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**Abstract**

We establish a new method for obtaining non-convex spectral enclosures for operators associated with second order differential equations $\ddot{z}(t) + D\dot{z}(t) + A_0 z(t) = 0$ in a Hilbert space. In particular, we succeed in establishing the existence of a spectral gap which is the first result of this kind since the seminal results of Krein and Langer for oscillations of damped systems. While the latter and other spectral bounds are confined to dampings $D$ that are symmetric and dominated by $A_0$, we allow for accretive $D$ of equal strength as $A_0$. To achieve these results we prove new abstract spectral inclusion results that are much more powerful than classical numerical range bounds. Two different applications, small transverse oscillations of a horizontal pipe carrying a steady-state flow of an ideal incompressible fluid and wave equations with strong (viscoelastic and frictional) damping, illustrate that our new bounds are explicit.

**Keywords** Abstract second order differential equation, damped system, spectrum, operator matrix, numerical range, quadratic numerical range.

**Mathematics Subject Classification** 47A10, 35P05, 47A12, 34G10, 47D06, 76B99.

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# 1 Introduction

The mathematical analysis of abstract second order Cauchy problems has been a vital field of research over the last decades. In fact, many linear stability problems in applications, in particular in elasticity theory and hydromechanics, are modeled by second order differential equations of the form

$$\ddot{z}(t) + D\dot{z}(t) + A_0 z(t) = 0 \tag{1}$$

in a Hilbert space $H$, where $A_0$ is a self-adjoint and uniformly positive operator in $H$ and $D$ is a linear operator in $H$ representing e.g. the damping of the underlying system, see e.g. [30, 31, 13, 10]. Here we consider the case that $A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}$ is bounded and accretive. Both $A_0$ and $D$ may be unbounded, $D$ may be equally strong as $A_0$ and need not be self-adjoint, and for some results, $D$ need not even be sectorial.

By means of the standard substitution $x = (z, \dot{z})^\top$, the second order differential equation (1) is equivalent to a first-order system

$$\dot{x}(t) = \mathcal{A}x(t) \tag{2}$$

in a suitably defined product Hilbert space. More precisely, if we equip the space $H_{\frac{1}{2}} := \mathcal{D}(A_0^{\frac{1}{2}})$ with the graph norm of $A_0^{\frac{1}{2}}$, then the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H$ associated with (1) is defined as

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix}, \quad \mathcal{D}(\mathcal{A}) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid A_0 z + Dw \in H \right\}. \tag{3}$$

Under stronger assumptions on the damping operator $D$ such as self-adjointness and/or stronger relative boundedness, operators of this form and applications in elasticity theory or hydromechanics have been studied intensively in the literature for more than 20 years, see e.g. [3, 4, 7, 8, 9, 21, 22, 24, 25, 26, 27, 32, 39, 42, 43]. In particular, it was proved that $\mathcal{A}$ is boundedly invertible, has spectrum in the closed left half-plane, and generates a strongly continuous semigroup of contractions on $H_{\frac{1}{2}} \times H$, see e.g. [42, Proposition 5.1]. Another widely studied special case is proportional damping $D = dA_0^\theta$ with $\theta \in [\frac{1}{2}, 1]$ and $d > 0$ in view of corresponding semi- or non-linear strongly damped wave equations, see e.g. [10, 6] and the references therein.

Another example for a differential equation (1) and corresponding operator $\mathcal{A}$ are abstract Klein-Gordon equations originating in quantum mechanics, see e.g. [38] and the references therein. In this case $A_0$ has the form $A_0 = H_0 - V^2$ and $D = 2V$ where $H_0$ is a self-adjoint uniformly positive operator, e.g. $-\Delta + mc^2$ on $\mathbb{R}^n$ with particle mass $m > 0$, and $V$ is a symmetric operator such that $VH_0^{-\frac{1}{2}}$ is bounded. By means of indefinite inner product methods, the spectrum of $\mathcal{A}$ was analyzed and criteria on $D$ were found ensuring that $\mathcal{A}$ generates a group of bounded unitary operators in a Pontryagin space in [37].

