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INTERMEDIATE AND EXTRAPOLATED SPACES FOR BI-CONTINUOUS OPERATOR SEMIGROUPS

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Abstract. We discuss the construction of the entire Sobolev (Hölder) scale for non-densely defined operators with rays of minimal growth on a Banach space. In particular, we give a construction for extrapolation- and Favard spaces of generators of (bi-continuous) semigroups, or which is essentially the same, Hille–Yosida operators on *Saks spaces*.

Introduction

Extrapolation spaces for generators of C_0*-semigroups* (used here synonymously to *"strongly continuous, one-parameter semigroups of bounded linear operators"*) on Banach spaces, or for more general operators, have been designed to study e.g., maximal regularity questions by Da Prato and Grisvard [7]; see also Walter [34], Amann [3], van Neerven [33], Nagel, Sinestrari [28], Nagel [26], Sinestrari [31], Magal, Ruan [24, Ch. 3]. These spaces (and the corresponding extrapolated operators) play a central role in recent abstract perturbation results, most prominently in boundary-type or domain perturbations, see e.g., Desch, Schappacher [9], Greiner [15], Staffans, Weiss [32], Adler, Bombieri, Engel [1], Hadd, Manzo, Rhandi [17]. Extrapolation spaces are also important in the theory of coupled operator matrices, see Engel [10].

In this paper, we concentrate on the construction of extrapolation spaces for linear operators having a non-empty resolvent set on a Banach space, but we do not assume the operator to fulfill the Hille–Yosida conditions or to be densely defined. In case the operator is densely defined such a construction is known from the seminal papers of Da Prato, Grisvard, [8], Amann [3] and Nagel, Sinestrari [28]. In the case of non-densely defined, sectorial operators there is a very general—*almost purely algebraic*—construction due to Haase [16] leading also to universal extrapolation spaces. Here, we present a slightly different construction of extrapolation and extrapolated Favard spaces, allowing the construction of *extrapolated semigroups* in the absence of strong continuity with respect to the norm. For a non-densely defined Hille–Yosida operator A on the Banach space X_0 such a construction is possible by taking the part of A in $\underline{X}_0 := \overline{D(A)}$, so that the restricted operator becomes the generator of a C_0-semigroup on $\underline{X}_0$, thus leading to an extrapolated semigroup on the extrapolation space $\underline{X}_{-1}$, see Nagel, Sinestrari [29]. But this semigroup will usually not leave the original Banach space X_0 invariant. This is why we restrict our attention to the situation where strong continuity of the semigroup is guaranteed with respect to some coarser locally convex topology τ on X_0. Here the framework of bi-continuous semigroups, or that of Saks spaces, (see Kühnemund [20] and Section 4 below) appears to be adequate. However, most of the results presented here are valid also for generators of other classes of semigroups: integrable semigroups of Kunze [21], "C-class" semigroups of Kraaij [19], π-semigroups of Priola [30], weakly continuous semigroups of Cerrai [5], to mention a few.

Given a Banach space X_0 and a Hausdorff locally convex topology τ on X_0 (with certain properties described in Section 4), and a bi-continuous semigroup $(T(t))_{t\geq 0}$ with generator A, we construct the full scale of abstract Sobolev (or Hölder) and Favard spaces X_α, $\underline{X}_\alpha$, F_α for $\alpha \in \mathbb{R}$, and the corresponding extrapolated semigroups $(T_\alpha(t))_{t\geq 0}$. (If τ is the norm topology, there is nothing new here, and everything can be found in [11, Section II.5].) These constructions, along with some applications, form the main content of this paper. Here we illustrate the results on the following well-known example (see also Nagel, Nickel, Romanelli [27] and Section 5 for

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details): Consider the Banach space $X_0 := \mathrm{C_b}(\mathbb{R})$ of bounded, continuous functions and the (left) translation semigroup $(S(t))_{t\geq 0}$ thereon, defined by $(S(t)f)(x) = f(x+t)$, $x \in \mathbb{R}$, $t \geq 0$, $f \in X_0$. For $\alpha \in (0,1)$ we have the continuous embeddings

$$\mathrm{C}^1_\mathrm{b}(\mathbb{R}) \hookrightarrow \mathrm{Lip_b}(\mathbb{R}) \hookrightarrow \mathrm{h}^\alpha_\mathrm{b}(\mathbb{R}) \hookrightarrow \mathrm{h}^\alpha_\mathrm{b,loc}(\mathbb{R}) \hookrightarrow \mathrm{C}^\alpha_\mathrm{b}(\mathbb{R}) \hookrightarrow \mathrm{UC_b}(\mathbb{R}) \hookrightarrow \mathrm{C_b}(\mathbb{R}) \hookrightarrow \mathrm{L}^\infty(\mathbb{R}),$$

where $\mathrm{C}^1_\mathrm{b}(\mathbb{R})$ is the space of differentiable functions with derivative in $\mathrm{C_b}(\mathbb{R})$, $\mathrm{Lip_b}(\mathbb{R})$ is the space of bounded, Lipschitz functions, $\mathrm{h}^\alpha_\mathrm{b}(\mathbb{R})$ is the space of bounded, little-Hölder continuous functions, $\mathrm{h}^\alpha_\mathrm{b,loc}(\mathbb{R})$ is the space of bounded, locally little-Hölder continuous functions, $\mathrm{C}^\alpha_\mathrm{b}(\mathbb{R})$ is the space of bounded, Hölder continuous functions, $\mathrm{UC_b}(\mathbb{R})$ is the space of bounded, uniformly continuous functions. In the abstract perspective and using the notation in this paper, this corresponds to the inclusions of Banach spaces:

$$X_1 \hookrightarrow F_1 \hookrightarrow \underline{X}_\alpha \hookrightarrow X_\alpha \hookrightarrow F_\alpha \hookrightarrow \underline{X}_0 \hookrightarrow X_0 \hookrightarrow F_0.$$

The extension of the previous diagram for the full scale $\alpha \in \mathbb{R}$ is possible by extrapolation. The (abstract) spaces $\underline{X}_\alpha$ and F_α ($\alpha \in (0,1)$) are well studied and we refer to the books by Lunardi [23] and Engel, Nagel [11, Section II.5] for a systematic treatment. However, the definition of X_α is new, and requires a recollection of results concerning the other spaces, $\underline{X}_\alpha$ and F_α.

Extrapolated Favard spaces are not only important for perturbation theory. They help to reduce problems concerning semigroups being not strongly continuous to the study of an underlying C_0-semigroup. This perspective is propagated by Nagel and Sinestrari in [29]: To any Hille–Yosida operator on X_0 one can construct a Banach space F_0 (the Favard class) containing X_0 as a closed subspace, and a semigroup $(T(t))_{t\geq 0}$ on F_0. (Note, however, that the semigroup $(T(t))_{t\geq 0}$ defined on F_0 may not leave X_0 invariant.) We adapt this point of view also in this paper. In particular, we provide an alternative (and short) proof of the Hille–Yosida type generation theorem for bi-continuous semigroups (due to Kühnemund [20]) by employing solely the C_0-theory.

Applications of the Sobolev (Hölder) scale, as presented here, to perturbation theory, in the spirit of the results of Desch, Schappacher [9], or of Jacob, Wegner, Wintermayr [18], will be presented in a forthcoming paper.

This work is organized as follows: In Section 1 we recall the standard constructions and results for extrapolation spaces for densely defined (invertible) operators. Moreover, we construct extrapolation spaces for not densely defined operators A with $D(A^2)$ dense in $D(A)$ for the norm of X_0. Our argument differs form the one in Haase [16] in that we build the space X_{-1} based on $\underline{X}_{-2}$ (which, in turn, arises from $\underline{X}_0$ and $\underline{X}_{-1}$), i.e., in a bottom-to-top and then back-to-bottom manner, resulting in the continuous inclusions

$$\underline{X}_0 \hookrightarrow X_0 \hookrightarrow \underline{X}_{-1} \hookrightarrow X_{-1} \hookrightarrow \underline{X}_{-2}.$$

(None of these inclusions is surjective in general.) This approach becomes convenient when we compare the arising extrapolation spaces X_{-1} and $\underline{X}_{-1}$ and construct the extrapolated semigroups thereon. In Section 2 we turn to intermediate spaces; the results there are classical, but are put in the general perspective of this paper. We also present a method for a "concrete" representation of extrapolation spaces (see Theorem 1.15). Section 3 discusses the Sobolev (Hölder) scale for semigroup generators, and has a survey character. In Section 4 we recall the concept of bi-continuous semigroups, construct the corresponding extrapolated semigroups and give a direct proof of the Hille–Yosida generation theorem (due to Kühnemund, see [20]) using extrapolation techniques. We conclude this paper with some examples in Section 5, where we determine the extrapolation spaces of concrete semigroup generators. In particular, we discuss the previously mentioned example of the translation semigroup (complementing results of Nagel, Nickel, Romanelli [27, Sec. 3.1, 3.2]) and then *left implemented semigroups* (cf. Alber [2]).

1. Sobolev and extrapolation spaces for invertible operators

In this section we construct abstract Sobolev (Hölder) and extrapolation spaces (the so-called Sobolev scale) for a boundedly invertible linear operator defined on a Banach space. Some of the results are well-known and even standard, but we chose to include them here for the sake of completeness and also because they are needed for the construction of spaces when we deal with not densely defined operators. The emphasis will be, however, on this latter case, when the construction is new, see Section 1.2 below. We also note that everything what follows is also valid for operators on Fréchet spaces.

The following is a standing assumption in this paper.

Assumption A. We suppose that $A : D(A) \to X_0$ is a (not necessarily densely defined) linear operator on a Banach space X_0 with 0 in the resolvent set $\rho(A)$.

As a matter of fact, it is only for convenience to suppose $0 \in \rho(A)$ instead of $\rho(A) \neq \emptyset$. Indeed, if $\lambda \in \rho(A)$ we may consider $A - \lambda$ and carry out the constructions for this new operator satisfying $0 \in \rho(A - \lambda)$. The arising spaces will not depend on $\lambda \in \rho(A)$ (up to isomorphism).

1.1. **Abstract Sobolev spaces.** The material presented here is standard, see Nagel [25], Nagel, Nickel, Romanelli [27] or Engel, Nagel [11, Section II.5], and some parts are valid even for operators on locally convex spaces, when one has to argue with a family of generating seminorms instead of one norm. We set $X_1 := D(A)$ which becomes a Banach space if endowed with the graph norm

$$\|x\|_A := \|x\| + \|Ax\|.$$

An equivalent norm is given by $\|x\|_{X_1} := \|Ax\|$ since we have assumed $0 \in \rho(A)$. Then we have the isometric isomorphism

$$A : X_1 \to X_0 \quad \text{with inverse} \quad A^{-1} : X_0 \to X_1.$$

Definition 1.1. Recall the assumption that $0 \in \rho(A)$, and take $n \in \mathbb{N}$, $n \geq 1$.

(a) We define

$$X_n := D(A^n) \quad \text{and} \quad \|x\|_{X_n} := \|A^n x\| \quad \text{for } x \in X_n.$$

If we want to stress the dependence on A, then we write $X_n(A)$ and $\|\cdot\|_{X_n(A)}$.

(b) Let

$$X_\infty(A) := \bigcap_{n \in \mathbb{N}} X_n,$$

often abbreviated as X_∞.

(c) We further set

$$\underline{X}_0 := \overline{D(A)}, \quad \underline{A} := A|_{\underline{X}_0},$$

the part of A in $\underline{X}_0$, i.e.,

$$D(\underline{A}) = \bigl\{x \in D(A) : Ax \in \underline{X}_0\bigr\}.$$

Moreover, we let

$$\underline{X}_n := D(\underline{A}^n), \quad \|x\|_{\underline{X}_n} := \|\underline{A}^n x\|.$$

To be specific about the underlying operator A we write $\underline{X}_n(A)$ and $\|x\|_{\underline{X}_n(A)}$.

(d) For $n \in \mathbb{N}$ we set $A_n := A|_{X_n}$, the part of A in X_n, in particular $A_0 = A$. Similarly, we let $\underline{A}_n := \underline{A}|_{\underline{X}_n}$, for example $\underline{A}_0 = \underline{A}$. By this notation we also understand implicitly that the surrounding space is $X_n(A)$ respectively $\underline{X}_n(A)$ with its norm, see Remark 1.2.

Remark 1.2. 1. By "underlining" we always indicate an object which is in some sense smaller than the one without underlining. The space $\underline{X}_0(A)$ is connected with the domain of $D(A)$, and the whole issue of distinguishing between X_0 and $\underline{X}_0$ becomes relevant only if A is not densely defined but its part $\underline{A}$ *is* (cf. Remark 1.5). We keep to the notation $\underline{A}$ for the part of the operator A instead of $A|_{\underline{X}_0}$.

2. If A is densely defined, then $X_n(A) = \underline{X}_n(A)$ for each $n \in \mathbb{N}$. In particular, if $\underline{X}_1(A) = D(\underline{A})$ is dense in $\underline{X}_0(A)$, then $\underline{X}_n(A) = \underline{X}_n(\underline{A})$ for each $n \in \mathbb{N}$.
3. For $n \in \mathbb{N}$ we evidently have $X_1(A^n) = X_n(A)$. Also $\underline{X}_1(A^n) = \underline{X}_n(A)$ holds, because $D(\underline{A^n}) = D(\underline{A}^n)$. Indeed, the inclusion "$D(\underline{A}^n) \subseteq D(\underline{A^n})$" is trivial. While for $x \in D(\underline{A^n})$ we have $x \in \underline{X}_0$ and $A^n x \in \underline{X}_0$, implying $A^{n-1}x \in D(\underline{A})$, and then recursively $x \in D(\underline{A}^n)$.
4. For $x \in D(A_n) = D(A^{n+1})$ we have $\|x\|_{X_1(A_n)} = \|A_n x\|_{X_n(A)} = \|A^{n+1}x\| = \|x\|_{X_{n+1}(A)}$. Similarly $D(\underline{A}_n) = D(\underline{A}^{n+1})$.

Proposition 1.3. *Suppose $\underline{A}$ is densely defined in $\underline{X}_0$.*
(a) *For $n \in \mathbb{N}$ the mappings $A^n : X_n \to X_0$ and $\underline{A}^n : \underline{X}_n \to \underline{X}_0$ are isometric isomorphisms.*
(b) *For $n \in \mathbb{N}$ the operators $A_n : X_{n+1} \to X_n$ and $\underline{A}_n : \underline{X}_{n+1} \to \underline{X}_n$ are isometric isomorphisms that intertwine A_{n+1} and A_n, respectively, $\underline{A}_{n+1}$ and $\underline{A}_n$.*
(c) *If $D(\underline{A})$ is dense in $\underline{X}_0$, then X_∞ is dense in $\underline{X}_n$ for each $n \in \mathbb{N}$. As a consequence, $\underline{X}_m$ is dense in $\underline{X}_n$ for each $m, n \in \mathbb{N}$ with $m \geq n$.*

Proof. The statements (a) and (b) are trivial by construction.

(c) This is [4, Thm. 6.2] due to Arendt, El-Mennaoui and Kéyantuo, because $\underline{A}$ is densely defined in $\underline{X}_0$. $\square$ $\square$

Remark 1.4. We note that the proof of the assertion (c) in [4, Thm. 6.2] is based on a Mittag-Leffler type result due to Esterle [12] which is valid in complete metric spaces. Hence the statements (a), (b) and (c) are all remain true for Fréchet spaces with verbatim the same proof as in [4].

Henceforth, another standing assumption will be the following (though not everywhere needed).

Assumption B. The operator $\underline{A} := A|_{\underline{X}_0} : D(\underline{A}) \to \underline{X}_0$ is densely defined, i.e.,

$$\overline{D(\underline{A})} = \underline{X}_0.$$

Remark 1.5. The condition of $D(\underline{A})$ being dense in $\underline{X}_0$ holds for example if there are $M, \omega > 0$ such that $(\omega, \infty) \subseteq \rho(A)$ and

$$\|\lambda R(\lambda, A)\| \leq M \quad \text{for all } \lambda > \omega. \tag{1.1}$$

Indeed, in this case we have for $x \in D(A)$

$$\|\lambda R(\lambda, A)x - x\| = \|R(\lambda, A)Ax\| \leq \frac{M\|Ax\|}{\lambda} \to 0 \quad \text{for } \lambda \to \infty.$$

Hence $D(A^2) \subseteq D(\underline{A})$ is dense in $D(A)$ for the norm of X_0, and this implies the density of $D(\underline{A})$ in $\underline{X}_0$. An operator A satisfying (1.1) is often said to have a *ray of minimal growth*, see, e.g., [23, Chapter 3], and also Section 2 below. Another term used is *"weak Hille–Yosida operator"*.

