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Birgit Jacob* Julia T. Kaiser†

Abstract

We consider the well-posedness of a class of hyperbolic partial differential equations on a one dimensional spatial domain. This class includes in particular infinite-dimensional networks of transport, wave and beam equations, or even combinations of these. Equivalent conditions for contraction semigroup generation are derived. In the first part we assume a finite interval and in the second part, we consider partial differential equations on the semi-axis.

Keywords: C_0-semigroups, contraction semigroup, hyperbolic partial differential equations, port-Hamiltonian differential equations, networks of partial differential equations.

Mathematics Subject Classification: 47D06, 35L40, 35L25, 37K99.

1 Introduction

We consider on an interval I a system of partial differential equations of the following form

$$\begin{aligned}\frac{\partial x}{\partial t}(\zeta,t) &= \left(\sum_{k=0}^{N} P_k \frac{\partial^k}{\partial \zeta^k}\right)(\mathcal{H}(\zeta)x(\zeta,t)), \qquad \zeta \in I, t \geq 0, \\ x(\zeta,0) &= x_0(\zeta),\end{aligned} \tag{1}$$

where P_N is an invertible operator on a Hilbert space H and $P_k \in \mathcal{L}(H)$, $k = 0, \cdots, N$, with $P_k^* = (-1)^{k+1} P_k$, $k = 1, \cdots, N$. $\mathcal{H}(\zeta)$ is a positive operator on H for a.e. $\zeta \in I$ satisfying $\mathcal{H}, \mathcal{H}^{-1} \in L^\infty(I; \mathcal{L}(H))$. Thereby, the interval I is either a finite interval or a semi-axis. Without lost of generality we consider the finite interval $[0,1]$ and the semi-axis $[0,\infty)$. This class of partial differential equations covers coupled wave and beam equations and in particular infinite networks of these equations. There has been enormous development in the study of the Cauchy problem (1) in the case of a finite-dimensional Hilbert space H and a finite interval I, see for example [BaCo16, En13,

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JaZw12, LeZwMa05, VaMa02, Vi07] and the references therein. These systems are also known as port-Hamiltonian systems, Hamiltonian partial differential equations or systems of linear conservation laws. In particular, contraction semigroup generation has been studied in [Au16, AuJa14, JaMoZw15, JaZw12, LeZwMa05]. In this paper we aim to generalize these results to the infinite-dimensional situation and to the semi-axis. In order to guarantee unique solutions of equation (1), we have to impose boundary conditions, which will be of the form

$$\hat{W}_B(\Phi(\mathcal{H}x))(\cdot,t) = 0. \tag{2}$$

In the case of the finite interval $I = [0,1]$, we assume $\hat{W}_B \in \mathcal{L}(H^N \times H^N, H^N)$ and that Φ is given by

$$\Phi : \mathcal{W}^{N,2}(I;H) \to H^{2N}, \qquad \Phi(x) := [\Phi_1(x), \Phi_0(x)],$$

where $\Phi_i(x) := \left[x(i), \dots, \frac{d^{N-1}x}{d\zeta^{N-1}}(i)\right]$ for $i \in \{0,1\}$ and $\mathcal{W}^{N,2}(I;H)$ denotes the Sobolev space of order N. If $I = [0,\infty)$, then $\hat{W}_B \in \mathcal{L}(H^N, \tilde{H}^N)$, where $\tilde{H}$ is a subspace of H, and Φ is given by

$$\Phi : \mathcal{W}^{N,2}(I;H) \to H^N, \qquad \Phi(x) := \Phi_0(x).$$

Clearly, whether or not equation (1) possesses unique and non-increasing solutions depend on the boundary conditions, or equivalently on the operator $\hat{W}_B$. The partial differential equation (1) with the boundary conditions (2) can be equivalently written as the abstract Cauchy problem

$$\dot{x}(t) = A\mathcal{H}x(t), \qquad x(0) = x_0,$$

where A is a linear operator on the Hilbert space $X := L^2(I;H)$ given by

$$Ax := \left(\sum_{k=0}^{N} P_k \frac{\partial^k}{\partial \zeta^k}\right)(x), \qquad x \in \mathcal{D}(A), \tag{3}$$

$$\mathcal{D}(A) := \left\{ x \in \mathcal{W}^{N,2}(I;H) \mid \hat{W}_B \Phi(x) = 0 \right\}. \tag{4}$$

We equip X with $\langle \cdot, \cdot \rangle_{L^2}$, the standard scalar product of $L^2(I;H)$. For convenience, we often write $\langle \cdot, \cdot \rangle$ instead.

The aim of the paper is to give equivalent conditions for the fact that $A\mathcal{H}$ generates a contraction semigroup on X. If $I = [0,1]$, then under a weak condition, we show that $A\mathcal{H}$ generates a contraction semigroup if and only if the operator A is dissipative. Moreover, equivalent conditions in terms of the operator $\hat{W}_B$ are presented. We note that the mentioned weak condition is in particular satisfied if the Hilbert space H is finite-dimensional. However, even if H is finite-dimensional, our result contains new equivalent conditions for the contraction semigroup characterization [AuJa14]. For the case $I = [0,\infty)$, the contraction semigroup property has been shown for some specific examples [EnNa99, I.4.16], [MuNoSe16], however, we are not aware of any general result. If $I = [0,\infty)$, $N = 1$ and $H = \mathbb{C}^d$ or $\mathbb{R}^d$, we provide a characterization of the contraction semigroup property of the operator $A\mathcal{H}$. Again $A\mathcal{H}$ generates a contraction semigroup if and only if the operator A is dissipative. The main difference to the case $I = [0,1]$ is that the number of boundary conditions depends on P_1. We conclude the paper with some examples to illustrate our results.

2 Main results

In this section, we formulate the main results of the paper for both cases $I = [0, 1]$ and $I = [0, \infty)$. The proof of all theorems and corollaries are given in Sections 3 and 4. We define

$$Q = (Q_{ij})_{\substack{1\le i\le N\\ 1\le j\le N}} = \begin{cases} (-1)^{i-1} P_{i+j-1} & \text{if } i+j \le N+1 \\ 0 & \text{else.} \end{cases} \tag{5}$$

Clearly, $Q_{ij} \in \mathcal{L}(H)$, i.e. $Q \in \mathcal{L}(H^N)$ and

$$Q = \begin{bmatrix} P_1 & P_2 & P_3 & \cdots & P_{N-1} & P_N \\ -P_2 & -P_3 & -P_4 & \cdots & -P_N & 0 \\ P_3 & P_4 & \iddots & \iddots & 0 & 0 \\ -P_4 & \iddots & \iddots & \iddots & & \vdots \\ \vdots & \iddots & \iddots & & & \vdots \\ (-1)^{N-1} P_N & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

Thus, $Q \in \mathcal{L}(H^N)$ is a selfadjoint block operator matrix and invertible due to the fact that P_N is invertible. Let

$$W_B := \begin{bmatrix} W_1 & W_2 \end{bmatrix} := \hat{W}_B \begin{bmatrix} Q & -Q \\ I & I \end{bmatrix}^{-1} \quad \text{and} \quad \Sigma := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathcal{L}(H^N \times H^N).$$

2.1 Main results for $I = [0, 1]$

In this subsection, we consider the operator $A\mathcal{H}$ on the Hilbert space $X = L^2(0, 1; H)$, where H is a (probably infinite-dimensional) Hilbert space.