The aim of this paper is to establish new enclosures for the spectrum of the operator $\mathcal{A}$ in (3) under rather weak assumptions on the damping operator $D$, allowing it to be as strong as $A_0$ so that even very general perturbation results such as [12] do not apply. To this end, we do not only use the classical numerical range $W(\mathcal{A})$ of $\mathcal{A}$, but also the so-called *quadratic numerical range* $W^2(\mathcal{A})$. The latter was introduced in 1998 for operator matrices with bounded off-diagonal entries in [36], shortly after studied in great detail for bounded operator matrices in [34, 35], and in 2009 generalized to diagonally dominant and off-diagonally dominant operator matrices in [40]. Unlike the numerical range, the quadratic numerical range is not convex: it may consist of two components which need not be convex either. Since the quadratic numerical range is always contained in the numerical range, see [41], it may give tighter spectral enclosures.

We show that this is indeed *always* the case here, for uniformly accretive, for sectorial and even for self-adjoint damping operator $D$ (see Figures 1–12 below).

If $D$ is only assumed to be uniformly accretive relative to $A_0$ in $H$ (and hence uniformly accretive in $H$) and no information on the imaginary part of $W(D)$ is available, then the numerical range $W(\mathcal{A})$ cannot provide a better spectral enclosure than the left half-plane since it is convex and contains the numerical ranges of the diagonal elements $D$ and 0 of $\mathcal{A}$. The quadratic numerical range $W^2(\mathcal{A})$ yields a non-convex enclosing set to the left of the imaginary axis and, under a certain additional condition, it provides a vertical strip free of spectrum, see Theorem 6.1.

If $D$ is assumed to be sectorial with angle $< \pi$ and uniformly accretive in $H$, then the quadratic numerical range $W^2(\mathcal{A})$ always yields an enclosure with corner at 0, whereas the numerical range $W(\mathcal{A})$ may still be a half-plane; if $D$ is uniformly accretive relative to $A_0$ in $H$, the former is even contained in a sector, while the latter only gives a parabolic enclosure, see Theorem 6.2 and Proposition 3.8. In fact, it was proved in [35] for the bounded case that, while every corner of the numerical range must belong to the spectrum $\sigma(\mathcal{A})$, corners of the quadratic numerical range may also belong to the spectrum of a diagonal entry of $\mathcal{A}$. Here $0 \notin \sigma(\mathcal{A})$, but 0 belongs to both numerical range and quadratic numerical range; hence 0 cannot be a corner of $W(\mathcal{A})$, but 0 may be, and indeed is, a corner of $W^2(\mathcal{A})$ since it belongs to the spectrum of the zero operator on the diagonal of $\mathcal{A}$.

Even for self-adjoint $D$, the difference between numerical range and quadratic numerical range is substantial. Whereas the imaginary part of the numerical range is always unbounded if $A_0$ is unbounded, the quadratic numerical range may have bounded imaginary part, may be partly confined to the negative real axis or may even be entirely real, see Theorem 7.2! In the latter case, under a certain additional condition, it may even consist of two disjoint real intervals.

There are two key problems we have to solve before we can take advantage of the quadratic numerical range. Firstly, the operator $\mathcal{A}$ in (3) is not an operator matrix itself since its domain does not decompose according to the decomposition $H_{\frac{1}{2}} \times H$ of the space in which $\mathcal{A}$ acts; in fact, $\mathcal{A}$ is merely the closure of the operator matrix $\mathcal{A}|_{H_1 \times H_1}$ and only the quadratic numerical range of $\mathcal{A}|_{H_1 \times H_1}$ is defined. Secondly, the operator matrix $\mathcal{A}|_{H_1 \times H_1}$ with its three unbounded entries $I : H \to H_{\frac{1}{2}}$, $A_0 : H_{\frac{1}{2}} \to H$, and $D : H \to H$, is neither diagonally dominant nor off-diagonally dominant; in fact, in the first column the stronger entry is the off-diagonal $A_0$, while in the second column the stronger entry is the diagonal $D$.