Proposition 1.6. *If $T \in \mathscr{L}(X_0)$ is a linear operator commuting with A^{-1}, then the spaces X_n and $\underline{X}_n$ are T-invariant, and $T \in \mathscr{L}(X_n)$ for $n \in \mathbb{N}$.*

Proof. The condition means that $Tx \in D(A)$ for each $x \in D(A)$ and for such x we have $ATx = TAx$. This implies the invariance of X_1 and that $\|Tx\|_{X_1(A)} \leq \|T\|\|x\|_{X_1(A)}$. Using the boundedness assumption we see that $\underline{X}_1$ remains invariant under T. For general $n \in \mathbb{N}$ we may argue by recursion, or simply invoke Remark 1.2. $\square$ $\square$

1.2. **Extrapolation spaces.** The construction for the extrapolation spaces here is standard if A is densely defined, or if A is a Hille–Yosida operator, see, e.g., [29].

For $x \in X_0$ we define $\|x\|_{\underline{X}_{-1}(A)} := \|A^{-1}x\|$. Then the surjective mapping

$$A : (D(A), \|\cdot\|) \to (X_0, \|\cdot\|_{\underline{X}_{-1}(A)})$$

becomes isometric, and hence has a uniquely continuous extension

$$\underline{A}_{-1} : (\underline{X}_0, \|\cdot\|) \to (\underline{X}_{-1}, \|\cdot\|_{\underline{X}_{-1}(\underline{A})}),$$

which is an isometric isomorphism, where $(\underline{X}_{-1}, \|\cdot\|_{\underline{X}_{-1}(\underline{A})})$ denotes the completion of $(\underline{X}_0, \|\cdot\|_{\underline{X}_{-1}(A)})$. By construction we obtain immediately:

Proposition 1.7. *The space X_0 is continuously and densely embedded in $\underline{X}_{-1}$. If $\underline{A}$ is densely defined in $\underline{X}_0$, then also X_∞ is dense in $\underline{X}_{-1}$. As a consequence $(\underline{X}_{-1}, \|\cdot\|_{\underline{X}_{-1}(\underline{A})})$ is the completion of $(\underline{X}_0, \|\underline{A}^{-1}\cdot\|)$.*

Proof. The space X_0 is dense in $\underline{X}_{-1}$ by construction. For $x \in X_0$ we have

$$\|x\|_{\underline{X}_{-1}(\underline{A})} = \|AA^{-1}x\|_{\underline{X}_{-1}(\underline{A})} = \|\underline{A}_{-1}A^{-1}x\|_{\underline{X}_{-1}(A)} \leq \|\underline{A}_{-1}\| \cdot \|A^{-1}x\| \leq \|\underline{A}_{-1}\| \cdot \|A^{-1}\| \cdot \|x\|,$$

showing the continuity of the embedding. The last assertion follows since X_∞ is dense in $D(A)$ with respect to $\|\cdot\|$. $\square$ $\square$

Of course one can iterate the whole procedure and obtain the following chain of dense and continuous embeddings

$$\underline{X}_0 \hookrightarrow \underline{X}_{-1} \hookrightarrow \underline{X}_{-2} \hookrightarrow \cdots \hookrightarrow \underline{X}_{-n} \quad \text{for } n \in \mathbb{N},$$

where for $n \geq 1$ the space $\underline{X}_{-n}$ is a completion of $\underline{X}_{-n+1}$ with respect to the norm $\|\cdot\|_{\underline{X}_{-n}(\underline{A})}$ defined by $\|x\|_{\underline{X}_{-n}(\underline{A})} = \|\underline{A}_{-n+1}^{-1}x\|_{\underline{X}_{-n+1}(\underline{A})}$ and

$$\underline{A}_{-n} : \underline{X}_{-n+1} \to \underline{X}_{-n}$$

is a unique continuous extension of $\underline{A}_{-n+1} : D(\underline{A}_{-n+1}) \to \underline{X}_{-n+1}$ to $\underline{X}_{-n}$.

These spaces, just as well the ones in the next definition, are called *extrapolation spaces* for the operator A, see, e.g., [29] or [11, Section II.5] for the case of semigroup generators. The spaces $\underline{X}_{-1}$, $\underline{X}_{-2}$ and the operator $\underline{A}_{-2}$ will be used to define the extrapolation space $X_{-1}(A)$. To this purpose we identify X_0 with a subspace of $\underline{X}_{-1}$ and of $\underline{X}_{-2}$.

Definition 1.8. Consider X_0 as a subspace of $\underline{X}_{-2}$, and define

$$X_{-1} := \underline{A}_{-2}(X_0) := \big\{\underline{A}_{-2}x : \; x \in X_0\big\} \quad \text{and} \quad \|x\|_{X_{-1}} := \|\underline{A}_{-2}^{-1}x\|.$$

Furthermore, we set $D(A_{-1}) := X_0$ and for $x \in X_0$ we define $A_{-1}x := \underline{A}_{-2}x$. To indicate the dependence on the operator A we write $X_{-1}(A)$ and $\|\cdot\|_{X_{-1}(A)}$.

Remark 1.9. It is easy to see that the operator A_{-1} is the part of $\underline{A}_{-2}$ in X_{-1}.

In what follows, we will define higher order extrapolation spaces and prove that all these spaces line up in a scale, where one can switch between the levels with the help of (a version) of the operator A (or A_{-1}).

Proposition 1.10. *The operator A_{-1} is an extension of $\underline{A}_{-1}$, $(X_{-1}, \|\cdot\|_{X_{-1}})$ is a Banach space, the norms of $\underline{X}_{-1}$ and X_{-1} coincide on $\underline{X}_{-1}$, and $\underline{X}_{-1}$ is a closed subspace of X_{-1}. The mapping $A_{-1} : X_0 \to X_{-1}$ is an isometric isomorphism.*

Proof. The first assertion is true because $\underline{A}_{-2}$ is an extension of $\underline{A}_{-1}$. That X_{-1} is a Banach space is immediate from the definition. Since $\underline{A}_{-2}^{-1}\underline{A}_{-1} = I$ on $\underline{X}_0$, we have $\underline{A}_{-1}^{-1}x \in \underline{X}_0 \subseteq X_0$ for $x \in \underline{X}_{-1}$, so that $\|\underline{A}_{-2}^{-1}x\| = \|\underline{A}_{-2}^{-1}\underline{A}_{-1}\underline{A}_{-1}^{-1}x\| = \|\underline{A}_{-1}^{-1}x\| = \|x\|_{\underline{X}_{-1}}$. This establishes that the norms coincide. Since $\underline{X}_{-1}$ is a Banach space (with its own norm), it is a closed subspace of X_{-1}. That A_{-1} is an isometric isomorphism follows from the definition. □ □

Remark 1.11. By construction we have $\underline{X}_{-1}(\underline{A}_{-n}) = \underline{X}_{-(n+1)}(\underline{A})$ and $X_{-1}(A_{-n}) = X_{-(n+1)}(A)$ for each $n \in \mathbb{N}$.

Proposition 1.12. *For $n \in \mathbb{Z}$ the operators $A_n : X_{n+1} \to X_n$ and $\underline{A}_n : \underline{X}_{n+1} \to \underline{X}_n$ are isometric isomorphisms that intertwine A_{n+1} and A_n, respectively, $\underline{A}_{n+1}$ and $\underline{A}_n$.*

Proof. For $n \in \mathbb{N}$ this is Proposition 1.12. So we assume $n < 0$. For $n = -1$ the statement about isometric isomorphisms is just the definition, and the intertwining property is also evident. By recursion we obtain the validity of the assertion for general $n \leq -1$ and for the operator $\underline{A}_n$. By Remark 1.11 it suffices to prove that A_{-1} intertwines A_{-1} and $A_0 = A$. For $x \in D(A_0) = D(A)$ we have $A_{-1}x \in X_0 = D(A_{-1})$ and $Ax = A_{-1}^{-1}A_{-1}A_{-1}x$. □ □

Thus for $n \in \mathbb{N}$ we have the following chain of embeddings (continuous and dense, denoted by $\hookrightarrow$) and inclusions as closed subspaces (denoted by $\subseteq$):

$$\cdots \hookrightarrow \underline{X}_n \subseteq X_n \hookrightarrow \underline{X}_0 \subseteq X_0 \hookrightarrow \underline{X}_{-1} \subseteq X_{-1} \hookrightarrow \underline{X}_{-2} \subseteq X_{-2} \hookrightarrow \cdots \underline{X}_{-n} \subseteq X_{-n} \hookrightarrow \cdots,$$

where in general the inclusions are strict (see the examples in Section 5). We also have the following chain of isometric isomorphisms

$$\cdots \longrightarrow \underline{X}_{n+1} \xrightarrow{\underline{A}_n^{-1}} \underline{X}_n \longrightarrow \cdots \longrightarrow \underline{X}_1 \xrightarrow{\underline{A}_0^{-1}} \underline{X}_0 \xrightarrow{\underline{A}_{-1}^{-1}} \underline{X}_{-1} \longrightarrow \cdots \longrightarrow \underline{X}_{-n+1} \xrightarrow{\underline{A}_{-n}^{-1}} \underline{X}_{-n} \longrightarrow \cdots$$

and

$$\cdots \longrightarrow X_{n+1} \xrightarrow{A_n^{-1}} X_n \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{A_0^{-1}} X_0 \xrightarrow{A_{-1}^{-1}} X_{-1} \longrightarrow \cdots \longrightarrow X_{-n+1} \xrightarrow{A_{-n}^{-1}} X_{-n} \longrightarrow \cdots.$$

Proposition 1.13. (a) $\underline{X}_1(\underline{A}_{-1}) = \underline{X}_0$ *and* $X_1(A_{-1}) = X_0$ *with the same norms.*
(b) $\underline{X}_{-1}(\underline{A}_1) = \underline{X}_0$ *with the same norms.*
(c) $(\underline{A}_1)_{-1} = \underline{A}$.
(d) $X_{-1}(A_1) = X_0$ *with the same norms, and* $(A_1)_{-1} = A$.

Proof. (a) By definition $X_1(A_{-1}) = D(A_{-1}) = X_0$ with the graph norm of A_{-1}. Since A_{-1} extends A, for $x \in X_0$ we have $\|A_{-1}x\|_{X_{-1}(A)} = \|Ax\|_{X_{-1}} = \|A^{-1}Ax\| = \|x\|$. The first statement then follows, because $\underline{X}_1(\underline{A}_{-1}) = X_1(\underline{A}_{-1}) = \overline{D(\underline{A})} = \underline{X}_0$ with the same norms.

(b) For $x \in \underline{X}_1(\underline{A}) = D(\underline{A}^2)$ we have

$$\|x\|_{\underline{X}_{-1}(\underline{A}_1)} = \|\underline{A}_1^{-1}x\|_{\underline{X}_1(\underline{A})} = \|\underline{A}\,\underline{A}_1^{-1}x\| = \|x\|,$$

which can be extended by density for all $x \in \underline{X}_0$, showing the equality of the spaces $\underline{X}_{-1}(\underline{A}_1) = \underline{X}_0$ (with the same norm).

(c) By construction the operator $(\underline{A}_1)_{-1} : \underline{X}_1(A) \to \underline{X}_{-1}(\underline{A}_1)$ is the unique continuous extension of

$$\underline{A}_1 : D(\underline{A}_1) = D(\underline{A}^2) \to \underline{X}_1(A),$$

and $(\underline{A}_1)_{-1}$ is an isometric isomorphism. For $x \in \underline{X}_1(A)$ we have $\|x\|_{\underline{X}_{-1}(A_1)} = \|\underline{A}_1^{-1}x\|_{\underline{X}_1(A)} = \|x\|$. But then it follows that $(\underline{A}_1)_{-1} = \underline{A} : D(\underline{A}) \to \underline{X}_0$.

(d) The space $X_{-1}(A_1)$ is defined by

$$X_{-1}(A_1) := (\underline{A}_1)_{-2}(X_1(A)) = \left((\underline{A}_1)_{-1}\right)_{-1}(X_1(A)) = \underline{A}_{-1}(X_1(A)) = AX_1(A) = X_0,$$

by part (c). For the norm equality let $x \in X_0$. Then

$$\|x\| = \|AA^{-1}x\| = \|A^{-1}x\|_{X_1(A)} = \|\underline{A}_{-1}^{-1}x\|_{X_1(A)} = \|(\underline{A}_1)_{-2}^{-1}x\|_{X_1(A)} = \|x\|_{X_{-1}(A_1)}.$$

For the last assertion we note that $(A_1)_{-1} = (\underline{A}_1)_{-2}|_{X_1(A)} = A$. □ □

Recall the standing assumption that $\underline{A} = A|_{\underline{X}_0}$ is densely defined in $\underline{X}_0 = \overline{D(A)}$. The following proposition plays the key role for the extension of operators to the extrapolation spaces, particularly for the construction of extrapolated semigroups in Section 3.

Proposition 1.14. (a) *Let $n \in \mathbb{N}$. If $T \in \mathscr{L}(X_0)$ is a linear operator commuting with A^{-1}, then the operator T has a unique continuous extension to $\underline{X}_{-n}$ denoted by $\underline{T}_{-n}$. The operator $\underline{T}_{-n}$ is the restriction of $\underline{T}_{-n-1}$. The space X_{-n} is invariant under $\underline{T}_{-n-1}$, whose restriction is denoted by T_{-n}, for which $T_{-n} \in \mathscr{L}(X_{-n})$. For $k, n \in -\mathbb{N}$ the operators $\underline{T}_n$, $\underline{T}_k$ are all similar; the same holds for T_n and T_k.*
(b) *Let $\underline{T} \in \mathscr{L}(\underline{X}_0)$ such that it leaves $D(A)$ invariant and commutes with $\underline{A}^{-1} = A^{-1}|_{X_0}$. Then $\underline{T}_{-1}x = A\underline{T}A^{-1}x$ for each $x \in X_0$, and as a consequence, $\underline{T}_{-1} : \underline{X}_{-1} \to \underline{X}_{-1}$ leaves X_0 invariant (and, of course, extends $\underline{T}$).*

Proof. (a) For $x \in X_0$ we have

$$\|Tx\|_{X_{-1}(A)} = \|A^{-1}Tx\| = \|TA^{-1}x\| \le \|T\| \cdot \|A^{-1}x\| = \|T\| \cdot \|x\|_{X_{-1}(A)}.$$

Therefore $T : (X_0, \|\cdot\|_{X_{-1}(A)}) \to (X_0, \|\cdot\|_{X_{-1}(A)})$ is continuous, and hence has a unique continuous extension $\underline{T}_{-1}$ to $\underline{X}_{-1}$. This extension commutes with $\underline{A}_{-1}^{-1}$, because T commutes with A^{-1} and $\underline{A}_{-1}^{-1}$ is the unique continuous extension of A^{-1}. By iteration we obtain the continuous extensions $\underline{T}_{-n}$ onto $\underline{X}_{-n}$, which then all commute with the corresponding $\underline{A}_{-n}^{-1}$. By construction $\underline{T}_{-n}$ is a restriction of $\underline{T}_{-n-1}$. We prove that X_{-1} is invariant under $\underline{T}_{-2}$. Let $x \in X_{-1}$, hence $x = \underline{A}_{-2}y$ for some $y \in X_0$. Then $Ty = \underline{T}_{-2}y = \underline{T}_{-2}\underline{A}_{-2}^{-1}x = \underline{A}_{-2}^{-1}\underline{T}_{-2}x$, hence $\underline{T}_{-2}x = \underline{A}_{-2}Ty \in X_{-1}$, i.e., the invariance of X_{-1} is proved. We have for $x \in X_{-1}$ that $\|T_{-1}x\|_{X_{-1}} = \|A_{-2}^{-1}T_{-1}x\| = \|A_{-2}^{-1}\underline{T}_{-2}x\| = \|\underline{T}_{-2}A_{-2}^{-1}x\| \le \|\underline{T}_2\| \cdot \|\underline{A}_{-2}^{-1}x\| = \|\underline{T}_2\| \cdot \|x\|_{X_{-1}}$, therefore $T_{-1} \in \mathscr{L}(X_{-1})$. The assertion about T_{-n} follows by recursion.

