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Abstract. We develop a comprehensive framework in which the existence of solutions to the semiclassical Einstein equation (SCE) in cosmological spacetimes is shown. Different from previous work on this subject, we do not restrict to the conformally coupled scalar field and we admit the full renormalization freedom. Based on a regularization procedure, which utilizes homogeneous distributions and is equivalent to Hadamard point-splitting, we obtain a reformulation of the evolution of the quantum state as an infinite-dimensional dynamical system with mathematical features that are distinct from the standard theory of infinite-dimensional dynamical systems (e.g. unbounded evolution operators). Nevertheless, applying new mathematical methods, we show existence of maximal/global (in time) solutions to the SCE for vacuum-like states, and of local solutions for thermal-like states. Our equations do not show the instability of the Minkowski solution described by other authors.

1 Introduction

The *semiclassical Einstein equation (SCE)* is the equation

$$G_{\mu\nu} = 8\pi \mathrm{G} \langle T^{\mathrm{ren}}_{\mu\nu} \rangle_{\omega}, \tag{1.1}$$

where $G_{\mu\nu}$ denotes the Einstein tensor for the (classical) metric $g_{\mu\nu}$, G is the gravitational constant, $m \geq 0$ the mass, and $\langle T^{\mathrm{ren}}_{\mu\nu} \rangle_{\omega}$ is the renormalized stress-energy tensor for a quantum field theory (QFT) in the state ω. That is, matter is described by a quantum field and gravity is described by a classical Lorentzian manifold. The SCE has been studied since the late 1960's by a number of authors, see [9, 21, 49] for an overview. It is typically introduced in an ad hoc manner as a minimal change of the classical Einstein equation by replacing the the stress-energy tensor of a classical field by that of a quantum field to take into account the quantum nature of matter. In particular, it is not considered a fundamental equation but rather an approximation of a more fundamental theory within some domain of validity that is sufficiently remote from the Planck scale. Some possible derivations of the SCE from a quantum gravity are critically discussed in Sect. II.B of [20].

Many equations in QFT are plagued by ultraviolet divergences and the SCE makes no difference, because (naïvely) the expectation value of the stress-energy tensor involves the evaluation of singular quantum fields at a point. As already discussed in [47], a renormalization of the stress-energy tensor needs to be coordinate independent and thus has to follow the principle of general covariance.

A procedure which satisfies these requirements is the point-splitting formalism by Christensen [11, 12]. Here one subtracts the singular part, given by the Hadamard parametrix (essentially a Lorentzian version of the heat kernel), from the two-point function of the state. For this reason one restricts the class of states to Hadamard states, viz., states with two-point functions that match the Hadamard parametrix up to smooth contributions. (Actually it is sufficient that the leading singularities of the two-point function match those of the Hadamard parametrix, as in the case of adiabatic states [27].)

Due to the ambiguity in the regularization procedure (satisfying certain conditions or axioms [23–25]), a renormalization freedom arises. Some terms renormalize the gravitational constant or the cosmological constant by a finite amount. Thus it can be seen that the widespread belief that quantum matter automatically leads to a very large cosmological constant is not correct; instead the cosmological constant corresponds to a renormalization freedom. Other terms contain higher than second order derivatives in the metric. Such terms are likely to change the entire characteristic of the SCE, especially with respect to the classical Einstein equation, which is only of second order, and there seems to be to be no justifiable reason why these higher order terms should be discarded [48]. We would like to mention, however, the method of order reduction by which higher order derivatives in the SCE can be replaced by lower order derivatives in a systematic way, see [20] and references therein.

Moreover, it was noted early on, see e.g. [11, 48], that the renormalized stress-energy tensor is not traceless, even if the classical action of the quantum field is conformally invariant. In fact, in this case one obtains the famous *trace (or conformal) anomaly*

$$\langle T^{\mathrm{ren}}\rangle = g^{\mu\nu}\langle T^{\mathrm{ren}}_{\mu\nu}\rangle = c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} + c_4 \Box R + \text{renormalization freedom},$$

where the constants $c_\bullet$ depend on spin of the (free) quantum field. Notably, the right-hand side does not depend on the quantum state at all, and contains higher than second order derivatives in the metric.

In the following we shall specialize our discussion to the special case of a free scalar field on flat *Friedmann–Lemaître–Robertson–Walker (FLRW)* spacetimes $M = I \times \mathbb{R}^3$, $I \subset \mathbb{R}$, with the metric (in conformal time τ and with signature convention $-{+}{+}{+}$)

$$g = a(\tau)^2\big(-\mathrm{d}\tau^2 + \mathrm{d}\vec{x}^2\big), \tag{1.2}$$

where $a(\tau) > 0$ is called the scale factor and $\mathrm{d}\vec{x}^2$ denotes the Euclidean metric on $\mathbb{R}^3$. This case already shows many features that distinguish it from QFT on the maximally symmetric Minkowski and de Sitter spacetime, or static spacetimes. Additionally, these spacetimes are of importance in cosmology as they represent the observed homogeneous and isotropic structure of our universe at the scale of several megaparsec, as well as the observed flatness [50].

Despite some mathematical problems in the pre-1990's literature on the SCE, by the beginning of the 1980's the approach has been developed to a stage where numerical solutions and cosmological applications of the SCE were in reach. This situation was exploited by Anderson in a series of four papers [1–4] starting with the conformally coupled and massless scalar field following up prior work by Starobinski [43]. Depending on the values of the aforementioned renormalization parameters, Anderson discovered a very rich behavior of the SCE ranging from big bang, big bounce to divergence of the scale factor to infinity in finite time. The non-conformally coupled case was investigated by analytical and numerical methods by Suen [45, 46]. His analysis revealed an instability of Minkowski spacetime as a global solution to the SCE. Further analytical and numerical work on the stability of the SCE has been performed by Hänsel and Verch [22].

The mathematical literature on the SCE starts with the formulation of an axiomatic framework for the definition of the stress-energy tensor by Wald along with the proof of uniqueness of this tensor up to renormalization degrees of freedom, see e.g. [49]. Mathematical research on quantum fields on curved spacetimes gathered further momentum by the work of Radzikowski [38, 39] who classified Hadamard states using methods of microlocal analysis. In particular, Radzikowski proved that a quasi-free state that is Hadamard on a time-slice around a Cauchy surface is Hadamard on the entire globally hyperbolic manifold containing this surface. From the perspective of the SCE, this result gives an important hint for the well-posedness of the former equation ensuring that the stress-energy tensor will remain well-defined on the entire spacetime manifold.

The proper description of the stress-energy tensor in the mathematically rigorous framework paved the way for new cosmological investigations on the SCE. Dappiaggi, Fredenhagen and Pinamonti [13] investigated the stability of the SCE on FLRW spacetimes using certain effective large-mass approximations of the quantum state. As a major milestone in the mathematical theory of the SCE, Pinamonti proved the existence of solutions for the trace equation for short times in the conformally coupled case for certain states defined on past null infinities [36]. This has been significantly extended by Pinamonti and one of us [37] to full solutions of the SCE on FLRW spacetimes, including the energy constraint with initial values specified on a Cauchy surface, but still limited to the conformally coupled scalar field, and a particular choice of renormalization parameters and quantum state.

We remark that in this paper we deal with the SCE without a classical background field. However, as background fields are important in inflationary cosmology (see e.g. [21, 32, 50], we give remarks on the minor changes that are needed to include background fields without changing the results of our paper.

After this introduction, in the second section we first outline the quantization of the (real, free) scalar field in curved spacetimes with emphasize on flat FLRW spacetimes. In particular, we present an initial value formulation for homogeneous isotropic states for the quantum scalar field in FLRW spacetimes [30]. Then, we develop a point-splitting regularization specially adapted to FLRW spacetimes and show its equivalence to the conventional Hadamard point-splitting.

In the third section, we briefly discuss the renormalized stress-energy tensor for the quantized scalar field. Since the trace and the energy component of the stress-energy tensor play an important role for the (semiclassical) Einstein equation in FLRW spacetimes, we present the corresponding expressions. We also discuss the problem posed by higher derivatives present in the semiclassical theory, partially due to regularization and renormalization.

Next, in the fourth section, we show how to formulate and solve a dynamical system for the coincidence limit of the regularized two-point function and its derivatives. This system shows various interesting physical and mathematical features: First, it hides the higher derivatives present in the regularization, thereby circumventing one of the challenges faced when solving the SCE. Next, it distinguishes a class of 'generalized' vacuum states from other classes of states such as thermal states. Finally, its evolution is given by a (generally unbounded) evolution operator which can be understood as acting between differently weighted sequence spaces. The fact that this evolution operator exists at all is intimately tied to the hyperbolicity of the Klein–Gordon operator, which expresses itself in this dynamical system through the nilpotence of a certain matrix. Depending on the choice of the weights in the sequence spaces, the evolution can be shown to exist for all time (geometrically growing weights) or for a finite amount of time (factorially growing weights).

Finally, in the fifth section, we use the just developed dynamical system for the quantum state to solve the SCE. In fact, we consider an abstract class of quasi-linear equations which includes as a special case the SCE. For this abstract equation we show existence and uniqueness of solutions, as well as continuous dependence on the initial values and parameters of the equation. Existence is shown in finite time-intervals with a priori bounds depending on the choice of initial values, in particular the initial values for the quantum state. We analyze this equation for various possible choices of parameters in the case of the SCE in FLRW spacetimes. In general the resulting equation is of fourth order but in the special case of conformal coupling it can reduce to a second order equation. We also remark on the instability of Minkowski spacetime stating that arbitrarily small perturbations in the initial conditions can lead to finite effects in the solution of the energy equation [45, 46] – no such instability appears in our approach based on the trace equation. At the end, we explain a method of constructing physical initial data and give a short outlook on future research topics.

In the appendix we list several auxiliary results (concerning homogeneous distributions, weighted sequence spaces, combinatorial inequalities and Synge's world function) used throughout this work.

2 Scalar field on flat FLRW

2.1 Klein–Gordon equation

Consider the (homogeneous) Klein–Gordon equation

$$K\phi := (\Box + \xi R + m^2)\phi = 0 \tag{2.1}$$

with mass $m \geq 0$ and curvature coupling ξ ($\xi = 0$ is called *minimal coupling* and $\xi = \frac{1}{6}$ *conformal coupling*). We define the d'Alembert operator as $\Box := -g^{\mu\nu}\nabla_\mu\nabla_\nu$, where ∇ is the covariant derivative and we employ Einstein summation convention.

In conformal time τ, the *flat* FLRW metric takes the form (1.2). The d'Alembertian for this metric is $\Box = a^{-3}(\partial_\tau^2 - a^{-1}\ddot{a} - \Delta)a$, where we denote derivatives with respect to conformal time by dots and Δ is the Laplace operator on $\mathbb{R}^3$. Introducing the conformally rescaled field φ and the potential V,

$$\varphi := a\phi, \quad V := (6\xi - 1)\frac{\ddot{a}}{a} + a^2 m^2,$$

the Klein–Gordon equation (2.1) can thus be written as $(\partial_\tau^2 - \Delta + V)\varphi = 0$, where we used $R = 6a^{-3}\ddot{a}$. Rewritten in Hamiltonian form, this equation becomes

$$\partial_\tau \begin{pmatrix} \varphi \\ \pi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta - V & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \pi \end{pmatrix}, \quad \pi := \dot{\varphi}. \tag{2.2}$$

2.2 Quantization

For the quantization of the (real, free) scalar field ϕ on a globally hyperbolic spacetime (M, g), we follow the algebraic approach, see e.g. [16, 21, 29]: We generate a (non-commutative, unital) $*$-algebra $\mathcal{A}$ by 'smeared' *quantum fields* $\hat{\phi}(f)$ for $f \in C_c(M)$ satisfying

(i) linearity: $\hat{\phi}(\alpha f + \beta f') = \alpha\hat{\phi}(f) + \beta\hat{\phi}(f')$,

(ii) hermiticity: $\hat{\phi}(f)^* = \hat{\phi}(\overline{f})$,

(iii) Klein–Gordon equation: $\hat{\phi}(Kf) = 0$,

(iv) canonical commutation relations (CCR): $[\hat{\phi}(f), \hat{\phi}(f')] = -\mathrm{i}\langle f | G^{\mathrm{PJ}} f'\rangle$,

where $f, f' \in C_c(M)$ and $\alpha, \beta \in \mathbb{C}$ Moreover, $\langle \cdot | \cdot \rangle$ is the canonical L^2 product on the spacetime, and G^{PJ} denotes the uniquely defined Pauli–Jordan propagator [8, 15], namely the difference of forward (retarded) and backward (advanced) propagator of the Klein–Gordon operator K.

A *state* on $\mathcal{A}$ is a linear functional $\omega : \mathcal{A} \to \mathbb{C}$, which is

(i) normalized ($\omega(\mathbb{1}) = 1$) and

(ii) positive ($\omega(a^* a) \geq 0$ for all $a \in \mathcal{A}$),

The two-point function ω_2 of ω is defined as $\omega_2(f, f') := \omega(\hat{\phi}(f)\hat{\phi}(f'))$. Due to the positivity of the state we have $\omega_2(\overline{f}, f) \geq 0$. Meanwhile, the canonical commutation relations imply

$$\omega_2(f, f') - \omega_2(f', f) = -\mathrm{i}\langle f | G^{\mathrm{PJ}} f'\rangle.$$

On Minkowski spacetime, and more generally on static spacetimes, one additionally imposes a spectrum condition (positive frequency condition) for the state, which distinguishes a vacuum state. On generic spacetimes, no (natural) preferred state exists [19] and (for free fields) the spectrum condition is replaced by a condition on the singular structure of the two-point function. Typically one requires that the state is a *Hadamard state*, viz., its two-point function satisfies the *microlocal spectrum condition* – a condition on the smooth wave-front set [38]. In applications it is sometimes useful to relax the microlocal spectrum condition. For example, on FLRW spacetimes one often considers the class of *adiabatic states* which are obtained via a WKB-type approach[27, 34]. These states satisfy a Sobolev version the microlocal spectrum condition [27].

Remark 2.1. Here and in the following we assume that the state ω has a vanishing one-point function $\omega(\hat{\phi}(f)) = 0$. Thus we do not distinguish between the two-point function and the connected two-point function. A non-vanishing one-point function $\phi^{\mathrm{bg}}(\tau,\vec{x}) := \omega(\hat{\phi}(\tau,\vec{x}))$ is interpreted as a classical background field. In the context of homogeneous and isotropic spacetimes, $\phi^{\mathrm{bg}}(\tau)$ does not depend on $\vec{x}$. Setting $\varphi^{\mathrm{bg}} := a\phi^{\mathrm{bg}}$ and $\pi^{\mathrm{bg}} = \partial_\tau \varphi^{\mathrm{bg}}$, we see that the dynamics of the additional two degrees of freedom introduced by the background field is given by (2.2), where the spatial Laplacian Δ can be omitted.

2.3 Two-point functions

Consider the two-point function ω_2 of a homogeneous and isotropic state. That is, it holds that

$$\omega_2\big((\tau,\vec{x}),(\tau',\vec{x}')\big) = \omega_2\big(\tau,\tau',r = |\vec{x}-\vec{x}'|\big). \tag{2.3}$$

Since the antisymmetric part of the two-point function is fixed by the commutator 'function' (viz., the Pauli–Jordan propagator G^{PJ}), ω_2 is completely determined by its symmetric part

$$\frac{1}{2}\big(\omega_2(\tau,\tau',r) + \omega_2(\tau',\tau,r)\big). \tag{2.4}$$

Therefore, the Cauchy data at conformal time τ of a homogeneous and isotropic state can be given by (2.4) and its first time derivatives.

We define

$$\mathcal{G}(\tau,r) := \begin{pmatrix} \mathcal{G}_{\varphi\varphi}(\tau,r) \\ \mathcal{G}_{(\varphi\pi)}(\tau,r) \\ \mathcal{G}_{\pi\pi}(\tau,r) \end{pmatrix} := \lim_{\tau'\to\tau} \begin{pmatrix} \mathbb{1} \\ \frac{1}{2}(\partial_\tau + \partial_{\tau'}) \\ \partial_\tau \partial_{\tau'} \end{pmatrix} a(\tau)a(\tau')\omega_2(\tau,\tau',r), \tag{2.5}$$

which represents the Cauchy data of the two-point function at τ. The fact that the two-point function is a solution to the Klein–Gordon equation in both variables can now be expressed as

$$\partial_\tau \mathcal{G} = \begin{pmatrix} 0 & 2 & 0 \\ \Delta_r - V & 0 & 1 \\ 0 & 2(\Delta_r - V) & 0 \end{pmatrix} \mathcal{G}, \tag{2.6}$$

where $\Delta_r := r^{-2}\partial_r r^2 \partial_r$ is the (three dimensional) radial Laplacian.

It is sometimes convenient to perform calculations in momentum space. We define the mode functions

$$\widehat{\mathcal{G}}(\tau,k) := \begin{pmatrix} \widehat{\mathcal{G}}_{\varphi\varphi}(\tau,k) \\ \widehat{\mathcal{G}}_{(\varphi\pi)}(\tau,k) \\ \widehat{\mathcal{G}}_{\pi\pi}(\tau,k) \end{pmatrix} := 4\pi \int_0^\infty \mathcal{G}(\tau,r) \frac{\sin(kr)}{kr} r^2 \,\mathrm{d}r \tag{2.7}$$

with $k \in [0,\infty)$. The mode functions are simply the Fourier transform of $\mathcal{G}(\tau, r = |\vec{x}|)$ in $\vec{x}$. Indeed, using the convention $\hat{f}(\vec{k}) := \int_{\mathbb{R}^3} f(\vec{x}) \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}} \,\mathrm{d}\vec{x}$ for the Fourier transform, we have for radial functions (or distributions) $f(\vec{x}) = f(r = |\vec{x}|)$

$$\hat{f}(\vec{k}) = \hat{f}(k = |\vec{k}|) = 4\pi \int_0^\infty f(r) \frac{\sin(kr)}{kr} r^2 \,\mathrm{d}r.$$

Conversely, the mode functions specify the two-point function by

$$\mathcal{G}(\tau, r) = \frac{1}{2\pi^2} \int_0^\infty \widehat{\mathcal{G}}(\tau, k) \frac{\sin(kr)}{kr} k^2 \, \mathrm{d}k. \tag{2.8}$$

It follows immediately from (2.6) that the mode functions solve the dynamical equations

$$\partial_\tau \widehat{\mathcal{G}} = \begin{pmatrix} 0 & 2 & 0 \\ -(k^2 + V) & 0 & 1 \\ 0 & -2(k^2 + V) & 0 \end{pmatrix} \widehat{\mathcal{G}}. \tag{2.9}$$

A straightforward computation shows that $\widehat{\mathcal{J}} := \widehat{\mathcal{G}}_{\varphi\varphi} \widehat{\mathcal{G}}_{\pi\pi} - \widehat{\mathcal{G}}_{(\varphi\pi)}^2$ is a conserved quantity. This reduces the degrees of freedom in (2.5) (and (2.9)) to two. Moreover, it follows from the positivity of the state that $\widehat{\mathcal{J}} \geq \frac{1}{4}$ with equality for pure states [30].