Our first main result is the so-called spectral inclusion property of the quadratic numerical range, i.e. the set of inclusions

$$\begin{aligned} \sigma_p(\mathcal{A}|_{H_1 \times H_1}) &\subset W^2(\mathcal{A}|_{H_1 \times H_1}), \quad & \sigma_{ap}(\mathcal{A}|_{H_1 \times H_1}) &\subset \overline{W^2(\mathcal{A}|_{H_1 \times H_1})}, \\ \sigma_p(\mathcal{A}) &\subset \overline{W^2(\mathcal{A}|_{H_1 \times H_1})}, & \sigma_{ap}(\mathcal{A}) &\subset \overline{W^2(\mathcal{A}|_{H_1 \times H_1})}, \end{aligned}$$

for the point and approximate point spectrum of $\mathcal{A}|_{H_1 \times H_1}$ and $\mathcal{A}$, respectively. As usual, one has to require the existence of at least one point of the resolvent set $\rho(\mathcal{A})$ in each component of $\mathbb{C} \setminus \overline{W^2(\mathcal{A}|_{H_1 \times H_1})}$ to obtain the full chain of spectral enclosures

$$\sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A}|_{H_1 \times H_1})} \subset \overline{W(\mathcal{A})}. \tag{4}$$

Although neither the numerical range nor the quadratic numerical range may be determined precisely, analytic estimates for either of them provide bounds for the spectrum via the enclosures (4). We derive an estimate for $W(\mathcal{A})$ and a series of estimates for $W^2(\mathcal{A}|_{H_1 \times H_1})$ in terms of various constants relating the "real part" of the operator $D$ to $A_0$ and, if $D$ is sectorial with angle $< \pi$, in terms of its sectoriality angle. In all cases, the quadratic numerical range yields tighter bounds than the numerical range since it allows for finer estimates. Moreover, we compare all the obtained estimates for $W^2(\mathcal{A}|_{H_1\times H_1})$ and combine them to further improve the enclosure for the spectrum.

Two different applications show the wide applicability and power of our new spectral bounds. The first example is a wave equation in a bounded domain $\Omega \subset \mathbb{R}^n$ with viscoelastic and frictional damping subject to Dirichlet conditions on $\partial\Omega$, where

$$A_0 = -\Delta, \quad D = -d\Delta + V$$

with $d \geq 0$ and $\operatorname{ess\,inf} \operatorname{Re} V \geq 0$ and certain minimal conditions on $V$; in particular, neither symmetry nor sectoriality are assumed. Secondly, we consider an operator of the form (3) arising in the investigation of small transverse oscillations of a pipe carrying steady-state flow of an ideal incompressible fluid. The corresponding second order equation (1) is of the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial r^2}\left[E\frac{\partial^2 u}{\partial r^2} + C\frac{\partial^3 u}{\partial r^2 \partial t}\right] + K\frac{\partial^2 u}{\partial t \partial x} = 0, \qquad r \in (0,1),\ t > 0, \tag{5}$$

where $u(r,t)$ denotes the transverse oscillation at time $t$ and position $r$, and $E$, $C$, $K$ are positive physical constants. Here both operator coefficients $A_0 = \frac{\partial^2}{\partial r^2} E \frac{\partial^2}{\partial r^2}$ and $D = \frac{\partial^2}{\partial r^2} C \frac{\partial^2}{\partial r^2} + K\frac{\partial}{\partial r}$ in $L^2(0,1)$ with appropriate domains are fourth order differential operators and hence have the same strength. Both applications demonstrate that all constants involved in our abstract results may be analytically estimated. In particular, in both cases we derive thresholds for the damping constants at which a spectral free strip opens up (see Figures 10–12 below), a phenomenon that could not be captured so far for non-symmetric damping.

Throughout this paper we use the following notation. For a closable densely defined linear operator $S$ in some Banach space $X$ we denote by $\rho(S)$ the resolvent set, by $\sigma_p(S)$ the point spectrum, and by $\sigma_{ap}(S)$ the *approximate point spectrum*, i.e. the set of all $\lambda \in \mathbb{C}$ for which there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(S)$ such that

$$\|x_n\| = 1, \quad \|(S-\lambda)x_n\| \to 0, \quad n \to \infty,$$

see e.g. [14, p. 242]. Clearly, the point spectrum is a subset of the approximate point spectrum; moreover, the boundary of the spectrum $\sigma(S)$ belongs to $\sigma_{ap}(S)$, see e.g. [14, Proposition 1.10].

## 2 Operator framework

In this section, we rigorously introduce the operator $\mathcal{A}$ in (3) associated with the second order differential equation (1). Throughout this paper $H$ is a Hilbert space and we assume the following.

**(A1)** The operator $A_0 : \mathcal{D}(A_0) \subset H \to H$ is a self-adjoint and uniformly positive linear operator on $H$ such that 0 is in the resolvent set of $A_0$.