It is enough to prove the similarity of $T_0 = T$ and T_{-1}, and the similarity of $\underline{T}_0$ and $\underline{T}_{-1}$. The latter assertions can be proved as follows: For $x \in D(A)$ we have

$$\underline{A}_{-1}^{-1}\underline{T}_{-1}\underline{A}_{-1}x = \underline{A}_{-1}^{-1}\underline{T}_{-1}Ax = \underline{A}_{-1}^{-1}TAx = \underline{A}_{-1}^{-1}ATx = \underline{A}_{-1}^{-1}A_{-1}Tx = \underline{T}x,$$

then by continuity and denseness the equality follows even for $x \in \underline{X}_0$. For the similarity of T and T_{-1} take $x \in X_0$. Then

$$A_{-1}^{-1}T_{-1}A_{-1}x = \underline{A}_{-2}^{-1}\underline{T}_{-2}\underline{A}_{-2}x = \underline{T}_{-1}x = Tx.$$

(b) Let $x \in X_0 \subseteq \underline{X}_{-1}$. Then there is a sequence (x_n) in $\underline{X}_0$ with $x_n \to x$ in $\underline{X}_{-1}$ (see Proposition 1.7). But then $A^{-1}x_n \to A^{-1}x$ in $\underline{X}_0$ and $\underline{T}x_n \to \underline{T}_{-1}x$ in $\underline{X}_{-1}$ by part (a). These imply $\underline{T}A^{-1}x_n = A^{-1}\underline{T}x_n \to \underline{A}_{-1}^{-1}\underline{T}_{-1}x$. Hence we conclude $\underline{T}A^{-1}x = \underline{A}_{-1}^{-1}\underline{T}_{-1}x$ and $A\underline{T}A^{-1}x = \underline{T}_{-1}x$ for $x \in X_0$. □ □

Haase in [16] and Wegner in [35] have constructed the so-called *universal extrapolation space* $X_{-\infty}$ as follows: Suppose A is densely defined (this assumption is *not* made by Haase), then $X_n = \underline{X}_n$ for each $n \in \mathbb{Z}$ and let $X_{-\infty}$ to be the inductive limit of the sequence of Banach spaces $(X_{-n})_{n\in\mathbb{N}}$ (algebraic inductive limit in [16]). One can extend the operator A to an operator $A_{-\infty} : X_{-\infty} \to X_{-\infty}$ such that

$$A_{-\infty}|_{X_n} = A_n, \quad n \in \mathbb{Z}.$$

We now look at a converse situation, and our starting point is the following: Let $\mathscr{E}$ be a locally convex space such that we can embed the Banach space X_0 continuously in $\mathscr{E}$, i.e., there is a continuous injective map $i : X_0 \to \mathscr{E}$, and so we can identify X_0 with a subspace of $\mathscr{E}$. We also assume that we have a continuous operator $\mathcal{A} : \mathscr{E} \to \mathscr{E}$ such that $\lambda - \mathcal{A} : i(X_0) \to \mathscr{E}$ is injective and that

$$D(A) = \{x \in X_0 : \ \mathcal{A} \circ i(x) \in i(X_0)\},$$

and

$$i \circ A = \mathcal{A} \circ i|_{D(A)}.$$

In the next theorem we use this setting to describe the extrapolation spaces $\underline{X}_{-n}$, X_{-n}. Notice that we do not assume that A is a Hille–Yosida operator or densely defined.

Theorem 1.15. *Let X_0 be a Banach space with a continuous embedding $i : X_0 \to \mathscr{E}$ into a locally convex space $\mathscr{E}$, let $A : D(A) \to X_0$ be a linear operator with $\lambda \in \rho(A)$ such that $A = \mathcal{A}|_{X_0}$ (after identifying X_0 with a subspace of $\mathscr{E}$ as described above). We suppose furthermore that $\lambda - \mathcal{A}$ is injective on X_0. Then there is a continuous embedding $i_{-1} : X_{-1} \to \mathscr{E}$ which extends i. After identifying X_{-1} with a subspace of $\mathscr{E}$ (under i_{-1}) we have*

$$X_{-1} = \{(\lambda - \mathcal{A})x : \ x \in X_0\}, \quad \underline{X}_{-1} = \{(\lambda - \mathcal{A})x : \ x \in \underline{X}_0\} \quad \textit{and} \quad A_{-1} = \mathcal{A}|_{X_{-1}}.$$

Proof. Without lost of generality we may assume that $\lambda = 0$. Recall that $A_{-1}|_{X_0} = A$ and A_{-1} is an isometric isomorphism $A_{-1} : X_0 \to X_{-1}$. We now define the embedding $i_{-1} : X_{-1} \to \mathscr{E}$ by

$$i_{-1} := \mathcal{A} \circ i \circ A_{-1}^{-1},$$

which is indeed injective and continuous by assumption. Of course, i_{-1} extends i since we have $i = \mathcal{A} \circ i \circ A^{-1}$. We can write

$$i_{-1} \circ A_{-1} = \mathcal{A} \circ i \circ A_{-1}^{-1} \circ A_{-1} = \mathcal{A} \circ i,$$

which yields the following commutative diagram:

$$\begin{tikzcd} X_0 \arrow[r, "i"] \arrow[d, "A_{-1}"] & \mathscr{E} \arrow[d, "\mathcal{A}"] \\ X_{-1} \arrow[r, "i_{-1}"'] & \mathscr{E} \end{tikzcd}$$

Now all assertions follow easily. □ □

The last corollary in this section can be proved by induction based on the previous facts.

Corollary 1.16. *Let $\mathcal{A}$, X_0, $\mathscr{E}$ and i be as in Theorem 1.15. Then $X_n \subseteq \mathscr{E}$ and $A_n = \mathcal{A}|_{X_n}$ for each $n \in \mathbb{Z}$ (after identifying X_n with a subspace of $\mathscr{E}$ under an embedding i_n compatible with i).*

2. Intermediate spaces for operators with rays of minimal growth

The following definition of intermediate, and as a matter of fact interpolation spaces, just as well many results in this section are standard, and we refer, e.g., to the book by Lunardi [23, Chapter 3], and to Engel, Nagel [11, Section II.5] for the case of semigroup generators. In this section we suppose the following.

Assumption C. The operator A on the Banach space X_0 has a *ray of minimal growth*, i.e., $(0,\infty) \subseteq \rho(A)$ and for some $M \geq 0$

$$\|\lambda R(\lambda, A)\| \leq M \quad \text{for all } \lambda > 0. \tag{2.1}$$

Definition 2.1. For $\alpha \in (0,1]$ and $x \in X_0$ we define

$$\|x\|_{F_\alpha(A)} := \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\|,$$

and the abstract Favard space of order α by

$$F_\alpha(A) := \big\{x \in X_0 : \|x\|_{F_\alpha(A)} < \infty\big\}.$$

In the literature the notation $D_A(\alpha,\infty)$ is also used, see, e.g., [23]. We further set

$$F_0(A) := F_1(A_{-1}),$$

see [11, Section II.5(b)] for the case of semigroup generators.

Proposition 2.2. (a) *The Favard space $F_\alpha(A)$ becomes a Banach space if endowed with the norm $\|\cdot\|_{F_\alpha(A)}$.*
(b) *The space X_0 is isomorphic to a closed subspace of $F_0(A)$.*

The statement that X_0 is a closed subspace of $F_0(A)$ when A is a Hille–Yosida operator is due to Nagel and Sinestrari [29, Proof of Prop. 2.7].

Proof. (a) is trivial.
(b) For $x \in X_0$ we have

$$\|\lambda A_{-1}R(\lambda, A_{-1})x\|_{X_{-1}(A)} = \|\lambda AR(\lambda, A)x\|_{X_{-1}(A)} = \|\lambda A^{-1}AR(\lambda, A)x\| \leq M\|x\|,$$

yielding

$$\|x\|_{F_0(A)} = \|x\|_{F_1(A_{-1})} \leq M\|x\|.$$

On the other hand, since A and A_{-1} are similar, we have $\sup_{\lambda>0}\|\lambda R(\lambda, A_{-1})\|_{X_{-1}(A)} \leq M'$ for some $M' \geq 0$ and for all $\lambda > 0$. In particular, by Remark 1.5, $\lambda R(\lambda, A_{-1})x \to x$ for each $x \in \underline{X}_{-1}$. From this we obtain for $x \in X_0$ that

$$\begin{aligned}\|x\| = \|A_{-1}x\|_{X_{-1}(A)} &= \Big\|\lim_{\lambda\to 0}\lambda R(\lambda, A_{-1})A_{-1}x\Big\|_{X_{-1}(A)} \leq \sup_{\lambda>0}\Big\|\lambda A_{-1}R(\lambda, A_{-1})x\Big\|_{X_{-1}(A)}\\ &= \|x\|_{F_1(A_{-1})} = \|x\|_{F_0(A)},\end{aligned}$$

showing the equivalence of the norms $\|\cdot\|$ and $\|x\|_{F_0(A)}$ on X_0. □ □

We also need the following well-known result, see, e.g., [23, Chapters 1 and 3], for which we give a short proof.

Proposition 2.3. *For $\alpha \in (0,1]$ we have $F_\alpha(A) \subseteq \overline{D(A)} = \underline{X}_0$.*

Proof. We have

$$AR(\lambda, A)x = \lambda R(\lambda, A)x - x,$$

so that

$$\|\lambda R(\lambda, A)x - x\| \leq \frac{\|x\|_{F_\alpha(A)}}{\lambda^\alpha} \to 0 \quad \text{as } \lambda \to \infty.$$

□ □

Definition 2.4. Let A be a linear operator on the Banach space X_0 satisfying (2.1). For $\alpha \in (0,1)$ we set

$$\underline{X}_\alpha(A) := \Big\{ x \in F_\alpha(A) : \lim_{\lambda\to\infty} \lambda^\alpha AR(\lambda,A)x = 0 \Big\},$$

and we recall from Section 1 that

$$\underline{X}_0(A) := \overline{D(A)}, \quad \underline{X}_1(A) = D(A|_{\underline{X}_0(A)}).$$

The proof of the next proposition is straightforward and well-known.

Proposition 2.5. *For $\alpha, \beta \in (0,1)$ with $\alpha > \beta$ we have*

$$\underline{X}_1(A) \hookrightarrow \underline{X}_\alpha(A) \subseteq F_\alpha(A) \hookrightarrow \underline{X}_\beta(A) \subseteq F_\beta(A) \hookrightarrow \underline{X}_0(A) \subseteq X_0(A)$$

with $\hookrightarrow$ denoting continuous and dense embeddings of Banach spaces, and $\subseteq$ denoting inclusion of closed subspaces.

Proof. For $x \in F_\alpha(A)$ we have

$$\|\lambda^\beta AR(\lambda,A)x\| = \lambda^{\beta-\alpha}\|\lambda^\alpha AR(\lambda,A)x\| \le \lambda^{\beta-\alpha}\|x\|_\alpha \to 0 \quad \text{as } \lambda\to\infty,$$

which also proves the continuity of $F_\alpha(A) \hookrightarrow \underline{X}_\beta(A)$. The other statements can be proved by similar reasonings. □ □

Proposition 2.6. (a) *The spaces $F_\alpha(A)$ and $\underline{X}_\alpha(A)$ are invariant under each $T \in \mathscr{L}(X_0)$ which commutes with A^{-1}.*
(b) *If $T \in \mathscr{L}(X_0)$ commutes with A^{-1}, then the space $F_0(A)$ is invariant under T_{-1}.*

Proof. (a) Suppose that $T \in \mathscr{L}(X_0)$ commutes with $R(\cdot,A)$ and let $x \in \underline{X}_\alpha(A)$. We have to show that $Tx \in \underline{X}_\alpha(A)$. Since T is assumed to be bounded, we obtain:

$$\|\lambda^\alpha AR(\lambda,A)Tx\| = \|\lambda^\alpha ATR(\lambda,A)x\| \le \|T\| \cdot \|\lambda^\alpha AR(\lambda,A)x\|.$$

This implies both assertions.

(b) Follows from part (a) applied to T_{-1} on the space X_{-1}. □ □

Definition 2.7. For $\alpha \in \mathbb{R}$ we write $\alpha = m + \beta$ with $m \in \mathbb{Z}$ and $\beta \in (0,1]$, and define

$$F_\alpha(A) := F_\beta(A_m),$$

with the corresponding norms. For $\alpha \notin \mathbb{Z}$ we define

$$\underline{X}_\alpha(A) := \underline{X}_\beta(A_m),$$

also with the corresponding norms.

In particular we have for $\alpha \in (0,1)$ that

$$\underline{X}_{-\alpha}(A) = \underline{X}_{1-\alpha}(A_{-1}) \quad \text{and} \quad F_{-\alpha}(A) = F_{1-\alpha}(A_{-1}).$$

This definition is consistent with Definitions 2.1 and 2.4. The following property of these spaces can be directly deduced from the definitions and the previous assertions (by induction):

Proposition 2.8. *For any $\alpha, \beta \in \mathbb{R}$ with $\alpha > \beta$ we have*

$$\underline{X}_\alpha(A) \subseteq F_\alpha(A) \hookrightarrow \underline{X}_\beta(A) \subseteq F_\beta(A)$$

with $\hookrightarrow$ denoting continuous and dense embeddings of Banach spaces, and $\subseteq$ denoting inclusion of closed subspaces.

Now we put these spaces in the context presented at the end of Section 1.

Proposition 2.9. (a) *For $\alpha \in (0,1]$ we have $A_{-1}F_\alpha = F_{\alpha-1}$ and $A_{-1}\underline{X}_\alpha = \underline{X}_{\alpha-1}$.*
(b) *For $\alpha \in (0,1]$ and $\mathcal{A}$, λ and $\mathscr{E}$ as in Theorem 1.15 we have*

$$F_{-\alpha} = \Big\{ (\lambda - \mathcal{A})y \in X_{-1} : \ y \in F_{1-\alpha} \Big\}.$$

If $\alpha \in (0,1)$, then

$$\underline{X}_{-\alpha} = \Big\{ (\lambda - \mathcal{A})y \in X_{-1} : \ y \in \underline{X}_{1-\alpha} \Big\}.$$

3. Intermediate and extrapolation spaces for semigroup generators

In this section we consider intermediate and extrapolation spaces when the linear operator $A : D(A) \to X_0$ is the generator of a semigroup $(T(t))_{t\geq 0}$ (meaning that $T : [0,\infty) \to \mathscr{L}(X_0)$ is a monoid homomorphism) in the sense described in the following.

Assumption 3.1. 1. Let X_0 be a Banach space, and let $Y \subseteq X_0'$ be a norming subspace, i.e.,

$$\|x\| = \sup_{y\in Y, \|y\|\leq 1} |\langle x, y\rangle| \quad \text{for each } x \in X_0.$$

2. Let $T : [0,\infty) \to \mathscr{L}(X_0)$ be a semigroup of contractions for which a *generator* $A : D(A) \to X_0$ exists in the sense that

$$R(\lambda, A)x = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)x \; \mathrm{d}s \tag{3.1}$$

exists for each $\lambda \geq 0$ as a weak integral with respect to the dual pair (X_0, Y), i.e., for each $y \in Y$ and $x \in X_0$

$$\langle R(\lambda, A)x, y\rangle = \int_0^\infty \mathrm{e}^{-\lambda s}\langle T(s)x, y\rangle \; \mathrm{d}s,$$

and $R(\lambda, A) \in \mathscr{L}(X_0)$ is the resolvent of a linear operator A (see [21] by Kunze).