Theorem 2.1. *Let A be given by (3)-(4). Further, assume*

$$\operatorname{ran}(W_1 - W_2) \subseteq \operatorname{ran}(W_1 + W_2). \tag{6}$$

Then the following statements are equivalent:

1. *The operator $A\mathcal{H}$ with domain*

   $$\mathcal{D}(A\mathcal{H}) = \mathcal{H}^{-1}\mathcal{D}(A) = \{x \in X \mid \mathcal{H}x \in \mathcal{W}^{N,2}(0, 1; H) \text{ and } \hat{W}_B \Phi(\mathcal{H}x) = 0\}$$

   generates a contraction semigroup on $(X, \langle \cdot, \mathcal{H} \cdot \rangle)$;

2. *A is dissipative, that is, $\operatorname{Re}\langle Ax, x\rangle \le 0$ for every $x \in \mathcal{D}(A)$;*

3. *$\operatorname{Re} P_0 \le 0$, $W_1 + W_2$ is injective and $W_B \Sigma W_B^* \ge 0$;*

4. *$\operatorname{Re} P_0 \le 0$, $W_1 + W_2$ is injective and there exists $V \in \mathcal{L}(H)$ with $\|V\| \le 1$ such that $W_B = \frac{1}{2}(W_1 + W_2)\begin{bmatrix} I + V & I - V \end{bmatrix}$;*

5. *$\operatorname{Re} P_0 \le 0$ and $u^* Q u - y^* Q y \le 0$ for every $\begin{bmatrix} u \\ y \end{bmatrix} \in \ker \hat{W}_B$.*

Remark 2.2. 1. Condition (6) is in general not satisfied: Let $H = \ell^2$ and $W_B = \begin{bmatrix} W_1 & W_2 \end{bmatrix} \in \mathcal{L}((\ell^2)^2, \ell^2)$ with $W_1 e_i := e_{i+1} + e_i$ and $W_2 e_i := e_{i+1} - e_i$, where $\{e_i\}_{i\in\mathbb{N}}$ is a orthonormal basis of ℓ^2. Then $\operatorname{ran}(W_1 - W_2) = \ell^2$ whereas $e_1 \notin \operatorname{ran}(W_1 + W_2)$.

2. We point out that the implications $1 \Rightarrow 2$, $4 \Rightarrow 3$, and the equivalence $2 \Leftrightarrow 5$ hold even without the additional condition (6). Moreover, condition (6) is not needed for the fact that 2 implies $W_1 + W_2$ is injective.

3. We note that W_B is not uniquely determined, only the kernel of W_B is. However, if W_B does not satisfy condition (6), then in general it is not possible to chose another operator instead of W_B with the same kernel such that condition (6) holds.

4. If H is finite-dimensional, then $A\mathcal{H}$ has a compact resolvent, see Theorem 2.3 in [AuJa14]. However, in general, $A\mathcal{H}$ has not a compact resolvent. Take for example $N = 1$, $P_1 = 1$, $P_0 = 0$, $H = \ell^2$, $\hat{W}_B = [I \ \ S]$ and $\mathcal{H}(\zeta) = 1$. Here S denotes the left shift on H, that is, $Se_j = e_{j+1}$. Thus, A generates the left shift semigroup on $X = L^2(0, 1; \ell^2)$, which is isometric isomorph to the left shift on $X = L^2(0, \infty)$. However, 0 is a spectral point of A, but not in the point spectrum.

As a corollary of Theorem 2.1 we receive the well-known contraction semigroup characterization for the case of a finite-dimensional Hilbert space H, see [AuJa14]. However, we remark that Conditions 3 and 4 are even new in the finite-dimensional situation.

Theorem 2.3. *Let A be given by* (3)-(4) *and assume that H is finite-dimensional. Then the following statements are equivalent:*

1. *$A\mathcal{H}$ with domain $\mathcal{D}(A\mathcal{H}) := \{x \in X \mid \mathcal{H}x \in \mathcal{D}(A)\} = \mathcal{H}^{-1}\mathcal{D}(A)$ generates a contraction semigroup on $(X, \langle \cdot, \mathcal{H}\cdot\rangle)$;*

2. *$\operatorname{Re}\langle Ax, x\rangle \le 0$ for every $x \in \mathcal{D}(A)$;*

3. *$\operatorname{Re} P_0 \le 0$, $W_1 + W_2$ is injective and $W_B \Sigma W_B^* \ge 0$;*

3'. *$\operatorname{Re} P_0 \le 0$, W_B surjective and $W_B \Sigma W_B^* \ge 0$;*

4. *$\operatorname{Re} P_0 \le 0$, $W_1 + W_2$ is injective and there exists $V \in \mathcal{L}(H)$ with $\|V\| \le 1$ such that $W_B = \frac{1}{2}(W_1 + W_2)\begin{bmatrix} I + V & I - V \end{bmatrix}$;*

4'. *$\operatorname{Re} P_0 \le 0$, W_B surjective and there exists $V \in \mathcal{L}(H)$ with $\|V\| \le 1$ such that $W_B = \frac{1}{2}(W_1 + W_2)\begin{bmatrix} I + V & I - V \end{bmatrix}$;*

5. *$\operatorname{Re} P_0 \le 0$ and $u^*Qu - y^*Qy \le 0$ for every $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \ker \hat{W}_B$.*

Remark 2.4. If H is infinite-dimensional, then in general Conditions 3' and 4' of the previous theorem are not equivalent to the fact that $A\mathcal{H}$ generates a contraction semigroup. Let $H = \ell^2(\mathbb{N})$, $N \in \mathbb{N}$, and P_i and $\mathcal{H}$ are operators satisfying the general assumptions. First, we consider $W_B = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ with $W_1 := \frac{3}{2}R$ and $W_2 := \frac{1}{2}R$, where R denotes the right shift on $\ell^2(\mathbb{N})$. Then $\operatorname{ran}(W_1 - W_2) = \operatorname{ran}(W_1 + W_2)$, $W_1 + W_2$ is injective and $W_B \Sigma W_B^* \ge 0$ but W_B is not surjective. Thus, $A\mathcal{H}$ generates a contraction semigroup on $(X, \langle \cdot, \mathcal{H}\cdot\rangle)$, but Conditions 3' and 4' are not satisfied. Conversely, for the choice $W_B = \begin{bmatrix} I - L & -I - L \end{bmatrix}$, where L denotes the left shift on $\ell^2(\mathbb{N})$. Surjectivity of W_B

holds, $\operatorname{ran}(W_1 - W_2) \subseteq \operatorname{ran}(W_1 + W_2)$ and $W_B \Sigma W_B^* \geq 0$, but $W_1 + W_2$ is not injective. Thus, for these boundary conditions Conditions 3' and 4' are satisfied, but $A\mathcal{H}$ does not generate a contraction semigroup on $(X, \langle \cdot, \mathcal{H} \cdot \rangle)$.

Next, we characterize the property of unitary group generation of $A\mathcal{H}$.

Theorem 2.5. *Let A be given by (3)-(4). Further assume*

$$\operatorname{ran}(W_1 - W_2) = \operatorname{ran}(W_1 + W_2). \tag{7}$$

Then the following statements are equivalent:

1. *$A\mathcal{H}$ with domain $\mathcal{D}(A\mathcal{H}) := \{x \in X \mid \mathcal{H}x \in \mathcal{D}(A)\} = \mathcal{H}^{-1}\mathcal{D}(A)$ generates a unitary C_0-group on $(X, \langle \cdot, \mathcal{H} \cdot \rangle)$;*
2. *$\operatorname{Re} \langle Ax, x \rangle = 0$ for every $x \in \mathcal{D}(A)$;*
3. *$\operatorname{Re} P_0 = 0$, $W_1 + W_2$ and $-W_1 + W_2$ are injective and $W_B \Sigma W_B^* = 0$;*
4. *$\operatorname{Re} P_0 = 0$, $W_1 + W_2$ and $-W_1 + W_2$ are injective and there exists $V \in \mathcal{L}(H)$ with $\|V\| = 1$ such that $W_B = \frac{1}{2}(W_1 + W_2) \begin{bmatrix} I + V & I - V \end{bmatrix}$;*
5. *$\operatorname{Re} P_0 = 0$ and $u^* Q u - y^* Q y = 0$ for every $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in \ker \hat{W}_B$.*