Assumption (A1) allows us to introduce Hilbert spaces $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$ by means of $A_0$ as follows. We define

$$H_{\frac{1}{2}} := \mathcal{D}(A_0^{\frac{1}{2}}), \quad \|\cdot\|_{H_{\frac{1}{2}}} := \|A_0^{\frac{1}{2}} \cdot\|_H,$$

and we set $H_{-\frac{1}{2}} := H_{\frac{1}{2}}^*$, where the duality is taken with respect to the pivot space $H$; in other words, $H_{-\frac{1}{2}}$ is the completion of $H$ with respect to

$$\|z\|_{H_{-\frac{1}{2}}} = \|A_0^{-\frac{1}{2}} z\|_H.$$

If we further define $H_1 := \mathcal{D}(A_0)$ with the norm $\|\cdot\|_{H_1} := \|A_0 \cdot\|_H$, then $A_0$ may be viewed as a bounded operator $A_0 : H_1 \to H$ and extends to a bounded operator $A_0 : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$; in both cases we keep the notation $A_0$.

If we denote the inner product on $H$ by $\langle \cdot, \cdot \rangle_H$ or $\langle \cdot, \cdot \rangle$ and the duality pairing on $H_{-\frac{1}{2}} \times H_{\frac{1}{2}}$ by $\langle \cdot, \cdot \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}$, then, for $(z', z)^\top \in H \times H_{\frac{1}{2}}$,

$$\langle z', z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle z', z \rangle_H.$$

**(A2)** The (damping) operator $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is bounded and the (bounded) operator $A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}$ is accretive in $H$, i.e.

$$\operatorname{Re}\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq 0, \qquad z \in H_{\frac{1}{2}}.$$

**(A3)** The operator $D$ maps the space $H_1 = \mathcal{D}(A_0)$ into $H$.

**Example 2.1** (Wave equation with strong damping)**.** We consider a wave equation subject to viscoelastic and frictional damping on a domain $\Omega \subset \mathbb{R}^n$, where either $\Omega = \mathbb{R}^n$ or $\Omega$ is bounded and has $C^2$-boundary; in the latter case we impose Dirichlet conditions on $\partial\Omega$, cf. [16, (1.2)], [17, (GM), p. 1]. More precisely, let

$$A_0 = -\Delta + b, \quad D = -d\Delta + V$$

with $b \in L^\infty(\Omega, \mathbb{R})$ such that $\operatorname{ess\,inf} b > 0$ if $\Omega = \mathbb{R}^n$ and $\operatorname{ess\,inf} b \geq 0$ if $\Omega$ is bounded, viscoelastic damping constant $d \geq 0$, and frictional damping $V$ such that $\operatorname{ess\,inf} \operatorname{Re} V \geq 0$ and, for dimension $n \geq 5$, $V = V_1 + V_2 \in L_{\mathrm{w}}^{n/2}(\Omega, \mathbb{C}) + L^\infty(\Omega, \mathbb{C})$. Note that, with $L_{\mathrm{w}}^p$, $p \in [1, \infty)$, denoting the weak $L^p$-space, a sufficient condition for the latter is $V \in L_{\mathrm{w}}^p(\Omega, \mathbb{C}) + L^\infty(\Omega, \mathbb{C})$ for $p \geq \frac{n}{2}$. In this situation $H = L^2(\Omega, \mathbb{C})$, $H_{\frac{1}{2}} = H_0^1(\Omega, \mathbb{C})$, $H_1 = H^2(\Omega, \mathbb{C}) \cap H_{\frac{1}{2}}$ (due to elliptic boundary regularity) and $H_{-\frac{1}{2}} = H^{-1}(\Omega, \mathbb{C})$.

Clearly, $A_0$ satisfies (A1). If $\Omega = \mathbb{R}^n$, then [29, Corollary 2.11] implies (A2) and (A3). Indeed, choosing $\alpha = 1$ in [29, Corollary 2.11] and noting $|V_1|^{1/2} \in L_{\mathrm{w}}^n(\mathbb{R}^n, \mathbb{R})$, one finds that $|V_1|^{1/2} : H^1(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C})$ is a bounded multiplication operator and hence so is $V_1 = |V_1|^{1/2} \cdot \operatorname{sgn} V_1 \cdot |V_1|^{1/2} : H^1(\mathbb{R}^n, \mathbb{C}) \to H^{-1}(\mathbb{R}^n, \mathbb{C})$, i.e. (A2) holds; the choice $\alpha = 2$ in [29, Corollary 2.11] yields (A3). If $\Omega$ is bounded with $C^2$-boundary, (A2) and (A3) follow from the above considerations since $\Omega$ has the extension property.