3. We also suppose that $T(t)$ commutes with A^{-1} for each $t \geq 0$.

If the semigroup $(T(t))_{t>0}$ is only exponentially bounded of type (M,ω), that is

$$\|T(t)\| \leq M\mathrm{e}^{\omega t} \quad \text{for all } t \geq 0,$$

then one can rescale the semigroup (consider$(\mathrm{e}^{-(\omega+1)t}T(t))_{t\geq 0}$), and renorm the Banach space such that the rescaled semigroup becomes a contraction semigroup. Moreover, the new semigroup has negative growth bound, meaning that $T(t) \to 0$ in norm exponentially fast as $t \to \infty$. Then it also has an invertible generator.

Remark 3.2. (i) There are several important classes of semigroups, satisfying Assumption 3.1, hence can be treated in a unified manner: π-semigroups of Priola [30], weakly continuous semigroups of Cerrai [5], bi-continuous semigroups of Kühnemund. We will concentrate on this latter class of semigroups in Section 4.

(ii) In this framework Kunze [21] introduced the notion of integrable semigroups, which we briefly describe next. Since we have

$$\|y\| = \sup_{x\in X_0, \|x\|\leq 1} |\langle x, y\rangle|$$

and, by the norming assumption,

$$\|x\| = \sup_{y\in Y, \|y\|\leq 1} |\langle x, y\rangle|,$$

the pair (X_0, Y) is called a norming dual pair. Kunze has worked out the theory of semigroups on such norming dual pairs in [21]. We recall at least the basic definitions here: assume without loss of generality that Y is a Banach space and consider the weak topology $\sigma = \sigma(X_0, Y)$ on X_0. An *integrable semigroup* of type (M,ω) on the pair (X_0, Y) is a semigroup $(T(t))_{t\geq 0}$ of σ-continuous linear operators satisfying the following.

1. $(T(t))_{t\geq 0}$ is a semigroup, i.e. $T(t+s) = T(t)T(s)$ and $T(0) = I$ for all $t, s \geq 0$.
2. For all λ with $\mathrm{Re}(\lambda) > \omega$, there exists an σ-continuous linear operator $R(\lambda)$ such that for all $x \in X_0$ and all $y \in Y$

$$\langle R(\lambda)x, y\rangle = \int_0^\infty \mathrm{e}^{-\lambda t}\langle T(t)x, y\rangle \; \mathrm{d}t.$$

Kunze defines the generator A of the semigroup as the (unique) operator $A : D(A) \to X_0$ (if it exists at all) with $R(\lambda) = (\lambda - A)^{-1}$, precisely as in Assumption 3.1. Note that σ-continuity of $T(t)$ can be used to assure that Y is invariant under $T'(t)$, cf. the next remark.

Remark 3.3. The semigroup $(T(t))_{t\geq 0}$ commutes with the inverse of the generator if Y can be chosen such that it is invariant under $T'(t)$ for each $t \geq 0$:

$$\begin{aligned}\langle A^{-1}T(t)x, y\rangle &= \int_0^\infty \langle T(s)T(t)x, y\rangle \;\mathrm{d}s = \int_0^\infty \langle T(s+t)x, y\rangle \;\mathrm{d}s \\ &= \Big\langle \int_0^\infty T(s)x \;\mathrm{d}s, T'(t)y\Big\rangle = \langle T(t)A^{-1}x, y\rangle,\end{aligned}$$

for each $x \in X_0$ and $y \in Y$.

Remark 3.4. 1. From (3.1) it follows that for each $x \in X_0$

$$T(t)x - x = A\int_0^t T(s)x \;\mathrm{d}s. \tag{3.2}$$

Indeed, we have by (3.1) that

$$\begin{aligned}x &= A\int_0^\infty T(s)x \;\mathrm{d}s \\ T(t)x &= A\int_0^\infty T(s)T(t)x \;\mathrm{d}s = \int_t^\infty T(s)x \;\mathrm{d}s.\end{aligned}$$

Subtracting the first of these equation from the second one we obtain the statement.
2. If moreover A commutes with $T(t)$ for each $t \geq 0$, then for each $x \in D(A)$ we have

$$T(t)x - x = \int_0^t T(s)Ax \;\mathrm{d}s. \tag{3.3}$$

Indeed, as in the above, we have by (3.1)

$$\begin{aligned}-x &= -A^{-1}Ax = \int_0^\infty T(s)Ax \;\mathrm{d}s \\ -T(t)x &= -A^{-1}T(t)Ax = \int_0^\infty T(s)T(t)Ax \;\mathrm{d}s = \int_t^\infty T(s)Ax \;\mathrm{d}s.\end{aligned}$$

By a simple subtraction we obtain the statement.

The next lemma and its proof are standard for various classes of semigroups.

Lemma 3.5. *If $(T(t))_{t\geq 0}$ is (locally) norm bounded, then*

$$X_{\mathrm{cont}} := \{x \in X_0 : t \mapsto T(t)x \textit{ is } \|\cdot\|\text{-}\textit{continuous}\}$$

is a closed a subspace of X_0 invariant under the semigroup. Under Assumption 3.1 we have

$$\underline{X}_0 = \overline{D(A)} = X_{\mathrm{cont}}.$$

Proof. The closedness and invariance of X_{cont} are evident. We first show $D(A) \subseteq X_{\mathrm{cont}}$, which implies $\overline{D(A)} \subseteq X_{\mathrm{cont}}$ by closedness of X_{cont}. By (3.3) we conclude for $x \in D(A)$ that $T(t)x - x = \int_0^t T(s)Ax \;\mathrm{d}s$. Since

$$\|T(t)x - x\| = \sup_{\|y\|\leq 1} |\langle T(t)x - x, y\rangle| \leq \sup_{\|y\|\leq 1} \int_0^t |\langle T(s)Ax, y\rangle| \;\mathrm{d}s \leq t\|Ax\| \to 0$$

as $t \to 0$, we obtain $D(A) \subseteq X_{\mathrm{cont}}$ and $\overline{D(A)} \subseteq X_{\mathrm{cont}}$. For the converse inclusion suppose that $x \in X_{\mathrm{cont}}$. Again by (3.2) we obtain that the sequence of vectors $x_n := n\int_0^{\frac{1}{n}} T(s)x \;\mathrm{d}s \in D(A)$ $(n \in \mathbb{N})$ converges to x. Indeed

$$\|x_n - x\| = \sup_{\|y\|\leq 1} |\langle x_n - x, y\rangle| \leq \sup_{\|y\|\leq 1} n\int_0^{\frac{1}{n}} |\langle T(s)x - x, y\rangle| \;\mathrm{d}s \leq n\int_0^{\frac{1}{n}} \|T(s)x - x\| \;\mathrm{d}s.$$

By the continuity of $s \mapsto T(s)x$ we obtain the inclusion $X_{\mathrm{cont}} \subseteq \overline{D(A)}$. □ □

Based on this lemma one can prove the following characterization of the Favard and Hölder spaces:

Proposition 3.6. *Let $(T(t))_{t\geq 0}$ be a semigroup satisfying Assumption 3.1 with negative growth bound and generator A. For $\alpha \in (0,1]$ define*

$$F_\alpha(T) := \Big\{x \in X_0 : \sup_{s>0} \frac{\|T(s)x - x\|}{s^\alpha} < \infty\Big\} = \Big\{x \in X_0 : \sup_{s\in(0,1)} \frac{\|T(s)x - x\|}{s^\alpha} < \infty\Big\}, \tag{3.4}$$

and for $\alpha \in (0,1)$ define

$$\begin{aligned}\underline{X}_\alpha(T) &:= \Big\{x \in X_0 : \sup_{s>0} \frac{\|T(s)x - x\|}{s^\alpha} < \infty \text{ and } \lim_{s\downarrow 0} \frac{\|T(s)x - x\|}{s^\alpha} = 0\Big\} \\ &= \Big\{x \in X_0 : \lim_{s\downarrow 0} \frac{\|T(s)x - x\|}{s^\alpha} = 0\Big\},\end{aligned} \tag{3.5}$$

which become Banach spaces if endowed with the norm

$$\|x\|_{F_\alpha(T)} := \sup_{s>0} \frac{\|T(s)x - x\|}{s^\alpha}.$$

The space $\underline{X}_\alpha(T)$ is a closed subspace of $F_\alpha(T)$. These spaces are invariant under the semigroup $(T(t))_{t\geq 0}$, and $\underline{X}_\alpha(T)$ is the space of $\|\cdot\|_{F_\alpha(T)}$-strong continuity in $F_\alpha(T)$. For $\alpha \in (0,1]$ we have $F_\alpha(A) = F_\alpha(T)$ and for $\alpha \in (0,1)$ we have $\underline{X}_\alpha(A) = \underline{X}_\alpha(T)$ with equivalent norms.

Proof. For $x \in F_\alpha(T)$ we have

$$\|T(t)x\|_{F_\alpha(T)} = \sup_{s>0} \frac{\|T(s)T(t)x - T(t)x\|}{s^\alpha} \leq \|T(t)\| \cdot \sup_{s>0} \frac{\|T(s)x - x\|}{s^\alpha} \leq M\|x\|_{F_\alpha(T)},$$

proving the invariance of $F_\alpha(T)$. Similar reasoning proves the invariance of $\underline{X}_\alpha$. Since $F_\alpha(T) \subseteq X_{\mathrm{cont}} = \underline{X}_0 = \overline{D(A)}$ and $F_\alpha(A) \subseteq \underline{X}_0 = \overline{D(A)}$, the rest of the assertions follow from the corresponding results concerning C_0-semigroups, see, e.g., [11, Sec. II.5]. □ □

To complete the picture we recall a result from [23, Chapter 5]. It is stated there only for C_0-semigroup, but Lunardi also remarks that it holds in greater generality. We require here the conditions from Assumption 3.1 and note that the proof is verbatim the same as for the C_0-case due to the formulas (3.2) and (3.3).

Proposition 3.7. *Let A generate the semigroup $(T(t))_{t>0}$ of negative growth bound as described in Assumption 3.1. Then for $p \in [1,\infty]$ and $\alpha \in (0,1)$ we have*

$$(X, D(A))_{\alpha,p} = \{x \in X : \ t \mapsto \psi_x(t) := t^{-\alpha}\|T(t)x - x\| \in \mathrm{L}^p_*(0,\infty)\},$$

where $\mathrm{L}^p_*(0,\infty)$ denotes the L^p-space with respect to the Haar measure $\frac{\mathrm{d}t}{t}$ on the multiplicative group $(0,\infty)$. Moreover, the norms $\|x\|_{\alpha,p}$ and

$$\|x\|^{**}_{\alpha,p} = \|x\| + \|\psi_x\|_{\mathrm{L}^p_*(0,\infty)}$$

are equivalent.

We conclude this section with the construction of the extrapolated semigroup as a direct consequence of Proposition 1.14.

Proposition 3.8. *Let A generate the semigroup $(T(t))_{t\geq 0}$ of negative growth bound in the sense of Assumption 3.1. Then there is an extension $(T_{-1}(t))_{t\geq 0}$ of the semigroup $(T(t))_{t\geq 0}$ on the extrapolated space X_{-1}, whose generator is A_{-1}.*

4. Intermediate and extrapolation spaces for bi-continuous semigroups

In this section we concentrate on extrapolation spaces for generators of *bi-continuous semigroups.* This class was introduced by Kühnemund in [20] and these semigroups possess generators as described in Section 3. The following assumptions, as proposed by Kühnemund, will be made during this section.

Assumption 4.1. Consider a triple $(X_0, \|\cdot\|, \tau)$ where X_0 is a Banach space, and

1. τ is a locally convex Hausdorff topology coarser than the norm-topology on X_0, i.e. the identity map $(X_0, \|\cdot\|) \to (X_0, \tau)$ is continuous;

2. τ is sequentially complete on the $\|\cdot\|$-closed unit ball;
3. The dual space of (X_0, τ) is norming for X_0, i.e.,

$$\|x\| = \sup_{\substack{\varphi \in (X_0,\tau)' \\ \|\varphi\| \le 1}} |\varphi(x)|, \quad x \in X_0. \tag{4.1}$$

Remark 4.2. 1. There is the related notion of so-called Saks spaces, see [6]. By definition a *Saks space* is a triple $(X_0, \|\cdot\|, \tau)$ such that X_0 is a vector space with a norm $\|\cdot\|$ and locally convex topology τ coarser than the $\|\cdot\|$-topology, but the closed unit ball is τ-complete. In this case, X_0 is a Banach space.
2. It follows from (4.1) that (X_0, Y) with $Y = (X_0, \tau)'$ is a norming dual pair.
3. Kraaij puts this setting in the more general framework of locally convex spaces with mixed topologies, see [19, Sec. 4], and also [13, App. A].
4. Assumption (4.1) is equivalent to the following: There is a set $\mathcal{P}$ of τ-continuous seminorms defining the topology τ such that

$$\|x\| = \sup_{p \in \mathcal{P}} p(x), \quad x \in X_0. \tag{4.2}$$

This description is also used by Kraaij in [19], cf. his Lemma 4.4. Note also that by this remark and by Lemma 3.1 in [6] a Saks space satisfies Assumption 4.1. Indeed, assume (4.1) and let $\mathcal{P}$ be the collection of *all* τ-continuous seminorms p such that $p(x) \le \|x\|$. Then $|\varphi(\cdot)| \in \mathcal{P}$ for each $\varphi \in (X_0, \tau)'$ with $\|\varphi\| \le 1$, and (4.2) is satisfied. If q is any τ-continuous seminorm, then $q(x) \le M\|x\|$ for some constant M and for all $x \in X_0$. So that $M^{-1} \cdot \in \mathcal{P}$, proving that $\mathcal{P}$ defines the topology τ. For the converse implication suppose that (4.2) holds. Then by the Hahn–Banach theorem we obtain (4.1).

Now we give the definition of a bi-continuous semigroup.

Definition 4.3 (Kühnemund [20])**.** Let X_0 be a Banach space with norm $\|\cdot\|$ together with a locally convex topology τ such that the conditions in Assumption 4.1 are satisfied. We call $(T(t))_{t\ge 0}$ a *bi-continuous semigroup* if
1. $T(t+s) = T(t)T(s)$ and $T(0) = I$ for all $s, t \ge 0$.
2. $(T(t))_{t\ge0}$ is strongly τ-continuous, i.e. the map $\varphi_x : [0,\infty) \to (X_0, \tau)$ defined by $\varphi_x(t) = T(t)x$ is continuous for every $x \in X_0$.
3. $(T(t))_{t\ge0}$ has type (M, ω) for some $M \ge 1$ and $\omega \in \mathbb{R}$.
4. $(T(t))_{t\ge0}$ is locally-bi-equicontinuous, i.e., if $(x_n)_{n\in\mathbb{N}}$ is a norm-bounded sequence in X_0 which is τ-convergent to 0, then also $(T(s)x_n)_{n\in\mathbb{N}}$ is τ-convergent to 0 uniformly for $s \in [0, t_0]$ for each fixed $t_0 \ge 0$.

Significant examples in this context are evolution semigroups on $\mathrm{C_b}(\mathbb{R}, X)$, semigroups induced by flows, adjoint semigroups and the Ornstein–Uhlenbeck semigroup on $\mathrm{C_b}(\mathbb{R}^d)$, to mention a few.

As in the case of C_0-semigroups we obtain the generator of a bi-continuous semigroup.