2.4 A point-splitting regularization

Many expressions of physical relevance in QFT involve expectation values of products of quantum fields at a point. Naïvely, such expressions are ill-defined because of the distributional nature of quantum fields. By restricting to a class of states which share a common singular structure, we can define a renormalization scheme that allows to make sense of these expressions.

Below we develop an regularization scheme for two-point functions on FLRW spacetimes (or, in fact, for the kernels $\mathcal{G}(\tau, r)$) which carries features of both the *Hadamard point-splitting* method [11, 12] and the WKB-type approach named *adiabatic regularization* [7, 10, 34]. More concretely, the aim of this subsection is to define kernels $\mathcal{H}_n(\tau, r)$ such that $\mathcal{G}(\tau, r) - \mathcal{H}_n(\tau, r)$ is sufficiently regular in the limit $r \to 0$.

Let μ be an arbitrary (but fixed) length scale. On $\mathbb{R}$, define the piecewise function

$$k_+^z := \begin{cases} k^z & \text{if } k > 0, \\ 0 & \text{if } k \leq 0. \end{cases}$$

For $z \in \mathbb{C} \setminus \{-1, -2, \dots\}$ this defines a distribution (by analytic continuation, cf. Chap. III.2 of [26]). It can be extended to all $z \in \mathbb{C}$ by defining for $n \in \mathbb{N}$,

$$\langle k_+^{-n}, f \rangle := \frac{1}{(n-1)!} \Big(-\int_0^\infty \log(\mu k) f^{(n)}(k) \, \mathrm{d}x + f^{(n-1)}(0) \sum_{j=1}^{n-1} \frac{1}{j} \Big), \quad f \in C_\mathrm{c}^\infty(\mathbb{R}).$$

We also define for $r \geq 0$ and $z \in \mathbb{C}$ the distributions

$$h_z(r) := \frac{\mathrm{e}^{\mathrm{i}z\pi/2}}{2\pi^2} \frac{r^{z-2}}{\Gamma(z)} \Big(\log\Big(\frac{r}{\mu}\Big) - \psi(z) \Big).$$

Note that $h_{-2} = -(\pi^2 r^4)^{-1}$ and $h_0 = 1/(2\pi^2 r^2)^{-1}$. The (homogeneous) distributions defined above are discussed in more detail in Sect. A.1, although without the constant length scale μ. Tacitly, the so defined distributions and the resulting regularization scheme depend on μ.

We make the Ansatz (equivalently either in position or momentum space with relations analogous to (2.7) and (2.8))

$$\mathcal{H}_n(\tau, r) := \begin{pmatrix} \mathcal{H}_{\varphi\varphi,n}(\tau, r) \\ \mathcal{H}_{(\varphi\pi),n}(\tau, r) \\ \mathcal{H}_{\pi\pi,n}(\tau, r) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma_{-1}(\tau) \end{pmatrix} h_{-2}(r) + \sum_{j=0}^{n} \begin{pmatrix} \alpha_j(\tau) \\ \beta_j(\tau) \\ \gamma_j(\tau) \end{pmatrix} h_{2j}(r), \tag{2.10a}$$

$$\widehat{\mathcal{H}}_n(\tau, k) := \begin{pmatrix} \widehat{\mathcal{H}}_{\varphi\varphi,n}(\tau, k) \\ \widehat{\mathcal{H}}_{(\varphi\pi),n}(\tau, k) \\ \widehat{\mathcal{H}}_{\pi\pi,n}(\tau, k) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma_{-1}(\tau) \end{pmatrix} k_+^1 + \sum_{j=0}^{n} \begin{pmatrix} \alpha_j(\tau) \\ \beta_j(\tau) \\ \gamma_j(\tau) \end{pmatrix} k_+^{-(2j+1)}, \tag{2.10b}$$

where we fix the lowest order parameters to

$$\alpha_0 = \frac{1}{2}, \quad \beta_0 = 0, \quad \gamma_{-1} = \frac{1}{2}. \tag{2.11}$$

We wish to find coefficients $\alpha_\bullet$, $\beta_\bullet$ and $\gamma_\bullet$ such that the two constraints

$$\partial_\tau \mathcal{H}_n - \begin{pmatrix} 0 & 2 & 0 \\ \Delta_r - V & 0 & 1 \\ 0 & 2(\Delta_r - V) & 0 \end{pmatrix} \mathcal{H}_n \overset{!}{=} \mathcal{O}(r^{2(n-1)}) \tag{2.12}$$

$$\widehat{\mathcal{H}}_{\varphi\varphi,n} \widehat{\mathcal{H}}_{\pi\pi,n} - \widehat{\mathcal{H}}^2_{(\varphi\pi),n} \overset{!}{=} \frac{1}{4} + \mathcal{O}(k^{-2(n+1)}) \tag{2.13}$$

are satisfied. That is, $\mathcal{H}_n$ satisfies the Klein–Gordon equation and has the properties of the two-point function of a pure state up to any desired order.

Using the homogeneity property (A.4) and (2.11), we find

$$\begin{aligned} &\partial_\tau \mathcal{H}_n - \begin{pmatrix} 0 & 2 & 0 \\ \Delta_r - V & 0 & 1 \\ 0 & 2(\Delta_r - V) & 0 \end{pmatrix} \mathcal{H}_n \\ &\quad = \sum_{j=0}^{n-1} \begin{pmatrix} \dot{\alpha}_j - 2\beta_j \\ \dot{\beta}_j + \alpha_{j+1} + V\alpha_j - \gamma_j \\ \dot{\gamma}_j + 2\beta_{j+1} + 2V\beta_j \end{pmatrix} h_{2j} + \begin{pmatrix} \dot{\alpha}_n - 2\beta_n, \\ \dot{\beta}_n + V\alpha_n - \gamma_n, \\ \dot{\gamma}_n + 2V\beta_n \end{pmatrix} h_{2n}. \end{aligned}$$

Consequently, the coefficients must satisfy the equations

$$\dot{\alpha}_j = 2\beta_j, \tag{2.14a}$$

$$\dot{\beta}_j = -\alpha_{j+1} - V\alpha_j + \gamma_j, \tag{2.14b}$$

$$\dot{\gamma}_j = -2\beta_{j+1} - 2V\beta_j. \tag{2.14c}$$

Moreover, the left-hand side of (2.13) evaluates to

$$\begin{aligned} &\widehat{\mathcal{H}}_{\varphi\varphi,n} \widehat{\mathcal{H}}_{\pi\pi,n} - \widehat{\mathcal{H}}^2_{(\varphi\pi),n} \\ &\quad = \alpha_0\gamma_{-1} + \sum_{j=1}^{n} \Big(\alpha_0\gamma_j + \alpha_{j+1}\gamma_{-1} - \beta_0\beta_j + \sum_{i=1}^{j} (\alpha_i\gamma_{j-i} - \beta_i\beta_{j-i}) \Big) k_+^{-2j} + \mathcal{O}\big(k_+^{-2(n+1)}\big) \\ &\quad = \frac{1}{4} + \frac{1}{2} \sum_{l=1}^{n} \Big(\alpha_{j+1} + \gamma_j + 2\sum_{i=1}^{j} (\alpha_i\gamma_{j-i} - \beta_i\beta_{j-i}) \Big) k_+^{-2j} + \mathcal{O}\big(k_+^{-2(n+1)}\big). \end{aligned}$$

Therefore, the coefficients must additionally satisfy the constraint

$$\alpha_{j+1} + \gamma_j = -2\sum_{i=1}^{j} (\alpha_i\gamma_{j-i} - \beta_i\beta_{j-i}). \tag{2.15}$$

It is a remarkable fact that (with this constraint) the differential equations (2.14) can be solved recursively without solving any integrals:

Proposition 2.2. *The differential equations* (2.14) *with initial values* (2.11) *and constraint* (2.15) *are solved by the recurrence relations*

$$\alpha_{j+1} = -\frac{1}{2}(V\alpha_j + \dot{\beta}_j) - \sum_{i=1}^{j} (\alpha_i\gamma_{j-i} - \beta_i\beta_{j-i}), \tag{2.16a}$$

$$\beta_{j+1} = -\frac{1}{4}\dot{V}\alpha_j - \frac{1}{4}\ddot{\beta}_j - V\beta_j, \tag{2.16b}$$

$$\gamma_j = \frac{1}{2}(V\alpha_j + \dot{\beta}_j) - \sum_{i=1}^{j} (\alpha_i\gamma_{j-i} - \beta_i\beta_{j-i}). \tag{2.16c}$$

Proof. To find (2.16b), we eliminate $\dot{\gamma}_j$ from (2.14c) by adding the derivative of (2.14b) and using (2.14a). The other two equations obtained from (2.15) and $-\alpha_{j+1}+\gamma_j = V\alpha_j + \dot{\beta}_j$, which follows from (2.14b), thus yield (2.16a) and (2.16c).

To see that (2.15) is consistent with (2.14), we first subtract (2.14a) and (2.14c) to obtain $\dot{\alpha}_{j+1}+\dot{\gamma}_j = -2V\beta_j$. Then we calculate

$$\begin{aligned}\partial_\tau \sum_{i=1}^{j}(\alpha_i\gamma_{j-i}-\beta_i\beta_{j-i}) &= \sum_{i=1}^{j}\big(\dot{\alpha}_i\gamma_{j-i}+\alpha_i\dot{\gamma}_{j-i}-2\dot{\beta}_i\beta_{j-i}\big)\\ &= 2\sum_{i=1}^{j}\big(\beta_i\gamma_{j-i}-\alpha_i(\beta_{j-i+1}+V\beta_{j-i})-(\gamma_i-\alpha_{i+1}-V\alpha_i)\beta_{j-i}\big)\\ &= 2\sum_{i=1}^{j}(\alpha_i\beta_{j-i+1}-\alpha_{i+1}\beta_{j-i})+2\gamma_0\beta_j = 2(\gamma_0-\alpha_1)\beta_j = V\beta_j,\end{aligned}$$

where, in the last step, we used that (2.14b) for $j=0$ implies $\gamma_0-\alpha_1 = \frac{1}{2}V$. □

2.5 Comparison to Hadamard point-splitting

The regularization procedure proposed in the previous subsection is equivalent to the typically considered Hadamard point-splitting method, in which a truncation of the Hadamard parametrix is subtracted [11, 12].

The truncated Hadamard parametrix (at order n and scale λ) for the Klein–Gordon equation (2.1) is defined as [18, 31]

$$H_n(x,x') := \lim_{\varepsilon\to 0^+}\frac{1}{8\pi^2}\left(\frac{\Delta(x,x')^{\frac{1}{2}}}{\sigma_\varepsilon(x,x')}+\sum_{j=0}^{n-1}v_j(x,x')\sigma(x,x')^j\log\frac{\sigma_\varepsilon(x,x')}{\lambda^2}\right),$$

where x, x' are in a geodesically convex neighbourhood,

$$\sigma_\varepsilon(x,x') := \sigma(x,x')+\mathrm{i}\varepsilon\big(t(x)-t(x')\big),\quad \varepsilon>0,$$

is the regularized Synge world function (i.e., half the signed squared geodesic distance) for a time-function t, $\Delta(x,x')$ is the van Vleck–Morette determinant, $v_j(x,x')$ are smooth coefficient functions satisfying the recurrence relations [18, 31]

$$(K\otimes\mathbb{1})\Delta^{\frac{1}{2}} = \big(-(\Box\otimes\mathbb{1})\sigma+\sigma(\Box\otimes\mathbb{1})-2\big)v_0, \tag{2.17a}$$

$$(K\otimes\mathbb{1})v_{j-1} = \big(-(\Box\otimes\mathbb{1})\sigma+\sigma(\Box\otimes\mathbb{1})+2j-2\big)jv_j \tag{2.17b}$$

for $j\in\mathbb{N}$. These relations are obtained by demanding that the Hadamard parametrix satisfies the Klein–Gordon equation up to an error of order σ^{n+1}.

Note that the coefficient functions v_j are entirely determined by the local geometry. For example, at lowest order v_1 is given by

$$[v_1] = \frac{1}{8}m^4+\frac{6\xi-1}{24}m^2R+\frac{(6\xi-1)^2}{288}R^2-\frac{1}{720}R_{\mu\nu}R^{\mu\nu}+\frac{1}{720}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+\frac{5\xi-1}{120}\Box R. \tag{2.18}$$

Since a FLRW spacetime is spatially isotropic and homogeneous, and the Hadamard parametrix is constructed from local geometric quantities, it inherits these properties so that

$$H_n\big((\tau,\vec{x}),(\tau',\vec{x}')\big) = H_n\big(\tau,\tau',r=|\vec{x}-\vec{x}'|\big),$$

analogously to (2.3). (Naturally, the same holds true for σ, $\Delta^{\frac{1}{2}}$ and v_j.) As before for the two-point function, this implies that the Hadamard parametrix can be represented by

$$\tilde{\mathcal{H}}_n(\tau,r) := \begin{pmatrix}\tilde{\mathcal{H}}_{\varphi\varphi,n}(\tau,r)\\ \tilde{\mathcal{H}}_{(\varphi\pi),n}(\tau,r)\\ \tilde{\mathcal{H}}_{\pi\pi,n}(\tau,r)\end{pmatrix} := \lim_{\tau'\to\tau}\begin{pmatrix}\mathbb{1}\\ \frac{1}{2}(\partial_\tau+\partial_{\tau'})\\ \partial_\tau\partial_{\tau'}\end{pmatrix}a(\tau)a(\tau')H_n(\tau,\tau',r).$$

It follows from results in [17], that the most singular terms of $\tilde{\mathcal{H}}_n$ and $\mathcal{H}_n$ agree, thus justifying the choice (2.11) of the lowest order coefficients. (The results of [17] are stated for cosmological time but a translation to conformal time is not difficult.) Equivalence of $\tilde{\mathcal{H}}$ and $\mathcal{H}$ to arbitrary order then follows from the fact that both are approximate solutions of the Klein–Gordon equation.

For $n \geq 2$, one can compute the expansions

$$\begin{aligned}
\tilde{\mathcal{H}}_{\varphi\varphi,n}(\tau,r) &= \frac{1}{4\pi^2 r^2} + \frac{V(\tau)}{8\pi^2}\left(\log(r) + \log\big(a(\tau)\big) - \frac{1}{2}\log\big(2\lambda^2\big)\right) + \frac{1}{48\pi^2}\frac{\ddot{a}(\tau)}{a(\tau)} \\
&\quad + \frac{3V(\tau)^2 + \ddot{V}(\tau)}{192\pi^2} r^2 \left(\log(r) + \log\big(a(\tau)\big) - \frac{1}{2}\log\big(2\lambda^2\big)\right) \\
&\quad + \frac{r^2}{5760\pi^2}\left(30V(\tau)\frac{\dot{a}(\tau)}{a(\tau)} + 11\Big(\partial_\tau \frac{\dot{a}(\tau)}{a(\tau)}\Big)^2 - 2\frac{\ddot{a}(\tau)^2}{a(\tau)^2} + 12\frac{\dot{a}(\tau)}{a(\tau)}\partial_\tau\frac{\ddot{a}(\tau)}{a(\tau)}\right) \\
&\quad + \mathcal{O}(r^4), \\
\tilde{\mathcal{H}}_{(\varphi\pi),n}(\tau,r) &= \frac{\dot{V}(\tau)}{16\pi^2}\left(\log(r) + \log\big(a(\tau)\big) - \frac{1}{2}\log\big(2\lambda^2\big)\right) \\
&\quad + \frac{1}{96\pi^2}\left(6V(\tau)\frac{\dot{a}(\tau)}{a(\tau)} + \partial_\tau\frac{\ddot{a}(\tau)}{a(\tau)}\right) + \mathcal{O}(r^2), \\
\tilde{\mathcal{H}}_{\pi\pi,n}(\tau,r) &= -\frac{1}{2\pi^2 r^4} + \frac{V(\tau)}{8\pi^2 r^2} + \frac{V(\tau)^2 + \ddot{V}(\tau)}{32\pi^2}\left(\log(r) + \log\big(a(\tau)\big) - \frac{1}{2}\log\big(2\lambda^2\big)\right) \\
&\quad + \frac{1}{192\pi^2}\left(3V(\tau)^2 - 6V(\tau)\frac{\dot{a}(\tau)^2}{a(\tau)^2} + 4V(\tau)\frac{\ddot{a}(\tau)}{a(\tau)} + 12\dot{V}(\tau)\frac{\dot{a}(\tau)}{a(\tau)} + \ddot{V}(\tau)\right) \\
&\quad + \frac{1}{960\pi^2}\left(\Big(\partial_\tau\frac{\dot{a}(\tau)}{a(\tau)}\Big)^2 + 2\frac{\ddot{a}(\tau)^2}{a(\tau)^2} + 4\partial_\tau^2\frac{\ddot{a}(\tau)}{a(\tau)}\right) + \mathcal{O}(r^2).
\end{aligned}$$

To obtain this result, one uses a covariant expansion of the coefficient functions $\Delta^{\frac{1}{2}}$ and v_j, see e.g. [11, 12, 14], together with an expansion of Synge's world function, see Sect. A.4.

