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Recently a new approach to the modeling of one-way wave propagation in Kerr media was proposed [1]. Within this approach the solution of the nonlinear Helmholtz equation is approximated by a series of solutions of iterative parabolic equations (IPEs). It was also shown that IPEs take the nonparaxial propagation effects into account. In this study we develop an efficient pseudospectral numerical method for solving the system of IPEs. The method is a generalization of an exponential time differencing (ETD) method for the nonlinear Schrödinger equation [2]. The ETD technique is well-suited for the system of IPEs, as it allows to reduce the order of the derivative in the input term.

1 Introduction

A new approach to the derivation of wide-angle parabolic approximations for the solution of the linear Helmholtz equation in the propagation problems of underwater acoustics was proposed in [3]. In a different context similar ideas appeared earlier in [5, 6]. This approach is based on the method of multiple scales, while typical propagation models in underwater acoustics involve parabolic equations obtained by various approximations of the operator square root [4]. An important advantage of the iterative parabolic approximations from [3] is that they can be easily generalized to the case of the nonlinear Helmholtz equation (NHE) [1] describing the propagation of light in a Kerr medium. In this case, the standard nonlinear Schrödinger equation (NSE) is obtained as a zero-order approximation, and higher-order equations of the IPE system provide the corrections to its solution. An an example, an exact solution of the NHE known as the nonparaxial soliton was considered in [1], and it was shown that the series of IPE solutions converges to the latter. A similar result was established in [3] for normal modes (a rigorous proof is given using the Banach fixed-point theorem). In this study we propose a numerical scheme based on the exponential time differencing (ETD) technique [7] for solving the IPE system. It can be considered as a generalization of the ETD2 scheme for the numerical solution of the NSE [7, 2].

2 Iterative parabolic equations in a Kerr medium

Consider the Helmholtz equation in a nonlinear Kerr medium [8]

$$\frac{\partial^2}{\partial z^2}E + \frac{\partial^2}{\partial x^2}E + k_0^2(1+\epsilon|E|^2)E = 0\,, \qquad (1)$$

where $E = E(x,z)$ denotes the electric field.

It is shown in [1] that the solution of the NHE (1) can be approximated by the truncated series of $N+1$ terms

$$E(x,z) \sim E_N(x,z) \equiv \exp(\mathrm{i}k_0 z)\sum_{j=0}^{N} A_j(x,z)\,, \qquad (2)$$

where the amplitudes A_j satisfy the equations

$$\begin{gathered} 2\mathrm{i}k_0 A_{0x} + A_{0zz} + \epsilon k_0^2|A_0|^2A_0 = 0\,, \\ 2\mathrm{i}k_0 A_{1x} + A_{1zz} + \epsilon k_0^2\left(2|A_0|^2A_1 + A_0^2A_1^*\right) + A_{0xx} = 0\,, \\ 2\mathrm{i}k_0 A_{2x} + A_{2zz} + \epsilon k_0^2\left(2|A_0|^2A_2 + A_0^2A_2^*\right) + \\ \epsilon k_0^2\left(2|A_1|^2A_0 + A_1^2A_0^*\right) + A_{1xx} = 0\,. \\ \ldots \end{gathered} \qquad (3)$$

Note that the first equation in (3) is simply the NSE which is often used as an approximation for the NHE [8]. In [1] it is shown that such approximation suffers from the phase error that is mounting with the propagation distance. This error can be corrected by taking higher-order terms of the series

$E_N(x,z)$ in (2) into account. The equation for A_s can be written in the following general form:

$$2\mathrm{i}k_0A_{s,x} + A_{s,zz} + \epsilon k_0^2 L_s + A_{s-1,xx} = 0\,, \qquad (4)$$

where $L_s = \sum\limits_{\substack{l+n+m=s\\ l,m,n<s}} A_l A_n A_M^*$.