For dimension $n \leq 4$, [29] is not applicable. In this case we have to assume more restrictively that $V_1 \in L^2(\Omega, \mathbb{C})$ if $n \leq 3$ and $V_1 \in L^{2+\epsilon}(\Omega, \mathbb{C})$ for some $\epsilon \in (0, 1)$ if $n = 4$. Then (A2) and (A3) follow from well-known Sobolev embeddings for $H^1(\Omega, \mathbb{C})$ and $H^2(\Omega, \mathbb{C})$, cf. [15, Section 2].

**Remark 2.2.** (a) Note that, by the closed graph theorem, (A3) implies that $D$ is a bounded operator from $H_1$ to $H$.

(b) The bounded operator $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ has $H_1$ as a core. If we view $D$ as an operator in $H$ with domain $H_1$, then it is densely defined and accretive by (A2), (A3),

$$W(D) := \{\langle Dg, g \rangle \mid g \in H_1, \ \|g\| = 1\} \subset \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\},$$

hence closable by [28, Theorem V.3.2]. In the following, we use the notation $D$ for both operators.

In the product Hilbert space $H_{\frac{1}{2}} \times H$ we now consider the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H$ given by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix}, \quad \mathcal{D}(\mathcal{A}) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid A_0 z + Dw \in H \right\}. \tag{6}$$

**Proposition 2.3.** *The operator $\mathcal{A}$ is closed with bounded inverse given by*

$$\mathcal{A}^{-1} = \begin{bmatrix} -A_0^{-1} D & -A_0^{-1} \\ I & 0 \end{bmatrix} \tag{7}$$

*in $H_{\frac{1}{2}} \times H$, $\mathcal{A}$ generates a $C_0$-semigroup of contractions, and $H_1 \times H_1$ is a core of $\mathcal{A}$, i.e.*

$$\overline{\mathcal{A}|_{H_1 \times H_1}} = \mathcal{A}.$$

*Moreover, $\mathcal{A}$ is bounded if and only if $A_0$ is a bounded operator in $H$.*

*Proof.* The formula (7) for the inverse of $\mathcal{A}$ is easy to check; it is also easy to see that all entries therein are bounded operators between the respective Hilbert spaces. Hence $\mathcal{A}$ is a closed operator. The semigroup property was shown e.g. in [19].

By (A3), we have $H_1 \times H_1 \subset \mathcal{D}(\mathcal{A})$. Hence $\overline{\mathcal{A}|_{H_1 \times H_1}} \subset \mathcal{A}$ since $\mathcal{A}$ is a closed operator. Thus it remains to be shown that $\mathcal{A} \subset \overline{\mathcal{A}|_{H_1 \times H_1}}$. Let $(z, w)^\top \in \mathcal{D}(\mathcal{A})$, i.e. $z, w \in H_{\frac{1}{2}}$ and $f := A_0 z + Dw \in H$. Since $H_1$ is dense in $H_{\frac{1}{2}}$, there exists a sequence $(w_n)_{n \in \mathbb{N}}$ in $H_1$ such that $w_n \to w$, $n \to \infty$, in $H_{\frac{1}{2}}$. If we define $z_n := A_0^{-1} f - A_0^{-1} D w_n \in H_1$, $n \in \mathbb{N}$, then $z_n \to z$, $n \to \infty$, in $H_{\frac{1}{2}}$ and $A_0 z_n + D w_n = f$ in $H$. This shows that $(z_n, w_n)^\top \in \mathcal{D}(\mathcal{A})$ and $\big(\mathcal{A}(z_n, w_n)^\top\big)_{n \in \mathbb{N}} = (w_n, -f)^\top$ converges in $H_{\frac{1}{2}} \times H$.