The formulas above suggest the convention

$$\lambda^2 = \frac{\mathrm{e}^{2\gamma-2}}{2}\mu^2, \tag{2.19}$$

where γ is the Euler–Mascheroni constant. With this convention, we find the differences ($n \geq 2$)

$$\tilde{\mathcal{H}}_{\varphi\varphi,n} - \mathcal{H}_{\varphi\varphi,n}\Big|_{r=0} = \frac{V}{8\pi^2}\log(a) + \frac{1}{48\pi^2}\frac{\ddot{a}}{a}, \tag{2.20a}$$

$$\tilde{\mathcal{H}}_{(\varphi\pi),n} - \mathcal{H}_{(\varphi\pi),n}\Big|_{r=0} = \frac{\dot{V}}{16\pi^2}\log(a) + \frac{1}{96\pi^2}\left(6V\frac{\dot{a}}{a} + \partial_\tau\frac{\ddot{a}}{a}\right), \tag{2.20b}$$

$$\begin{aligned}
\tilde{\mathcal{H}}_{\pi\pi,n} - \mathcal{H}_{\pi\pi,n}\Big|_{r=0} &= \frac{V^2 + \ddot{V}}{32\pi^2}\log(a) + \frac{1}{960\pi^2}\left(\Big(\partial_\tau\frac{\dot{a}}{a}\Big)^2 + 2\frac{\ddot{a}^2}{a^2} + 4\partial_\tau^2\frac{\ddot{a}}{a}\right) \\
&\quad + \frac{1}{192\pi^2}\left(3V^2 - 6V\frac{\dot{a}^2}{a^2} + 4V\frac{\ddot{a}}{a} + 12\dot{V}\frac{\dot{a}}{a} + \ddot{V}\right),
\end{aligned} \tag{2.20c}$$

$$\begin{aligned}
\Delta_r(\tilde{\mathcal{H}}_{\varphi\varphi,n} - \mathcal{H}_{\varphi\varphi,n})\Big|_{r=0} &= \frac{3V^2 + \ddot{V}}{32\pi^2}\left(\frac{5}{6} + \log(a)\right) + \frac{V}{32\pi^2}\frac{\dot{a}^2}{a^2} \\
&\quad + \frac{1}{960\pi^2}\left(11\Big(\partial_\tau\frac{\dot{a}}{a}\Big)^2 - 2\frac{\ddot{a}^2}{a^2} + 12\frac{\dot{a}}{a}\partial_\tau\frac{\ddot{a}}{a}\right).
\end{aligned} \tag{2.20d}$$

We will use these differences in the following section.

3 Semiclassical Einstein equation

On FLRW spacetimes, it is sufficient to look at the traced SCE

$$-R = 8\pi G\langle T^{\mathrm{ren}}\rangle_\omega, \tag{3.1}$$

where $\langle T^{\mathrm{ren}}\rangle_\omega := g^{\mu\nu}\langle T^{\mathrm{ren}}_{\mu\nu}\rangle_\omega$ is the trace of the quantum stress-energy tensor, and at the energy-component (or 00- or tt-component) $G_{00} = 8\pi G\langle T^{\mathrm{ren}}_{00}\rangle_\omega$. The latter equation can be regarded as a constraint equation, which, if imposed at a fixed time, holds everywhere because of the covariant conservation $\nabla^\mu\langle T^{\mathrm{ren}}_{\mu\nu}\rangle_\omega = 0$ of the stress-energy tensor [25, 31].

Henceforth we will ignore the energy equation (for the most part) and suppose that it has been solved by choosing consistent initial conditions. We will focus our attention on the traced equation (3.1), which completely determines the dynamics of the geometry, given by the scale factor.

3.1 Renormalized stress-energy tensor

The (classical) stress-energy tensor for a (real) free scalar field is

$$\begin{aligned} T_{\mu\nu} &= (1-2\xi)(\nabla_\mu\phi)(\nabla_\nu\phi) - \frac{1}{2}(1-4\xi)g_{\mu\nu}(\nabla^\sigma\phi)(\nabla_\sigma\phi) - \frac{1}{2}g_{\mu\nu}m^2\phi^2 \\ &\quad + \xi\big(G_{\mu\nu}\phi^2 - 2\phi\nabla_\mu\nabla_\nu\phi - 2g_{\mu\nu}\phi\Box\phi\big). \end{aligned} \tag{3.2}$$

To quantize this expression, we replace products of (derivatives of) the classical fields by their renormalized quantum counterparts. That is, using Hadamard point-splitting, we have for the expectation value of the stress-energy tensor in a state ω (cf. [25, 31, 42]):

$$\begin{aligned} \langle T^{\mathrm{ren}}_{\mu\nu}\rangle_\omega &= (1-2\xi)[(\nabla_\mu\otimes\nabla_\nu)\omega_2^{\mathrm{reg}}] - \frac{1}{2}(1-4\xi)g_{\mu\nu}[(\nabla^\sigma\otimes\nabla_\sigma)\omega_2^{\mathrm{reg}}] - \frac{1}{2}g_{\mu\nu}m^2[\omega_2^{\mathrm{reg}}] \\ &\quad + \xi\big(G_{\mu\nu}[\omega_2^{\mathrm{reg}}] - 2[(\mathbb{1}\otimes\nabla_\mu\nabla_\nu)\omega_2^{\mathrm{reg}}] - 2g_{\mu\nu}[(\mathbb{1}\otimes\Box)\omega_2^{\mathrm{reg}}]\big) \\ &\quad + \frac{1}{4\pi^2}g_{\mu\nu}[v_1] + c_1m^4g_{\mu\nu} + c_2m^2G_{\mu\nu} + c_3I_{\mu\nu} + c_4J_{\mu\nu}, \end{aligned} \tag{3.3}$$

where, for fixed but arbitrary $n \geq 1$, we set $\omega_2^{\mathrm{reg}} := \omega_2 - H_n$, $[\,\cdot\,]$ denotes the coincidence limit (e.g., $[v_1](x) = v_1(x,x)$ is the coincidence limit of the Hadamard coefficient v_1) with implicit parallel transport, $c_\bullet$ are renormalization constants, and $I_{\mu\nu}$, $J_{\mu\nu}$ are the two fourth order conserved curvature tensors:

$$\begin{aligned} I_{\mu\nu} &:= 2RR_{\mu\nu} - 2\nabla_\mu\nabla_\nu R - \frac{1}{2}g_{\mu\nu}(R^2 + 4\Box R), \\ J_{\mu\nu} &:= 2R^{\rho\sigma}R_{\rho\mu\sigma\nu} - \nabla_\mu\nabla_\nu R - \Box R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R_{\rho\sigma}R^{\rho\sigma} + \Box R). \end{aligned}$$

These tensors can be obtained as the variations with respect to the metric of R^2 and $R_{\mu\nu}R^{\mu\nu}$ [48]. For conformally flat spacetimes like FLRW, it holds $I_{\mu\nu} = 3J_{\mu\nu}$.

The term involving the Hadamard coefficient v_1 in (3.3) ensures that the quantized stress-energy tensor satisfies $\nabla^\mu\langle T^{\mathrm{ren}}_{\mu\nu}\rangle_\omega = 0$ [25, 31]. The term $c_1m^4g_{ab}$ is a renormalization of the cosmological constant, and $c_2m^2G_{ab}$ corresponds to a renormalization of Newton's gravitational constant G; the remaining two renormalization terms have no classical interpretation. A change of the scale λ in the Hadamard parametrix corresponds to a change of the renormalization constants [21], and thus we are in principle at liberty to set λ to any value if we change the renormalization constants accordingly.

We also remark that in case the state ω includes a non-zero one-point function (background field), the corresponding contribution to the two point function is $\phi^{\mathrm{bg}}(x)\phi^{\mathrm{bg}}(x')$, that is, in (3.3) we have to replace $\omega_2^{\mathrm{reg}}(x,x')$ by $\omega_2^{\mathrm{reg}}(x,x') + \phi^{\mathrm{bg}}(x)\phi^{\mathrm{bg}}(x')$. After performing the coincidence limit, this leads to an additional contribution given by the classical stress-energy tensor (3.2).

3.2 Trace of the stress-energy tensor

Taking the trace of (3.3) yields[1]

$$\langle T^{\mathrm{ren}}\rangle_\omega = \big((6\xi-1)(\xi R+m^2)-m^2\big)[\omega_2^{\mathrm{reg}}]+(6\xi-1)g^{\mu\nu}[(\nabla_\mu\otimes\nabla_\nu)\omega_2^{\mathrm{reg}}] \\ -\frac{9\xi-2}{2\pi^2}[v_1]+4c_1m^4-c_2m^2R-(6c_3+2c_4)\Box R. \tag{3.4}$$

where $c_\bullet$ are the same constants as above, and we used that $4\pi^2[(\mathbb{1}\otimes K)\omega_2^{\mathrm{reg}}]=3[v_1]$.

For homogeneous states and isotropic states (i.e., satisfying (2.3)) on FLRW spacetimes, we thus find

$$\langle T^{\mathrm{ren}}\rangle_\omega = \big((6\xi-1)(\xi R+m^2)-m^2\big)[\omega_2^{\mathrm{reg}}]-\frac{6\xi-1}{a^2}\big([(\Delta\otimes\mathbb{1})\omega_2^{\mathrm{reg}}]+[(\partial_\tau\otimes\partial_\tau)\omega_2^{\mathrm{reg}}]\big) \\ -\frac{9\xi-2}{2\pi^2}[v_1]+4c_1m^4-c_2m^2R-(6c_3+2c_4)\Box R, \tag{3.5}$$

where

$$R=6\frac{\ddot a}{a},\quad \Box R=36\frac{\ddot a\dot a^2}{a^7}-18\frac{\ddot a^2}{a^6}-24\frac{a^{(3)}\dot a}{a^6}+6\frac{a^{(4)}}{a^5}$$

and (2.18) specializes to

$$[v_1]=\frac{m^4}{8}+\frac{1}{60}\left(\frac{\dot a^4}{a^8}-\frac{\ddot a\dot a^2}{a^7}\right)+\frac{(6\xi-1)m^2}{4}\frac{\ddot a}{a^3}+\frac{(6\xi-1)^2}{8}\frac{\ddot a^2}{a^6} \\ +\frac{5\xi-1}{20}\left(6\frac{\ddot a\dot a^2}{a^7}-3\frac{\ddot a^2}{a^6}-4\frac{a^{(3)}\dot a}{a^6}+\frac{a^{(4)}}{a^5}\right).$$

In the case of a non-vanishing one-point function yielding the background field ϕ^{bg}, the following expression needs to be added to $\langle T^{\mathrm{ren}}\rangle_\omega$:

$$\big((6\xi-1)(\xi R+m^2)-m^2\big)\big(\phi^{\mathrm{bg}}\big)^2-\frac{6\xi-1}{a^2}\big(\dot\phi^{\mathrm{bg}}\big)^2. \tag{3.6}$$

3.3 Energy component of the stress-energy tensor

For completeness, we also state the energy component of the stress-energy tensor in a homogeneous and isotropic state on an FLRW spacetime:

$$\langle T^{\mathrm{ren}}_{00}\rangle_\omega = \frac{1}{2}[(\partial_\tau\otimes\partial_\tau)\omega_2^{\mathrm{reg}}]-\frac{1}{2}[(\mathbb{1}\otimes\Delta)\omega_2^{\mathrm{reg}}]+\frac{1}{2}a^2m^2[\omega_2^{\mathrm{reg}}] \\ +\xi\left(G_{00}[\omega_2^{\mathrm{reg}}]+6\frac{\dot a}{a}[(\mathbb{1}\otimes\partial_\tau)\omega_2^{\mathrm{reg}}]\right) \\ -\frac{a^2}{4\pi^2}[v_1]-c_1a^2m^4+c_2m^2G_{00}+(3c_3+c_4)J_{00}, \tag{3.7}$$

where

$$G_{00}=3\frac{\dot a^2}{a^2},\quad J_{00}=-24\frac{\ddot a\dot a^2}{a^5}-6\frac{\ddot a^2}{a^4}+12\frac{a^{(3)}\dot a}{a^4}.$$

If a non-vanishing one-point function is present, it contributes to $\langle T^{\mathrm{ren}}_{00}\rangle_\omega$ the following expression:

$$\frac{1}{2}\big(\dot\phi^{\mathrm{bg}}\big)^2+\frac{1}{2}a^2m^2\big(\phi^{\mathrm{bg}}\big)^2+\xi\left(G_{00}\big(\phi^{\mathrm{bg}}\big)^2+3\frac{\dot a}{a}\partial_\tau\big(\phi^{\mathrm{bg}}\big)^2\right).$$

[1] Note that the factor in front of v_1 in the corresponding Eq. (22) of [17] is incorrect.

3.4 Problem of higher derivatives

Note that the SCE contains, via the renormalized stress-energy tensor, up to fourth order derivatives of the metric (i.e., in the case of a FLRW spacetime, fourth order derivatives of the scale factor). This is in striking contrast to the classical Einstein equation, which contains only second order derivatives of the metric. Therefore, it can be argued that the SCE has solutions which diverge significantly from similar solutions with similar initial data for the classical Einstein equation, see e.g. the discussion in [20].

A naïve strategy to solve the SCE for FLRW spacetimes, i.e., the equation $-R = \langle T^{\mathrm{ren}} \rangle_\omega$ coupled with the dynamics of the quantum state, would be to move the highest order derivatives of the scale factor to the left-hand side and all remaining terms to the right-hand side. Unfortunately this is not possible (in the case of non-conformal coupling) because the regularization includes singular terms with fourth order derivatives of the scale factor, and thus it is not clear how to proceed, cf. [44] where the same problem is discussed. For this reason, earlier approaches to solving the SCE, e.g. [37], focused on the conformally coupled case, where this problem can be avoided. Below we suggest an alternative approach which relies on a dynamical system for the regularized two-point function and its derivatives in the coincidence limit.

4 Dynamical system for sequences of coincidence limits

4.1 Dynamics

For $n \in \mathbb{N}_0$, we define

$$\mathcal{M}_n := \Delta_r^n (\mathcal{G} - \mathcal{H}_l)\Big|_{r=0}, \quad l \geq n+1. \tag{4.1}$$

That this definition is indeed independent of l follows immediately from (A.4) and the definition of $\mathcal{H}_n$:

Proposition 4.1. *For $j \geq l \geq n+1$, we have*

$$\Delta_r^n (\mathcal{H}_j - \mathcal{H}_l)\Big|_{r=0} = 0.$$

Occasionally, we call $\mathcal{M}_n$ *moments*. To understand this nomenclature, consider the momentum space representation of $\mathcal{M}_n$:

$$\mathcal{M}_n = (-1)^n \int_0^\infty k^{2(n+1)} (\hat{\mathcal{G}} - \hat{\mathcal{H}}_l)\, \mathrm{d}k.$$

That is, the sequence of $\mathcal{M}_n$ is given by the moments of $\hat{\mathcal{G}} - \hat{\mathcal{H}}$. Note, however, that $\hat{\mathcal{G}} - \hat{\mathcal{H}}$ cannot be expected to be positive.

To formulate a dynamics for $\mathcal{M}_n$, recall (2.6) and (2.12). It follows that

$$\partial_\tau \mathcal{M}_n = \partial_\tau \Delta_r^n (\mathcal{G} - \mathcal{H}_l)\Big|_{r=0} = A\mathcal{M}_n + B\mathcal{M}_{n+1},$$

where we defined

$$A := \begin{pmatrix} 0 & 2 & 0 \\ -V & 0 & 1 \\ 0 & -2V & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Sometimes we write $A(\tau)$ to emphasize the dependence on τ for a fixed potential V. Henceforth we shall suppose that V is an arbitrary (but sufficiently regular) function and can forget its relation to the Klein–Gordon equation.

Considering $\mathcal{M}_n$ as a sequence $\mathcal{M} = (\mathcal{M}_n)$, we can also write the equation above as

$$\partial_\tau \mathcal{M}(\tau) = \big(A(\tau) \otimes \mathbb{1} + B \otimes L\big)\mathcal{M}(\tau), \tag{4.2}$$

where L denotes the left-shift operator. Below we study this equation on the weighted sequence spaces

$$\vec{\ell}^p(w) := \mathbb{R}^3 \otimes \ell^p(w), \tag{4.3}$$

where $p \geq 1$ and w is a sequence of weights; we denote the norms by $\|\cdot\|_{p,w}$. See Sect. A.2 for an introduction and our conventions. Note in particular that in our convention the weights appear as inverses in the norm and thus they directly translate to the maximum growth rate of elements in the sequence space.

In the following two subsections, we calculate $\mathcal{M}$ in two relevant examples to motivate the choice of weights later on. Afterwards, the remainder of this section is concerned with solving (4.2). The solution is given by a mathematically rigorous definition of the time-ordered exponential of $A \otimes \mathbb{1} + B \otimes L$.

Finally, we wish to remark that the set of possible sequences $\mathcal{M}$ contains many unphysical examples, and, furthermore, not all possible Hadamard states can be represented as a sequence in (4.3) for the weights considered below.

4.2 $\mathcal{M}$ for the massive vacuum state on Minkowski spacetime

First, let us recall the expansions of the modified Bessel functions, see e.g. Chap. 10 of [33]:

$$I_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^j}{j!\Gamma(\nu+j+1)},$$

$$K_n(z) = \tfrac{1}{2}\left(\tfrac{1}{2}z\right)^{-n} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!}\left(-\tfrac{1}{4}z^2\right)^j + (-1)^{n+1}\log\left(\tfrac{1}{2}z\right) I_n(z)$$
$$+ (-1)^n \tfrac{1}{2}\left(\tfrac{1}{2}z\right)^n \sum_{j=0}^{\infty}\Big(\psi(j+1)+\psi(n+j+1)\Big)\frac{\left(\frac{1}{4}z^2\right)^j}{j!(n+j)!},$$

for $\nu \in \mathbb{C}$ and $n = 0, 1, 2, \ldots$, where ψ denotes the Digamma function.