Expressing A_s and applying the Fourier transform with respect to x we replace (4) with the following (infinite) coupled system of ordinary differential equations (ODE)

$$\hat{A}_{s,z} = c\hat{A}_s + b\hat{L}_s + a\hat{A}_{s-1,zz}\,, \qquad (5)$$

where ξ is the dual variable for x, $a = \frac{i}{2k_0}$, $b = \frac{\epsilon k_0 i}{2}$, $c = -\xi^2 a$, and $\hat{F}(\xi)$ denotes the Fourier transform of a function $F(x)$.

3 Complex absorbing potential for simulating wave propagation in an unbounded medium

In order to simulate the propagation of nonlinear waves in an unbounded medium numerically, we have to truncate the computational domain by introducing some artificial boundaries $x = x_0$ and $x = x_1$. At such boundaries one can either set up some artificial boundary conditions (BCs) (see, e.g., [9]), or use absorbing layers in order to suppress the reflection of incident waves at $x = x_0$ and $x = x_1$. Although the theory of artificial (or transparent) BCs for the IPEs in the linear case was developed in [10], it is so far unclear if it can be generalized to the case of the system (3). In this study we used more flexible but somewhat less efficient domain truncation technique based on the so-called complex absorbing potential (CAP) [11]. The equation (4) on the interval $[x_0, x_1]$ is replaced by

$$2ik_0A_{s,z} + A_{s,xx} + \epsilon k_0^2 L_s + VA_s + A_{s-1,zz} = 0 \qquad (6)$$

$$V = V(x,z) = -i\sigma(x) \qquad (7)$$

$$\sigma(x) = \delta^{-2}\begin{cases} (x-x_0)^2, & \tilde{x}_0 \leqslant x < x_0\\ 0, & x_0 \leqslant x \leqslant x_1\\ (x-x_1)^2, & x_1 < x \leqslant \tilde{x}_1\end{cases} \qquad (8)$$

$$\delta \in \mathbb{R}$$

on the interval $[\tilde{x}_0, \tilde{x}_1]$. The waves propagating outwards from the domain $x \in [x_0, x_1]$ are absorbed inside the layers $[\tilde{x}_0, x_0]$ and $[x_1, \tilde{x}_1]$, and the solution inside the domain is not corrupted by the waves reflected by the artificial boundaries $x = x_0$ and $x = x_1$.

Introducing the function P_s by the formula $P_s = \epsilon k_0^2 L_s + VA_s$ and applying the Fourier transform, we can recast (7) in a form similar to (5)

$$\hat{A}_{s,z} = c\hat{A}_s + a\hat{P}_s + a\hat{A}_{s-1,zz}\,. \qquad (9)$$

4 ETD2 numerical scheme for the solution of IPEs

In this section we propose a ETD2 z-marching numerical scheme for the coupled system of ODEs (5) and (9). Consider a uniform grid $z_0, z_1, z_2, \ldots$ where $h = z_{i+1} - z_i$. Let us multiply (5) by e^{-cz} and integrate from z_i to z_{i+1}

$$\hat{A}_s(z_{i+1}) = e^{ch}\hat{A}_s(z_i) + be^{ch}\int_0^h e^{-c\zeta}\hat{L}_s(z_i+\zeta)\,d\zeta + ae^{ch}\int_0^h e^{-c\zeta}\hat{A}_{s-1,\zeta\zeta}(z_i+\zeta)\,d\zeta\,. \qquad (10)$$

Integrating the last term of (10) by parts we obtain

$$\hat{A}_s(z_{i+1}) = e^{ch}\hat{A}_s(z_i) + be^{ch}\int_0^h e^{-ch}\hat{L}_s(z_i+\zeta)\,d\zeta + a\Big(\hat{A}_{s-1,z}(z_{i+1}) - e^{ch}\hat{A}_{s-1,z}(z_i) + c\Big(\hat{A}_{s-1}(z_{i+1}) - e^{ch}\hat{A}_{s-1}(z_i) + ce^{ch}\int_0^h e^{-ch}\hat{A}_{s-1}(z_i+\zeta)\,d\zeta\Big)\Big)\,. \qquad (11)$$