Clearly, if $\mathcal{A}$ is bounded in $H_{\frac{1}{2}} \times H$, then so is $A_0 : H_{\frac{1}{2}} \to H$. This is equivalent to $A_0^{\frac{1}{2}} : H \to H$ being bounded which implies that $A_0$ is bounded in $H$. Vice versa, the boundedness of $A_0$ implies that $\mathcal{D}(A_0) = H$ and $H_{\frac{1}{2}} = H = H_{-\frac{1}{2}}$ with all norms being equivalent. Then also the entries $I : H \to H_{\frac{1}{2}}$ and $D : H \to H$ in $\mathcal{A}$ are bounded and hence so is $\mathcal{A}$. $\square$

**Remark 2.4.** Proposition 2.3 implies that $\sigma(\mathcal{A})$ is contained in the closed left half-plane and that $0 \in \rho(\mathcal{A})$. However, otherwise the spectrum of $\mathcal{A}$ may be quite arbitrary, see [25, Example 3.2].

In the following sections we will establish new, tighter enclosures for the spectrum of $\mathcal{A}$ in terms of its entries $A_0$ and $D$; particular attention will be paid to the case of sectorial and self-adjoint damping operator $D$.

# 3 The numerical range of $\mathcal{A}$

In this section we investigate the numerical range of the operator $\mathcal{A}$ in (6), which is defined as

$$W(\mathcal{A}) := \big\{ \langle \mathcal{A}x, x \rangle_{H_{\frac{1}{2}} \times H} \mid x \in \mathcal{D}(\mathcal{A}), \ \|x\| = 1 \big\}.$$

By the Toeplitz-Hausdorff Theorem, $W(\mathcal{A})$ is always a convex subset of $\mathbb{C}$, see [28, Theorem V.3.1], and it has the so-called spectral inclusion property

$$\sigma_p(\mathcal{A}) \subset W(\mathcal{A}), \quad \sigma_{ap}(\mathcal{A}) \subset \overline{W(\mathcal{A})}. \tag{8}$$

Since $\mathcal{A}$ is unbounded in general, additional assumptions are needed to ensure $\sigma(\mathcal{A}) \subset \overline{W(\mathcal{A})}$: if a component $\Omega$ of $\mathbb{C} \setminus \overline{W(\mathcal{A})}$ contains a point of $\rho(\mathcal{A})$, then $\Omega \subset \rho(\mathcal{A})$, see [28, Theorem V.3.2].

The following constants will play an important role throughout this paper, see also [25, 26]:

$$\begin{aligned}
\beta_0 &:= \inf_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\operatorname{Re}\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\|^2} \in [0, \infty), \\
\gamma_0 &:= \sup_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\operatorname{Re}\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\|^2} \in [0, \infty], \\
\delta_0 &:= \inf_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\operatorname{Re}\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\|^2_{H_{\frac{1}{2}}}} \in [0, \infty), \\
\mu_0 &:= \inf_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\operatorname{Re}\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\| \|z\|_{H_{\frac{1}{2}}}} \in [0, \infty).
\end{aligned} \tag{9}$$

By (A1), the operator $A_0$ is uniformly positive, i.e. there exists a constant $a_0 > 0$ such that $\langle A_0 z, z\rangle \geq a_0^2 \|z\|^2$ for $z \in \mathcal{D}(A_0)$. In other words,

$$\|z\|_{H_{\frac{1}{2}}} = \|A_0^{\frac{1}{2}} z\| \geq a_0 \|z\|, \quad z \in H_{\frac{1}{2}}; \tag{10}$$

note that one may choose $a_0 = (\min \sigma(A_0))^{\frac{1}{2}} = \|A_0^{-\frac{1}{2}}\|^{-1}$. Altogether, we have the following estimates between the constants in (9):

$$\mu_0^2 \geq \beta_0 \delta_0, \quad \gamma_0 \geq \beta_0 \geq a_0 \mu_0, \quad \mu_0 \geq a_0 \delta_0. \tag{11}$$

Note that $\beta_0 > 0$ means that $D$ is uniformly accretive as an operator in $H$ with domain $H_1$, $\operatorname{Re} W(D) \geq \beta$, while $\delta_0 > 0$ means that $D$ is uniformly accretive relative to $A_0$ in $H$, i.e. the numerical range of the linear operator pencil $L$ in $H$, cf. [35], given by $L(\lambda) := D - \lambda A_0$, $\mathcal{D}(L(\lambda)) = H_1$, $\lambda \in \mathbb{C}$, satisfies $\operatorname{Re} W(L) \geq \delta_0$. Note that both $\mu_0 > 0$ and $\delta_0 > 0$ imply $\beta_0 > 0$.