The two-point function for the massive vacuum state on Minkowski spacetime has the initial data:

$$\begin{aligned}\mathcal{G}_{\varphi\varphi}(r) &= \frac{m}{4\pi^2 r} K_1(mr) \\ &= \frac{1}{4\pi^2 r^2} + \frac{m^2}{8\pi^2}\sum_{j=0}^{\infty}\Big(\log\left(\tfrac{1}{2}mr\right) - \tfrac{1}{2}\big(\psi(j+1)+\psi(j+2)\big)\Big)\frac{\left(\frac{1}{2}mr\right)^{2j}}{j!(j+1)!},\end{aligned} \tag{4.4a}$$

$$\mathcal{G}_{(\varphi\pi)}(r) = 0, \tag{4.4b}$$

$$\begin{aligned}\mathcal{G}_{\pi\pi}(r) &= -\frac{m^2}{4\pi^2 r^2} K_2(mr) \\ &= -\frac{1}{2\pi^2 r^4} + \frac{m^2}{8\pi^2 r^2} \\ &\quad + \frac{m^4}{16\pi^2}\sum_{j=0}^{\infty}\Big(\log\left(\tfrac{1}{2}mr\right) - \tfrac{1}{2}\big(\psi(j+1)+\psi(j+3)\big)\Big)\frac{\left(\frac{1}{2}mr\right)^{2j}}{j!(j+2)!},\end{aligned} \tag{4.4c}$$

Now we determine $\mathcal{H}_n$ on Minkowski spacetime. For this purpose, we need to solve the recurrence relations (2.16) for constant $V = m^2$. Hence, also the coefficients $\alpha_\bullet, \beta_\bullet$ and $\gamma_\bullet$ are constant, and (2.16) imply $\beta_\bullet = 0$,

$$\alpha_{j+1} - \gamma_j = -m^2\alpha_j, \quad \alpha_{j+1} + \gamma_j = -2\sum_{i=1}^{j}\alpha_i\gamma_{j-i}. \tag{4.5}$$

This recurrence relation can be solved in closed form:

Proposition 4.2. *If $V = m^2$ is constant,*

$$\alpha_j = \frac{1}{2}\binom{-\frac{1}{2}}{j} m^{2j}, \quad \beta_j = 0, \quad \gamma_{j-1} = \frac{1}{2}\binom{\frac{1}{2}}{j} m^{2j}. \tag{4.6}$$

Proof. Inserting (4.6) on the left-hand side of (4.5) yields

$$\alpha_{j+1} - \gamma_j = \frac{1}{2}\left(\binom{-\frac{1}{2}}{j+1} - \binom{\frac{1}{2}}{j+1}\right) m^{2(j+1)} = -\frac{1}{2}\binom{-\frac{1}{2}}{j} m^{2(j+1)} = -m^2\alpha_j,$$

$$\alpha_{j+1} + \gamma_j = \frac{1}{2}\left(\binom{-\frac{1}{2}}{j+1} + \binom{\frac{1}{2}}{j+1}\right) m^{2(j+1)} = \frac{1}{2}\left(-\sum_{i=0}^{j+1}\binom{-\frac{1}{2}}{i}\binom{\frac{1}{2}}{j-i+1} + \binom{\frac{1}{2}}{j+1}\right) m^{2(j+1)}$$

$$= -\frac{1}{2} m^{2(j+1)} \sum_{i=1}^{j}\binom{-\frac{1}{2}}{i}\binom{\frac{1}{2}}{j-i+1} = -2\sum_{i=1}^{j}\alpha_i\gamma_{j-i},$$

where in the second step of the second equation we used the Chu–Vandermonde identity. □

Then, with the coefficients (4.6), we find

$$\mathcal{H}_{\varphi\varphi,n}(r) = \frac{1}{4\pi^2 r^2} + \frac{m^2}{8\pi^2}\sum_{j=0}^{n-1}\big(\log(r) - \psi(2j+2)\big)\frac{\left(\frac{1}{2}mr\right)^{2j}}{j!(j+1)!}, \tag{4.7a}$$

$$\mathcal{H}_{(\varphi\pi),n}(r) = 0, \tag{4.7b}$$

$$\mathcal{H}_{\pi\pi,n}(r) = -\frac{1}{2\pi^2 r^4} + \frac{m^2}{8\pi^2 r^2} + \frac{m^4}{16\pi^2}\sum_{j=0}^{n-1}\big(\log(r) - \psi(2j+2)\big)\frac{\left(\frac{1}{2}mr\right)^{2j}}{j!(j+2)!}. \tag{4.7c}$$

Therefore, subtracting (4.7) (of sufficiently high order) from (4.4) and differentiating n times with Δ, we obtain in the coinciding point limit $r \to 0$:

$$\mathcal{M}_{\varphi\varphi,n} = \frac{1}{2\pi^2}\left(\tfrac{1}{2}m\right)^{2n+2}\Big(\log(\tfrac{1}{2}m) + \psi(2n+2) - \tfrac{1}{2}\big(\psi(n+1) + \psi(n+2)\big)\Big)\binom{2n+1}{n+1}, \tag{4.8a}$$

$$\mathcal{M}_{(\varphi\pi),n} = 0, \tag{4.8b}$$

$$\mathcal{M}_{\pi\pi,n} = \frac{1}{\pi^2}\left(\tfrac{1}{2}m\right)^{2n+4}\Big(\log(\tfrac{1}{2}m) + \psi(2n+2) - \tfrac{1}{2}\big(\psi(n+1) + \psi(n+3)\big)\Big)\frac{(2n+1)!}{n!(n+2)!}. \tag{4.8c}$$

Let $r \geq \frac{3}{2}$. It follows from the properties of the Digamma function (e.g., (5.4.14) and (5.4.15) of [33]) that

$$\psi(2n+2) - \frac{1}{2}\big(\psi(n+1) + \psi(n+r)\big) \leq \log(2)$$

and the bound is approached as $n \to \infty$. Moreover, by the duplication formula for the Gamma function,

$$\frac{(2n+1)}{n!(n+r)!} \leq \frac{2^{2n+1}}{\sqrt{\pi}}.$$

Consequently, we have

$$|\mathcal{M}_{\varphi\varphi,n}| \leq \frac{|\log(m)|}{4\pi^{5/2}} m^{2n+2}, \quad |\mathcal{M}_{\pi\pi,n}| \leq \frac{|\log(m)|}{8\pi^{5/2}} m^{2n+4},$$

whence $\mathcal{M} \in \vec{\ell}^p(w)$ for $p \geq 1$ and $w_n = \omega^n$ with $\omega > m^2$.

As a further consistency check, we show that $(A \otimes \mathbb{1} + B \otimes L)\mathcal{M} = 0$ for $V = m^2$. Indeed, inserting (4.8) into the left-hand side, a straightforward calculation shows that $-m^2\mathcal{M}_{\varphi\varphi,n} + \mathcal{M}_{\pi\pi,n} + \mathcal{M}_{\varphi\varphi,n+1} = 0$.

Finally, we note that in the massless case $m = 0$ all moments vanish exactly: $\mathcal{M} = 0$.

4.3 $\mathcal{M}$ for the massless thermal state on Minkowski spacetime

The two-point function for the massless KMS-state on Minkowski spacetime at inverse temperature β has the initial data

$$\mathcal{G}_{\varphi\varphi}(r) = \frac{1}{4\pi r\beta}\coth\Big(\frac{\pi r}{\beta}\Big) = \frac{1}{4\pi^2 r^2} + \frac{1}{2\pi^2\beta^2}\sum_{j=0}^{\infty}(-1)^j\zeta(2j+2)\Big(\frac{r}{\beta}\Big)^{2j}, \tag{4.9a}$$

$$\mathcal{G}_{(\varphi\pi)}(r) = 0, \tag{4.9b}$$

$$\begin{aligned}\mathcal{G}_{\pi\pi}(r) &= -\frac{\pi}{2r\beta^3}\coth\Big(\frac{\pi r}{\beta}\Big)\operatorname{csch}\Big(\frac{\pi r}{\beta}\Big)^2 \\ &= -\frac{1}{2\pi^2 r^4} + \frac{1}{2\pi^2\beta^4}\sum_{j=0}^{\infty}(-1)^j(2j+2)(2j+3)\zeta(2j+4)\Big(\frac{r}{\beta}\Big)^{2j},\end{aligned} \tag{4.9c}$$

where ζ denotes the (Riemann) Zeta function. These data can be obtained after an appropriate Bogoliubov transformation from the two-point function of the vacuum state. The coefficients for $\mathcal{H}_n$ with $V = 0$ are all zero except for α_0 and γ_{-1}. Therefore,

$$\mathcal{H}_{\varphi\varphi,n}(r) = \frac{1}{4\pi^2 r^2}, \quad \mathcal{H}_{(\varphi\pi),n}(r) = 0, \quad \mathcal{H}_{\pi\pi,n}(r) = -\frac{1}{2\pi r^4}. \tag{4.10a}$$

Subtracting (4.10) from (4.9) and differentiating n times with Δ, we obtain in the coinciding point limit $r \to 0$:

$$\begin{aligned}\mathcal{M}_{\varphi\varphi,n} &= (-1)^n\beta^{-2n-2}\zeta(2n+2)(2n+1)!,\\ \mathcal{M}_{(\varphi\pi),n} &= 0,\\ \mathcal{M}_{\pi\pi,n} &= (-1)^n\beta^{-2n-4}\zeta(2n+4)(2n+3)!.\end{aligned}$$

Note that $\zeta(2) = \frac{1}{6}\pi^2$, $\zeta(4) = \frac{1}{90}\pi^4$ and the Zeta function $\zeta(s)$ is monotonically decreasing for $s > 1$ with limit 1. Hence $\mathcal{M} \in \vec{\ell}^p(w)$ for $p \geq 1$ and $w_n = (2n)!\omega^{2n}$ with $\omega > \beta^{-1}$.

As a further consistency check, we show that $(A \otimes \mathbb{1} + B \otimes L)\mathcal{M} = 0$ for $V = 0$. Indeed, it is easily checked that $\mathcal{M}_{\pi\pi,n} + \mathcal{M}_{\varphi\varphi,n+1} = 0$.

4.4 Properties of the matrices *A* and *B*

The matrix B is nilpotent: $B^3 = 0$. For this reason, many products of the matrices A and B vanish.

We are interested in the non-zero words in A and B with the largest number of B-factors. It is already an easy consequence of the nilpotency of B, that products of length $3n$ contain at most $2n$ B-factors (e.g., $(B^2A)^n$). This implies that a non-zero word of length n contains at most $\lceil \frac{2n}{3} \rceil$ B-factors. This estimate can be substantially improved using further relations between the matrices A and B.

In the following, we denote by dots the non-zero entries of a 3×3 matrix. For instance, $(\cdot\!:\!\cdot)$ denotes any 3×3 matrix (a_{ij}) whose only non-zero entries are a_{12}, a_{21}, a_{23}, a_{32}. The matrix A introduced in the previous subsection is an example of such a matrix and we write $A = (\cdot\!:\!\cdot)$. Another example is $B = (\cdot.\ )$. Products and powers of 3×3 matrices will also be represented in this notation. For example, we write $(\cdot\!:\!\cdot)^3$ for $A(t_1)A(t_2)A(t_3)$. Note that $(\cdot.\ )^3 = 0$.

A quick calculation shows the following additional relations pertaining to products of the matrices A and B:

$$\begin{aligned}&(\cdot.\ )(\cdot\!:\!\cdot)(\cdot.\ ) = (\cdot.\ ), \quad (\cdot\!:\!\cdot)(\cdot.\ )(\cdot.\ ) = (\cdot\ \ ), \quad (\cdot.\ )(\cdot.\ )(\cdot\!:\!\cdot) = (\ .\ ),\\ &(\cdot\!:\!\cdot)(\cdot.\ )(\cdot\!:\!\cdot) = (\cdot\!:\!\cdot), \quad (\cdot.\ )(\cdot\!:\!\cdot)(\cdot\!:\!\cdot) = (\cdot.\cdot), \quad (\cdot\!:\!\cdot)(\cdot\!:\!\cdot)(\cdot.\ ) = (\cdot\!:\ ).\end{aligned} \tag{4.11}$$

Using these relations we can show that non-zero words in A and B of length n contain at most $\lceil \frac{n+1}{2} \rceil$ B-factors. More generally, we have:

Proposition 4.3. *A non-zero word in* $(\cdot\!:\!\cdot)$ *and* $(\cdot\,.\,)$ *of length n contains at most* $\lceil \frac{n+1}{2} \rceil$ $(\cdot\,.\,)$*-factors.*

Proof. By induction, it follows easily from (4.11), that a non-zero word of odd length with a maximal number of $(\cdot\,.\,)$-factors has the shape $(\cdot\,.\,)$, $(\cdot\;\;)$ or $(\;.\;)$. This also shows that such a $(\cdot\,.\,)$-maximal word contains at most one occurrence of two consecutive $(\cdot\,.\,)$-factors. Hence we find that a non-zero word of length $2n+1$ contains at most $n+1$ factors of the shape $(\cdot\,.\,)$. It is then obvious, that a non-zero word of length $2n$ can also not contain more than $n+1$ factor of the shape $(\cdot\,.\,)$. This completes the proof. $\square$

We conclude this subsection on the properties of the matrices A and B by noting that their matrix (max) norms are given by:

$$\|A\| = 2\sqrt{1+V^2}, \quad \|B\| = 2. \tag{4.12}$$

4.5 Evolution operator

In this subsection, we define the evolution operator for

$$S(\tau) := A(\tau) \otimes \mathbb{1} + B \otimes L \tag{4.13}$$

as an operator on the weighted sequence spaces $\vec{\ell}^p(w)$ (see (4.3) and Sect. A.2 for the definition of these spaces) for certain weight sequences w and $p \geq 1$.

The evolution operator for $S(\tau)$ can formally be defined via the time-ordered exponential

$$U(\tau, \tau_0) := \operatorname{Texp}\left(\int_{\tau_0}^{\tau} S(\tau')\,\mathrm{d}\tau'\right).$$

That is, defining for $n > 0$

$$U_n(\tau, \tau_0) := \int_{\tau_0}^{\tau}\int_{\tau_0}^{\tau_1}\cdots\int_{\tau_0}^{\tau_{n-1}} S(\tau_1)\cdots S(\tau_n)\,\mathrm{d}\tau_n \cdots \mathrm{d}\tau_1, \tag{4.14}$$

it is given by the Dyson series

$$U(\tau, \tau_0) = \begin{cases} \mathbb{1} + \sum_{n=1}^{\infty} U_n(\tau, \tau_0) & \text{for } \tau \geq \tau_0 \\ \mathbb{1} + \sum_{n=1}^{\infty} (-1)^n U_n(\tau_0, \tau) & \text{for } \tau < \tau_0. \end{cases} \tag{4.15}$$

Note that this is simply the solution of the Picard iteration method.

Motivated by the examples in Subsects. 4.2 and 4.3, in the following two subsections we show that the series (4.15) converges as an operator

(W1) on $\vec{\ell}^p(w)$ for $w_n = c\omega^n$ with $c, \omega > 0$,

(W2) from $\vec{\ell}^p(w)$ to $\vec{\ell}^p(v)$ for $w_n = (2n)!\,\omega^{2n}$ and $v_n = (2n)!\,\upsilon^{2n}$ with $\upsilon > \omega \geq 1$ in bounded time intervals,

and that it has the properties of an evolution operator, i.e.,

(E1) $U(\tau, \tau) = \mathbb{1}$,

(E2) $U(\tau, \tau_0) = U(\tau, \tau_1)U(\tau_1, \tau_0)$,

(E3) $\partial_\tau U(\tau, \tau_0) = S(\tau)U(\tau, \tau_0)$

for τ, τ_0, τ_1 in an appropriately chosen interval. Note that (E2) implies $U(\tau, \tau_0)^{-1} = U(\tau_0, \tau)$ and, together with (E3), we thus find $\partial_{\tau_0} U(\tau, \tau_0) = -U(\tau, \tau_0)S(\tau_0)$.

Once we have obtained an evolution operator $U(\tau, \tau_0)$, we can solve the dynamical equation (4.2) for $\mathcal{m}$. Namely, given initial data $\mathcal{m}(\tau_0)$ at a fixed time τ_0, we have

$$\mathcal{m}(\tau) = U(\tau, \tau_0)\mathcal{m}(\tau_0).$$

4.6 Evolution operator for geometrically growing weights

In this subsection we consider the case (W1), i.e., the evolution operator on $\vec{\ell}^p(w)$ for the geometrically growing weights $w_n = c\omega^n$ with $c, \omega > 0$.

In this case, $S(\tau)$ is a bounded operator on $\vec{\ell}^p(w)$ with

$$\|S(\tau)\mathcal{M}\|_{p,w} \leq (\|A(\tau)\| + \|B\|\omega)\|\mathcal{M}\|_{p,w} \leq 2\big(\sqrt{1+V(\tau)^2} + \omega\big)\|\mathcal{M}\|_{p,w}.$$

Setting

$$C_\omega(\tau,\tau_0) := \omega + \left|\int_{\tau_0}^{\tau} \sqrt{1+V(\tau')^2}\,\mathrm{d}\tau'\right|, \tag{4.16}$$

we then calculate

$$\|U(\tau,\tau_0)\mathcal{M}\|_{p,w} \leq \sum_{n=0}^{\infty} \frac{2^n}{n!} C_\omega(\tau,\tau_0)^n \|\mathcal{M}\|_{p,w} = \mathrm{e}^{2C_\omega(\tau,\tau_0)}\|\mathcal{M}\|_{p,w},$$

where, in the first step, we used (4.12) and the fact that the volume of the standard n-simplex is $1/n!$.