Definition 4.4. Let $(T(t))_{t\ge0}$ be a bi-continuous semigroup on X_0. The generator A is defined by

$$Ax := \tau\lim_{t\to0} \frac{T_0(t)x - x}{t}$$

with the domain

$$D(A) := \Big\{ x \in X_0 : \ \tau\lim_{t\to0} \frac{T(t)x - x}{t} \text{ exists and } \sup_{t\in(0,1]} \frac{\|T(t)x - x\|}{t} < \infty \Big\}.$$

This generator has a couple of important properties which are summarized in the next theorem (see [20], [14]):

Theorem 4.5. *Let $(T(t))_{t\ge0}$ be a bi-continuous semigroup with generator A. Then the following hold:*

(a) *The operator A is bi-closed, i.e., whenever $x_n \xrightarrow{\tau} x$ and $Ax_n \xrightarrow{\tau} y$ and both sequences are norm-bounded, then $y \in D(A)$ and $Ax = y$.*

(b) *The domain* $D(A)$ *is bi-dense in* X_0, *i.e., for each* $x \in X_0$ *there exists a norm-bounded sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $D(A)$ *such that* $x_n \xrightarrow{\tau} x$.
(c) *For* $x \in D(A)$ *we have* $T(t)x \in D(A)$ *and* $T(t)Ax = AT(t)x$ *for all* $t \geq 0$.
(d) *For* $t > 0$ *and* $x \in X_0$ *one has*

$$\int_0^t T(s)x \,\mathrm{d}s \in D(A) \quad \text{and} \quad A\int_0^t T(s)x \,\mathrm{d}s = T(t)x - x. \tag{4.3}$$

(e) *For* $\lambda > \omega$ *one has* $\lambda \in \rho(A)$ *(thus* A *is closed) and:*

$$R(\lambda, A)x = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)x \,\mathrm{d}s, \quad x \in X, \tag{4.4}$$

where the integral is a τ*-improper integral.*

Recall the following result of Kühnemund from [20], whose proof is originally based on integrated semigroups. We present here a different proof using extrapolation spaces.

Theorem 4.6 (Kühnemund)**.** *Let* $(X_0, \|\cdot\|, \tau)$ *be a triple satisfying Assumption 4.1, and let* A *be a linear operator on the Banach space* X_0. *The following are equivalent:*
(i) *The operator* A *is the generator of a bi-continuous semigroup* $(T(t))_{t\geq 0}$ *of type* (M, ω).
(ii) *The operator* A *is a Hille–Yosida operator of type* (M, ω), *i.e.,*

$$\|R(s, A)^k\| \leq \frac{M}{(s-\omega)^k}$$

for all $k \in \mathbb{N}$ *and for all* $s > \omega$. *Moreover,* A *is bi-densely defined and the family*

$$\left\{(s-\alpha)^k R(s, A)^k : \ k \in \mathbb{N}, \ s \geq \alpha\right\} \tag{4.5}$$

is bi-equicontinuous for each $\alpha > \omega$, *meaning that for each norm bounded* τ*-null sequence* (x_n) *one has* $(s-\alpha)^k R(s, A)^k x_n \to 0$ *in* τ *uniformly for* $k \in \mathbb{N}$ *and* $s \geq \alpha$ *as* $n \to \infty$.

In this case, we have the Euler formula

$$T(t)x := \tau\lim_{m\to\infty} \left(\frac{m}{t} R\left(\frac{m}{t}, A\right)\right)^m x \quad \text{for each } x \in X_0.$$

Moreover, the subspace $\underline{X}_0 := \overline{D(A)} \subseteq X_0$ *is the space of norm strong continuity for* $(T(t))_{t\geq 0}$, *it is invariant under the semigroup, and* $(\underline{T}(t))_{t\geq 0} := (T(t)|_{\underline{X}_0})_{t\geq 0}$ *is the strongly continuous semigroup on* $\underline{X}_0$ *generated by the part* $\underline{A}$ *of* A *in* $\underline{X}_0$.

Proof. It follows from Lemma 3.5 that $\underline{X}_0$ is the space of norm strong continuity for a bi-continuous semigroup $(T(t))_{t\geq 0}$.

We only prove the implication (ii) ⇒ (i) and the Euler formula; the other implication is easy. We may suppose that $\omega < 0$. Since A is a Hille–Yosida operator, the part $\underline{A}$ of A in $\underline{X}_0$ generates a C_0-semigroup $(\underline{T}(t))_{t\geq 0}$ of type (M, ω) on the space $\underline{X}_0 := \overline{D(A)}$. Define the function

$$F(s) := \begin{cases} \frac{1}{s} R(\frac{1}{s}, A) & \text{for } s > 0, \\ I & \text{for } s = 0, \end{cases}$$

which is strongly continuous on $\underline{X}_0$ by Remark 1.5. Moreover, we have the Euler formula

$$\underline{T}_0(t)x = \lim_{m\to\infty} F\left(\tfrac{t}{m}\right)^m x$$

for $x \in \underline{X}_0$ with convergence being uniform for t in compact intervals $[0, t_0]$, see, e.g., [11, Section III.5(a)]. Since $R(\lambda, A)|_{\underline{X}_0} = R(\lambda, \underline{A})$ and since $D(A)$ is bi-dense in X_0, by the local bi-equicontinuity assumption in (4.5) we conclude that for $x \in X_0$ and $t > 0$ the limit

$$S(t)x := \tau\lim_{m\to\infty} F\left(\tfrac{t}{m}\right)^m x \tag{4.6}$$

exists, and the convergence is uniform for t in compact intervals $[0, t_0]$. It follows that $t \mapsto S(t)x$ is τ-strongly continuous for each $x \in X_0$. The operator family $(S(t))_{t\geq 0}$ is locally bi-equicontinuous because of the bi-equicontinuity assumption in (4.5).

Next, we prove that $\underline{T}(t)$ leaves $D(A)$ invariant. Let $x \in D(A)$, so that $x = A^{-1}y$ for some $y \in X_0$, and insert x in the formula (4.6) to obtain

$$\underline{T}(t)x = S(t)A^{-1}y = \tau\lim_{m\to\infty} F\big(\tfrac{t}{m}\big)^m A^{-1}y = A^{-1}\ \tau\lim_{m\to\infty} F\big(\tfrac{t}{m}\big)^m y = A^{-1}S(t)y \in D(A), \tag{4.7}$$

where we have used the bi-continuity of A^{-1} and the boundedness of $(\big[\frac{m}{t}R\big(\frac{m}{t},A\big)\big]^m y)_{m\in\mathbb{N}}$. By Proposition 1.14 (b) we can extend $\underline{T}(t)$ to X_0 by setting $T(t) := A\underline{T}(t)A^{-1} \in \mathscr{L}(X_0)$. It follows that $(T(t))_{t\geq 0}$ is a semigroup. By formula (4.7), we have $T(t)y = A\underline{T}(t)A^{-1}y = AA^{-1}S(t)y = S(t)y$ for each $y \in X_0$. So that $(T(t))_{t\geq 0}$, coinciding with $(S(t))_{t\geq 0}$, is locally bi-equicontinuous, and hence a bi-continuous semigroup.

It remains to show that the generator of $(T(t))_{t\geq 0}$ is A. Let B denote the generator of $(T(t))_{t\geq 0}$. Then, for large $\lambda > 0$ and $x \in \underline{X}_0$, we have

$$R(\lambda,B)x = \int_0^\infty \mathrm{e}^{-\lambda s}T(s)x\ \mathrm{d}s = \int_0^\infty \mathrm{e}^{-\lambda s}\underline{T}(s)x\ \mathrm{d}s = R(\lambda,\underline{A}_0)x = R(\lambda,A)x.$$

Since $R(\lambda,B)$ and $R(\lambda,A)$ are sequentially τ-continuous on norm bounded sets and since $D(A)$ is bi-dense in X_0, we obtain $R(\lambda,B) = R(\lambda,A)$. This finishes the proof. ☐ ☐

The first statement in the next proposition is proved by Nagel and Sinestrari, see [26] and [28], while the second one follows directly from the results in Section 1.

Proposition 4.7. *Let A be a Hille–Yosida operator on the Banach space X_0 with domain $D(A)$. Denote by $(\underline{T}(t))_{t\geq 0}$ the C_0-semigroup on $\underline{X}_0 = \overline{D(A)}$ generated by the part $\underline{A}$ of A.*

(a) *There is a one-parameter semigroup $(\overline{T}(t))_{t\geq 0}$ on $F_0(A)$ which extends $(\underline{T}(t))_{t\geq 0}$. This semigroup is strongly continuous for the $\|\cdot\|_{X_{-1}(\underline{A})}$ norm.*
(b) *Suppose that for each $t \geq 0$ the operator $\underline{T}(t)$ leaves $D(A)$ invariant. Then the space X_0 is invariant under the semigroup operators $\overline{T}(t)$ for every $t \geq 0$, i.e., for $T(t) := \overline{T}(t)|_{X_0}$ we have $T(t) \in \mathscr{L}(X_0)$.*

4.1. **Extrapolated semigroups.** In this subsection we extend a bi-continuous semigroup on X_0 to the extrapolation space X_{-1} as a bi-continuous semigroup. We have to handle two topologies, and the next proposition leads to an additional locally convex topology on X_{-1} still satisfying Assumption 4.1.

Proposition 4.8. *Let the triple $(X_0,\|\cdot\|,\tau)$ satisfy Assumption 4.1, let $\mathcal{P}$ be as in Remark 4.2.4, let E be a vector space over $\mathbb{C}$, and let $B : X_0 \to E$ be a bijective linear mapping. We define for $e \in E$ and $p \in \mathcal{P}$*

$$\|e\|_E := \|B^{-1}e\| \quad \text{and} \quad p_E(e) := p(B^{-1}e).$$

Then the following assertions hold:

(a) *$\|\cdot\|_E$ is a norm, p_E is a seminorm for each $p \in \mathcal{P}$.*
(b) *For the topology τ_E generated by $\mathcal{P}_E := \{p_E : p \in \mathcal{P}\}$ the triple $(E,\|\cdot\|_E,\tau_E)$ satisfies the conditions in Assumption 4.1.*
(c) *If $(T(t))_{t\geq 0}$ is a bi-continuous semigroup on X_0 with respect to the topology τ, then $T_E(t) := BT(t)B^{-1}$ defines a bi-continuous semigroup on E. If A is the generator of $(T(t))_{t\geq 0}$, then BAB^{-1} is the generator of $(T_E(t))_{t\geq 0}$.*

Proof. Assertion (a) is evident. The conditions (1) and (2) from Assumption 4.1 are satisfied by the definition of $\|\cdot\|_E$ and p_E. Since

$$\|e\|_E = \|B^{-1}e\| = \sup_{p\in\mathcal{P}} p(B^{-1}e) = \sup_{p_E\in\mathcal{P}_E} p_E(e),$$

and by Remark 4.2 (3) in Assumption 4.1 is fulfilled. The proof of (b) is complete.

(c) For $e \in E$ we have $\|T_E(t)\|_E = \|B^{-1}BT(t)B^{-1}e\| = \|T(t)B^{-1}e\| \leq \|T(t)\|\cdot\|e\|_E$, which shows that $T_E(t) \in \mathscr{L}(E)$. Clearly, $(T_E(t))_{t\geq 0}$ satisfies the semigroup property. For $e \in E$ and $p_E \in \mathcal{P}_E$ we have

$$p_E(T_E(t)e - e) = p(B^{-1}BT(t)B^{-1}e - B^{-1}e) = p(T(t)B^{-1}e - B^{-1}e) \to 0 \quad \text{for } t \to 0,$$

showing the τ_E-strong continuity of $(T_E(t))_{t\geq 0}$. If (e_n) is a $\|\cdot\|_E$-bounded, τ_E-null sequence, then $(B^{-1}e_n)$ is a $\|\cdot\|$-bounded τ-null sequence, so that by assumption $T_E(t)e_n = T(t)B^{-1}e_n \to 0$ uniformly for t in compact intervals. If A is the generator of $(T(t))_{t\geq 0}$, then by means of (4.4) we can conclude that $B^{-1}AB$ is the generator of $(T_E(t))_{t\geq 0}$. □ □

Definition 4.9. Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup in X_0 with generator A.

(a) For $B = A^{-1} : X_0 \to X_1$ and $E = X_1$ in Proposition 4.8 define $\mathcal{P}_1 := \mathcal{P}_E$, $\tau_1 := \tau_E$, $(T_1(t))_{t\geq 0} := (T_E(t))_{t\geq 0}$.

(b) For $B = A_{-1} : X_0 \to X_{-1}$ and $E = X_{-1}$ in Proposition 4.8 define $\mathcal{P}_{-1} := \mathcal{P}_E$, $\tau_{-1} := \tau_E$, $(T_{-1}(t))_{t\geq 0}; := (T_E(t))_{t\geq 0}$.

We obtain immediately the next result.

Proposition 4.10. *The semigroups $(T_1(t))_{t\geq 0}$ and $(T_{-1}(t))_{t\geq 0}$ are bi-continuous with generators $A_1 = A|_{D(A)}$ and A_{-1}, respectively.*

Iterating the procedure in Definition 4.9 we obtain the full scale of (extrapolated) semigroups $(T_n(t))_{t\geq 0}$ for $n \in \mathbb{Z}$.

Definition 4.11. Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X_0 with generator A and suppose that $(T_{\pm n}(t))_{t\geq 0}$ and $\mathcal{P}_{\pm n}$ have been defined for some $n \in \mathbb{N}$ already.

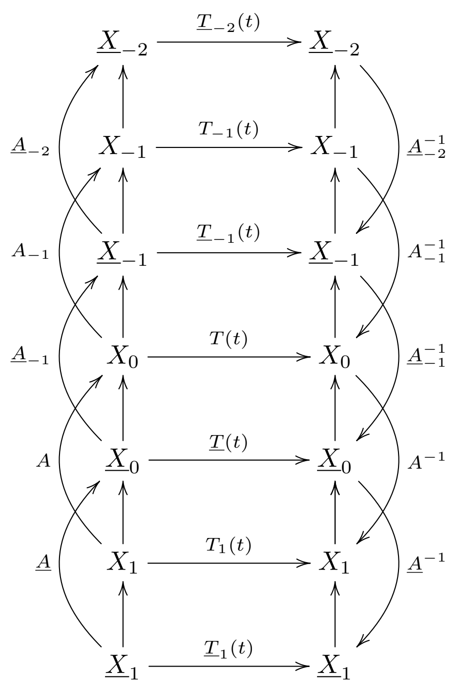
(a) For $B = A_n^{-1} : X_n \to X_{n+1}$, $E = X_{n+1}$ and the semigroup $(T_n(t))_{t\geq 0}$ in Proposition 4.8 define $\mathcal{P}_{n+1} := \mathcal{P}_E$, $\tau_{n+1} := \tau_E$, $(T_{n+1}(t))_{t\geq 0} := (T_E(t))_{t\geq 0}$.

(b) For $B = A_{-n-1} : X_{-n} \to X_{-n-1}$, $E = X_{-n-1}$ and the semigroup $(T_{-n}(t))_{t\geq 0}$ in Proposition 4.8 define $\mathcal{P}_{-n-1} := \mathcal{P}_E$, $\tau_{-n-1} := \tau_E$, $(T_{-n-1}(t))_{t\geq 0} := (T_E(t))_{t\geq 0}$.

Proposition 4.12. *For each $n \in \mathbb{Z}$ the semigroup $(T_n(t))_{t\geq 0}$ is bi-continuous on $(X_n, \|\cdot\|_n, \tau_n)$ with generator $A_n : X_{n+1} \to X_n$. Its space of norm strong continuity is $\underline{X}_n$.*

Proof. The first statement follows directly from Proposition 4.10 by induction. For $n = 0$ the second assertion is the content of Lemma 3.5, for general $n \in \mathbb{Z}$ one can argue inductively. □ □

The following diagram summarizes the situation:



The spaces $\underline{X}_{n+1}$ are bi-dense in X_n for the topology τ_n and dense in $\underline{X}_n$ for the norm $\|\cdot\|_{X_n}$. The semigroups $(T_n(t))_{t\geq 0}$ are bi-continuous on X_n, while $(\underline{T}_n(t))_{t\geq 0}$ are C_0-semigroups (strongly continuous for the norm) on $\underline{X}_n$.

