$$

and the immigration mechanism is given by

$$\nu(dz) = \mathbb{1}_{\left\{z \in \mathbb{R}_+^d \ | \ |z| \leq 1\right\}}(z) \frac{dz}{|z|^{d+\alpha}} + \nu'(dz), \quad \alpha \in (0,1),$$

where ν' is any measure supported on $\mathbb{R}_+^d$ satisfying $\nu'(\{0\}) = 0$ and $\int_{\mathbb{R}_+^d} |z| \nu'(dz) < \infty$. Then Theorem 3 is applicable with $I = \emptyset$ and $\alpha_1 = \cdots = \alpha_d = \alpha$, provided $\alpha > \frac{\gamma_0}{1+\gamma_0}\gamma_0$.

8 Appendix

Below we prove some simple estimates on the moments of stochastic integrals with respect to Poisson random measures. Similar results for the Lévy noise case were obtained in [DF13, Lemma 5.2].

Lemma 19. *Let $N(du,dz)$ be a Poisson random measure with compensator $\widehat{N}(du,dz) = dum(dz)$ on $\mathbb{R}_+ \times E$, where $m(dz)$ is a σ-finite measure on some Polish space E. The following assertions hold.*

(a) Let $0 < \eta \leq \gamma$ and $1 \leq \gamma \leq 2$. Then there exists a constant $C > 0$ such that, for any predictable process $H(u,z)$ and $0 \leq s \leq t \leq s+1$,

$$\mathbb{E}\left[\left|\int_s^t \int_E H(u,z)\widetilde{N}(du,dz)\right|^{\eta}\right] \leq C(t-s)^{\eta/\gamma} \sup_{u\in[s,t]} \mathbb{E}\left[\int_E |H(u,z)|^{\gamma} m(dz)\right]^{\eta/\gamma},$$

provided the stochastic integral is well-defined.

(b) Let $0 < \eta \leq \gamma \leq 2$. Then there exists a constant $C > 0$ such that, for any predictable process $H(u,z)$ and $0 \leq s \leq t \leq s+1$,

$$\begin{aligned}\mathbb{E}\left[\left|\int_s^t \int_E H(u,z)N(du,dz)\right|^{\eta}\right] &\leq C(t-s)^{\eta/\gamma} \sup_{u\in[s,t]} \mathbb{E}\left[\int_E |H(u,z)|^{\gamma} m(dz)\right]^{\eta/\gamma} \\ &\quad + C\mathbb{1}_{\gamma\in(1,2]}(t-s)^{\eta/\gamma} \sup_{u\in[s,t]} \mathbb{E}\left[\left(\int_E |H(u,z)| m(dz)\right)^{\gamma}\right]^{\eta/\gamma},\end{aligned}$$

provided the stochastic integral is well-defined.

Proof. (a) If $\eta \geq 1$, then by the BDG-inequality, sub-additivity of $x \longmapsto x^{\frac{\gamma}{2}}$ and Hölder inequality we obtain

$$\begin{aligned}&\mathbb{E}\left[\left|\int_s^t \int_E H(u,z)\widetilde{N}(du,dz)\right|^{\eta}\right] \leq C\mathbb{E}\left[\left|\int_s^t \int_E |H(u,z)|^2 N(du,dz)\right|^{\eta/2}\right] \\ &\leq C\mathbb{E}\left[\left|\int_s^t \int_E |H(u,z)|^{\gamma} N(du,dz)\right|^{\eta/\gamma}\right] \leq C\mathbb{E}\left[\int_s^t \int_E |H(u,z)|^{\gamma} dum(dz)\right]^{\eta/\gamma} \\ &\leq C(t-s)^{\frac{\eta}{\gamma}} \sup_{u\in[s,t]} \mathbb{E}\left[\int_E |H(u,z)|^{\gamma} m(dz)\right]^{\eta/\gamma}.\end{aligned}$$

If $0 < \eta \leq 1 \leq \gamma \leq 2$, then the Hölder inequality and previous estimates imply

$$\mathbb{E}\left[\left|\int_s^t \int_E H(u,z)\widetilde{N}(du,dz)\right|^\eta\right] \leq \mathbb{E}\left[\left|\int_s^t \int_E H(u,z)\widetilde{N}(du,dz)\right|\right]^\eta$$
$$\leq C(t-s)^{\frac{\eta}{\gamma}} \sup_{u\in[s,t]} \mathbb{E}\left[\int_E |H(u,z)|^\gamma m(dz)\right]^{\eta/\gamma}.$$

(b) If $\gamma \in (0,1]$, then by sub-additivity of $x \longmapsto x^\gamma$ and Hölder inequality we get

$$\mathbb{E}\left[\left|\int_s^t \int_E H(u,z)N(du,dz)\right|^\eta\right] \leq \mathbb{E}\left[\left|\int_s^t \int_E |H(u,z)|^\gamma N(du,dz)\right|^{\eta/\gamma}\right]$$
$$\leq \mathbb{E}\left[\int_s^t \int_E |H(u,z)|^\gamma du m(dz)\right]^{\eta/\gamma} \leq (t-s)^{\frac{\eta}{\gamma}} \sup_{u\in[s,t]} \mathbb{E}\left[\int_E |H(u,z)|^\gamma m(dz)\right]^{\eta/\gamma}.$$

If $\gamma \in (1,2]$, then

$$\mathbb{E}\left[\left|\int_s^t \int_E H(u,z)N(du,dz)\right|^\eta\right] \leq C\mathbb{E}\left[\left|\int_s^t \int_E H(u,z)\widetilde{N}(du,dz)\right|^\eta\right] + C\mathbb{E}\left[\left|\int_s^t \int_E H(u,z)du m(dz)\right|^\eta\right].$$

The stochastic integral can be estimated by part (a), and the second term by

$$\mathbb{E}\left[\left|\int_s^t \int_E H(u,z)du m(dz)\right|^\eta\right] \leq \mathbb{E}\left[\left|\int_s^t \int_E H(u,z)du m(dz)\right|^\gamma\right]^{\eta/\gamma}$$
$$\leq (t-s)^\eta \sup_{u\in[s,t]} \mathbb{E}\left[\left(\int_E |H(u,z)|m(dz)\right)^\gamma\right]^{\eta/\gamma},$$

which proves the assertion since $t-s \leq 1$ and $\gamma \geq 1$. □

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