The following theorem is easily shown, see e.g. Thm. X.69 in [40] (the proofs of the analogous Thm. 4.9 in the next subsection can also be adapted to the case of geometrically growing weights discussed here):

Theorem 4.4. *Let $c, \omega > 0$ and $p \geq 1$. Suppose that $I \subsetneq \mathbb{R}$ and $V \in C(I)$. The evolution operator $U(\tau,\tau_0)$, defined by (4.15), is the unique bounded operator on $\vec{\ell}^p(w)$ with weights $w_n = c\omega^n$, such that the properties (E1)–(E3) hold for $\tau, \tau_0, \tau_1 \in I$. Moreover, we have the bound*

$$\|U(\tau,\tau_0)\|_{p,w} \leq \mathrm{e}^{2C_\omega(\tau,\tau_0)},$$

and it is norm-continuous $\lim_{\tau\to\tau_0} \|U(\tau,\tau_0) - \mathbb{1}\|_{p,w} = 0$.

In particular, this theorem implies the following:

Remark 4.5. If we are given initial data of geometrically growing moments (e.g., the moments of the vacuum state on Minkowski spacetime), they will continue to be geometrically growing under time-evolution independent of the concrete potential $V(t)$.

Remark 4.6. Given a continuous potential V, the solution exists in $\vec{\ell}^p(w)$ for arbitrarily large time intervals I.

The following perturbation result is standard (cf. Lem. 6.4.4 of [35]) and can be shown along the same lines as Thm. 4.10:

Theorem 4.7. *In addition to the assumptions of Thm. 4.4, suppose that $\tilde{V} \in C(I) \cap L^1(I)$. Denote by $U(\tau,\tau_0)$ and $\tilde{U}(\tau,\tau_0)$ the evolution operators for V and $\tilde{V}$. If $\mathcal{M}, \tilde{\mathcal{M}} \in \vec{\ell}^p(w)$, we have*

$$\begin{aligned}&\|U(\tau,\tau_0)\mathcal{M} - \tilde{U}(\tau,\tau_0)\tilde{\mathcal{M}}\|_{p,w} \\ &\qquad \leq \mathrm{e}^{2C_\omega(\tau,\tau_0)}\|\mathcal{M} - \tilde{\mathcal{M}}\|_{p,w} + 2\mathrm{e}^{2C_\omega(\tau,\tau_0)+2\tilde{C}_\omega(\tau,\tau_0)}\|\tilde{\mathcal{M}}\|_{p,w}\int_{\tau_0}^{\tau} |V(\tau') - \tilde{V}(\tau')|\,\mathrm{d}\tau'.\end{aligned}$$

4.7 Evolution operator for factorially growing weights

For weights growing faster than the geometric growth considered in the previous subsection, the left-shift operator and also $S(\tau)$ are unbounded (see also (A.5)). Moreover, the following shows that a construction based on the standard Hille–Yosida theory of C_0-semigroups can not be used to resolve this:

Proposition 4.8. *If the weights grow faster than geometrically, the resolvent set of $S(\tau)$ is empty.*

Proof. For any $\lambda, \xi \in \mathbb{C}$,

$$\mathcal{n}_n = \frac{1}{4^n}\big(4V(\tau)+\lambda^2\big)^n \begin{pmatrix} 4\xi \\ 2\lambda\xi \\ \lambda^2\xi \end{pmatrix}$$

is in $\vec{\ell}^p(w)$ because it grows at most geometrically, and a short calculation shows

$$(S(\tau)-\lambda)\mathcal{n} = \big((A(\tau)-\lambda)\otimes \mathbb{1} + B\otimes L\big)\mathcal{n} = 0. \qquad \square$$

However, even then it is sometimes possible to understand $S(\tau)$ and, somewhat surprisingly, also the evolution operator (4.15) as a bounded operator *between two different weighted sequence spaces* $\vec{\ell}^p(v)$ and $\vec{\ell}^p(w)$. In other words, both $S(\tau)$ and the evolution operator $U(\tau,\tau_0)$ can be considered as unbounded operators on $\vec{\ell}^p(v)$ with domain $\vec{\ell}^p(w)$.

Let us consider the case (W2), i.e., the weights $w_n = (2n)!\,\omega^{2n}$ and $v_n = (2n)!\,\upsilon^{2n}$ with $\upsilon > \omega \geq 1$. We apply Prop. 4.3, (4.12) and (4.14) to obtain for $n > 0$ the bound

$$\begin{aligned}\|U_n(\tau,\tau_0)\mathcal{m}\|_{p,v} &\leq \frac{1}{n!}\sum_{m=0}^{\lceil\frac{n+1}{2}\rceil}\binom{n}{m}\left|\int_{\tau_0}^{\tau}\|A(\tau')\|\,\mathrm{d}\tau'\right|^{n-m}\|B\|^m\|(\mathbb{1}\otimes L^m)\mathcal{m}\|_{p,v}\\ &\leq \frac{2^n}{n!}\sum_{m=0}^{\lceil\frac{n+1}{2}\rceil}\binom{n}{m}r_m\left|\int_{\tau_0}^{\tau}\sqrt{1+V(\tau')^2}\,\mathrm{d}\tau'\right|^{n-m}\|\mathcal{m}\|_{p,w},\end{aligned}$$

where we set

$$r_m = \sup_n \frac{w_{n+m}}{v_n} = \sup_n \frac{(2m+2n)!}{(2n)!}\frac{\omega^{2m+2n}}{\upsilon^{2n}}.$$

We compute $r_0 = 1$ and, for $m > 0$,

$$\begin{aligned}\frac{r_m}{m!} &< \frac{(2m)!}{\sqrt{\pi m}\,m!}\left(\frac{\upsilon}{\omega}\right)^{\frac12}\left(\frac{\upsilon\omega}{\upsilon-\omega}\right)^{2m} = \frac{\Gamma(m+\frac12)}{\pi\sqrt{m}}\left(\frac{\upsilon}{\omega}\right)^{\frac12}\left(\frac{2\upsilon\omega}{\upsilon-\omega}\right)^{2m}\\ &< \frac{(m-1)!}{\pi}\left(\frac{\upsilon}{\omega}\right)^{\frac12}\left(\frac{2\upsilon\omega}{\upsilon-\omega}\right)^{2m},\end{aligned} \tag{4.17}$$

where we applied Prop. A.2, the duplication formula for the Gamma function, and Gautschi's inequality (A.7).

In Prop. A.3, we show that

$$\sum_{m=1}^{\lceil\frac{n+1}{2}\rceil}\frac{(m-1)!}{(n-m)!} \leq \begin{cases}\frac32 & \text{odd } n,\\ \frac{n}{2}+1 & \text{even } n.\end{cases}$$

Thus, estimating the remaining factors in the sum by their supremum and recalling the definition (4.16) of $C_\omega(\tau,\tau_0)$, we obtain for $n > 0$

$$\|U_n(\tau,\tau_0)\mathcal{m}\|_{p,v} < \big(2C_0(\tau,\tau_0)\big)^n\left(\frac{1}{n!}+\frac{1}{\pi}\left(\frac{n}{2}+1\right)\left(\frac{\upsilon}{\omega}\right)^{\frac12}\left(\frac{2\upsilon\omega}{\upsilon-\omega}\right)^{n+2}\right)\|\mathcal{m}\|_{p,w}. \tag{4.18}$$

Consequently, the time-ordered exponential (4.15) exists and can be bounded for sufficiently small time intervals by

$$\begin{aligned}\|U(\tau,\tau_0)\mathcal{m}\|_{p,v} &\leq \mathrm{e}^{2C_0(\tau,\tau_0)}\|\mathcal{m}\|_{p,w}\\ &\quad+\frac{C_0(\tau,\tau_0)K(\upsilon,\omega)^3}{\pi}\left(\frac{\upsilon}{\omega}\right)^{\frac12}\frac{3-4C_0(\tau,\tau_0)K(\upsilon,\omega)}{\big(1-2C_0(\tau,\tau_0)K(\upsilon,\omega)\big)^2}\|\mathcal{m}\|_{p,w},\end{aligned} \tag{4.19}$$

where we defined

$$K(\upsilon,\omega) := \frac{2\upsilon\omega}{\upsilon-\omega}.$$

Note that these inequalities only guarantee the existence of the exponential for a bounded interval of time, even if $\omega = 1$ and υ is arbitrarily large.

Using the result above, we can now show:

Theorem 4.9. *Let $\upsilon > \omega \geq 1$ and $p \geq 1$. Suppose that $I = [\tau_0, \tau_1]$ and $V \in C(I)$ such that*

$$2C_0(\tau_0,\tau_1)K(\upsilon,\omega) < 1.$$

The evolution operator $U(\tau,\tau_0)$, defined by (4.15), is the unique bounded operator from $\vec{\ell}^p(w)$ to $\vec{\ell}^p(v)$ with weights $w_n = (2n)!\omega^{2n}$ and $v_n = (2n)!\upsilon^{2n}$, such that the properties (E1)–(E3) hold in the interval I (in the strong sense from $\vec{\ell}^p(w)$ to $\vec{\ell}^p(v)$). Moreover, for $\mathcal{M} \in \vec{\ell}^p(w)$, we have the bound (4.19) and the strong continuity property

$$\lim_{\tau\to\tau_0} \|U(\tau,\tau_0)\mathcal{M} - \mathcal{M}\|_{p,v} = 0.$$

Proof. We have already shown that the evolution operator $U(\tau,\tau_0)$ is well-defined as an operator from $\vec{\ell}^p(w)$ to $\vec{\ell}^p(v)$. In fact, the same estimates show that, for sufficiently small $\varepsilon > 0$, $U(\tau,\tau_0)$ is bounded from $\vec{\ell}^p(w)$ to $\vec{\ell}^p(v_\varepsilon)$, where $v_{\varepsilon,n} = (2n)!(\upsilon-\varepsilon)^{2n}$. Moreover, applying these estimates again, we find

$$\begin{aligned}
&\|U(\tau,\tau_0)\mathcal{M} - \mathcal{M}\|_{p,v} \\
&\quad\leq \left(e^{2C_0(\tau,\tau_0)} + \frac{C_0(\tau,\tau_0)K(\upsilon,\omega)^3}{\pi}\left(\frac{\upsilon}{\omega}\right)^{\frac{1}{2}} \frac{3-4C_0(\tau,\tau_0)K(\upsilon,\omega)}{\big(1-2C_0(\tau,\tau_0)K(\upsilon,\omega)\big)^2} - 1\right)\|\mathcal{M}\|_{p,w} \to 0
\end{aligned}$$

as $\tau \to \tau_0$.

Property (E1) is obvious and (E2) is shown (as for groups generated by bounded operators) by multiplying the series (4.15) for $U(\tau, r)$ and $U(r,\tau_0)$. The final property (E3) is proven by differentiating (4.15) term by term using the formal relation $\partial_\tau U_n(\tau,\tau_0) = S(\tau)U_{n-1}(\tau,\tau_0)$. The resulting expression is well-defined because $U(\tau,\tau_0)$ is bounded from $\vec{\ell}^p(w)$ to $\vec{\ell}^p(v_\varepsilon)$ and $S(\tau)$ is bounded from $\vec{\ell}^p(v_\varepsilon)$ to $\vec{\ell}^p(v)$. □

Finally, we prove a perturbation result:

Theorem 4.10. *In addition to the assumptions of Thm. 4.9, suppose that $\tilde{V} \in C(I)$ such that*[2]

$$C_0(\tau,\tau_0) \geq \left| \int_{\tau_0}^{\tau} \sqrt{1+\tilde{V}(\tau')^2}\,d\tau' \right| \quad \textit{and} \quad 4C_0(\tau,\tau_0)K(\upsilon,\omega) < 1.$$

Denote by $U(\tau,\tau_0)$ and $\tilde{U}(\tau,\tau_0)$ the evolution operators for V and $\tilde{V}$. If $\mathcal{M},\tilde{\mathcal{M}} \in \vec{\ell}^p(w)$, we have

$$\|U(\tau,\tau_0)\mathcal{M} - \tilde{U}(\tau,\tau_0)\tilde{\mathcal{M}}\|_{p,v} \leq c\|\mathcal{M}-\tilde{\mathcal{M}}\|_{p,w} + 2c^2\|\tilde{\mathcal{M}}\|_{p,w}\int_{\tau_0}^{\tau} |V(\tau')-\tilde{V}(\tau')|\,d\tau',$$

where

$$c = e^{2C_0(\tau,\tau_0)} + \frac{8C_0(\tau,\tau_0)K(\upsilon,\omega)^3}{\pi}\left(\frac{\upsilon}{\omega}\right)^{\frac{1}{2}} \frac{3-8C_0(\tau,\tau_0)K(\upsilon,\omega)}{\big(1-4C_0(\tau,\tau_0)K(\upsilon,\omega)\big)^2}.$$

[2]Note that this condition adds no additional restriction because the role of V and $\tilde{V}$ can be exchanged.

Proof. Set

$$\tilde{w}_n = (2n)!\tilde{\omega}^{2n}, \quad \tilde{\omega} = \frac{2\upsilon\omega}{\upsilon+\omega}.$$

Observe that

$$\frac{\tilde{\omega}\omega}{\tilde{\omega}-\omega} = \frac{2\upsilon\omega}{\upsilon-\omega} = \frac{\upsilon\tilde{\omega}}{\upsilon-\tilde{\omega}}$$

and

$$\frac{\tilde{\omega}}{\omega} = \frac{2\upsilon}{\upsilon+\omega} < \frac{\upsilon}{\omega}, \quad \frac{\upsilon}{\tilde{\omega}} = \frac{\upsilon+\omega}{2\omega} < \frac{\upsilon}{\omega}.$$

Further, note that $4C_0(\tau,\tau_0)K(\upsilon,\omega) < 1$ by assumption. Therefore,

$$\begin{aligned}
\|U(\tau,\tau_0)\mathcal{m}\|_{p,\nu} &\le c\|\mathcal{m}\|_{p,\tilde{w}}, \quad \|\tilde{U}(\tau,\tau_0)\mathcal{m}\|_{p,\nu} \le c\|\mathcal{m}\|_{p,\tilde{w}},\\
\|U(\tau,\tau_0)\mathcal{m}\|_{p,\tilde{w}} &\le c\|\mathcal{m}\|_{p,w}, \quad \|\tilde{U}(\tau,\tau_0)\mathcal{m}\|_{p,\tilde{w}} \le c\|\mathcal{m}\|_{p,w}.
\end{aligned}$$

Now, note the elementary identity

$$U(\tau,\tau_0)\mathcal{m} - \tilde{U}(\tau,\tau_0)\tilde{\mathcal{m}} = U(\tau,\tau_0)(\mathcal{m}-\tilde{\mathcal{m}}) + \big(U(\tau,\tau_0)-\tilde{U}(\tau,\tau_0)\big)\tilde{\mathcal{m}}.$$

For the first summand we find

$$\|U(\tau,\tau_0)(\mathcal{m}-\tilde{\mathcal{m}})\|_{p,\nu} \le \|U(\tau,\tau_0)(\mathcal{m}-\tilde{\mathcal{m}})\|_{p,\tilde{w}} \le c\|\mathcal{m}-\tilde{\mathcal{m}}\|_{p,w}.$$

For the second summand we find, using the fundamental theorem of calculus (for Banach space-valued integrals),

$$\big(U(\tau,\tau_0)-\tilde{U}(\tau,\tau_0)\big)\tilde{\mathcal{m}} = \int_{\tau_0}^{\tau} U(\tau,\tau_0)\big(\tilde{S}(\tau')-S(\tau')\big)\tilde{U}(\tau',\tau_0)\tilde{\mathcal{m}}\,\mathrm{d}\tau',$$

and thus

$$\begin{aligned}
\|U(\tau,\tau_0)\tilde{\mathcal{m}} - \tilde{U}(\tau,\tau_0)\tilde{\mathcal{m}}\|_{p,\nu} &\le \int_{\tau_0}^{\tau} \Big\|U(\tau,\tau_0)\big(\tilde{S}(\tau')-S(\tau')\big)\tilde{U}(\tau',\tau_0)\tilde{\mathcal{m}}\Big\|_{p,\nu}\,\mathrm{d}\tau'\\
&\le 2c\int_{\tau_0}^{\tau} |V(\tau')-\tilde{V}(\tau')|\|\tilde{U}(\tau',\tau_0)\tilde{\mathcal{m}}\|_{p,\tilde{w}}\,\mathrm{d}\tau'\\
&\le 2c^2\|\tilde{\mathcal{m}}\|_{p,w}\int_{\tau_0}^{\tau} |V(\tau')-\tilde{V}(\tau')|\,\mathrm{d}\tau'. \qquad \square
\end{aligned}$$

5 Abstract semiclassical Einstein equation

The SCE on FLRW spacetimes contains only one geometric degree of freedom – the scale factor $a = a(\tau)$. Therefore, as described in Sect. 3, it turns into an ODE for the scale factor, coupled to a dynamical system describing the evolution of the state. In this section, we develop a scheme to solve such systems, encompassing also a large class of modifications of the SCE. Here we always solve the SCE forward in time but equivalent results hold for solutions backward in time.

For fixed $k \in \mathbb{N}$ and $\tau \in I \subset \mathbb{R}$, consider the initial value problem for the quasi-linear system

$$\begin{cases} \partial_\tau a^{(k)} = f(\tau, J^k a, \mathcal{m}), & (5.1a)\\ \partial_\tau \mathcal{m} = S(\tau, J^k a)\mathcal{m}, & (5.1b)\end{cases}$$

where

- $a = a(\tau) \in \mathbb{R}$ is the scale factor with k-jet $J^k a := (a, \dot{a}, \ldots, a^{(k)})$,

- $\mathcal{M} = \mathcal{M}(\tau) \in \vec{\ell}^p(v)$ is the sequence of coincidence limits, as defined in Sect. 4, for some $p \geq 1$ and sequence of weights v,
- $S(\tau, J^k a) = S\big(\tau, J^k a(\tau)\big)$ is the generator of the dynamics of $\mathcal{M}$, as defined in (4.13), for a potential $V(\tau) = V\big(\tau, J^k a(\tau)\big)$ which depends on the jet of the scale factor and may also have an explicit time-dependence, and
- $f(\tau, J^k a, \mathcal{M}) = f\big(\tau, J^k a(\tau), \mathcal{M}(\tau)\big)$ specifies the dynamics of the scale factor including a possible explicit time-dependence and a back-reaction by the quantum field via $\mathcal{M}$.