4.2. **Hölder spaces of bi-continuous semigroups.** Suppose A generates the bi-continuous semigroup $(T(t))_{t\geq 0}$ of negative growth bound on X_0. Recall from Theorem 4.6 that the restricted operators $\underline{T}(t) := T(t)|_{\underline{X}_0}$ form a C_0-semigroup $(\underline{T}(t))_{t\geq 0}$ on $\underline{X}_0$. Also recall from Proposition 3.6 that for $\alpha \in (0,1]$

$$F_\alpha(A) = F_\alpha(T) = \Big\{x \in \underline{X}_0 : \sup_{t>0} \frac{\|\underline{T}(t)x - x\|}{t^\alpha} < \infty\Big\} = \Big\{x \in X_0 : \sup_{t>0} \frac{\|T(t)x - x\|}{t^\alpha} < \infty\Big\}$$

with the norm

$$\|x\|_{F_\alpha} = \sup_{t>0} \frac{\|\underline{T}(t)x - x\|}{t^\alpha},$$

and for $\alpha \in (0,1)$

$$\underline{X}_\alpha(A) := \Big\{x \in \underline{X}_0 : \lim_{t\to 0} \frac{\|\underline{T}(t)x - x\|}{t^\alpha} = 0\Big\} = \Big\{x \in X_0 : \lim_{t\to 0} \frac{\|T(t)x - x\|}{t^\alpha} = 0\Big\}.$$

We have the (continuous) inclusions

$$\underline{X}_1 \hookrightarrow X_1 \to \underline{X}_\alpha(A) \hookrightarrow F_\alpha(A) \to \underline{X}_0 \hookrightarrow X_0;$$

all these spaces are invariant under $(T(t))_{t\geq 0}$.We now extend this diagram by a space which lies between $\underline{X}_\alpha$ and F_α.

Definition 4.13. Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup of negative growth bound on a Banach space X_0 with respect to a locally convex topology τ that is generated by a family $\mathcal{P}$ of seminorms satisfying (4.2). For $\alpha \in (0,1)$ we define the space

$$X_\alpha := X_\alpha(T) := \Big\{x \in X_0 : \tau\lim_{t\to 0} \frac{T(t)x - x}{t^\alpha} = 0 \text{ and } \sup_{t>0} \frac{\|T(t)x - x\|}{t^\alpha} < \infty\Big\}, \tag{4.8}$$

and endow it with the norm $\|\cdot\|_{F_\alpha}$, We further equip F_α and X_α with the locally convex topology τ_{F_α} generated by the family of seminorms $\mathcal{P}_{F_\alpha} := \{p_{F_\alpha} : p \in \mathcal{P}\}$, where $p_{F\alpha}$ is defined as

$$p_{F_\alpha}(x) := \sup_{t>0} \frac{p(T(t)x - x)}{t^\alpha}. \tag{4.9}$$

It is easy to see that X_α is a Banach space, i.e., as closed subspace of F_α. By construction we have that indeed $\underline{X}_\alpha(A) \subseteq X_\alpha \subseteq F_\alpha(A)$. Next we discuss some properties of this space.

Lemma 4.14. (a) *Let (x_n) be a $\|\cdot\|_{F_\alpha}$-norm bounded sequence in F_α with $x_n \to x \in X_0$ in the topology τ. Then $x \in F_\alpha$.*
(b) *The triple $(F_\alpha, \|\cdot\|_{F_\alpha}, \tau_{F_\alpha})$ satisfies the conditions in Assumption 4.1.*
(c) *X_α is bi-closed in F_α, i.e., every $\|\cdot\|_{F_\alpha}$-bounded an τ_{F_α}-convergent sequence in X_α has its limit in X_α.*

Proof. (a) The statement follows from the fact that the norm $\|\cdot\|_{F_\alpha}$ is lower semicontinuous for the topology τ. If

$$\frac{\|T(t)x_n - x_n\|}{t^\alpha} \leq \|x_n\|_{F_\alpha} \leq M$$

for each $n \in \mathbb{N}$, $t > 0$ and for some $M \geq 0$ we can estimate

$$\begin{aligned}\sup_{t>0} \frac{\|T(t)x - x\|}{t^\alpha} &= \sup_{t>0} \sup_{p\in\mathcal{P}} p\Big(\frac{T(t)x - x}{t^\alpha}\Big) = \sup_{t>0} \sup_{p\in\mathcal{P}} \lim_{n\to\infty} p\Big(\frac{T(t)x_n - x_n}{t^\alpha}\Big)\\ &\leq \sup_{t>0} \sup_{p\in\mathcal{P}} \limsup_{n\to\infty} \Big\|\frac{T(t)x_n - x_n}{t^\alpha}\Big\| \leq \sup_{t>0} \sup_{n\in\mathbb{N}} \Big\|\frac{T(t)x_n - x_n}{t^\alpha}\Big\| \leq M.\end{aligned}$$

(b) We have for $p \in \mathcal{P}$ and $x \in F_\alpha$ that

$$p_{F_\alpha}(x) = \sup_{t>0} \frac{p(T(t)x - x)}{t^\alpha} \leq \sup_{t>0} \frac{\|T(t)x - x\|}{t^\alpha} = \|x\|_{F_\alpha}.$$

This proves that τ_{F_α} is coarser than the $\|\cdot\|_{F_\alpha}$-topology, but is still Hausdorff by construction. For the second property of Assumption 4.1 let $(x_n)_{n\in\mathbb{N}}$ be a τ_{F_α}-Cauchy sequence in F_α such that there exists $M > 0$ with $\|x_n\|_{F_\alpha} \leq M$ for each $n \in \mathbb{N}$. Since τ is coarser than τ_{F_α}, we conclude that (x_n) is τ-Cauchy sequence which is also bounded in $\|\cdot\|_{F_\alpha}$, hence in $\|\cdot\|$. By assumption there is $x \in X_0$ such that $x_n \to x$ in τ. By part (a) we obtain $x \in F_\alpha$. It remains to prove that $x_n \to x$ in τ_{F_α}. Let $\varepsilon > 0$, and take $N \in \mathbb{N}$ such that for each $n, m \in \mathbb{N}$ with $n, m \geq N$ we have $p_{F_\alpha}(x_n - x_m) < \varepsilon$. For $t > 0$

$$p\Big(\frac{T(t)(x_n - x) - (x_n - x)}{t^\alpha}\Big) = \lim_{m\to\infty} p\Big(\frac{T(t)(x_n - x_m) - (x_n - x_m)}{t^\alpha}\Big) \leq p_{F_\alpha}(x_n - x_m) < \varepsilon$$

for each $n \geq N$. Taking the supremum in $t > 0$ we obtain $p_{F_\alpha}(x - x_n) \leq \varepsilon$ for each $n \geq N$.

The norming property in (4.1) follows again from Remark 4.2 and the fact that the family $\mathcal{P}$ is norming by assumption.

(c) Let $(x_n)_{n\in\mathbb{N}}$ be a $\|\cdot\|_{F_\alpha}$-bounded and τ_{F_α}-convergent sequence in X_α with limit $x \in X_0$. For $p \in \mathcal{P}$ we then have

$$\sup_{t>0} p\Big(\frac{T(t)(x_n - x) - (x_n - x)}{t^\alpha}\Big) \to 0.$$

Since $x_n \in X_\alpha$ for each $n \in \mathbb{N}$, we have

$$\lim_{t\to 0} p\Big(\frac{T(t)x_n - x_n}{t^\alpha}\Big) = 0, \quad \text{and} \quad \sup_{t>0} \Big\|\frac{T(t)x_n - x_n}{t^\alpha}\Big\| < \infty.$$

We now conclude for a fixed $p \in \mathcal{P}$

$$\begin{aligned} p\Big(\frac{T(t)x - x}{t^\alpha}\Big) &= p\Big(\frac{T(t)(x - x_n) - (x - x_n) + T(t)x_n - x_n}{t^\alpha}\Big) \\ &\leq p\Big(\frac{T(t)(x - x_n) - (x - x_n)}{t^\alpha}\Big) + p\Big(\frac{T(t)x_n - x_n}{t^\alpha}\Big) \\ &\leq p_{F_\alpha}(x - x_n) + p\Big(\frac{T(t)x_n - x_n}{t^\alpha}\Big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

where we first fix $n \in \mathbb{N}$ such that $p_{F_\alpha}(x - x_n) < \frac{\varepsilon}{2}$, and then we take $\delta > 0$ such that $0 < t < \delta$ implies $p(\frac{T(t)x_n - x_n}{t^\alpha}) < \frac{\varepsilon}{2}$. □ □

The next goal is to verify that $(T(t))_{t\geq 0}$ can be restricted to X_α to obtain a bi-continuous semigroup with respect to the topology τ_{F_α}.

Lemma 4.15. *If $(T(t))_{t\geq 0}$ is a bi-continuous semigroup, then X_α is invariant under the semigroup.*

Proof. We notice that in order to prove

$$\tau\lim_{s\to 0} \frac{T(s)x - x}{s^\alpha} = 0$$

we only have to check that

$$\frac{p(T(s_n)x - x)}{s_n^\alpha} \to 0$$

for $n \to \infty$ for every null-sequence $(s_n)_{n\in\mathbb{N}}$ in $[0, \infty)$ and for each $p \in \mathcal{P}$. Let $x \in X_\alpha$. Then we have that $y_n := \frac{T(s_n)x - x}{s_n^\alpha}$ converges to 0 with respect to τ if $(s_n)_{n\in\mathbb{N}}$ is any null-sequence and $n \to \infty$. Moreover, this sequence $(y_n)_{n\in\mathbb{N}}$ is $\|\cdot\|$-bounded by the assumption that $x \in X_\alpha$. Whence we conclude

$$\tau\lim_{n\to\infty} T(t)y_n = \tau\lim_{n\to\infty} \frac{T(s_n)T(t)x - T(t)x}{s_n^\alpha} = 0,$$

so that $T(t)x \in X_\alpha$. □ □

We now prove that $(T(t))_{t\geq 0}$ is bi-continuous on X_α and notice first that the local boundedness and the semigroup property are trivial.

Lemma 4.16. *If $(T(t))_{t\geq 0}$ is a bi-continuous semigroup on X_0 and $\alpha \in (0, 1)$, then $(T(t))_{t\geq 0}$ is strongly τ_{F_α}-continuous on X_α.*

Proof. We have to show that $p_{F_\alpha}(T(t_n)x - x) \to 0$ for all $p \in \mathcal{P}$ whenever $t_n \downarrow 0$. Let $s_n, t_n > 0$ be with $s_n, t_n \to 0$. Then

$$\begin{aligned}\frac{p(T(s_n)T(t_n)x - T(s_n)x - T(t_n)x + x)}{s_n^\alpha} &\leq \frac{p(T(t_n)T(s_s)x - T(t_n)x)}{s_n^\alpha} + \frac{p(T(s_n)x - x)}{s_n^\alpha} \\ &= \frac{p(T(t_n)(T(s_n)x - x))}{s_n^\alpha} + \frac{p(T(s_n)x - x)}{s_n^\alpha}.\end{aligned} \tag{4.10}$$

The sequence (y_n) given by $y_n := \frac{T(s_n)x-x}{s_n^\alpha}$ is $\|\cdot\|$-bounded and τ-convergent to 0, because $x \in X_\alpha$. So that the last term in the previous equation (4.10) converges to 0. But since $\{T(t_n) : n \in \mathbb{N}\}$ is bi-equicontinuous, also the first term in (4.10) converges to 0. This proves strong continuity with respect to τ_{F_α}. □ □

Lemma 4.17. *Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X_0. Then $(T(t))_{t\geq 0}$ is locally bi-equicontinuous on F_α.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a $\|\cdot\|_{F_\alpha}$-bounded sequence which converges to zero with respect to τ_{F_α} and assume that $(T(t)x_n)_{n\in\mathbb{N}}$ does not converge to zero uniformly for $t \in [0, t_0]$ for some $t_0 > 0$. Hence there exists $p \in \mathcal{P}$, $\delta > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers such that

$$p_{F_\alpha}(T(t_n)x_n) > \delta$$

for all $n \in \mathbb{N}$. As a consequence there exists a null-sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$\frac{p(T(s_n)T(t_n)x_n - T(t_n)x_n)}{s_n^\alpha} > \delta$$

for each $n \in \mathbb{N}$. Now notice that the sequence $(y_n)_{n\in\mathbb{N}}$ defined by $y_n := \frac{T(s_n)x_n - x_n}{s_n^\alpha}$ is a τ-null sequence since

$$\frac{q(T(s_n)x_n - x_n)}{s_n^\alpha} \leq \sup_{s>0} \frac{q(T(s)x_n - x_n)}{s^\alpha}, \quad q \in \mathcal{P},$$

and the term on the right hand side converges to zero as $n \to \infty$ by assumption. Using the local bi-equicontinuity of the semigroup $(T(t))_{t\geq 0}$ with respect to τ, we conclude that $\frac{T(t)T(s_n)x_n - T(t)x_n}{s_n}$ converges to zero uniformly for $t \in [0, t_0]$, which is a contradiction. Hence $(T(t))_{t\geq 0}$ is locally bi-equicontinuous on X_α. □ □

Remark 4.18. Notice that the local bi-equicontinuity with respect to τ_{F_α} holds on the whole space F_α, while strong τ_{F_α}-continuity holds on X_α only. In particular, we will see in Theorem 4.20 that X_α is the space of strong τ_{F_α}-continuity.

We can summarize the previous results in the following theorem.

Theorem 4.19. *Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X_0. Then the restricted operators $T_\alpha(t) := T(t)|_{X_\alpha}$ on X_α form a bi-continuous semigroup. Moreover, the generator A_α of $(T_\alpha(t))_{t\geq 0}$ is the part of A in X_α.*

Proof. Because of the previous lemmas it remains to prove that the part of A in X_α generates the restricted semigroup on X_α. We can argue as in the proof of the proposition in [11, Chap. II, Par. 2.3]. Since the embedding $X_\alpha \subseteq X_0$ is continuous for the topologies τ_{F_α} and τ, we conclude that $A_\alpha \subseteq A|_{X_\alpha}$. For the converse let C denote the generator of $(T_\alpha(t))_{t\geq 0}$ and take $\lambda \in \mathbb{R}$ large enough such that

$$R(\lambda, C)x = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)x \, \mathrm{d}s = R(\lambda, A)x, \quad x \in X_\alpha.$$

For $x \in D(A|_{X_\alpha})$ we obtain

$$x = R(\lambda, A)(\lambda - A)x = R(\lambda, C)(\lambda - A)x \in D(C)$$

and hence $A|_{X_\alpha} \subseteq A_\alpha$. This proves that the part of A in X_α generates the restricted semigroup. □ □

By similar reasoning as in Lemma 3.5 one can prove the following.

Theorem 4.20. *Let $\alpha \in (0,1)$ and let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X. Then $D(A)$ is τ_{F_α}-bi-dense in X_α and*

$$X_\alpha = \{x \in F_\alpha : \ \tau_{F_\alpha} \lim_{t\to 0} T(t)x = x\}, \tag{4.11}$$

i.e., for $x \in F_\alpha$ the mapping $t \mapsto T(t)x$ is τ_{F_α}-continuous if and only if $x \in X_\alpha$.

Proof. Denote by $X_{\alpha,\mathrm{cont}}$ the right-hand side of (4.11), i.e., the space of τ_{F_α}-strong continuity. Notice that $D(A) \subseteq \underline{X}_\alpha \subseteq X_\alpha \subseteq X_{\alpha,\mathrm{cont}}$.

Suppose $x \in X_{\alpha,\mathrm{cont}}$. For each $n \in \mathbb{N}$ we have

$$x_n := n\int_0^{\frac{1}{n}} T_\alpha(t)x \ \mathrm{d}t = n\int_0^{\frac{1}{n}} T(t)x \ \mathrm{d}t \in D(A)$$

as a τ- and τ_{F_α}-convergent Riemann integral. Whence it follows that $x_n \overset{\tau_{F_\alpha}}{\to} x$, whereas the $\|\cdot\|_{F_\alpha}$-boundedness of $(x_n)_{n\in\mathbb{N}}$ clear. We conclude that $x \in X_\alpha$ (because X_α is bi-closed in F_α), implying $X_{\alpha,\mathrm{cont}} \subseteq X_\alpha$. As a byproduct we also obtain that $D(A)$ is bi-dense in X_α. □ □

Proposition 4.21. *For $0 \leq \alpha < \beta \leq 1$ we have*

$$X_1 = D(A) \hookrightarrow F_\beta \hookrightarrow \underline{X}_\alpha \subseteq X_\alpha,$$

where the embeddings are continuous for the respective norms and for the respective topologies τ_1, τ_{F_β}, τ_{F_α}. The space $D(A)$ bi-dense in X_α, and as a consequence X_β is bi-dense in X_α.

4.3. **Characterization of Hölder spaces by generators.** Analogously to Proposition 3.6 we characterize the Hölder space X_α by means of the semigroup generator.