Corollary 2.6. *Let A be given by (3)-(4) and assume that H is finite-dimensional. Then the following statements are equivalent:*

1. *$A\mathcal{H}$ with domain $\mathcal{D}(A\mathcal{H}) := \{x \in X \mid \mathcal{H}x \in \mathcal{D}(A)\} = \mathcal{H}^{-1}\mathcal{D}(A)$ generates a unitary C_0-group on $(X, \langle \cdot, \mathcal{H} \cdot \rangle)$;*
2. *$\operatorname{Re} \langle Ax, x \rangle = 0$ for every $x \in \mathcal{D}(A)$;*
3. *$\operatorname{Re} P_0 = 0$, $W_1 + W_2$ and $-W_1 + W_2$ is injective and $W_B \Sigma W_B^* = 0$;*

3'. *$\operatorname{Re} P_0 = 0$, W_B surjective and $W_B \Sigma W_B^* = 0$;*

4. *$\operatorname{Re} P_0 = 0$, $W_1 + W_2$ and $-W_1 + W_2$ is injective and there exists $V \in \mathcal{L}(H)$ with $\|V\| = 1$ such that $W_B = \frac{1}{2}(W_1 + W_2) \begin{bmatrix} I + V & I - V \end{bmatrix}$;*

4'. *$\operatorname{Re} P_0 = 0$, W_B surjective and there exists $V \in \mathcal{L}(H)$ with $\|V\| = 1$ such that $W_B = \frac{1}{2}(W_1 + W_2) \begin{bmatrix} I + V & I - V \end{bmatrix}$;*

5. *$\operatorname{Re} P_0 = 0$ and $u^* Q u - y^* Q y = 0$ for every $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \ker \hat{W}_B$.*

2.2 Main results for $I = [0, \infty)$

In this subsection, we choose $I = [0, \infty)$, $N = 1$ and $H = \mathbb{F}^d$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, that is, we consider the operator $A\mathcal{H}$,

$$A\mathcal{H}x = P_0 \mathcal{H}x + P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}x) \text{ with} \tag{8}$$

$$\mathcal{D}(A\mathcal{H}) = \left\{ x \in L^2(0, \infty; \mathbb{F}^d) \mid \mathcal{H}x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{F}^d), \hat{W}_B(\mathcal{H}x(0)) = 0 \right\} \tag{9}$$

on the space $X = L^2(0,\infty;\mathbb{F}^d)$. Here P_1 is an invertible Hermitian $d \times d$-matrix, $P_0 \in \mathbb{F}^{d\times d}$, $\hat{W}_B \in \mathbb{F}^{k\times d}$ with $k \in \{0,1,\cdots,d\}$ and $\mathcal{H}(\zeta) \in \mathbb{F}^{d\times d}$ is positive definite for a.e. $\zeta \in [0,\infty)$ satisfying $\mathcal{H}, \mathcal{H}^{-1} \in L^\infty(0,\infty;\mathbb{F}^{d\times d})$. Since P_1 is an invertible, Hermitian matrix, its eigenvalues are real and non zero.

We denote by n_1 the number of positive and by $n_2 = d - n_1$ the number of negative eigenvalues of P_1 and write

$$P_1 = S^{-1}\Delta S = S^{-1}\begin{bmatrix}\Lambda & 0\\ 0 & \Theta\end{bmatrix}S, \tag{10}$$

with a unitary matrix $S \in \mathbb{F}^{d\times d}$, a positive definite, diagonal matrix $\Lambda \in \mathbb{R}^{n_1\times n_1}$ and a negative definite, diagonal matrix $\Theta \in \mathbb{R}^{n_2\times n_2}$. We define $\Delta = \left[\begin{smallmatrix}\Lambda & 0\\ 0 & \Theta\end{smallmatrix}\right]$.

Theorem 2.7. *Assume $A\mathcal{H}$ is given by (8)-(9), $\hat{W}_B \in \mathbb{F}^{k\times d}$ with $k \le n_2$ has full row rank. Then the following statements are equivalent:*

1. *$A\mathcal{H}$ generates a contraction semigroup on $(X, \langle\cdot,\mathcal{H};\cdot\rangle)$;*
2. *$\operatorname{Re}\langle Ax, x\rangle \le 0$ for every $x \in \mathcal{D}(A)$;*
3. *$\operatorname{Re} P_0 \le 0$, $k \le n_2$ and $y^*P_1y \ge 0$ for every $y \in \ker \hat{W}_B$;*
4. *$\operatorname{Re} P_0 \le 0$, $k = n_2$ and $\hat{W}_B = B\begin{bmatrix}U & I\end{bmatrix}S$, with $B \in \mathbb{F}^{n_2\times n_2}$ invertible, $U \in \mathbb{F}^{n_2\times n_1}$, $\Lambda + U^*\Theta U \ge 0$.*

Further, we are able to characterize the property of unitary group generation in the case $I = [0,\infty)$.

Theorem 2.8. *Let $A\mathcal{H}$ be given by (8)- (9), $\hat{W}_B \in \mathbb{F}^{k\times d}$ with $k \le \min\{n_1, n_2\}$ has full row rank. Then the following statements are equivalent:*

1. *$A\mathcal{H}$ generates a unitary C_0-group on $(X, \langle\cdot,\mathcal{H}\cdot\rangle)$;*
2. *$\operatorname{Re}\langle Ax, x\rangle = 0$ for every $x \in \mathcal{D}(A)$;*
3. *$\operatorname{Re} P_0 = 0$ and $y^*P_1y = 0$ for every $y \in \ker \hat{W}_B$;*
4. *$k = n_1 = n_2$, $\operatorname{Re} P_0 = 0$ and $\hat{W}_B = \begin{bmatrix}U_1 & U_2\end{bmatrix}S$; where $U_1, U_2 \in \mathbb{F}^{n_1\times n_1}$ invertible with $\Lambda + U_1^*U_2^{-*}\Theta U_2^{-1}U_1 = 0$.*

3 Proofs of the main results: $I = [0,1]$

Throughout this section we will assume that $I = [0,1]$, $X = L^2(0,1;H)$, A is given by (3)-(4), and W_B and Σ are defined as in Section 2. In order to prove the main statements it is convenient to introduce the following linear combinations of the boundary values [LeZwMa05].

Definition 3.1. *For $x \in \mathcal{H}^{-1}\mathcal{W}^{N,2}(0,1;H)$ we define so called boundary port variables namely* boundary flow *and* boundary effort *by*

$$\begin{bmatrix}f_{\partial,\mathcal{H}x}\\ e_{\partial,\mathcal{H}x}\end{bmatrix} := \frac{1}{\sqrt{2}}\begin{bmatrix}Q & -Q\\ I & I\end{bmatrix}\Phi(\mathcal{H}x) = R_{ext}\Phi(\mathcal{H}x), \tag{11}$$

where Q is defined by (5) *and $R_{ext} := \frac{1}{\sqrt{2}}\begin{bmatrix}Q & -Q\\ I & I\end{bmatrix} \in \mathcal{L}(H^{2N})$.*

Remark 3.2. Thanks to the invertibility of Q, the operator R_{ext} is invertible. Thus, we can use the boundary port variables to reformulate the domain of the operator $A\mathcal{H}$:

$$\begin{aligned}\mathcal{D}(A\mathcal{H}) &= \left\{x \in X \mid \mathcal{H}x \in \mathcal{W}^{N,2}(0,1;H) \text{ and } \hat{W}_B\Phi(\mathcal{H}x) = 0\right\} \\ &= \left\{x \in X \mid \mathcal{H}x \in \mathcal{W}^{N,2}(0,1;H) \text{ and } W_B \begin{bmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x}\end{bmatrix} = 0\right\},\end{aligned}$$

where $W_B = \hat{W}_B R_{ext}^{-1}$.