We note that the results in this section generalize to states including classical background fields, if one simply includes the additional degrees of freedom from Rem. 2.1 in the system (5.1) and adds (3.6) to $f(\tau, J^k a, \mathcal{M})$. These modifications do not change the structure of the proofs below.

5.1 Existence of solutions

Existence of solutions to (5.1) can be shown by a (partial) linearization and employing a fixed-point argument via the construction of a contraction map. With the preparatory results from the previous sections at hand, this theorem is a relatively straightforward adaption of standard results (see e.g. [28]) on quasi-linear systems to the case of Eq. (5.1).

Theorem 5.1. *Let $\vec{a}(\tau_0) \in \mathbb{R}^{k+1}$ and $\mathcal{M}(\tau_0) \in \vec{\ell}^p(w)$ for some weight sequence w. Let $R > 0$ and set $\mathcal{B} := \big\{\vec{b} \in \mathbb{R}^{k+1} \;\big|\; \|\vec{b} - \vec{a}(\tau_0)\| \leq R\big\}$. Suppose that there is $I := [\tau_0, \tau_1] \subset \mathbb{R}$ and a weight sequence v such that the following holds:*

(i) *For each $\vec{a} \in C(I; \mathcal{B})$ there exists an evolution operator $U_{\vec{a}}(\tau, \tau_0)$ for $S(\tau, \vec{a})$ from $\vec{\ell}^p(w)$ to $\vec{\ell}^p(v)$ such that (E1)–(E3) hold (in the strong sense).*

(ii) *There is $\mu_{\mathcal{M}} > 0$ such that $\mathcal{M}_{\vec{a}}(\tau) := U_{\vec{a}}(\tau, \tau_0)\mathcal{M}(\tau_0)$ satisfies the inequality $\|\mathcal{M}_{\vec{a}}(\tau)\|_{p,v} \leq \mu_{\mathcal{M}}$ uniformly in $\tau \in I$ and $\vec{a} \in C(I; \mathcal{B})$.*

(iii) *There is $L_{\mathcal{M}} > 0$ such that, for all $\vec{a}_1, \vec{a}_2 \in C(I; \mathcal{B})$,*

$$\|\mathcal{M}_{\vec{a}_1}(\tau) - \mathcal{M}_{\vec{a}_2}(\tau)\|_{p,v} \leq L_{\mathcal{M}} \int_{\tau_0}^{\tau} \|\vec{a}_1(\tau') - \vec{a}_2(\tau')\| \, \mathrm{d}\tau'.$$

(iv) *For each $\vec{b}_1, \vec{b}_2 \in \mathcal{B}$ and $\mathcal{M}_{\vec{b}_1}, \mathcal{M}_{\vec{b}_2} \in \vec{\ell}^p(v)$ with $\|\mathcal{M}_\bullet\|_{p,v} \leq \mu_{\mathcal{M}}$, the map $\tau \mapsto f(\tau, \vec{b}_1, \mathcal{M}_{\vec{b}_1})$ is continuous for $\tau \in I$, and there are $\mu_f > 0$, $L_f > 0$ such that*

$$|f(\tau, \vec{b}_1, \mathcal{M}_{\vec{b}_1})| \leq \mu_f,$$
$$|f(\tau, \vec{b}_1, \mathcal{M}_{\vec{b}_1}) - f(\tau, \vec{b}_2, \mathcal{M}_{\vec{b}_2})| \leq L_f\big(\|\vec{b}_1 - \vec{b}_2\| + \|\mathcal{M}_{\vec{b}_1} - \mathcal{M}_{\vec{b}_2}\|_{p,v}\big)$$

uniformly for $\tau \in I$.

Then there exists $\tau_2 \in (\tau_0, \tau_1]$ such that (5.1) has a unique (local) solution $(a, \mathcal{M})$ with initial values $(\vec{a}(\tau_0), \mathcal{M}(\tau_0))$ at τ_0, where $a \in C^{k+1}[\tau_0, \tau_2]$ and $\mathcal{M} \in C^1([\tau_0, \tau_2]; \vec{\ell}^p(w))$ such that $J^k a(\tau) \in \mathcal{B}$ for all $\tau \in [\tau_0, \tau_2]$.

Proof. By assumption (i), for every $\vec{a} \in C(I; \mathcal{B})$, it holds that $\mathcal{M}_{\vec{a}}(\tau) := U_{\vec{a}}(\tau, \tau_0)\mathcal{M}(\tau_0)$ is the solution of $\partial_\tau \mathcal{M}_{\vec{a}} = S(\tau, \vec{a})\mathcal{M}_{\vec{a}}$ with initial value $\mathcal{M}_{\vec{a}}(\tau_0) = \mathcal{M}(\tau_0)$.

Hence, linearizing (5.1a) in the scale factor, we shall consider the system $\partial_\tau \vec{a} = F(\tau, \vec{a})$, where $\tau \mapsto \vec{a}(\tau) \in \mathbb{R}^{k+1}$ and

$$F(\tau, \vec{a}) := \big(a_0(\tau), a_1(\tau), a_2(\tau), \dots, a_k(\tau), f(\tau, \vec{a}, \mathcal{M}_{\vec{a}})\big) \in \mathbb{R}^{k+1}, \quad \vec{a} = (a_0, a_1, a_2, \dots, a_k).$$

Clearly, its solution is

$$\vec{a}(\tau) = \vec{a}(\tau_0) + \int_{\tau_0}^{\tau} F(\tau', \vec{a})\,\mathrm{d}\tau'. \tag{5.2}$$

By assumptions (ii) and (iii), we have

$$\|\vec{a}(\tau) - \vec{a}(\tau_0)\| \le \int_{\tau_0}^{\tau} \|F(\tau', \vec{a})\|\,\mathrm{d}\tau' \le \tau(\|\vec{a}(\tau_0)\| + R + \mu_f),$$

which proves that $\vec{a}(\tau) \in \mathcal{B}$ for $\tau \le \tau_2 \in (\tau_0, \tau_1]$ if τ_2 is sufficiently small.

Thus (5.2) defines a mapping $\vec{a} \mapsto \Phi\vec{a}$ of $C([\tau_0, \tau_2]; \mathcal{B})$ into itself. Note that $C([\tau_0, \tau_2]; \mathcal{B})$ is a Banach space; we denote its norm by $\|\cdot\|_\infty$.

If $\vec{a}_1, \vec{a}_2 \in C(I; \mathcal{B})$, we calculate (using assumptions (iii) and (iv)) for $\tau \in [\tau_0, \tau_2]$

$$\begin{aligned}
\left\|\Phi\vec{a}_1(\tau) - \Phi\vec{a}_2(\tau)\right\| &\le \int_{\tau_0}^{\tau} \|F(\tau', \vec{a}_1) - F(\tau', \vec{a}_2)\|\,\mathrm{d}\tau' \\
&\le \int_{\tau_0}^{\tau} \Big((1 + L_f)\|\vec{a}_1(\tau') - \vec{a}_2(\tau')\| + L_f \|\mathcal{M}_{\vec{a}_1}(\tau') - \mathcal{M}_{\vec{a}_2}(\tau')\|_{p,\nu}\Big)\,\mathrm{d}\tau' \\
&\le (1 + L_f)(\tau_2 - \tau_0)\|\vec{a}_1 - \vec{a}_2\|_\infty \\
&\qquad + L_f L_{\mathcal{M}} \int_{\tau_0}^{\tau} \int_{\tau_0}^{\tau'} \|\vec{a}_1(\tau'') - \vec{a}_2(\tau'')\|\,\mathrm{d}\tau''\,\mathrm{d}\tau' \\
&\le \Big((1 + L_f)(\tau_2 - \tau_0) + \tfrac{1}{2} L_f L_{\mathcal{M}}(\tau_2 - \tau_0)^2\Big)\|\vec{a}_1 - \vec{a}_2\|_\infty,
\end{aligned}$$

and thus

$$\|\Phi\vec{a}_1 - \Phi\vec{a}_2\|_\infty \le \Big((1 + L_f)(\tau_2 - \tau_0) + \tfrac{1}{2} L_f L_U(\tau_2 - \tau_0)^2\Big)\|\vec{a}_1 - \vec{a}_2\|_\infty.$$

which shows that Φ is a contraction map for sufficiently small τ_2.

It now follows by the Banach fixed-point theorem that F has a unique fixed point $\vec{a} \in C^1([\tau_0, \tau_2]; \mathbb{R}^{k+1})$ and there is $a \in C^{k+1}[\tau_0, \tau_2]$ such that $J^k a = \vec{a}$. This yields the unique solution to (5.1) with the properties stated in the theorem. □

In particular, we can apply the above theorem to the cases studied in the previous section. In the case of geometrically growing weights we can even obtain maximal or global solutions.

Proposition 5.2. *Consider Thm. 5.1 with the weights*

(a) $\nu_n = w_n = \omega^n$ *with* $\omega > 0$ *(geometrically growing weights), or*

(b) $\nu_n = (2n!)\upsilon^{2n}$ *and* $w_n = (2n!)\omega^{2n}$ *with* $\upsilon, \omega > 0$ *(factorially growing weights).*

Further, suppose that $\tau \mapsto V(\tau, \vec{b})$ *is continuous for all* $\vec{b} \in \mathcal{B}$*, and there is* $L_V > 0$ *such that*

$$|V(\tau, \vec{b}_1) - V(\tau, \vec{b}_2)| \le L_V \|\vec{b}_1 - \vec{b}_2\|$$

for each $\vec{b}_1, \vec{b}_2 \in \mathcal{B}$*, uniformly for* $\tau \in [\tau_0, \tau']$*. Then there exists* $\tau_1 \in (\tau_0, \tau']$ *such that the assumptions* (i)–(iii) *of Thm. 5.1 are satisfied.*

Proof. The assertion follows from Thms. 4.4 and 4.7 for (a), and from Thms. 4.9 and 4.10 for (b). □

Proposition 5.3. *If the assumptions of Prop. 5.2 hold with geometrically growing weights (a), then the unique local solution* $(a, \mathcal{M})$ *of Thm. 5.1 extends to a maximal solution (the solution exists up to a singularity of* V *or* f*) or to a global solution (the solution exists for arbitrarily large times).*

Proof. This is shown by gluing local solutions of the initial value problem (using Thm. 5.1 and Prop. 5.2). □

Remark 5.4. In the case of factorially growing weights, the situation is considerably more complicated because, a priori, already the solution for the dynamics of the moments $\mathcal{M}$ exists only for a bounded time interval.

In the case of the SCE as described in Sect. 3, V and f (as we will see below) are rational functions in the derivatives of the scale factor a and $\log(a)$. Therefore, the assumptions of Thm. 5.1 imply that we can solve (5.1) with $J^k a(\tau)$ inside a ball $\mathcal{B}$ and thus away from the poles of V and f. In the case of geometrically growing weights, we even have maximal (or global) solution.

5.2 Continuous dependence on initial data and parameters

To study the continuous dependence of the abstract SCE (5.1) on its initial data, the potential V and the 'back-reaction' function f, we consider a second quasi-linear equation of the form (5.1):

$$\begin{cases} \partial_\tau \tilde{a}^{(k)} = \tilde{f}(\tau, J^k\tilde{a}, \tilde{\mathcal{M}}), & (5.3a) \\ \partial_\tau \tilde{\mathcal{M}} = \tilde{S}(\tau, J^k\tilde{a})\tilde{\mathcal{M}}, & (5.3b) \end{cases}$$

with initial conditions $J^k\tilde{a}(\tau_0) \in \mathbb{R}^{k+1}$ and $\tilde{\mathcal{M}}(\tau_0) \in \vec{\ell}^p(w)$. Above, $\tilde{S}(\tau, J^k\tilde{a})$ is given by (4.13) for a potential $\tilde{V}(\tau, J^k\tilde{a})$.

Theorem 5.5. *Suppose that* (5.3) *satisfies the assumptions* (i),(ii),(iv) *of Thm. 5.1 for the same constants. Furthermore, (with the obvious notation) assume that*

(iii') *There are $L_{\mathcal{M}}, L_V > 0$ such that, for all $\vec{a}_1, \vec{a}_2 \in C(I;\mathcal{B})$,*

$$\|\mathcal{M}_{\vec{a}_1}(\tau) - \tilde{\mathcal{M}}_{\vec{a}_2}(\tau)\|_{p,v} \le L_{\mathcal{M}}\left(\|\mathcal{M}(\tau_0) - \tilde{\mathcal{M}}(\tau_0)\|_{p,w} + \int_{\tau_0}^{\tau} |V(\tau', \vec{a}_1) - \tilde{V}(\tau', \vec{a}_2)|\,\mathrm{d}\tau'\right),$$
$$|V(\tau, \vec{a}_1) - V(\tau, \vec{a}_2)| \le L_V \|\vec{a}_1(\tau) - \vec{a}_1(\tau)\|.$$

Then there exist $\tau_2 \in (\tau_0, \tau_1]$ and $\kappa \in [0,1)$ such that

$$\begin{aligned}(1-\kappa)\|J^k a - J^k\tilde{a}\|_\infty \le{}& \|J^k a(\tau_0) - J^k\tilde{a}(\tau_0)\| + (\tau_2 - \tau_0) L_f L_{\mathcal{M}} \|\mathcal{M}(\tau_0) - \tilde{\mathcal{M}}(\tau_0)\|_{p,w} \\ &+ L_f L_{\mathcal{M}} \int_{\tau_0}^{\tau_2}\int_{\tau_0}^{\tau} |V(\tau', J^k a) - \tilde{V}(\tau', J^k a)|\,\mathrm{d}\tau'\,\mathrm{d}\tau \\ &+ \int_{\tau_0}^{\tau_2} |f(\tau, J^k a, \mathcal{M}) - \tilde{f}(\tau, J^k a, \mathcal{M})|\,\mathrm{d}\tau.\end{aligned}$$

Proof. As in the proof of Thm. 5.1, we solve (5.1) resp. (5.3) by defining maps Φ and $\tilde{\Phi}$ which are contractions for sufficiently small $\tau_2 \in (\tau_0, \tau_1]$ with common contraction constant $\kappa < 1$. Then $J^k a$ and $J^k\tilde{a}$ are fixed-points of Φ and $\tilde{\Phi}$, respectively, and we find

$$\begin{aligned}\|J^k a - J^k\tilde{a}\|_\infty &= \|\Phi(J^k a) - \tilde{\Phi}(J^k\tilde{a})\|_\infty \\ &\le \|\Phi(J^k a) - \tilde{\Phi}(J^k a)\|_\infty + \|\tilde{\Phi}(J^k a) - \tilde{\Phi}(J^k\tilde{a})\|_\infty \\ &\le \|\Phi(J^k a) - \tilde{\Phi}(J^k a)\|_\infty + \kappa\|J^k a - J^k\tilde{a}\|_\infty.\end{aligned}$$

Using the same notation as in the proof of Thm. 5.1, we have

$$\|\Phi\vec{a} - \tilde{\Phi}\vec{a}\|_\infty \le \|J^k a(\tau_0) - J^k\tilde{a}(\tau_0)\| + \int_{\tau_0}^{\tau_2} \|F(\tau, \vec{a}) - \tilde{F}(\tau, \vec{a})\|\,\mathrm{d}\tau$$

for any $\vec{a} \in C(I;\mathcal{B})$, where

$$\begin{aligned}
\|F(\tau,\vec{a}) - \tilde{F}(\tau,\vec{a})\| &= |f(\tau,\vec{a},\mathcal{M}_{\vec{a}}) - \tilde{f}(\tau,\vec{a},\tilde{\mathcal{M}}_{\vec{a}})| \\
&\leq |f(\tau,\vec{a},\mathcal{M}_{\vec{a}}) - f(\tau,\vec{a},\tilde{\mathcal{M}}_{\vec{a}})| + |f(\tau,\vec{a},\tilde{\mathcal{M}}_{\vec{a}}) - \tilde{f}(\tau,\vec{a},\tilde{\mathcal{M}}_{\vec{a}})|, \\
|f(\tau,\vec{a},\mathcal{M}_{\vec{a}}) - f(\tau,\vec{a},\tilde{\mathcal{M}}_{\vec{a}})| &\leq L_f L_{\mathcal{M}} \Big(\|\mathcal{M}(\tau_0) - \tilde{\mathcal{M}}(\tau_0)\|_{p,w} \\
&\qquad + \int_{\tau_0}^{\tau} |V(\tau',\vec{a}) - \tilde{V}(\tau',\vec{a})| \, \mathrm{d}\tau' \Big).
\end{aligned}$$

Putting everything together, this shows that

$$\begin{aligned}
(1-\kappa)\|J^k a - J^k \tilde{a}\|_\infty \leq \|J^k a(\tau_0) - J^k \tilde{a}(\tau_0)\| &+ (\tau_2 - \tau_0) L_f L_{\mathcal{M}} \|\mathcal{M}(\tau_0) - \tilde{\mathcal{M}}(\tau_0)\|_{p,w} \\
&+ L_f L_{\mathcal{M}} \int_{\tau_0}^{\tau_2} \int_{\tau_0}^{\tau} |V(\tau', J^k a) - \tilde{V}(\tau', J^k a)| \, \mathrm{d}\tau' \, \mathrm{d}\tau \\
&+ \int_{\tau_0}^{\tau_2} |f(\tau, J^k a, \mathcal{M}) - \tilde{f}(\tau, J^k a, \mathcal{M})| \, \mathrm{d}\tau. \qquad \square
\end{aligned}$$

5.3 Application to the traced SCE

In this subsection we show how the results above may be applied to the SCE as discussed in Sect. 3.