Theorem 4.22. *Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup with negative growth bound and generator A. For $\alpha \in (0,1)$ we have*

$$X_\alpha = \Big\{x \in X_0 : \ \tau\lim_{\lambda\to\infty} \lambda^\alpha AR(\lambda,A)x = 0 \ \textit{and} \ \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda,A)x\| < \infty\Big\}. \tag{4.12}$$

Proof. Suppose $x \in X_\alpha$. From Proposition 3.6 we deduce immediately

$$\sup_{\lambda>0} \|\lambda^\alpha AR(\lambda,A)x\| < \infty.$$

Let now $\varepsilon > 0$ be arbitrary. For $x \in X_\alpha$ and $p \in \mathcal{P}$ we can find $\delta > 0$ such that $0 \leq t < \delta$ implies $\frac{p(T(t)x-x)}{t^\alpha} < \varepsilon$. Recall the following formula:

$$\lambda^\alpha AR(\lambda,A)x = \lambda^{\alpha+1}\int_0^\infty \mathrm{e}^{-\lambda s}(T(s)x - x) \ \mathrm{d}s.$$

From this we deduce

$$\begin{aligned}
p(\lambda^\alpha AR(\lambda,A)x) &\leq \lambda^{\alpha+1}\int_0^\infty \mathrm{e}^{-\lambda s}\cdot\frac{p(T(s)x-x)}{s^\alpha}s^\alpha \ \mathrm{d}s\\
&= \lambda^{\alpha+1}\int_0^\delta \mathrm{e}^{-\lambda s}\cdot\frac{p(T(s)x-x)}{s^\alpha}s^\alpha \ \mathrm{d}s + \lambda^{\alpha+1}\int_\delta^\infty \mathrm{e}^{-\lambda s}\cdot\frac{p(T(s)x-x)}{s^\alpha}s^\alpha \ \mathrm{d}s\\
&< \lambda^{\alpha+1}\varepsilon\int_0^\delta \mathrm{e}^{-\lambda s}s^\alpha \ \mathrm{d}s + \lambda^{\alpha+1}\int_\delta^\infty \mathrm{e}^{-\lambda s}\cdot\frac{\|T(s)x-x\|}{s^\alpha}s^\alpha \ \mathrm{d}s\\
&\leq \lambda^{\alpha+1}\varepsilon\int_0^\delta \mathrm{e}^{-\lambda s}s^\alpha \ \mathrm{d}s + \|x\|_{F_\alpha}\lambda^{\alpha+1}\int_\delta^\infty \mathrm{e}^{-\lambda s}\cdot s^\alpha \ \mathrm{d}s\\
&= \varepsilon\int_0^{\lambda\delta} \mathrm{e}^{-t}t^\alpha \ \mathrm{d}t + \|x\|_{F_\alpha}\int_{\lambda\delta}^\infty \mathrm{e}^{-t}t^\alpha \ \mathrm{d}t\\
&\leq L\varepsilon + \|x\|_{F_\alpha}\int_{\lambda\delta}^\infty \mathrm{e}^{-t}t^\alpha \ \mathrm{d}t
\end{aligned}$$

where $L := \int_0^\infty \mathrm{e}^{-\lambda s}s^\alpha \ \mathrm{d}s < \infty$. Notice that the last part of the sum tends to zero if $\lambda \to \infty$ since $\delta > 0$ is fixed. So we obtain $\tau\lim_{\lambda\to\infty} \lambda^\alpha AR(\lambda,A)x = 0$.

For the converse inclusion suppose that $\tau\lim_{\lambda\to\infty}\lambda^\alpha AR(\lambda,A)x=0$ and $\sup_{\lambda>0}\|\lambda^\alpha AR(\lambda,A)x\|<\infty$, the latter immediately implying $\|x\|_{F_\alpha(T)}<\infty$ (see Proposition 3.6). We have to show that $\tau\lim_{t\to0}\frac{T(t)x-x}{t^\alpha}=0$. For $\lambda>0$ define $x_\lambda=\lambda R(\lambda,A)$ and $y_\lambda=AR(\lambda,A)$, then we have

$$x=\lambda R(\lambda,A)x-AR(\lambda,A)x=x_\lambda-y_\lambda.$$

First notice that for $p\in\mathcal{P}$

$$\frac{p(T(t)x_\lambda-x_\lambda)}{t^\alpha}\le\frac{1}{t^\alpha}p(T(t)\lambda R(\lambda,A)x-\lambda R(\lambda,A)x)\le\frac{\lambda^{1-\alpha}}{t^\alpha}\int_0^t p(T(s)\lambda^\alpha AR(\lambda,A)x)\,\mathrm{d}s. \tag{4.13}$$

By assumption the term $\lambda^\alpha AR(\lambda,A)x$ is norm-bounded and converges in the topology τ to zero as $\lambda\to\infty$, hence by the local bi-equicontinuity we conclude that $p(T(s)\lambda^\alpha AR(\lambda,A)x)\to0$ uniformly for $s\in[0,1]$. Now let $\varepsilon>0$ and $\lambda_0>1$ so large that for $\lambda>\lambda_0$ and $s\in[0,1]$ we have $p(T(s)\lambda^\alpha AR(\lambda,A)x)<\varepsilon$. If $t<\frac{1}{\lambda_0}$, then $\lambda:=\frac1t>\lambda_0$ and we obtain that the expression in (4.13) becomes smaller than ε.

For the estimate of the second part involving y_λ we observe that

$$\frac{p(T(t)y_\lambda-y_\lambda)}{t^\alpha}\le\frac{1}{(t\lambda)^\alpha}p(T(t)\lambda^\alpha AR(\lambda,A)x)+\frac{1}{(t\lambda)^\alpha}p(\lambda^\alpha AR(\lambda,A)x).$$

By taking $t<\frac{1}{\lambda_0}$ and $\lambda:=\frac1t$ we obtain the estimate

$$\frac{p(T(t)y_\lambda-y_\lambda)}{t^\alpha}\le p(T(\tfrac1\lambda)\lambda^\alpha AR(\lambda,A)x)+p(\lambda^\alpha AR(\lambda,A)x)<\varepsilon+\varepsilon, \tag{4.14}$$

by the choice of λ_0. Altogether we obtain for $t<\frac{1}{\lambda_0}$ that $\frac{p(T(t)x-x)}{t^\alpha}<3\varepsilon$, showing

$$\tau\lim_{t\to0}\frac{T(t)x-x}{t^\alpha}=0,$$

i.e., $x\in X_\alpha$ as required. □ □

Remark 4.23. It is possible to define the space $X_\alpha(A)$ as in (4.12) also for operators which are not necessarily generators of bi-continuous semigroups. However, we have to suppose that the resolvent fulfills certain continuity assumptions with respect to a topology satisfying, say, Assumption 4.1.

Again, we put our spaces X_α in the general context of Theorem 1.15.

Proposition 4.24. *For $\alpha\in(0,1)$ and $\mathcal{A}$, λ and $\mathscr{E}$ as in Theorem 1.15 we have*

$$X_{-\alpha}=\Big\{(\lambda-\mathcal{A})y\in X_{-1}:\ \sup_{t>0}\frac{\|T(t)y-y\|}{t^{1-\alpha}}<\infty,\ \tau\lim_{t\to0}\frac{T(t)y-y}{t^{1-\alpha}}=0\Big\}.$$

Finally, we extend the scale of spaces X_α to the whole range $\alpha\in\mathbb{R}$.

Definition 4.25. For $\alpha\in\mathbb{R}\setminus\mathbb{Z}$ we write $\alpha=m+\beta$ with $m\in\mathbb{Z}$ and $\beta\in(0,1]$, and define

$$X_\alpha(A):=X_\beta(A_m),$$

with the corresponding norms. The locally convex topology on X_α comes from X_β via the mapping A_m.

Remark 4.26. We summarize all previous results in the following diagram

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] \arrow[rrrrr, bend left, "A"] & \underline{X}_\alpha \arrow[r] \arrow[d] & X_\alpha \arrow[r] \arrow[d] \arrow[rrrrr, bend left, "A_{\alpha-1}"] & F_\alpha \arrow[r] \arrow[d] & \underline{X}_0 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & \underline{X}_{\alpha-1} \arrow[r] \arrow[d] & X_{\alpha-1} \arrow[r] \arrow[d] & F_{\alpha-1} \arrow[r] \arrow[d] & X_{-1} \arrow[d] \\
X_1 \arrow[r] & \underline{X}_\alpha \arrow[r] \arrow[rrrrr, bend right, "\underline{A}_{\alpha-1}"'] & X_\alpha \arrow[r] & F_\alpha \arrow[r] \arrow[rrrrr, bend right, "A_{-1}|_{F_\alpha}"'] & \underline{X}_0 \arrow[r] & X_0 \arrow[r] \arrow[rrrr, bend right, "A_{-1}"'] & \underline{X}_{\alpha-1} \arrow[r] & X_{\alpha-1} \arrow[r] & F_{\alpha-1} \arrow[r] & X_{-1}
\end{tikzcd}$$

where $\alpha \in (0,1)$. Here $A_{\alpha-1}$ and $\underline{A}_{\alpha-1}$ are defined to be the part of A_{-1} in $X_{\alpha-1}$ and the part of $\underline{A}_{-1}$ in $\underline{X}_{\alpha-1}$, respectively. They are all continuous with respect to the norms and topologies on these spaces. In addition, we recall that $X_{\alpha-1}$ and $\underline{X}_{\alpha-1}$ are the extrapolation spaces of $X_\alpha(A_{-1})$ and $\underline{X}_\alpha(A_{-1})$, respectively. All horizontal arrows represent continuous inclusions, while the vertical arrows represent the action(s) of the semigroup(s). All the spaces are dense in the underlined ones containing them, while the spaces with underlining are bi-dense in each of the bigger ones.

5. Examples

In this section we present examples for extrapolation and intermediate spaces for (generators of) bi-continuous semigroups. We will use Theorem 1.15 and its variants to identify the space X_α for $\alpha < 0$.

5.1. **The translation semigroup.** Let $X_0 = \mathrm{C_b}(\mathbb{R})$ be the space of bounded and continuous functions on $\mathbb{R}$, equipped with the supremum norm $\|\cdot\|_\infty$ and consider thereon the compact-open topology τ_{co} generated by the family of seminorms $\mathcal{P} = \{p_K :\ K \subseteq \mathbb{R} \text{ compact}\}$, where

$$p_K(f) = \sup_{x\in K} |f(x)|, \quad f \in \mathrm{C_b}(\mathbb{R}).$$

The left translation semigroup $(T(t))_{t\geq 0}$ defined by

$$T(t)f(x) = f(x+t), \quad t \geq 0$$

is bi-continuous on X_0 with respect to τ_{co}. The generator A of this semigroup is the first derivative $Af = f'$ on the domain (see [20])

$$D(A) = \{f \in \mathrm{C_b}(\mathbb{R}) :\ f \quad \text{is differentiable} \quad f' \in \mathrm{C_b}(\mathbb{R})\}.$$

The space of strong continuity is $\underline{X}_0 = \mathrm{UC_b}(\mathbb{R})$, the space of all bounded, uniformly continuous functions. We use Theorem 1.15 to determine the corresponding extrapolation spaces. To this purpose let $\mathscr{E} = \mathscr{D}'(\mathbb{R})$ be the space of all distributions on $\mathbb{R}$, let $\mathcal{A} := D : \mathscr{D}'(\mathbb{R}) \to \mathscr{D}'(\mathbb{R})$ be the distributional derivative, and let $i : \mathrm{C_b}(\mathbb{R}) \to \mathscr{D}'(\mathbb{R})$ be the regular embedding. From Theorem 1.15 it then follows that

$$\underline{X}_{-1} = \{F \in \mathscr{D}'(\mathbb{R}) :\ F = f - Df \text{ for some } f \in \mathrm{UC_b}(\mathbb{R})\},$$
$$X_{-1} = \{F \in \mathscr{D}'(\mathbb{R}) :\ F = f - Df \text{ for some } f \in \mathrm{C_b}(\mathbb{R})\}.$$

For the Favard and Hölder spaces we have

$$F_\alpha = \Big\{f \in \mathrm{C_b}(\mathbb{R}) :\ \sup_{\substack{x,y\in\mathbb{R}\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty\Big\} = \mathrm{C}^\alpha_{\mathrm{b}}(\mathbb{R}),$$

$$\underline{X}_\alpha = \Big\{f \in \mathrm{UC_b}(\mathbb{R}) :\ \lim_{t\to 0} \sup_{\substack{x,y\in\mathbb{R}\\ 0<|x-y|<t}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} = 0\Big\} = \mathrm{h}^\alpha_{\mathrm{b}}(\mathbb{R}).$$

Hence F_α is the space of bounded α-Hölder-continuous functions and $\underline{X}_\alpha$ with the so-called little Hölder space $\mathrm{h}^\alpha_{\mathrm{b}}(\mathbb{R})$ (see also [23]). The abstract Hölder space X_α corresponding to the bi-continuous semigroup yields the local version $\mathrm{h}^\alpha_{\mathrm{b,loc}}(\mathbb{R})$ of the little Hölder space

$$\mathrm{h}^\alpha_{\mathrm{b,loc}} = \Big\{f \in \mathrm{C}^\alpha_{\mathrm{b}}(\mathbb{R}) :\ \lim_{t\to 0} \sup_{\substack{x,y\in K\\ 0<|x-y|<t}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} = 0 \text{ for each } K \subseteq \mathbb{R} \text{ compact}\Big\}.$$

Then $X_\alpha = h^\alpha_{\mathrm{b,loc}}(\mathbb{R})$.

It is easy to see $\underline{X}_\alpha \subsetneq X_\alpha \subsetneq F_\alpha$. The extrapolated Favard class F_0 can be identified with $\mathrm{L}^\infty(\mathbb{R})$. To prove this we argue as follows: We know from the general theory that $F_0(T) = (1-D)F_1(T)$ where $F_1(T)$ are precisely the bounded Lipschitz functions on $\mathbb{R}$. Now using the fact that $\mathrm{Lip_b}(\mathbb{R}) = \mathrm{W}^{1,\infty}(\mathbb{R})$ with equivalent norms we obtain that indeed $F_0 = \mathrm{L}^\infty(\mathbb{R})$. For an alternative proof of this fact we refer to [11, Chapter II.5(b)].

Moreover, from Corollary 2.9 we obtain for $\alpha \in (0,1)$

$$F_{-\alpha} = \Big\{ f \in \mathscr{D}'(\mathbb{R}) : \quad F = f - Df \text{ for } f \in \mathrm{C}_{\mathrm{b}}^{1-\alpha}(\mathbb{R}) \Big\},$$

and

$$X_{-\alpha} = \Big\{ f \in \mathscr{D}'(\mathbb{R}) : \quad F = f - Df \text{ for } f \in \mathrm{h}_{\mathrm{b,loc}}^{1-\alpha}(\mathbb{R}) \Big\}.$$

We summarize this example by the diagram:

$$\mathrm{C}_{\mathrm{b}}^1(\mathbb{R}) \hookrightarrow \mathrm{Lip}_{\mathrm{b}}(\mathbb{R}) \hookrightarrow \mathrm{h}_{\mathrm{b}}^{\alpha}(\mathbb{R}) \hookrightarrow \mathrm{h}_{\mathrm{b,loc}}^{\alpha}(\mathbb{R}) \hookrightarrow \mathrm{C}_{\mathrm{b}}^{\alpha}(\mathbb{R}) \hookrightarrow \mathrm{UC}_{\mathrm{b}}(\mathbb{R}) \hookrightarrow \mathrm{C}_{\mathrm{b}}(\mathbb{R}) \hookrightarrow \mathrm{L}^{\infty}(\mathbb{R})$$

according to the abstract chain of spaces

$$X_1 \hookrightarrow F_1 \hookrightarrow \underline{X}_\alpha \hookrightarrow X_\alpha \hookrightarrow F_\alpha \hookrightarrow \underline{X}_0 \hookrightarrow X_0 \hookrightarrow F_0$$

for $\alpha \in (0,1)$. For the higher order spaces we have

$$\begin{aligned} X_n := D(A^n) &= \Big\{ f \in \mathrm{C}_{\mathrm{b}}(\mathbb{R}) : \ f \text{ is } n\text{-times differentiable and } f^{(n)} \in \mathrm{C}_{\mathrm{b}}(\mathbb{R}) \Big\} \\ &= \Big\{ f \in \mathrm{C}_{\mathrm{b}}(\mathbb{R}) : \ f^{(k)} \in \mathrm{C}_{\mathrm{b}}(\mathbb{R}), \ k = 1, \dots, n \Big\} = \mathrm{C}_{\mathrm{b}}^n(\mathbb{R}) \end{aligned}$$

for $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $\alpha \in [0,1)$

$$F_{n+\alpha} = \Big\{ f \in \mathrm{C}_{\mathrm{b}}^n(\mathbb{R}) : \ \sup_{\substack{x,y \in \mathbb{R} \\ x \neq y}} \frac{|f^{(n)}(x) - f^{(n)}(y)|}{|x-y|^\alpha} < \infty \Big\} = \mathrm{C}_{\mathrm{b}}^{n,\alpha}(\mathbb{R}).$$

This example complements the corresponding one in Nagel, Nickel, Romanelli [27, Sec. 3.2].