Next, we determine the adjoint operator of A. We define $\tilde{Q} = -Q$ and

$$\begin{bmatrix} \tilde{f}_{\partial,\mathcal{H}x} \\ \tilde{e}_{\partial,\mathcal{H}x}\end{bmatrix} = \tilde{R}_{ext}\Phi(\mathcal{H}x) \text{ with } \tilde{R}_{ext} = \frac{1}{\sqrt{2}}\begin{bmatrix} \tilde{Q} & -\tilde{Q} \\ I & I\end{bmatrix}.$$

Lemma 3.3. *The adjoint operator of the operator A defined in (3) with domain (4) and a boundary operator W_B of the form $W_B = S\begin{bmatrix} I+V & I-V\end{bmatrix}$ where $S, V \in \mathcal{L}(H^N)$ and S is left invertible, is given by*

$$A^*y = P_0^*y - \sum_{k=1}^{N} P_k \frac{d^k}{d\zeta^k}y, \quad y \in \mathcal{D}(A^*), \tag{12}$$

$$\mathcal{D}(A^*) = \left\{y \in \mathcal{W}^{N,2}(0,1;H) : S\begin{bmatrix} I+V^* & I-V^*\end{bmatrix}\begin{bmatrix} \tilde{f}_{\partial,y} \\ \tilde{e}_{\partial,y}\end{bmatrix} = 0\right\}. \tag{13}$$

Proof. The statement can be proved in a similar manner as Proposition 3.4.3 in [Au16], where the statement is shown for finite-dimensional Hilbert spaces H. □

Definition 3.4. *We define the operators $A_0 : \mathcal{D}(A_0) \subset X \to X$ and $(A^*)_0 : \mathcal{D}((A^*)_0) \subset X \to X$ by*

$$\begin{aligned}A_0x &:= \left(\sum_{k=0}^{N} P_k \frac{\partial^k}{\partial\zeta^k}\right)(x), \quad (A^*)_0y := P_0^*y - \sum_{k=1}^{N} P_k\frac{d^k}{d\zeta^k}y \\ \mathcal{D}(A_0) &= \mathcal{D}(A_0^*) = \mathcal{W}^{N,2}(0,1;H).\end{aligned}$$

Remark, that A_0 and $(A^*)_0$ are extensions of A and A^*, respectively. Integration by parts yields the following lemma.

Lemma 3.5. *We have for $x \in \mathcal{W}^{N,2}(0,1;H)$*

$$\begin{aligned}\operatorname{Re}\langle A_0x, x\rangle &= \operatorname{Re}\langle f_{\partial,x}, e_{\partial,x}\rangle_{H^N} + \operatorname{Re}\langle P_0x, x\rangle \\ &= \Phi_1(x)^*Q\Phi_1(x) - \Phi_0(x)^*Q\Phi_0(x) + \operatorname{Re}\langle P_0x, x\rangle, \\ \operatorname{Re}\langle (A^*)_0x, x\rangle &= \operatorname{Re}\langle \tilde{f}_{\partial,x}, \tilde{e}_{\partial,x}\rangle_{H^N} + \operatorname{Re}\langle P_0x, x\rangle \\ &= \Phi_1(x)^*\tilde{Q}\Phi_1(x) - \Phi_0(x)^*\tilde{Q}\Phi_0(x) + \operatorname{Re}\langle P_0x, x\rangle.\end{aligned}$$

Furthermore, we need some technical results. First, we give a generalization of the technical Lemma 7.3.2 in [JaZw12] for $N \geq 1$ and arbitrary Banach spaces Z.

Lemma 3.6. *Let Z be a Banach space and $V \in \mathcal{L}(Z)$. Then it yields*

$$\ker \begin{bmatrix} I+V & I-V \end{bmatrix} = \operatorname{ran} \begin{bmatrix} I-V \\ -I-V \end{bmatrix},$$

where $\begin{bmatrix} I+V & I-V \end{bmatrix} \in \mathcal{L}(Z \times Z, Z)$ and $\left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right] \in \mathcal{L}(Z, Z \times Z)$.

Proof. Assume $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \ker \begin{bmatrix} I+V & I-V \end{bmatrix}$. Thus, it yields

$$x + Vx + y - Vy = 0.$$

For $l := \frac{1}{2}(x-y) \in Z$ we get

$$(I-V)l = \frac{1}{2}(x-y) - \frac{1}{2}V(x-y) = x \text{ and } (-I-V)l = -\frac{1}{2}(x-y) - \frac{1}{2}V(x-y) = y.$$

Thus, it follows $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \operatorname{ran} \left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right]$. Conversely, assume $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \operatorname{ran} \left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right]$. Then, we have

$$\begin{bmatrix} I+V & I-V \end{bmatrix} \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \begin{bmatrix} I+V & I-V \end{bmatrix} \begin{bmatrix} I-V \\ -I-V \end{bmatrix} l = 0$$

for some $l \in Z$ and the lemma is proved. □

Lemma 3.7. *[KuZw15, Lemma 2.4] Let $W = \begin{bmatrix} W_1 & W_2 \end{bmatrix} \in \mathcal{L}(H^{2N}, H^N)$ such that $W_1 + W_2$ is injective and*

$$\operatorname{ran}(W_1 - W_2) \subseteq \operatorname{ran}(W_1 + W_2).$$

Then there exist an unique operator $V \in \mathcal{L}(H^N)$ such that

$$W = \begin{bmatrix} W_1 & W_2 \end{bmatrix} = \frac{1}{2}(W_1 + W_2) \begin{bmatrix} I+V & I-V \end{bmatrix}. \tag{14}$$

Moreover,

$$\ker \begin{bmatrix} W_1 & W_2 \end{bmatrix} = \ker \begin{bmatrix} I+V & I-V \end{bmatrix},$$

and

$$\begin{bmatrix} W_1 & W_2 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} W_1 & W_2 \end{bmatrix}^* \geq 0 \Leftrightarrow VV^* \leq I.$$

Lemma 3.8. *Let A_0 be defined as in Definition 3.4. For an arbitrary element $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in H^N \times H^N$ exist a function $x \in \mathcal{D}(A_0)$ such that $\Phi(x) = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$.*

Proof. We give a constructive proof: Consider $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in H^N \times H^N$ where

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix},$$

with entries $u_1, \ldots, u_N, v_1, \ldots, v_N \in H$. To construct a proper function $\Phi(x)$, we define two polynomials $P_u(\zeta)$ and $P_v(\zeta)$ by

$$P_u(\zeta) := \sum_{i=0}^{N} \frac{u_{i+1}}{i!}(\zeta - 1)^i \text{ and } P_v(\zeta) := \sum_{i=0}^{N} \frac{v_{i+1}}{i!}\zeta^i.$$

Furthermore, we define the functions $\varphi_0 \in \mathcal{C}^\infty[0,1]$ and $\varphi_1 \in \mathcal{C}^\infty[0,1]$ such that $\varphi_0|_{[0,\varepsilon]} = 0$ and $\varphi_0|_{[1-\varepsilon,1]} = 1$ and analogously $\varphi_1|_{[0,\varepsilon]} = 1$ and $\varphi_1|_{[1-\varepsilon,1]} = 0$ hold. Thus, for

$$x := (\varphi_0 \cdot P_u + \varphi_1 \cdot P_v) I_{H^N} \in \mathcal{C}^\infty([0,1]; H^N) \subset \mathcal{D}(A_0)$$

we get $\Phi(x) = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$. □

Lemma 3.9. *Let A be defined by (3)-(4). Then A is dissipative if and only if $A - P_0$ is dissipative and it holds* $\operatorname{Re} P_0 \leq 0$.