The last integral can be discretized using the trapezoidal rule:

$$\int_0^h e^{-c\zeta}\hat{A}_{s-1}(z_i+\zeta)\,d\zeta \approx \frac{h}{2}\left(e^{-ch}\hat{A}_{s-1}(z_{i+1}) + \hat{A}_{s-1}(z_i)\right)\,. \qquad (12)$$

Next, we can express the derivative $\hat{A}_{0,z}$ using the NSE as

$$\hat{A}_{0,z}(z) = c\hat{A}_s(z) + b\hat{A}_0(z)\,|\hat{A}_0(z)|^2\,, \qquad (13)$$

while the derivatives of higher-order terms $\hat{A}_{s-1,z}$, $s-1>0$ can be computed by central differences:

$$\hat{A}_{s-1,z}\left(z_{i+1}\right)=\frac{\hat{A}_{s-1}\left(z_{i+2}\right)-\hat{A}_{s-1}\left(z_{i}\right)}{2h}\,, \quad (14)$$

$$\hat{A}_{s-1,z}\left(z_{i}\right)=\frac{\hat{A}_{s-1}\left(z_{i+1}\right)-\hat{A}_{s-1}\left(z_{i-1}\right)}{2h}\,. \quad (15)$$

The first integral in (11) can be approximated as shown in [7]

$$e^{ch}\int\limits_0^h e^{-c\zeta}\hat{L}_s\left(z_i+\zeta\right)d\zeta\approx \left(e^{ch}\left(1+(ch)^{-1}\right)-(ch)^{-1}-2\right)c^{-1}\hat{L}_s\left(z_i\right)+ \left(-e^{ch}\left(ch\right)^{-1}+(ch)^{-1}+1\right)c^{-1}\hat{L}_s\left(z_{i-1}\right)= \mu_1\hat{L}_s\left(z_i\right)+\mu_2\hat{L}_s\left(z_{i-1}\right)\,. \quad (16)$$

Finally we obtain the following second-order numerical scheme:

$$\hat{A}_s\left(z_{i+1}\right)=e^{ch}\hat{A}_s\left(z_i\right)+ b\left(\mu_1\hat{L}_s\left(z_i\right)+\mu_2\hat{L}_s\left(z_{i-1}\right)\right)+ a\Bigg(\hat{A}_{s-1,z}\left(z_{i+1}\right)-e^{ch}\hat{A}_{s-1,z}\left(z_i\right)+ c\Bigg(\hat{A}_{s-1}\left(z_{i+1}\right)\left(1+\frac{ch}{2}\right)+ e^{ch}\hat{A}_{s-1}\left(z_i\right)\left(\frac{ch}{2}-1\right)\Bigg)\Bigg)\,. \quad (17)$$

Similarly, for the ODE system (9) the numerical scheme can be written as

$$\hat{A}_s\left(z_{i+1}\right)=e^{ch}\hat{A}_s\left(z_i\right)+ a\Bigg(\mu_1\hat{P}_s\left(z_i\right)+\mu_2\hat{P}_s\left(z_{i-1}\right)+ \hat{A}_{s-1,z}\left(z_{i+1}\right)-e^{ch}\hat{A}_{s-1,z}\left(z_i\right)+ c\Bigg(\hat{A}_{s-1}\left(z_{i+1}\right)\left(1+\frac{ch}{2}\right)+ e^{ch}\hat{A}_{s-1}\left(z_i\right)\left(\frac{ch}{2}-1\right)\Bigg)\Bigg)\,. \quad (18)$$