The following simple observations will be useful in the following.

**Remark 3.1.** (i) In the definition (9) of $\beta_0$, $\gamma_0$, $\delta_0$, $\mu_0$, the infimum and supremum, respectively, may equivalently be taken over $z \in H_1 \setminus \{0\}$ since $H_1$ is a core for $D$, see Remark 2.2.

(ii) In applications the explicit values of $\beta_0$, $\mu_0$ and $\delta_0$ may not be known but only lower bounds, see Example 8.1. All the following spectral enclosures are therefore formulated in terms of (non-negative) bounds $\beta \leq \beta_0$, $\mu \leq \mu_0$ and $\delta \leq \delta_0$, respectively; in each case, the best enclosures are obtained for $\beta = \beta_0$, $\mu = \mu_0$ and $\delta = \delta_0$.

**Lemma 3.2.** *If $\gamma_0 < \infty$ and $\mu_0 > 0$, then $A_0$ is a bounded operator in $H$ with $\|A_0\| \leq \frac{\gamma_0^2}{\mu_0^2}$; the same holds if $\gamma_0 < \infty$ and $\delta_0 > 0$.*

*Proof.* By definition of $\mu_0$, we have $\operatorname{Re}\langle Dz, z\rangle \geq \mu_0 \|z\| \|z\|_{H_{\frac{1}{2}}}$ for all $z \in H_1 \setminus \{0\}$ and hence, because $\gamma_0 < \infty$ and $\mu_0 > 0$,

$$\|A_0^{\frac{1}{2}} z\| = \|z\|_{H_{\frac{1}{2}}} \leq \frac{1}{\mu_0} \frac{\operatorname{Re}\langle Dz, z\rangle}{\|z\|^2} \|z\| \leq \frac{\gamma_0}{\mu_0} \|z\|;$$

note that $\delta_0 > 0$ implies $\mu_0 > 0$ by (11). □

**Remark 3.3.** We have $\lambda \in W(\mathcal{A})$ if and only if there is $(f,g)^\top \in \mathcal{D}(\mathcal{A})$, $\|f\|^2_{H_{\frac12}} + \|g\|^2 = 1$, with

$$\begin{aligned}\lambda = \left\langle \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix} \right\rangle_{H_{\frac12}\times H} &= \langle g, f\rangle_{H_{\frac12}} - \langle A_0 f + Dg, g\rangle_{H_{-\frac12}\times H_{\frac12}} \\ &= \langle g, f\rangle_{H_{\frac12}} - \langle A_0 f, g\rangle_{H_{-\frac12}\times H_{\frac12}} - \langle Dg, g\rangle_{H_{-\frac12}\times H_{\frac12}} \\ &= -2\mathrm{i}\,\mathrm{Im}\langle A_0^{\frac12} f, A_0^{\frac12} g\rangle - \langle Dg, g\rangle_{H_{-\frac12}\times H_{\frac12}}.\end{aligned} \tag{12}$$

**Proposition 3.4.** (i) *The numerical range $W(\mathcal{A})$ of $\mathcal{A}$ is contained in the closed left half-plane and*

$$W(-D) \cup \{0\} \subset W(\mathcal{A}). \tag{13}$$

(ii) *The real part* $\mathrm{Re}\, W(\mathcal{A})$ *satisfies*

$$\inf\,(\mathrm{Re}\, W(\mathcal{A})) = -\gamma_0, \quad \max\,(\mathrm{Re}\, W(\mathcal{A})) = 0; \tag{14}$$

*in particular,* $\mathrm{Re}\, W(\mathcal{A})$ *is bounded if and only if $\gamma_0 < \infty$.*

(iii) *The imaginary part* $\mathrm{Im}\, W(\mathcal{A})$ *is bounded if and only if $A_0$ is a bounded operator in $H$; in this case, also $D$ is bounded and*

$$|\mathrm{Im}\, W(\mathcal{A})| \le \|A_0^{\frac12}\| + \|\mathrm{Im}\, D\|.$$

*Proof.* (i) By Remark 3.3, assumption (A2) ensures that $W(\mathcal{A})$ is contained in the closed left half-plane.