Proof. "'⇒"': Let A be dissipative. Hence, the operator $A - P_0$ is dissipative if $\operatorname{Re} P_0 \leq 0$ holds. We will prove $\operatorname{Re}\langle P_0 z, z\rangle \leq 0$ for all $z \in H$: Let $z \in H$ and $\Psi(\zeta) \in \mathcal{C}_c^\infty(0,1)$ with $\zeta \in [0,1]$ an arbitrary, scalar-valued function with $\Psi \not\equiv 0$. We define

$$x := \Psi(\zeta) z \in \mathcal{C}_c^\infty(0,1; H) \subseteq \mathcal{D}(A)$$

and it yields, since the derivation equals zero at the boundary,

$$\begin{aligned} 0 \geq \operatorname{Re}\langle Ax, x\rangle_{L^2} &= \operatorname{Re}\langle P_0 x, x\rangle_{L^2} = \operatorname{Re}\langle P_0 \Psi z, \Psi z\rangle_{L^2} \\ &= \operatorname{Re} \int_0^1 |\Psi(\zeta)|^2 \langle P_0 z, z\rangle_H d\zeta \\ &= \|\Psi\|_{L^2}^2 \operatorname{Re}\langle P_0 z, z\rangle_H . \end{aligned}$$

"'⇐"': We assume $\operatorname{Re} P_0 \leq 0$ and $\operatorname{Re}\langle (A - P_0)x, x\rangle_{L^2} \leq 0$ for all $x \in \mathcal{D}(A)$. Thus, we get for $x \in \mathcal{D}(A)$

$$\operatorname{Re}\langle Ax, x\rangle = \operatorname{Re}\langle (A - P_0)x, x\rangle_{L^2} + \operatorname{Re}\langle P_0 x, x\rangle_{L^2} \leq 0 \, . \qquad \square$$

We are now in the position to prove the main results for $I = [0,1]$.

Proof of Theorem 2.1. Without loss of generality we may assume $\mathcal{H} = I$, see [JaZw12, Lemma 7.2.3]. The implication $1 \Rightarrow 2$ follows by the Lumer-Phillips Theorem, c.f. [EnNa06, Theorem II.3.15], and the equivalence $3 \Leftrightarrow 4$ has been shown in Lemma 3.7.

Next, we prove the equivalence $2 \Leftrightarrow 5$: Lemma 3.5 implies for $x \in \mathcal{D}(A)$

$$\operatorname{Re}\langle Ax, x\rangle = \Phi_1(x)^* Q \Phi_1(x) - \Phi_0(x)^* Q \Phi_0(x) + \operatorname{Re}\langle P_0 x, x\rangle.$$

Note that $x \in \mathcal{W}^{N,2}(0,1;H)$ satisfies $x \in \mathcal{D}(A)$ if and only if $\left[\begin{smallmatrix} \Phi_1(x) \\ \Phi_0(x) \end{smallmatrix}\right] \in \ker \hat{W}_B$. This proves the implication $5 \Rightarrow 2$. We now assume that 2 holds. Then Lemma 3.9 shows that $\operatorname{Re} P_0 \leq 0$ and that $A - P_0$ is dissipative, that is,

$$\Phi_1(x)^* Q \Phi_1(x) - \Phi_0(x)^* Q \Phi_0(x) \leq 0$$

for every $x \in \mathcal{W}^{N,2}(0,1;H)$ satisfying $\left[\begin{smallmatrix} \Phi_1(x) \\ \Phi_0(x) \end{smallmatrix}\right] \in \ker \hat{W}_B$. Further, by Lemma 3.8, for an arbitrary element $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in \ker \hat{W}_B$ there exists a function $x \in \mathcal{D}(A)$ such that $\left[\begin{smallmatrix} \Phi_1(x) \\ \Phi_0(x) \end{smallmatrix}\right] = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$. This proves 5.

Next, we prove the implication $2 \Rightarrow 4$: Lemma 3.9 shows that $\operatorname{Re} P_0 \leq 0$ and that $A - P_0$ is dissipative, that is, using Lemma 3.5

$$\operatorname{Re}\langle f_{\partial,x}, e_{\partial,x}\rangle_{H^N} \leq 0, \qquad x \in \mathcal{D}(A). \tag{15}$$

For an arbitrary element $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] \in \ker W_B \subset H^N \times H^N$ a function $x \in \mathcal{D}(A)$ exists due to Lemma 3.8 such that $R_{ext}\Phi(x) = \left[\begin{smallmatrix} f_{\partial,x} \\ e_{\partial,x} \end{smallmatrix}\right] = \left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right]$. With equation (15) we get $e^*f + f^*e \leq 0$ for all $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] \in \ker W_B$, where $W_B := \begin{bmatrix} W_1 & W_2 \end{bmatrix}$. For $y \in \ker(W_1+W_2)$ we have $W_B \left[\begin{smallmatrix} y \\ y \end{smallmatrix}\right] = 0$ and thus $y^*y + yy^* \leq 0$. Since the norm of an element is non negative, it follows $y = 0$ and therefore $\ker(W_1 + W_2) = \{0\}$, which shows the injectivity of $W_1 + W_2$. Due to this fact, by Lemma 3.7 there exists an operator V satisfying (14). It remains to show that $\|V\| \leq 1$. Let $l \in H^N$ be arbitrarily. By Lemma 3.6 we obtain $\left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right] l \in \ker W_B$.

From Lemma 3.8 we conclude that a function $x \in \mathcal{D}(A_0)$ exists, such that $R_{ext}\Phi(x) = \left[\begin{smallmatrix} f_{\partial,x} \\ e_{\partial,x} \end{smallmatrix}\right] = \left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right] l$. Therefore, $\left[\begin{smallmatrix} f_{\partial,x} \\ e_{\partial,x} \end{smallmatrix}\right] \in \ker W_B$ and even $x \in \mathcal{D}(A)$. In conclusion, we obtain with (15)

$$\begin{aligned} 2\mathrm{Re}\,\langle f_{\partial,x}, e_{\partial,x}\rangle_{H^N} &= \langle f_{\partial,x}, e_{\partial,x}\rangle_{H^N} + \langle e_{\partial,x}, f_{\partial,x}\rangle_{H^N} \\ &= \langle (I-V)l, (-I-V)l\rangle_{H^N} + \langle (-I-V)l, (I-V)l\rangle_{H^N} \\ &= 2\langle l, (-I+V^*V)l\rangle_{H^N} \leq 0 \end{aligned} \tag{16}$$

and therefore $\|V\| \leq 1$.

Finally, we show the implication $4 \Rightarrow 1$: A is a closed operator, see [Au16, Lemma 3.2.2]. To prove that A generates a contraction semigroup, it is sufficient to verify that A and A^* are dissipative; c.f. [JaZw12, Theorem 6.1.8]. Let $x \in \mathcal{D}(A)$. Then, we have $\left[\begin{smallmatrix} f_{\partial,x} \\ e_{\partial,x} \end{smallmatrix}\right] \in \ker W_B$ and from Lemma 3.6 it follows the existence of $l \in H^N$ such that $\left[\begin{smallmatrix} f_{\partial,x} \\ e_{\partial,x} \end{smallmatrix}\right] = \left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right] l$. Using Lemma 3.5 and Lemma 3.7, we obtain

$$\begin{aligned} 2\mathrm{Re}\,\langle Ax, x\rangle_{L^2} &= 2\mathrm{Re}\,\langle f_{\partial,x}, e_{\partial,x}\rangle_{H^N} + 2\langle P_0 x, x\rangle \\ &\leq 2\langle l, (-I+V^*V)l\rangle_{H^N} \leq 0. \end{aligned}$$