Recall that $\omega_2^{\mathrm{reg}} = \omega_2 - H_n$, where n is sufficiently large. It follows by a straightforward calculation from the definitions of Sect. 2 that

$$\begin{aligned}
[\omega_2^{\mathrm{reg}}] &= \frac{1}{a^2}\big(\mathcal{M}_{\varphi\varphi,0} + \mathcal{H}_{\varphi\varphi,1} - \tilde{\mathcal{H}}_{\varphi\varphi,1}\big)\Big|_{r=0}, \\
[(\mathbb{1}\otimes\Delta)\omega_2^{\mathrm{reg}}] &= \frac{1}{a^2}\big(\mathcal{M}_{\varphi\varphi,1} + \Delta_r(\mathcal{H}_{\varphi\varphi,2} - \tilde{\mathcal{H}}_{\varphi\varphi,2})\big)\Big|_{r=0}, \\
[(\partial_\tau\otimes\partial_\tau)\omega_2^{\mathrm{reg}}] &= \frac{1}{a^2}\big(\mathcal{M}_{\pi\pi,0} + \mathcal{H}_{\pi\pi,2} - \tilde{\mathcal{H}}_{\pi\pi,2}\big) + \frac{\dot{a}^2}{a^4}\big(\mathcal{M}_{\varphi\varphi,0} + \mathcal{H}_{\varphi\varphi,1} - \tilde{\mathcal{H}}_{\varphi\varphi,1}\big) \\
&\quad - 2\frac{\dot{a}}{a^3}\big(\mathcal{M}_{(\varphi\pi),0} + \mathcal{H}_{(\varphi\pi),2} - \tilde{\mathcal{H}}_{(\varphi\pi),2}\big)\Big|_{r=0}.
\end{aligned}$$

Thus, combining the results from Sects. 2 and 3, the traced SCE (3.1) can be expanded to the rather long equation

$$\begin{aligned}
0 = &\left(-12(3c_3 + c_4) - \frac{1}{480\pi^2} + \frac{6\xi - 1}{48\pi^2} + \frac{(6\xi-1)^2}{16\pi^2}\log(a)\right)\left(\frac{a^{(4)}}{a^5} - 4\frac{a^{(3)}\dot{a}}{a^6} - 3\frac{\ddot{a}^2}{a^6} + 6\frac{\ddot{a}\dot{a}^2}{a^7}\right) \\
&+ \frac{(6\xi-1)^2}{32\pi^2}\left(4\frac{a^{(3)}\dot{a}}{a^6} + 3\frac{\ddot{a}^2}{a^6} - 10\frac{\ddot{a}\dot{a}^2}{a^7}\right) + \frac{1}{240\pi^2}\left(-\frac{\ddot{a}\dot{a}^2}{a^7} + \frac{\dot{a}^4}{a^8}\right) \\
&+ \left(\frac{6}{8\pi G} + m^2\Big(-6c_2 + \frac{1}{48\pi^2} + \frac{6\xi-1}{8\pi^2}\big(1 + \log(a)\big)\Big)\right)\frac{\ddot{a}}{a^3} \\
&+ \frac{(6\xi-1)m^2}{16\pi^2}\frac{\dot{a}^2}{a^4} + m^4\left(4c_1 + \frac{1}{32\pi^2} + \frac{1}{8\pi^2}\log(a)\right) - \frac{m^2}{a^2}\mathcal{M}_{\varphi\varphi,0} \\
&+ (6\xi - 1)\left(\Big(6\xi\frac{\ddot{a}}{a^5} - \frac{\dot{a}^2}{a^6} + \frac{m^2}{a^2}\Big)\mathcal{M}_{\varphi\varphi,0} + 2\frac{\dot{a}}{a^5}\mathcal{M}_{(\varphi\pi),0} - \frac{1}{a^4}\big(\mathcal{M}_{\pi\pi,0} + \mathcal{M}_{1,\varphi\varphi}\big)\right).
\end{aligned}$$

We remark that the first line of this equation is due to terms proportional to $\Box R$.

In the general case, this equation can be rewritten as quasi-linear fourth order equation of the form

$$\partial_\tau a^{(3)} = f(a, \dot{a}, \ddot{a}, a^{(3)}, \mathcal{M}_0, \mathcal{M}_1), \tag{5.4}$$

which can be solved using Thm. 5.1 (see also Props. 5.2 and 5.3). Note that the right-hand side has poles at $a = 0$ and for a such that

$$30(6\xi - 1)^2 \log(a) = 11 + 5760\pi^2(3c_3 + c_4) - 60\xi, \tag{5.5}$$

which must be taken into account for the choice of the ball $\mathcal{B}$ in Thm. 5.1. We do not see the instability near the Minkowski solution described in [45, 46], but in the non-conformally coupled case the singularity (5.5) appears. It can, however, be seen that the position of this singularity is an artifact of the convention (2.19) relating the length scales μ and λ, which is arbitrary.

There is only one special case in which this equation reduces to a lower than fourth order equation: if

$$\xi = \frac{1}{6} \quad \text{and} \quad 3c_3 + c_4 = -\frac{1}{5760\pi^2}, \tag{5.6}$$

the fourth and third order terms drop out and the equation can be rewritten as

$$\ddot{a} = \left(\frac{\dot{a}^2}{a^4} - 1440\pi^2(8\pi G)^{-1} + (1440\pi^2 c_2 - 5)m^2\right)^{-1} \times \left(\frac{\dot{a}^4}{a^5} + \frac{1}{2}m^4 a^3\big(1920\pi^2 c_1 + 15 + 60\log(a)\big) - 240\pi^2 m^2 a \mathcal{M}_{\varphi\varphi,0}\right)$$

and thus has a the correct form to be solved by the methods above. This equation is equivalent to that already considered in [37]. Note that the right-hand side has poles at $a = 0$ (big bang/crunch) and

$$\frac{\dot{a}^2}{a^4} = 1440\pi^2(8\pi G)^{-1} - (1440\pi^2 c_2 - 5)m^2, \tag{5.7}$$

viz., for a certain value of the square of the Hubble parameter ($\dot{a}/a^2$ in conformal time), which must be taken into account for the choice of the ball $\mathcal{B}$ in Thm. 5.1. Also the instability (5.7) should be considered irrelevant because $c_2 m^2$ must be small as it corresponds to a renormalization of Newton's gravitational constant, which has already been measured, and thus the singularity occurs for a Hubble parameter close to the inverse Planck time (very many orders of magnitude larger than the currently observed value).

We sum up the results of this subsection:

Theorem 5.6. *The traced SCE is of the form* (5.1) *and can be locally solved using Thm. 5.1 and Props. 5.2, 5.3. In the case of geometrically growing weights for the moments $\mathcal{M}$ (i.e., vacuum-like states), a maximal or global solution exists. In the generic fourth order case, the maximal solution exists up to a big bang/crunch $a = 0$, the logarithmic singularity* (5.5)*, or a blow-up $a \to \infty$ (big rip). In the second order case (i.e.,* (5.6) *are satisfied), the maximal solutions exists up to a big bang/crunch, the singularity* (5.7)*, or a big rip.*

5.4 Choice of initial data

It is not clear how to give initial values for a (Hadamard) state unless the scale function is known in a neighbourhood of the Cauchy surfaces. This is another reason for why it is not clear how to pose a satisfying initial value problem for the SCE. Our use of the moments $\mathcal{M}$ instead of a state does not completely solve this problem as it is not clear which sequences of moments belong to *positive* two-point functions of *physical* states.

What we do know very well is how to construct a wide variety of quantum states on Minkowski spacetime. For these states it is often possible to calculate the moments $\mathcal{M}$ without too much difficulty, see e.g. Sects. 4.2 and 4.3. This suggests the following approach:

Proposition 5.7. *For some $\varepsilon > 0$ and $\tau_0 \in \mathbb{R}$, let χ be a smooth switching function such that*

$$\chi(\tau) = \begin{cases} 0 & \tau \in (-\infty, \tau_0 + \varepsilon], \\ 1 & \tau \in [\tau_0 + 2\varepsilon, \infty), \end{cases}$$

Then, multiplying the right-hand side of the traced SCE (5.4) by χ, the quasilinear system

$$\begin{cases} \partial_\tau a^{(3)} = \chi(\tau) f(a, \dot{a}, \ddot{a}, a^{(3)}, \mathcal{M}_0, \mathcal{M}_1), & (5.8a) \\ \partial_\tau \mathcal{M} = S(a, \ddot{a}) \mathcal{M} & (5.8b) \end{cases}$$

has a local solution with initial values $\vec{a}(\tau_0) = (1, 0, 0, 0)$ and $\mathcal{M}(\tau_0) \in \ell^p(w)$ given by a Hadamard state on Minkowski spacetime. In the case of geometrically growing weights w, the local solution extends to a maximal or global solution.

Proof. This equation with its explicit time-dependence still fits into the class of equations (5.1) considered before, and we can apply Thm. 5.1 or rather Props. 5.2 and 5.3. □

For proper physical initial data it should be required that the initial data for the state (resp. the moments) satisfy the energy (constraint) equation given by (3.7). That is, for initial data on a Minkowski spacetime as required by the construction above, the following relation needs to hold:

$$\begin{aligned} 0 &= \frac{1}{2}[(\partial_\tau \otimes \partial_\tau)\omega_2^{\mathrm{reg}}](\tau_0) - \frac{1}{2}[(\mathbb{1} \otimes \Delta)\omega_2^{\mathrm{reg}}](\tau_0) + \frac{m^2}{2}[\omega_2^{\mathrm{reg}}](\tau_0) - \left(\frac{1}{32\pi^2} + c_1\right) m^4 \\ &= \frac{1}{2}\mathcal{M}_{\pi\pi,0}(\tau_0) - \frac{1}{2}\mathcal{M}_{\varphi\varphi,1}(\tau_0) + \frac{m^2}{2}\mathcal{M}_{\varphi\varphi,0}(\tau_0) - \left(\frac{5}{64\pi^2} + c_1\right) m^4 \end{aligned}$$

For example, the Minkowski vacuum satisfies this equation for $c_1 = -5/64\pi^2$.

If the solution of (5.8) exists in a time interval $[\tau_0, \tau_1]$ with $\tau_1 > \tau_0 + \varepsilon$, it yields proper physical initial data for the normal (unswitched) SCE, as the energy constraint is preserved for a covariantly conserved energy momentum tensor.

By varying χ, we can construct a large set of physical initial data. Moreover, we see that the set of physical solutions to the SCE is non-empty. Whether all initial conditions $a(\tau_0), \dots, a^{(3)}(\tau_0)$ can be realized for any physical $\mathcal{M}$ needs to be investigated.

5.5 Reconstruction of the quantum state

There is no obvious way to directly relate a sequence of moments $\mathcal{M}$ to a quasi-free state. Note that this is not a classical moment problem, as the degree l of the counter terms in (4.1) is neither fixed nor unique. Here we can easily circumvent this problem: Suppose that in addition to the initial data for the moments $\mathcal{M}$ we are given initial data for the associated state, or rather the two-point function. (Concretely, this is possible, for instance, in the setup of Prop. 5.7.) Then, once we have solved the quasilinear system (5.1) for the scale factor and the moments, we can evolve the initial data for the two-point function with the obtained scale factor. Necessarily, the evolved two-point function is compatible with the evolved moments. Alternatively, we can augment (5.1) by adding a third equation for the two-point function and directly co-evolve it with the moments.

6 Outlook

The higher derivatives of the metric appearing in the SCE can potentially lead to runaway solutions which deviate significantly from classical solutions of the Einstein equation. Therefore, sometimes the order-reduced SCE (cf. [20]) is suggested as a better behaved approximation of the interaction between quantum matter and classical spacetime geometry. It would be interesting to study also the order-reduced SCE within the approach presented above and compare the resulting solutions with those of the standard SCE.

Here we restricted ourselves to the SCE with a free scalar field on flat cosmological spacetimes. An extension of our approach to other non-interacting types of matter (e.g., the Dirac field) and non-flat cosmological spacetime seems feasible. The former requires some modifications of the moment spaces, while the latter requires a careful treatment of homogeneous distributions on maximally symmetric spaces. Also, the introduction of potential energy of the field (neglecting self-interaction) $-\lambda\langle{:}\phi^4{:}\rangle_\omega$ would lead to quadratic terms in $\mathcal{M}_{00}$ and further correction terms in the trace equation that nevertheless still fall in the class of the abstract SCE (5.1). Such modifications could be of interest in the cosmology of the early universe on time scales well above the Planck time but below characteristic times of nuclear reactions, see [5, 6] for related work.

A much more ambitious would be the passage to non-cosmological spacetimes. Space-dependent germs of distributional tensor structures that remain closed under successive applications of the Klein–Gordon operator would have to replace the expansion in radially symmetric homogeneous distributions. The point-splitting limit of such an approach would result in an infinite hierarchy of coupled PDEs for the germ coefficients. It is not obvious, if such a system can be set up or even be solved.

Potentially, our new formulation of the SCE also has numerical consequences, as it avoids time integration of rapidly oscillating modes of the quantum state and integration in momentum space. Theorem 5.5 establishes certain bounds for the change of the solution under modification of the initial conditions. This can be used to truncate $\mathcal{M}$ at a certain order with a controlled error for the solution of the SCE. As the space of moments with zero entries after a prescribed order is invariant under the dynamics (4.2), such approximated initial conditions give rise to a finite dimensional ODE which can be numerically integrated in time using standard methods. However, convergence rates as a function of the time interval need to be carefully examined to judge numerical viability.

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A Appendix

A.1 Homogeneous distributions

Roughly following Chap. 3.2 of [26], we define for $z \in \mathbb{C}$ with $\operatorname{Re} z > -1$ the function on $\mathbb{R}$

$$k_+^z := \begin{cases} k^z & \text{if } k > 0, \\ 0 & \text{if } k \leq 0. \end{cases} \tag{A.1}$$

Since this function is locally integrable, it defines a distribution. It can be extended to $z \in \mathbb{C} \setminus \{-1, -2, \dots\}$ by analytic continuation. To further extend k_+^z to all $z \in \mathbb{C}$, we define for $n \in \mathbb{N}$ and any test function $f \in C_c^\infty(\mathbb{R})$

$$\langle k_+^{-n}, f\rangle := \frac{1}{(n-1)!}\Big(-\int_0^\infty \log(k) f^{(n)}(k)\,\mathrm{d}x + f^{(n-1)}(0)\sum_{j=1}^{n-1}\frac{1}{j}\Big). \tag{A.2}$$

Defined in this way, k_+^z satisfies for $z \in \mathbb{C}$ the homogeneity property $\langle k_+^z, kf\rangle = \langle k_+^{z+1}, f\rangle$. Note that (A.2) is not the only possible extension of (A.1) to all of $z \in \mathbb{C}$ – different extensions differ by derivatives of the delta distribution at zero.

To calculate the (inverse) Fourier transform of k_+^{-n}, note first that

$$\langle k_+^{-1}, \mathrm{e}^{\mathrm{i}kx}\rangle = \lim_{\nu\to 0}\int_0^\infty k^{\nu-1}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x - \frac{1}{\nu}.$$

Then we compute

$$\begin{aligned}\int_0^\infty k^{\nu-1}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x &= \Gamma(\nu)(-\mathrm{i}x)^{-\nu} = \big(\nu^{-1}-\gamma+\mathcal{O}(\nu)\big)\big(1-\nu\log(-\mathrm{i}x+0)+\mathcal{O}(\nu^2)\big)\\ &= \nu^{-1}-\gamma-\log(-\mathrm{i}x+0)+\mathcal{O}(\nu)\end{aligned}$$

Therefore, we find

$$\langle k_+^{-1}, \mathrm{e}^{\mathrm{i}kx}\rangle = -\gamma-\log(-\mathrm{i}x+0) = -\gamma-\log|x|+\frac{\mathrm{i}\pi}{2}\operatorname{sgn}(x),$$

which, by (A.2), immediately implies that, for $n\in\mathbb{N}$,

$$\langle k_+^{-n}, \mathrm{e}^{\mathrm{i}kx}\rangle = \frac{(\mathrm{i}x)^{n-1}}{\Gamma(n)}\left(\psi(n)-\log|x|+\frac{\mathrm{i}\pi}{2}\operatorname{sgn}(x)\right), \tag{A.3}$$

where we used the relation between the harmonic numbers and the digamma function

$$\psi(n) = \sum_{j=1}^{n-1}\frac{1}{j}-\gamma, \quad n\in\mathbb{N}.$$

This motivates the definition of the distributions (for $r\geq 0$)

$$h_z(r) := \frac{\mathrm{e}^{\mathrm{i}z\pi/2}}{2\pi^2}\frac{r^{z-2}}{\Gamma(z)}\big(\log(r)-\psi(z)\big),$$

which extends analytically to $z\in\mathbb{C}$. In particular, we have for $n\in\mathbb{N}_0$

$$h_{-n}(r) = \frac{\mathrm{i}^n n!}{2\pi^2 r^{n+2}}$$

because both $1/\Gamma(z)$ and $\psi(z)/\Gamma(z)$ are entire functions with $1/\Gamma(-n)=0$ and $\psi(-n)/\Gamma(-n) = (-1)^{n+1}\Gamma(n+1)$.

Taking the imaginary part of (A.3), we find

$$h_n(r) = \frac{1}{2\pi^2 r}\operatorname{Im}\langle k_+^{-n}, k\mathrm{e}^{\mathrm{i}kr}\rangle = \frac{1}{2\pi^2 r}\operatorname{Im}\langle k_+^{-n+1}, \sin(kr)\rangle$$

for even n. Moreover, h_z satisfies the homogeneity property

$$-\Delta_r h_{z+2}(r) = h_z(r), \tag{A.4}$$

where we recall that Δ_r denotes the (three dimensional) radial Laplacian.

A.2 Weighted sequence spaces

Let $w=(w_n)$ be a sequence of strictly positive numbers, called the *weights*. By $\ell^p(w)$, $p\geq 1$, we denote the space of complex sequences $x=(x_n)$ with convergent norm

$$\|x\|_{p,w} := \Big(\sum_n |w_n^{-1}x_n|^p\Big)^{1/p}.$$

If $p=\infty$, we denote by $\ell^\infty(w)$ the space of complex sequences with convergent norm

$$\|x\|_{\infty,w} := \sup_n w_n^{-1}|x_n|.$$

These are the *weighted* ℓ^p *spaces*. If $w_n = 1$, we omit the weight and denote by ℓ^p with norm $\|\cdot\|_p$ the ordinary ℓ^p spaces. Note that $\ell^p(w)$ is reflexive for $1 < p < \infty$.