5 A numerical example

In this example we perform a numerical simulation of nonparaxial solitons propagation. The NHE admits an analytical solution $E(x,z)$ that has an envelope function $A(x,z)=E(x,z)\exp(-\mathrm{i}k_0z)$ given by

$$A(x,z)=\frac{\sigma}{k_0}\sqrt{\frac{2}{\epsilon}}\mathrm{sech}\left(\frac{\sigma k_0 x+\sigma v z}{\sqrt{k_0^2+v^2}}\right)\times \exp\left(-\mathrm{i}vqx+\mathrm{i}k_0z\left(q-1\right)\right)\,, \quad (19)$$

where $q=\sqrt{\frac{k_0^2+\sigma^2}{k_0^2+v^2}}$. It contains two independent parameters v and σ, and they can be expressed as

$$\sigma=\sqrt{2}\frac{\eta}{w_0}\,,\quad v=\sqrt{2}\frac{V}{w_0}\,,$$

where V is the transverse velocity of the soliton, η is an amplitude parameter, and w_0 is the width of a nonparaxial soliton [12]. Note that the expression $E(x,y)=A(x,z)\exp(\mathrm{i}kz)$ satisfies the NHE (1) exactly. Our goal is to demonstrate the convergence of the series (2), where the terms A_s are computed using the numerical scheme (18). In this case CAP layers are essential for accurate an simulation of the nonparaxial soliton (19).

In this study we consider only the propagation of the nonparaxial soliton with zero transverse velocity $V=0$. The exact solution (19) of the NHE is used as a reference. Cauchy problems for the equations (4) are solved numerically using the scheme (18) on the domain $\Omega=\{(x,z)|x_0\le x\le x_1, 0\le z\le z_{max}\}$, where $x_0=-4\cdot10^{-5}$ m, $x_1=4\cdot10^{-5}$ m, $z_{max}=4\cdot10^{-3}$ m. The computational domain is expanded by the two CAP layers $[\tilde{x}_0,x_0]$ and $[x_1,\tilde{x}_1]$ as suggested in [11]. We introduce the computational grid with $n_x=512$ points in x and $n_z=8\cdot10^5$ points in z. The periodicity condition is imposed at $x=\tilde{x}_0$ and $x=\tilde{x}_1$.

The initial condition for A_0 at $z=0$ is obtained from the formula (19):

$$A_0|_{z=0}=A(x,0)=\frac{\sigma}{k_0}\sqrt{\frac{2}{\epsilon}}\mathrm{sech}\left(\sigma x\right)\,, \quad (20)$$

and zero Cauchy data $A_s|_{z=0}=0$ is used at $z=0$ for $s>0$.

The computational results are shown in Figs. 1-2. From the figures it is clear that while the zero-order approximation is inherently out of phase with the analytical solution, high-order corrections significantly improve the agreement with the latter. In this example, the third-order approximation is almost identical to the analytical solution.

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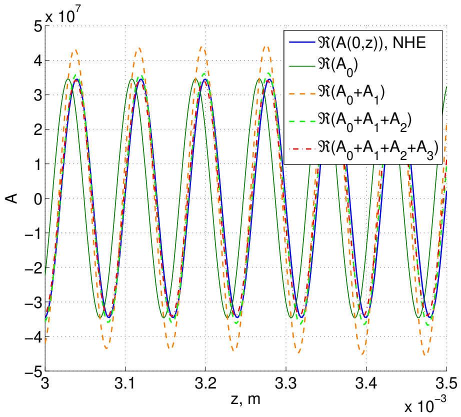

Figure 1: Real part $\Re(A(0,z))$ of the NHE analytical solution (19) as a function of z on the interval $z \in [3\text{ mm}, 3.5\text{ mm}]$ and its iterative parabolic approximations of the order $N = 0, 1, 2, 3$.

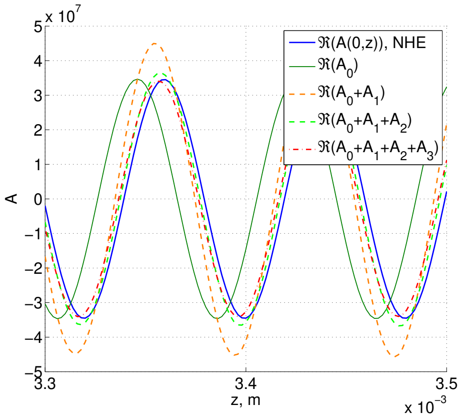

Figure 2: Same as Fig. 1, a comparison for $z \in [3.3\text{ mm}, 3.5\text{ mm}]$ in detail.

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