Now we consider the adjoint operator A^*: Let $y \in \mathcal{D}(A^*)$. Due to Lemma 3.3, it yields $\left[\begin{smallmatrix} \tilde{f}_{\partial,y} \\ \tilde{e}_{\partial,y} \end{smallmatrix}\right] \in \ker S \begin{bmatrix} I+V^* & I-V^* \end{bmatrix}$. We apply Lemma 3.6 and Lemma 3.7 to the operator V^* and obtain the existence of $m \in H^N$ such that $\left[\begin{smallmatrix} \tilde{f}_{\partial,x} \\ \tilde{e}_{\partial,x} \end{smallmatrix}\right] = \left[\begin{smallmatrix} I-V^* \\ -I-V^* \end{smallmatrix}\right] m$. Using again Lemma 3.5 we get

$$2\mathrm{Re}\,\langle A^*y, y\rangle_{L^2} \leq 2\langle m, (-I+VV^*)m\rangle_{H^N} \leq 0, \tag{17}$$

which concludes the proof. □

Proof of Theorem 2.3. If $\dim H < \infty$, then $W_1 + W_2$ injective implies the surjectivity of $W_1 + W_2$ and hence condition (6). We want to apply Theorem 2.1 for the proof of Theorem 2.3. Therefore, we have to check condition (6). Due to Remark 2.2.2 and the fact that injectivity of $W_1 + W_2$ implies condition (6), Part 1, 2, 3, 4 and 5 of Theorem 2.3 are equivalent. The implications $3 \Rightarrow 3'$ and $4 \Rightarrow 4'$ follows, since we have $W_1 + W_2$ injective, and thus, $W_1 + W_2$ is also surjective. Clearly, it follows W_B surjective. It remains to show that $3' \Rightarrow 3$ and $4' \Rightarrow 3'$. A straightforward calculation shows the implication $4' \Rightarrow 3'$. In order to show $3' \Rightarrow 3$ we prove that in the finite-dimensional setting W_B surjective and $W_B\Sigma W_B \geq 0$ implies the injectivity of $W_1 + W_2$: It yields $W_B = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$. Thus,

$$W_B\Sigma W_B \geq 0 \Leftrightarrow W_2W_1^* + W_1W_2^* \geq 0,$$

which shows

$$W_1W_1^*+W_2W_1^*+W_2W_2^*+W_1W_2^* = (W_1+W_2)(W_1+W_2)^* \geq (W_1-W_2)(W_1-W_2)^* \geq 0.$$

Let x be in $\ker(W_1+W_2)^*$. Then it yields $x \in \ker(W_1-W_2)(W_1-W_2)^*$. With

$$\begin{aligned}\|(W_1-W_2)^*x\|^2 &= \langle (W_1-W_2)^*x, (W_1-W_2)^*x\rangle \\ &= \langle x, (W_1-W_2)(W_1-W_2)^*x\rangle = \langle x, 0\rangle = 0\end{aligned}$$

we get $x \in \ker(W_1-W_2)^*$ and thus, $x \in \ker W_1^* \cap W_2^*$. Since W_B is surjective, W_B^* injective and thus it follows $x=0$. This implies W_1+W_2 is injective. □

Proof of Theorem 2.5. Without loss of generality we consider again just the case $\mathcal{H}=I$. In the following proof we will apply often Theorem 2.1 to the operators A and $-A$. So, first of all, we have to verify, that also the boundary condition operator $\bar{W}_B$ of $-A$ satisfies the condition (6).

We define analogously to (11) the boundary flow and the boundary effort for $-A$:

$$\begin{bmatrix}\bar{f}_{\partial,x}\\ \bar{e}_{\partial,x}\end{bmatrix} := \frac{1}{\sqrt{2}}\begin{bmatrix}-Q & Q\\ I & I\end{bmatrix}\Phi(\mathcal{H}x). \tag{18}$$

Therefore, it yields $\bar{f}_{\partial,x} = -f_{\partial,x}$ and $\bar{e}_{\partial,x} = e_{\partial,x}$. Due to $\mathcal{D}(A)=\mathcal{D}(-A)$, we get

$$\begin{aligned}\mathcal{D}(A) &= \left\{x \in \mathcal{W}^{N,2}(0,1;H) | W_B\begin{bmatrix}f_{\partial,x}\\ e_{\partial,x}\end{bmatrix} = 0\right\}\\ = \mathcal{D}(-A) &= \left\{x \in \mathcal{W}^{N,2}(0,1;H) | \bar{W}_B\begin{bmatrix}\bar{f}_{\partial,x}\\ \bar{e}_{\partial,x}\end{bmatrix} = 0\right\}\\ &= \left\{x \in \mathcal{W}^{N,2}(0,1;H) | \bar{W}_B\begin{bmatrix}-f_{\partial,x}\\ e_{\partial,x}\end{bmatrix} = 0\right\}\end{aligned}$$

and thus,

$$\bar{W}_B = \begin{bmatrix}-W_1 & W_2\end{bmatrix}. \tag{19}$$

It is easy to check that under condition (7) the operator $\bar{W}_B$ satisfied (6).

Then the equivalences $1 \Leftrightarrow 2 \Leftrightarrow 5$ follow by Theorem 2.1 applied for A and $-A$.

$1 \Rightarrow 4$: Let A be the generator of a unitary group. Then, due to Theorem [JaZw12, Theorem 6.2.5] A and $-A$ are generators of contraction semigroups. It follows $\operatorname{Re} P_0 = 0$, W_1+W_2 and $-W_1+W_2$ are injective and $\operatorname{Re}\langle Ax,x\rangle = 0 \;\forall\; x \in \mathcal{D}(A)$ by Theorem 2.1. Thus, we get with the estimation (16)

$$0 = 2\operatorname{Re}\langle Ax,x\rangle = 2\langle l, (-I+V^*V)l)\rangle_{H^N} \forall\; l \in H^N \tag{20}$$

and therefore $\|V\|=1$.

$4 \Rightarrow 3$: Let $\operatorname{Re} P_0 = 0$, $\|V\|=1$, W_1+W_2 and $-W_1+W_2$ injective. Define $S := \frac{1}{2}(W_1+W_2)$ and with the technical Lemma 3.7 (Lemma 2.4 in [KuZw15]) it yields

$$\begin{aligned}W_B\Sigma W_B^* &= S\begin{bmatrix}I+V & I-V\end{bmatrix}\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}(S\begin{bmatrix}I+V & I-V\end{bmatrix})^*\\ &= S(2I-2VV^*)S^* = 0.\end{aligned}$$

The implication $3 \Rightarrow 1$ follows analogously to the proof of $3 \Rightarrow 1$ in Theorem 2.1 for the operator $-A$. However, instead of the boundary effort and the boundary flow for A we need to consider them for $-A$ and have to determine the boundary condition operator $\bar{W}_B$ for $-A$. □

4 Proofs of the main results: $I = [0, \infty)$

Throughout this section we will assume that $I = [0, \infty)$, A is given by (8)-(9). For the proof of the main statements we need the following two technical assertions.

Lemma 4.1. *Assume $\Lambda \in \mathbb{R}^{n_1 \times n_1}$ is a positive, invertible diagonal matrix and $y \in L^2(0, \infty; \mathbb{F}^{n_1})$. Then the function*

$$x(t) := \int_0^\infty e^{-s\Lambda^{-1}} \Lambda^{-1} y(s+t)\, ds, \qquad t \geq 0, \tag{21}$$

satisfies $x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{F}^{n_1})$ and $x - \Lambda x' = y$.

Proof: $\Lambda > 0$ and $y \in L^2(0, \infty; \mathbb{F}^{n_1})$ imply that $x(t)$ is well defined for every $t \geq 0$. Minkowski's integral inequality shows $x \in L^2(0, \infty; \mathbb{F}^{n_1})$. Further, the solution of $x - \Lambda x' = y$, or equivalently, of $x' = \Lambda^{-1} x - \Lambda^{-1} y$ is given by

$$x(t) = e^{t\Lambda^{-1}} x(0) - \int_0^t e^{(t-s)\Lambda^{-1}} \Lambda^{-1} y(s)\, ds, \qquad t \geq 0.$$

The choice of $x(0) = \int_0^\infty e^{-s\Lambda^{-1}} \Lambda^{-1} y(s)\, ds$, implies (21). Moreover, $x' = \Lambda^{-1} x - \Lambda^{-1} y$ and hence $x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{F}^{n_1})$. □

Lemma 4.2. *Let $\Theta \in \mathbb{R}^{n_2 \times n_2}$ be a negative, invertible diagonal matrix, $y \in L^2(0, \infty; \mathbb{F}^{n_2})$ and $x_0 \in \mathbb{F}^{n_2}$. Then the differential equation*

$$x - \Theta x' = y, \quad x(0) = x_0, \tag{22}$$

has a unique solution satisfying $x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{F}^{n_2})$.