Consider two weight sequences v, w. Then it is easily seen that

$$\|x\|_{p,v} \le \sup_n \frac{w_n}{v_n} \|x\|_{p,w}.$$

One of the most important operators on sequence spaces is the left-shift operator L, formally defined by $L(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots)$. If w are weights such that the sequence (w_{n+1}/w_n) of ratios of consecutive weights is bounded, the left-shift operator is bounded on $\ell^p(w)$. Indeed,

$$\|Lx\|_{p,w} \le \sup_n \frac{w_{n+1}}{w_n} \|x\|_{p,w}. \tag{A.5}$$

However, if (w_{n+1}/w_n) is unbounded, also the left-shift operator is unbounded.

More generally, for $m \in \mathbb{N}$ and two sequences v, w weights, we have

$$\|L^m x\|_{p,v} \le \sup_n \frac{w_{n+m}}{v_n} \|x\|_{p,w}.$$

Finally, note that, if L is unbounded on $\ell^p(w)$, its resolvent set is empty and its point spectrum fills the entire complex plane.

A.3 Some inequalities

Lemma A.1. *For $0 \le m \le n \in \mathbb{N}_0$ and $p \in (0,1)$,*

$$\binom{n}{m}(1-p)^n \le \binom{\lfloor m/p \rfloor}{m}(1-p)^{\lfloor m/p \rfloor},$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

Proof. The case $m = 0$ is obvious. For $m > 0$, we calculate the ratio of successive terms on the left-hand side:

$$\frac{\binom{n+1}{m}(1-p)^{n+1}}{\binom{n}{m}(1-p)^n} = \frac{n+1}{n+1-m}(1-p).$$

Then we observe that

$$\frac{n+1}{n+1-m}(1-p) \ge 1$$

if and only if $n + 1 \le m/p$ to find the maximum at $n = \lfloor m/p \rfloor$. □

Note that this lemma can also be stated in the language of probability theory: $n = \lfloor m/p \rfloor$ maximizes the probability of getting exactly m failures for a random variable following the binomial distribution with parameters n (number of trials) and p (probability of success).

An important inequality, accurately describing the asymptotics of the Gamma function, is Stirling's inequality

$$\sqrt{2\pi}\, x^{x+\frac{1}{2}} \mathrm{e}^{-x} < \Gamma(x+1) < x^{x+\frac{1}{2}} \mathrm{e}^{-x+1}, \quad x > 0.$$

This double inequality can be improved in various ways, e.g., for $x \ge 1$,

$$\sqrt{2\pi}\, x^{x+\frac{1}{2}} \mathrm{e}^{-x+(12x+1)^{-1}} < \Gamma(x+1) < \sqrt{2\pi}\, x^{x+\frac{1}{2}} \mathrm{e}^{-x+(12x)^{-1}}, \tag{A.6}$$

which can be obtained from [41], where also sharper bounds are presented.

Another useful inequality for the Gamma function is Gautschi's inequality. For $x > 0$ and $s \in (0,1)$, we have

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}, \tag{A.7}$$

see e.g. (5.6.4) of [33], which follows from the strict log-convexity of the Gamma function.

Combining the inequalities above, we find

Proposition A.2. *For $m \in \mathbb{N}$ and $p \in (0,1)$,*

$$\max_n \binom{n+m}{m} p^m (1-p)^n \leq \min\left\{1, \frac{2}{\sqrt{2\pi m}}(1-p)^{-\frac{1}{2}}\right\} \tag{A.8}$$

and, as $m \to \infty$,

$$\max_n \binom{n+m}{m} p^m (1-p)^n \sim \frac{1}{\sqrt{2\pi m}}(1-p)^{-\frac{1}{2}}. \tag{A.9}$$

Proof. Denote by $\{x\} = x - \lfloor x \rfloor$ the fractional part of a real number x. We calculate

$$\begin{aligned}
\max_n \binom{n+m}{m} p^m (1-p)^n &= \binom{\lfloor m/p \rfloor}{m} p^m (1-p)^{\lfloor m/p \rfloor - m} \\
&\leq \binom{m/p}{m} p^m (1-p)^{m/p-m} \left(1 + \frac{p}{m(1-p)}\right)^{\{m/p\}} \\
&< \frac{1}{\sqrt{2\pi m}}(1-p)^{-\frac{1}{2}} \left(1 + \frac{p}{m(1-p)}\right)^{\{m/p\}} \\
&\leq \frac{1}{\sqrt{2\pi m}}(1-p)^{-\frac{1}{2}} \left(1 + \left\{\frac{m}{p}\right\} \frac{p}{m(1-p)}\right),
\end{aligned}$$

where we applied (in this order) Lem. A.1, Gautschi's inequality (A.7), Stirling's inequality (A.6) and Bernoulli's inequality. Finally, an application of the inequality

$$\left\{\frac{m}{p}\right\} \frac{p}{m(1-p)} = \left\{\frac{m(1-p)}{p}\right\} \frac{p}{m(1-p)} \leq \min\left\{1, \frac{p}{m(1-p)}\right\}$$

yields (A.8), and from

$$\lim_{m \to \infty} \left\{\frac{m}{p}\right\} \frac{p}{m(1-p)} = 0$$

we obtain (A.9). □

Proposition A.3. *Let $n \in \mathbb{N}$. We have*

$$\sum_{m=1}^{\lceil \frac{n+1}{2} \rceil} \frac{(m-1)!}{(n-m)!} \leq \begin{cases} \frac{3}{2} & \text{odd } n, \\ \frac{n}{2} + 1 & \text{even } n. \end{cases}$$

Proof. We consider only the odd case, the proof for the even case proceeds analogously. The smallest summand in the sum for n is $1/(n-1)!$, which is larger than the second smallest summand $1/n!$ in the sum for $n+2$. Therefore we can bound all sums uniformly in n by summing up the smallest summands for each n. If we proceed like that for $n \geq 5$, we obtain

$$1 + \frac{1}{6} + \sum_{n=2}^{\infty} \frac{1}{(2n)!} = \cosh 1 - \frac{1}{3} < \frac{3}{2}.$$

For $n = 1$ and $n = 3$, the sums yield 1 and $\frac{3}{2}$, respectively. □

A.4 Expansion of Synge's world function

To compute the (truncated) Hadamard parametrix in a given spacetime, it is necessary (among other things) to find Synge's world function σ. The world function $\sigma(x, x')$ is defined as half the square signed geodesic distance between the points x and x' (if the two points lie in a geodesically convex neighbourhood) and satisfies the relation

$$2\sigma = \big((\nabla^\mu \otimes \mathbb{1})\sigma\big)\big((\nabla_\mu \otimes \mathbb{1})\sigma\big). \tag{A.10}$$

In fact, the relation (A.10) together with the coincidence limits

$$[\sigma]=0,\quad [(\nabla_\mu\otimes\mathbb{1})\sigma]=0,\quad [(\nabla_\mu\nabla_\nu\otimes\mathbb{1})\sigma]=g_{\mu\nu} \tag{A.11}$$

uniquely defines Synge's world function.

An expansion of Synge's world function $\sigma(x,x')$ in terms of the coordinate distance δx between the points x and x' can be obtained in the following way [42]: We make the Ansatz (in the sense of formal power series)

$$\sigma(x,x')=\sum_n \frac{1}{n!}\varsigma_{\mu_1\cdots\mu_n}(x)\delta x^{\mu_1}\delta x^{\mu_n}.$$

As a consequence of (A.10) and (A.11), we find the recurrence relation

$$\begin{aligned}2(1-n)\varsigma_{\mu_1\cdots\mu_n}=&\sum_{j=2}^{n-2}\binom{n}{j}g^{\nu\rho}\big(\partial_\nu\varsigma_{(\mu_1\cdots\mu_j|}-\varsigma_{(\mu_1\cdots\mu_j|\nu}\big)\big(\partial_\rho\varsigma_{|\mu_{j+1}\cdots\mu_n)}-\varsigma_{|\mu_{j+1}\cdots\mu_n)\rho}\big)\\&-2n\partial_{(\mu_1}\varsigma_{\mu_2\cdots\mu_n)}\end{aligned}$$

together with 'initial' coefficients $\varsigma=0,\varsigma_\mu=0,\varsigma_{\mu\nu}=g_{\mu\nu}$.

References

[1] Anderson, P.R.: Effects of Quantum Fields on Singularities and Particle Horizons in the Early Universe. Phys. Rev. D **28**, 271–285 (1983)

[2] Anderson, P.R.: Effects of Quantum Fields on Singularities and Particle Horizons in the Early Universe. II. Phys. Rev. D **29**, 615–627 (1984)

[3] Anderson, P.R.: Effects of Quantum Fields on Singularities and Particle Horizons in the Early Universe. III. The Conformally Coupled Massive Scalar Field. Phys. Rev. D **32**, 1302–1315 (1985)

[4] Anderson, P.R.: Effects of Quantum Fields on Singularities and Particle Horizons in the Early Universe. IV. Initially Empty Universes. Phys. Rev. D **33**, 1567–1575 (1986)

[5] Anderson, P.R., Molina-París, C., Evanich, D., Cook, G.B.: Study of The Preheating Phase of Chaotic Inflation. Phys. Rev. D **78**, 083514 (2008)

[6] Anderson, P.R., Molina-París, C., Sanders, D.H.: Validity of the Semiclassical Approximation during the Preheating Phase of Chaotic Inflation. In: K. Rosquist, R.T. Jantzen, R. Ruffini (eds.) *The Thirteenth Marcel Grossmann Meeting*, pp. 1500–1503. World Scientific, (2015)

[7] Anderson, P.R., Parker, L.E.: Adiabatic Regularization in Closed Robertson–Walker Universes. Phys. Rev. D **36**, 2963–2969 (1987)

[8] Bär, C., Ginoux, N., Pfäffle, F.: *Wave Equations on Lorentzian Manifolds and Quantization*. ESI Lectures in Mathematical Physics. European Mathematical Society, (2007). `arXiv:0806.1036 [math.DG]`

[9] Birrell, N.D., Davies, P.C.W.: *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, (1984)

[10] Bunch, T.S.: Adiabatic Regularisation for Scalar Fields with Arbitrary Coupling to the Scalar Curvature. J. Phys. A: Math. Gen. **13**, 1297–1310 (1980)

[11] Christensen, S.M.: Vacuum Expectation Value of the Stress Tensor in an Arbitrary Curved Background: The Covariant Point-Separation Method. Phys. Rev. D **14**, 2490–2501 (1976)

[12] Christensen, S.M.: Regularization, Renormalization, and Covariant Geodesic Point Separation. Phys. Rev. D **17**, 946–963 (1978)

[13] Dappiaggi, C., Fredenhagen, K., Pinamonti, N.: Stable Cosmological Models Driven by a Free Quantum Scalar Field. Phys. Rev. D **77**, 104015 (2008). `arXiv:0801.2850 [gr-qc]`

[14] Décanini, Y., Folacci, A.: Off-Diagonal Coefficients of the DeWitt–Schwinger and Hadamard Representations of the Feynman propagator. Phys. Rev. D **73**, 044027 (2006). `arXiv:gr-qc/0511115`

[15] Dereziński, J., Siemssen, D.: An Evolution Equation Approach to the Klein–Gordon Operator on Curved Spacetime. `arXiv:1709.03911 [math-ph]`

[16] Dimock, J.: Algebras of Local Observables on a Manifold. Commun. Math. Phys. **77**, 219–228 (1980)

[17] Eltzner, B., Gottschalk, H.: Dynamical Backreaction in Robertson–Walker Spacetime. Rev. Math. Phys. **23**, 531–551 (2011). `arXiv:1003.3630 [math-ph]`

[18] Fewster, C.J., Smith, C.J.: Absolute Quantum Energy Inequalities in Curved Spacetime. Ann. Henri Poincaré **9**, 425–455 (2008). `arXiv:gr-qc/0702056`

[19] Fewster, C.J., Verch, R.: Dynamical Locality and Covariance: What Makes a Physical Theory the Same in All Spacetimes? Ann. Henri Poincaré **13**, 1613–1674 (2012). `arXiv:1106.4785 [math-ph]`

[20] Flanagan, É.É., Wald, R.M.: Does Back Reaction Enforce the Averaged Null Energy Condition in Semiclassical Gravity? Phys. Rev. D **54**, 6233–6283 (1996)

[21] Hack, T.P.: *Cosmological Applications of Algebraic Quantum Field Theory in Curved Spacetimes*. No. 6 in SpringerBriefs in Mathematical Physics. Springer, (2016). `arXiv:1506.01869 [gr-qc]`

[22] Hänsel, M.: Qualitative Analysis of Solutions to the Semiclassical Einstein Equation in Homogeneous and Isotropic Spacetimes. Ph.D. thesis, Universität Leipzig, (2018)

[23] Hollands, S., Wald, R.M.: Local Wick Polynomials and Time Ordered Products of Quantum Fields in Curved Spacetime. Commun. Math. Phys. **223**, 289–326 (2001). `arXiv:gr-qc/0103074`

[24] Hollands, S., Wald, R.M.: Existence of Local Covariant Time Ordered Products of Quantum Fields in Curved Spacetime. Commun. Math. Phys. **231**, 309–345 (2002). `arXiv:gr-qc/0111108`

[25] Hollands, S., Wald, R.M.: Conservation of the Stress Tensor in Perturbative Interacting Quantum Field Theory in Curved Spacetimes. Rev. Math. Phys. **17**, 277–311 (2005). `arXiv:gr-qc/0404074`

[26] Hörmander, L.: *Distribution Theory and Fourier Analysis*, *Grundlehren der mathematischen Wissenschaften*, vol. 1. Springer, 2 edn., (1990)

[27] Junker, W., Schrohe, E.: Adiabatic Vacuum States on General Spacetime Manifolds: Definition, Construction, and Physical Properties, Construction, and Physical Properties. Ann. Henri Poincaré **3**, 1113–1181 (2002). `arXiv:math-ph/0109010`

[28] Kato: Quasi-linear equations of evolution, with applications to partial differential equations. In: W.N. Everitt (ed.) *Spectral Theory and Differential Equations*, no. 448 in Lecture Notes in Mathematics. Springer, (1975)

[29] Khavkine, I., Moretti, V.: *Algebraic QFT in Curved Spacetime and Quasifree Hadamard States: An Introduction*, chap. 5, pp. 191–251. Mathematical Physics Studies. Springer, (2015)

[30] Lüders, C., Roberts, J.E.: Local Quasiequivalence and Adiabatic Vacuum States. Commun. Math. Phys. **134**, 29–63 (1990)

[31] Moretti, V.: Comments on the Stress-Energy Tensor Operator in Curved Spacetime. Commun. Math. Phys. **232**, 189–221 (2003). `arXiv:gr-qc/0109048`

[32] Mukhanov, V.F.: *Physical Foundations of Cosmology*. Cambridge University Press, (2005)

[33] Olver, F.W.J., Loizer, D.W., Boisvert, R.F., Clark, C.W. (eds.): *NIST Handbook of Mathematical Functions*. Cambridge University Press, (2010). `http://dlmf.nist.gov/`

[34] Parker, L.E.: Quantized Fields and Particle Creation in Expanding Universes. I. Phys. Rev. **183**, 1057–1068 (1969)

[35] Pazy, A.: *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences. Springer, (1983)

[36] Pinamonti, N.: On the Initial Conditions and Solutions of the Semiclassical Einstein Equations in a Cosmological Scenario. Commun. Math. Phys. **305**, 563–604 (2011). `arXiv:1001.0864 [gr-qc]`

[37] Pinamonti, N., Siemssen, D.: Global Existence of Solutions of the Semiclassical Einstein Equation for Cosmological Spacetimes. Commun. Math. Phys. **334**, 171–191 (2015). `arXiv:1309.6303 [math-ph]`

[38] Radzikowski, M.J.: Micro-Local Approach to the Hadamard Condition in Quantum Field Theory on Curved Space-Time. Commun. Math. Phys. **179**, 529–553 (1996)

[39] Radzikowski, M.J., Verch, R.: A Local-to-Global Singularity Theorem for Quantum Field Theory on Curved Space-Time. Commun. Math. Phys. **180**, 1–22 (1996)

[40] Reed, M., Simon, B.: *Fourier Analysis, Self-Adjointness*, vol. 2. Academic Press, (1975)

[41] Shi, X., Liu, F., Hu, M.: A new asymptotic series for the Gamma function. J. Comput. Appl. Math. **195**, 135–154 (2006)

[42] Siemssen, D.: The Semiclassical Einstein Equation on Cosmological Spacetimes. Ph.D. thesis, Università di Genova, (2015). `arXiv:1503.01826 [math-ph]`

[43] Starobinsky, A.A.: A New Type of Isotropic Cosmological Models without Singularity. Phys. Lett. B **91**, 99–102 (1980)

[44] Suen: Back-Reaction Calculation of Quantum Fields in Curved Spacetime with Ararbitry Curvature Coupling. Phys. Rev. D **35**, 1793–1797 (1987)

[45] Suen: Minkowski Spacetime is Unstable in Semiclassical Gravity. Phys. Rev. Lett. **62**, 2217–2220 (1989)

[46] Suen, W.M.: Stability of the Semiclassical Einstein Equation. Phys. Rev. D **40**, 315–326 (1989)

[47] Wald, R.M.: The Back Reaction Effect in Particle Creation in Curved Spacetime. Commun. Math. Phys. **54**, 1–19 (1977)

[48] Wald, R.M.: Trace Anomaly of a Conformally Invariant Quantum Field in Curved Spacetime. Phys. Rev. D **17**, 1477–1484 (1978)

[49] Wald, R.M.: *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*. University of Chicago Press, (1994)

[50] Weinberg, S.: *Cosmology*. Oxford University Press, (2008)