5.2. **The multiplication semigroup.** Let Ω be a locally compact space and $X_0 = \mathrm{C}_{\mathrm{b}}(\Omega)$. Let $q : \Omega \to \mathbb{C}$ be continuous such that $\sup_{x\in\Omega} \mathrm{Re}(q(x)) < 0$. We define the multiplication operator $M_q : D(M_q) \to \mathrm{C}_{\mathrm{b}}(\Omega)$ by $M_q f = qf$ on the maximal domain

$$D(M_q) = \{ f \in \mathrm{C}_{\mathrm{b}}(\Omega) : \ qf \in \mathrm{C}_{\mathrm{b}}(\Omega) \}.$$

This operator generates the semigroup $(T_q(t))_{t\geq 0}$ defined by

$$(T_q(t)f)(x) = \mathrm{e}^{tq(x)} f(x), \quad t \geq 0, x \in \Omega, f \in \mathrm{C}_{\mathrm{b}}(\Omega),$$

which is bi-continuous on $\mathrm{C}_{\mathrm{b}}(\Omega)$ with respect to the compact-open topology. Now let $\mathscr{E} = \mathrm{C}(\Omega)$ the space of all continuous functions on Ω, let $\mathcal{M}_q : \mathrm{C}(\Omega) \to \mathrm{C}(\Omega)$ be the multiplication operator $\mathcal{M}_q f := qf$ and $i : \mathrm{C}_{\mathrm{b}}(\Omega) \to \mathrm{C}(\Omega)$ the identical embedding. Then by Theorem 1.15 we obtain

$$X_{-1} = \{ g \in \mathrm{C}(\Omega) : \ q^{-1} g \in \mathrm{C}_{\mathrm{b}}(\Omega) \}.$$

For $\alpha \in (0,1)$, the (abstract) Favard space is

$$F_\alpha = \{ f \in \mathrm{C}_{\mathrm{b}}(\Omega) : \ |q|^\alpha f \in \mathrm{C}_{\mathrm{b}}(\Omega) \}. \tag{5.1}$$

To see this suppose first that $f \in F_\alpha$, which means

$$\sup_{t>0} \sup_{x\in\Omega} \frac{|\mathrm{e}^{tq(x)} f(x) - f(x)|}{t^\alpha} < \infty.$$

By taking supremum only for $t = \frac{1}{|q(x)|}$ we obtain

$$\sup_{x\in\Omega} \Big| \mathrm{e}^{\frac{q(x)}{|q(x)|}} - 1 \Big| \cdot |f(x)| \cdot |q(x)|^\alpha < \infty,$$

since

$$\frac{|\mathrm{e}^{tq(x)} f(x) - f(x)|}{t^\alpha} = \frac{|\mathrm{e}^{tq(x)} - 1| \cdot |f(x)| |q(x)|^\alpha}{|q(x)|^\alpha t^\alpha}. \tag{5.2}$$

Hence $|q|^\alpha f \in \mathrm{C}_{\mathrm{b}}(\Omega)$, so that the inclusion "$\subseteq$" in (5.1) is established. For the converse assume that $|q|^\alpha f \in \mathrm{C}_{\mathrm{b}}(\Omega)$. Since the function $g(z) = \frac{|\mathrm{e}^z - 1|}{|z|^\alpha}$ is bounded on the left half-plane, we obtain that $f \in F_\alpha$ by (5.2). This proves the equality. We also conclude that $F_\alpha = X_\alpha$ since

$$\sup_{x\in K} \left| \frac{\mathrm{e}^{tq(x)} f(x) - f(x)}{t^\alpha} \right| = \sup_{x\in K} \left| \frac{\mathrm{e}^{tq(x)} - 1}{tq(x)} \right| \cdot |f(x)| \cdot |q(x)|^\alpha \, t^{1-\alpha}$$

for each compact set $K \subseteq \Omega$. The extrapolated Favard spaces are then given by

$$F_{-\alpha} = \left\{ f \in \mathrm{C_b}(\Omega) : \ |q|^{1-\alpha} f \in \mathrm{C_b}(\Omega) \right\} = X_{-\alpha}.$$

The spaces $\underline{X}_\alpha$ are more difficult to describe in general since the space of strong continuity $\underline{X}_0$ depends substantially on the choice of q. For example, if $\frac{1}{q} \in \mathrm{C}_0(\Omega)$, then $\underline{X}_0 = \mathrm{C}_0(\Omega)$. To see this notice that $\mathrm{C}_0(\Omega) \subseteq \underline{X}_0$ trivially. On the other hand

$$|f| = \left| \frac{1}{q} \right| \cdot |fq|$$

which shows that $D(M_q) \subseteq \mathrm{C}_0(\Omega)$ and hence that $\underline{X}_0 \subseteq \mathrm{C}_0(\Omega)$. For $\alpha \in [0,1]$ this yields

$$\underline{X}_\alpha = \{ f \in \mathrm{C}_0(\Omega) : |q|^\alpha f \in \mathrm{C}_0(\Omega) \},$$

and

$$\underline{X}_{-\alpha} = \{ qf : f \in \mathrm{C}_0(\Omega),\ |q|^{1-\alpha} f \in \mathrm{C}_0(\Omega) \} = \{ f \in \mathrm{C}(\Omega) : |q|^{-\alpha} f \in \mathrm{C}_0(\Omega) \}.$$

This example extends Section 3.2 in [27] by Nagel, Nickel and Romanelli.

5.3. **The Gauß-Weierstraß semigroup.** On $X_0 = \mathrm{C_b}(\mathbb{R}^d)$ $(d \geq 1)$ we consider the Gauß–Weierstraß semigroup defined by $T(0) = I$ and

$$T(t)f(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4t}} f(y)\, \mathrm{d}y, \quad t > 0, \quad x \in \mathbb{R}^d. \tag{5.3}$$

If we equip $\mathrm{C_b}(\mathbb{R}^d)$ with the compact-open topology τ_{co}, then $(T(t))_{t\geq 0}$ becomes a bi-continuous semigroup, and its space of strong continuity is $\mathrm{UC_b}(\mathbb{R}^d)$. From [22, Proposition 2.3.6] we know that the generator A of this semigroup is given $Af = \Delta f$ on the maximal domain

$$D(A) = \{ f \in \mathrm{C_b}(\mathbb{R}^d) : \quad \Delta f \in \mathrm{C_b}(\mathbb{R}^d) \},$$

where Δ is the distributional Laplacian. Now the extrapolation space can again be obtained by Theorem 1.15. If we take $\mathscr{E} = \mathscr{D}'(\mathbb{R}^d)$, $\mathcal{A} = \Delta$ and $i : \mathrm{C_b}(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$ the regular embedding, we then have

$$X_{-1} = \{ F \in \mathscr{D}'(\mathbb{R}^d) : \ F = f - \Delta f \text{ for some } f \in \mathrm{C_b}(\mathbb{R}^d) \}.$$

The domain of the generator can be given explicitly, see, e.g., [22] or [23]. For $d = 1$ it is

$$D(A) = \mathrm{C}^2_{\mathrm{b}}(\mathbb{R}),$$

while for $d \geq 2$

$$D(A) = \left\{ f \in \mathrm{C_b}(\mathbb{R}^d) \cap \mathrm{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d), \text{ for all } p \in [1,\infty) \text{ and } \Delta f \in \mathrm{C_b}(\mathbb{R}^d) \right\}.$$

For $\alpha \in (0,1) \setminus \{\frac{1}{2}\}$ the Favard spaces are

$$F_\alpha = \mathrm{C}^{2\alpha}_{\mathrm{b}}(\mathbb{R}^d),$$

while for $\alpha = \frac{1}{2}$ one obtains

$$F_{\frac{1}{2}} = \left\{ f \in \mathrm{C_b}(\mathbb{R}^d) : \ \sup_{x \neq y} \frac{|f(x) + f(y) - 2f(\frac{x+y}{2})|}{|x-y|} < \infty \right\}.$$

From Corollary 2.9 it follows that for $\alpha \in (0,1)$, $\alpha \neq \frac{1}{2}$

$$F_{-\alpha} = \left\{ F \in \mathscr{D}'(\mathbb{R}^d) : \ F = f - \Delta f \text{ for some } f \in \mathrm{C}^{2(1-\alpha)}_{\mathrm{b}}(\mathbb{R}^d) \right\},$$

and

$$F_{-\frac{1}{2}} = \left\{ F \in \mathscr{D}'(\mathbb{R}^d) : \ F = f - \Delta f \text{ for some } f \in F_{\frac{1}{2}} \right\}.$$

5.4. **The left implemented semigroup.** Let $X_0 := \mathscr{L}(E)$ be the space of bounded linear operators on a Banach space E. We equip $\mathscr{L}(E)$ with the operator norm and the strong topology τ_{stop} generated by the family of seminorms $\mathcal{P} = \{p_x : \ x \in E\}$ where

$$p_x(B) = \|Bx\|, \quad B \in \mathscr{L}(E).$$

Let $(S(t))_{t\geq 0}$ be a C_0-semigroup with negative growth bound on the Banach space E. The semigroup $(\mathcal{U}(t))_{t\geq 0}$ on X_0 defined by

$$\mathcal{U}(t)B = S(t)B, \quad B \in X_0, \quad t \geq 0,$$

is called the semigroup left implemented by $(S(t))_{t\geq 0}$. Note that $(\mathcal{U}(t))_{t\geq 0}$ still has negative growth bound and is a bi-continuous semigroup. We determine the intermediate and extrapolation spaces for this semigroup. We can write

$$\begin{aligned}\|B\|_{F_\alpha(\mathcal{U})} &= \sup_{t>0} \frac{\|\mathcal{U}(t)B - B\|}{t^\alpha} = \sup_{t>0} \frac{\|S(t)B - B\|}{t^\alpha} \\ &= \sup_{t>0} \sup_{\|x\|\leq 1} \frac{\|S(t)Bx - Bx\|}{t^\alpha} = \sup_{\|x\|\leq 1} \sup_{t>0} \frac{\|S(t)Bx - Bx\|}{t^\alpha} = \sup_{\|x\|\leq 1} \|Bx\|_{F_\alpha(S)}.\end{aligned}$$

From this we conclude the following.

Proposition 5.1. *Let $(\mathcal{U}_L(t))_{t\geq 0}$ be the semigroup which is left implemented by $(S(t))_{t\geq 0}$. Then*

$$F_\alpha(\mathcal{U}) = \mathscr{L}(E, F_\alpha(S)) \quad \textit{for } \alpha \in (0,1]$$

with the same norms.

From the definition we obtain that

$$\begin{aligned}X_\alpha(\mathcal{U}) &= \Big\{B \in \mathscr{L}(E) : \ \tau\lim_{t\to 0} \frac{\mathcal{U}_L(t)B - B}{t^\alpha} = 0, \ \|B\|_{F_\alpha(\mathcal{U})} < \infty\Big\} \\ &= \Big\{B \in \mathscr{L}(E) : \ \lim_{t\to 0} \frac{\|S(t)Bx - Bx\|}{t^\alpha} = 0 \quad \text{for all } x \in E\Big\}, \\ \underline{X}_\alpha(\mathcal{U}) &= \Big\{B \in \mathscr{L}(E) : \ \lim_{t\to 0} \frac{S(t)B - B}{t^\alpha} = 0\Big\}.\end{aligned}$$

Proposition 5.2. *Let $(\mathcal{U}(t))_{t\geq 0}$ be the semigroup which is left implemented by $(S(t))_{t\geq 0}$. Then*

$$X_\alpha(\mathcal{U}) = \mathscr{L}(E, X_\alpha(S))$$

with the same norms.

We now turn to the extrapolation spaces. For the C_0-semigroup $(\underline{\mathcal{U}}(t))_{t\geq 0}$ on the space $\underline{X}_0$ these have been studied by Alber in [2]. He has shown that the generator $\mathcal{G}$ of $(\mathcal{U}(t))_{t\geq 0}$ is given by

$$\mathcal{G}V = A_{-1}V$$

on

$$D(\mathcal{G}) = \{V \in \mathscr{L}(E) : \quad A_{-1}V \in \mathscr{L}(E)\},$$

where A_{-1} denotes the generator of the extrapolated C_0-semigroup $(S_{-1}(t))_{t\geq 0}$ on E_{-1}. The extrapolation spaces X_{-1} and $\underline{X}_{-1}$ can now be obtained by Theorem 1.15. For that let

$$\mathscr{E} = \big\{S : E \to E_{-\infty} : \ \text{linear and continuous}\big\},$$

where $E_{-\infty}$ is the universal extrapolation space of $(S(t))_{t\geq 0}$ (see the paragraph preceding Theorem 1.15), and let $i : \mathscr{L}(E) \to \mathscr{E}$ be the identity. Consider the operator-valued multiplication operator

$$\mathcal{A}V = A_{-\infty}V, \quad V \in \mathscr{E},$$

where $A_{-\infty}x = A_{-(n-1)}x$ for $x \in E_{-n}$. Notice that $\lambda - \mathcal{A} : X_0 \to \mathscr{E}$ is injective for $\lambda > 0$ since $A_{-\infty}$ and A_{-1} coincide on E. Hence by applying Theorem 1.15 we obtain

$$X_{-1} = \{A_{-1}V : \quad V \in \mathscr{L}(E)\}$$

and

$$\underline{X}_{-1} = \{A_{-1}V : \quad V \in \underline{X}_0\}.$$

From this we conclude that

$$X_{-1} = \Big\{ V \in \mathscr{L}(E, E_{-1}) :\ \exists (V_n)_{n\in\mathbb{N}} \subseteq \mathscr{L}(E) \text{ with } V_n \to V \text{ strongly} \Big\} = \overline{\mathscr{L}(E)}^{\mathscr{L}_{\mathrm{stop}}(E,E_{-1})}.$$

Since for any $C \in \mathscr{L}(E, E_{-1})$ we have $nR(n, A_{-1})C \in \mathscr{L}(E)$ and $nR(n, A_{-1})C \to C$ strongly as $n \to \infty$, we obtain

$$X_{-1} = \mathscr{L}(E, E_{-1}).$$

For $\underline{X}_{-1}$ we have:

$$\underline{X}_{-1} = \Big\{ V \in \mathscr{L}(E, E_{-1}) :\ \exists (V_n)_{n\in\mathbb{N}} \subseteq \mathscr{L}(E) \text{ with } V_n \to V \text{ in } \mathscr{L}(E, E_{-1}) \Big\} = \overline{\mathscr{L}(E)}^{\mathscr{L}(E,E_{-1})}.$$

This last statement is a result of Alber, see [2], which we could recover as a simple consequence of the abstract techniques described in this paper. Finally, we obtain by Corollary 2.9 and Remark 4.26 that for $\alpha \in [0, 1)$

$$F_{-\alpha}(\mathcal{U}) = A_{-1}\mathscr{L}(E, F_{1-\alpha}(S)) = \mathscr{L}(E, F_{-\alpha}(S))$$

and

$$X_{-\alpha}(\mathcal{U}) = A_{-1}\mathscr{L}(E, X_{1-\alpha}(S)) = \mathscr{L}(E, X_{-\alpha}(S)).$$

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