Proof: We first note that (22) is equivalent to $x' = \Theta^{-1} x - \Theta^{-1} y$. Now the statement of the lemma follows from ODE-Theory for linear stable systems, since $\Theta < 0$ and $y \in L^2(0, \infty; \mathbb{F}^{n_2})$. □

Proof of Theorem 2.7. Thanks to [JaZw12, Lemma 7.2.3] and the Theorem of Lumer-Phillips Part 1 implies Part 2.
Next, we show the implication $2 \Rightarrow 3$. For $x \in D(A)$ we have

$$\operatorname{Re} \langle Ax, x \rangle = -x(0)^* P_1 x(0) + 2\operatorname{Re} \int_0^\infty x(\zeta)^* P_0 x(\zeta)\, d\zeta, \tag{23}$$

since $\lim_{\zeta \to \infty} x(\zeta) = 0$ for $x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{F}^d)$. Choosing $x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{F}^d) \backslash \{0\}$ with $x(0) = 0$, we obtain $\operatorname{Re} P_0 \leq 0$. For every $y \in \mathbb{F}^d$ and every $\varepsilon > 0$ there exists a function in $x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{F}^d)$ such that $x(0) = y$ and the L^2-norm of x is less than ε. Choosing this function in equation (23) and letting ε go to zero implies the second assertion in 3.
In order to prove the implication $3 \Rightarrow 4$, for $x \in D(A)$ we define $\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right] := Sx(0)$. Using (10), the second condition in 3 can be written as

$$\begin{bmatrix} f_1^* & f_2^* \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \geq 0, \quad \text{for } \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \ker \hat{W}_B S^{-1}. \tag{24}$$

Since $\hat{W}_B S^{-1}$ is a full row rank $k \times d$-matrix with $k \leq n_2$, its kernel has dimension $d-k$. By the assumptions on Λ and Θ, we have $d-k \leq n_1$, or equivalently, $k \geq n_2$. Thus $k = n_2$.

We write $\hat{W}_B S^{-1} = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ with $U_1 \in \mathbb{F}^{n_2 \times n_1}$ and $U_2 \in \mathbb{F}^{n_2 \times n_2}$. Assuming U_2 is not invertible, there exists $u \in \mathbb{F}^{n_2}$ such that $\left[\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right] \in \ker \hat{W}_B S^{-1}$ which is in contradiction to (24), since $\Theta < 0$. Thus, the matrix $\hat{W}_B S^{-1}$ is of the form $B \begin{bmatrix} U & I \end{bmatrix}$, with $U \in \mathbb{F}^{n_2 \times n_1}$ and $B \in \mathbb{F}^{n_2 \times n_2}$ invertible. Hence, (24) is equivalent to

$$\begin{bmatrix} f_1^* & f_2^* \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \geq 0 \quad \text{and } U f_1 + f_2 = 0, \text{ for } \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathbb{F}^{n_1+n_2} \tag{25}$$

which is equivalent to $\Lambda + U^* \Theta U \geq 0$. This shows Part 4.
It remains to show that Part 4 implies Part 1. By [JaZw12, Lemma 7.2.3] it is sufficient to prove that A generates a contraction semigroup on $(X, \langle \cdot, \cdot \rangle)$. Due to the fact that $\operatorname{Re} P_0 \leq 0$, and bounded, dissipative perturbations of generators of contraction semigroups, again generate a contraction semigroup, see [EnNa99, Theorem III.2.7], without loss of generality we may assume $P_0 = 0$.

First, we prove the dissipativity of the operator A. Let $x \in \mathcal{D}(A)$ and define $\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right] := Sx(0)$, where the unitary matrix S is given by (10). This implies $U f_1 + f_2 = 0$ as $\hat{W}_B = B \begin{bmatrix} U & I \end{bmatrix} S$.

Thus, it yields

$$\begin{aligned} \operatorname{Re} \langle Ax, x \rangle &= -\langle x(0), P_1 x(0) \rangle_{\mathbb{F}^d} = -\langle x(0), S^{-1} \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix} S x(0) \rangle_{\mathbb{F}^d} \\ &= -\langle Sx(0), \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix} S x(0) \rangle_{\mathbb{F}^d} = -(f_1^* \Lambda f_1 + f_2^* \Theta f_2) \\ &= -(f_1^* \Lambda f_1 + f_1^* U^* \Theta U f_1) \leq 0 \end{aligned}$$

by the last assertion of Part 4.

Further, thanks to the Theorem of Lumer-Phillips it remains to show that for every $y \in L^2(0, \infty; \mathbb{F}^d)$ there exists $x \in D(A)$ such that $x - Ax = y$. Equivalently, by (10) it is sufficient to show that for every $y_1 \in L^2(0, \infty; \mathbb{F}^{n_1})$ and $y_2 \in L^2(0, \infty; \mathbb{F}^{n_2})$ there exist functions $x_1 \in \mathcal{W}^{1,2}(0, \infty; \mathbb{F}^{n_1})$ and $x_2 \in \mathcal{W}^{1,2}(0, \infty; \mathbb{F}^{n_2})$ such that

$$x_1 - \Lambda x_1' = y_1, \;\; x_2 - \Theta x_2' = y_2 \quad \text{and} \quad U x_1(0) + x_2(0) = 0.$$

Let $y_1 \in L^2(0, \infty; \mathbb{F}^{n_1})$ and $y_2 \in L^2(0, \infty; \mathbb{F}^{n_2})$ be arbitrarily. Lemma 4.1 implies the existence of $x_1 \in \mathcal{W}^{1,2}(0, \infty; \mathbb{F}^{n_1})$ with $x_1(0) = \int_0^\infty e^{-s\Lambda^{-1}} \Lambda^{-1} y_1(s)\, ds$ and $x_1 - \Lambda x_1' = y_1$. Finally, Lemma 4.2 shows that there exists a function $x_2 \in \mathcal{W}^{1,2}(0, \infty; \mathbb{F}^{n_1})$ with $x_2(0) = -U x_1(0)$ and $x_2 - \Theta x_2' = y_2$. This concludes the proof. □

Proof of Theorem 2.8. Since $A\mathcal{H}$ generates a unitary C_0-group if and only if $A\mathcal{H}$ and $-A\mathcal{H}$ generate contraction semigroups c.f. [JaZw12, Theorem 6.2.5], the equivalence of Part 1, Part 2, and Part 3 follows directly from Theorem 2.7 for $-A\mathcal{H}$ and $A\mathcal{H}$.

Fomulating Part 4 of Theorem 2.7 for $-A$, we get Re $-P_0 \leq 0$, $k = n_1$,

$$\hat{W}_B = \bar{B} \begin{bmatrix} I & \bar{U} \end{bmatrix} S$$

and $\Theta + \bar{U}^* \Lambda \bar{U} \leq 0$, where $\bar{B} \in K^{n_1 \times n_1}$ is invertible. Thus, Part 4 of Theorem 2.7 for $-A$ and A is equivalent to $\operatorname{Re} P_0 = 0$, $k = n_1 = n_2$ and $\hat{W}_B = \bar{B} \begin{bmatrix} I & \bar{U} \end{bmatrix} S = B \begin{bmatrix} U & I \end{bmatrix} S$ with B and $\bar{B}$ invertible. It yields $\bar{B} = BU$ and $B = \bar{B}\bar{U}$ with $B, \bar{B}$ invertible. Therefore, we get $\bar{U}U = I$ and $\bar{U}, U$ invertible. Thus, we have $\Theta + \bar{U}^* \Lambda \bar{U} \leq 0 \Leftrightarrow U^* \Theta U + \Lambda \leq 0$. Choosing $U_1 = BU$ and $U_2 = B$ we get the assertion. □

5 Examples

In this section we now illustrate our results by a number of examples. Networks of discrete partial differential equations on infinite-dimensional networks are also considered in [Mu14]. Examples of infinite-dimensional networks are given in Figure 1 and 2.

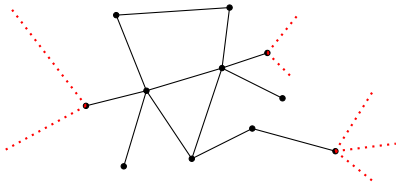

Figure 1: Arbitrary infinite-dimensional network

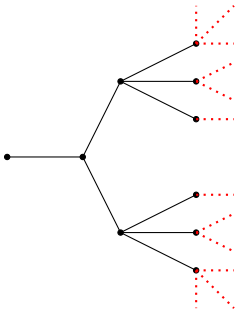

Figure 2: Infinite-dimensional tree

Examples with $I = [0, 1]$ and a finite-dimensional Hilbert space H can be found in [JaZw12].

Example 5.1. *We choose $I = [0, 1]$, $H = \ell^2(\mathbb{N})$ and consider the operator A given by*

$$Af = \frac{\partial}{\partial \zeta} f \tag{26}$$

on the domain

$$\mathcal{D}(A) = \left\{ f \in \mathcal{W}^{1,2}(0, 1; \ell^2(\mathbb{N})) | W_B \Phi(f) = 0 \right\} \tag{27}$$

on a line graph, see Figure 3.

Figure 3: Line graph

Clearly, A denotes a port-Hamiltonian operator with $N = 1$, $P_1 = I$ and $P_0 = 0$. We consider the operator A without boundary condition, such that W_B is just containing the information about the graph. We get $W_B = \begin{bmatrix} I + L & I - L \end{bmatrix}$, where L denotes the left shift and $L^* = R$ the right shift with $L : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is defined by $L(x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)$ and $R : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is given as $R(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$. Clearly, it yields $W_1 + W_2 = 2I$, and thus, condition (6) is fulfilled. Therefore, we can apply

Theorem 2.1 and check Condition 3: $W_1 + W_2$ is injective and

$$\begin{aligned} W_B \Sigma W_B &= \begin{bmatrix} I+L & I-L \end{bmatrix} \Sigma \begin{bmatrix} I+L & I-L \end{bmatrix}^* = \begin{bmatrix} I-L & I+L \end{bmatrix} \begin{bmatrix} I+L^* & I-L^* \end{bmatrix} \\ &= (I-L)(I+L^*) + (I+L)(I-L^*) = 2I - 2LL^* = 0. \end{aligned}$$

Hence, A generates a contraction semigroup. In the finite-dimensional setting we would expect that A also generates a unitary C_0-group, since $W_B \Sigma W_B = 0$. However, it can be shown that A does not generate a unitary C_0- group and Theorem 2.5 is not applicable as (7) is not satisfied.

Example 5.2. *Let $I = [0, \infty)$ and A be given by (8)- (9).*

1. *Let with $P_1 < 0$, that is, $n_2 = d$, and $\operatorname{Re} P_0 \leq 0$. In this situation $A\mathcal{H}$ with domain*

$$\mathcal{D}(A\mathcal{H}) = \mathcal{H}^{-1}\mathcal{D}(A) = \{x \in X \mid \mathcal{H}x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{C}^d) \text{ and } (\mathcal{H}x)(0) = 0\}$$

   generates a contraction semigroup on $(X, \langle \cdot, \mathcal{H}; \cdot \rangle)$.

2. *Let $P_1 > 0$, that is, $n_2 = 0$ and $\operatorname{Re} P_0 \leq 0$. Then $A\mathcal{H}$ with domain*

$$\mathcal{D}(A\mathcal{H}) = \mathcal{H}^{-1}\mathcal{D}(A) = \{x \in X \mid \mathcal{H}x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{C}^d)\}$$

   generates a contraction semigroup on $(X, \langle \cdot, \mathcal{H}; \cdot \rangle)$;

3. *An (undamped) vibrating string can be modelled by*

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left(T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right), \qquad t \geq 0, \zeta \in (0, \infty), \tag{28}$$

   where $\zeta \in [0, 1]$ is the spatial variable, $w(\zeta, t)$ is the vertical position of the string at place ζ and time t, $T(\zeta) > 0$ is the Young's modulus of the string, and $\rho(\zeta) > 0$ is the mass density, which may vary along the string. We assume that T and ρ are positive functions satisfying $\rho, \rho^{-1}, T, T^{-1} \in L^\infty(0, \infty)$. By choosing the state variables $x_1 = \rho \frac{\partial w}{\partial t}$ (momentum) and $x_2 = \frac{\partial w}{\partial \zeta}$ (strain), the partial differential equation (28) can equivalently be written as

$$\begin{aligned} \frac{\partial}{\partial t} \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} \right) \\ &= P_1 \frac{\partial}{\partial \zeta} \left(\mathcal{H}(\zeta) \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} \right), \end{aligned} \tag{29}$$

   where $P_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\mathcal{H}(\zeta) = \left[\begin{smallmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{smallmatrix}\right]$.

   The boundary conditions for (29) are

$$\hat{W}_B (\mathcal{H}x)(0, t) = 0,$$

   where $\hat{W}_B$ is a $k \times 2$-matrix with rank $k \in \{0, 1, 2\}$, or equivalently, the partial differential equation (28) is equipped with the boundary conditions

$$\hat{W}_B \begin{bmatrix} \frac{\partial w}{\partial t}(0, t) \\ T \frac{\partial w}{\partial \zeta}(0, t) \end{bmatrix} = 0.$$

The matrix P_1 can be factorized as

$$P_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix},$$

This implies $n_2 = 1$. Thus, by Theorem 2.7 the corresponding operator

$$\begin{aligned}(A\mathcal{H}x)(\zeta) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} x(\zeta) \right); \\ D(A\mathcal{H}) &= \left\{ x \in \mathcal{W}^{1,2}(0,1;\mathbb{F}^2) \mid \hat{W}_B(\mathcal{H}x)(0,t) = 0 \right\},\end{aligned}$$

generates a contraction semigroup on $(L^2(0,1;\mathbb{C}^2), \langle \cdot, \mathcal{H}; \cdot \rangle)$ if and only if

$$\hat{W}_B = \frac{b}{2} \begin{bmatrix} u-1 & u+1 \end{bmatrix}$$

for $b \in \mathbb{F} \backslash \{0\}$ and $u \in \mathbb{F}$. More precisely, the partial differential equation

$$\begin{aligned}\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left(T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta,t) \right), \qquad t \geq 0, \zeta \in (0,\infty), \\ (u-1)\frac{\partial w}{\partial t}(0,t) + (u+1)T(0)\frac{\partial w}{\partial \zeta}(0,t) &= 0, \qquad t \geq 0, \\ \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,0) &= z_0(\zeta), \qquad \zeta \geq 0, \\ \frac{\partial w}{\partial \zeta}(\zeta,0) &= z_1(\zeta), \qquad \zeta \geq 0,\end{aligned}$$

where $u \in \mathbb{F}$ and $z_0, z_1 \in L^2(0,\infty)$, possesses a unique solution satisfying

$$\int_0^\infty \rho(\zeta) \left[\frac{\partial w}{\partial t}(\zeta,t) \right]^2 + T(\zeta) \left[\frac{\partial w}{\partial \zeta}(\zeta,t) \right]^2 d\zeta \leq \int_0^\infty \frac{z_0^2(\zeta)}{\rho(\zeta)} + T(\zeta) z_1^2(\zeta) d\zeta$$

for $t > 0$.

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