If we choose $g = 0$ and $f \in H_1$ with $\|f\|_{H_{\frac12}} = 1$, then $(f,0)^\top \in \mathcal{D}(\mathcal{A})$ and Remark 3.3 shows that $0 \in W(\mathcal{A})$. If we choose $f = 0$ and $g \in H_1$ with $\|g\| = 1$, then $(0,g)^\top \in \mathcal{D}(\mathcal{A})$ and Remark 3.3 shows that $-\langle Dg, g\rangle \in W(\mathcal{A})$.

(ii) The second equality in (14) is immediate from (i). The inclusion (13) implies that $\inf\,(\mathrm{Re}\, W(\mathcal{A})) \le -\gamma_0$, cf. Remark 3.1; the opposite inequality follows since for $\lambda \in W(\mathcal{A})$, by Remark 3.3, there exists $g \in H_{\frac12}$ with $\|g\| \le 1$ such that either $\lambda = 0 \ge -\gamma_0$ if $g = 0$ or else, if $g \ne 0$,

$$\mathrm{Re}\lambda = -\|g\|^2 \frac{\mathrm{Re}\langle Dg, g\rangle_{H_{-\frac12}\times H_{\frac12}}}{\|g\|^2} \ge -\sup_{z\in H_{\frac12}\setminus\{0\}} \frac{\mathrm{Re}\langle Dz, z\rangle_{H_{-\frac12}\times H_{\frac12}}}{\|z\|^2} = -\gamma_0.$$

(iii) If $A_0$ is a bounded operator in $H$, then $\mathcal{A}$ is a bounded operator in $H_{\frac12} \times H$ by Proposition 2.3 and so $W(\mathcal{A})$ is bounded.

Vice versa, suppose that $A_0$, and hence $A_0^{\frac12}$, is unbounded. Then, since $H_1$ is a core for $A_0^{\frac12}$, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $H_1$ with $\|g_n\| = \frac{1}{\sqrt2}$ such that $\langle A_0^{\frac12} g_n, g_n\rangle \to \infty$, $n \to \infty$. For $n \in \mathbb{N}$, we set

$$f_n := \begin{cases} \mathrm{i}\, A_0^{-\frac12} g_n & \text{if } \mathrm{Im}\langle Dg_n, g_n\rangle = 0, \\ \mathrm{i}\dfrac{\mathrm{Im}\langle Dg_n, g_n\rangle}{|\mathrm{Im}\langle Dg_n, g_n\rangle|} A_0^{-\frac12} g_n & \text{otherwise.} \end{cases}$$

Obviously, $\|f_n\|^2_{H_{\frac12}} + \|g_n\|^2 = 1$ and, by (A3), $(f_n, g_n)^\top \in H_1 \times H_1 \subset \mathcal{D}(\mathcal{A})$. From Remark 3.3 we deduce

$$\mathrm{Im}\left\langle \mathcal{A}\begin{bmatrix} f_n \\ g_n \end{bmatrix}, \begin{bmatrix} f_n \\ g_n \end{bmatrix}\right\rangle_{H_{\frac12}\times H} = \left(-2\langle g_n, A_0^{\frac12} g_n\rangle - |\mathrm{Im}\langle Dg_n, g_n\rangle|\right) \frac{\mathrm{Im}\langle Dg_n, g_n\rangle}{|\mathrm{Im}\langle Dg_n, g_n\rangle|}$$

if $\mathrm{Im}\langle Dg_n, g_n\rangle \neq 0$ and

$$\mathrm{Im}\left\langle \mathcal{A}\begin{bmatrix} f_n \\ g_n\end{bmatrix}, \begin{bmatrix} f_n \\ g_n\end{bmatrix}\right\rangle_{H_{\frac{1}{2}}\times H} = -2\langle g_n, A_0^{\frac{1}{2}} g_n\rangle$$

if $\mathrm{Im}\langle Dg_n, g_n\rangle = 0$, which shows that the imaginary part of $W(\mathcal{A})$ is unbounded.

The last claim follows from Remark 3.3 if we use that $A_0$ bounded implies $D$ bounded and that in (12) we can estimate $|2\langle A_0^{\frac{1}{2}} f, A_0^{\frac{1}{2}} g\rangle| \leq 2\|A_0^{\frac{1}{2}}\|\|f\|_{H_{\frac{1}{2}}}\|g\| \leq \|A_0^{\frac{1}{2}}\|(\|f\|^2_{H_{\frac{1}{2}}} + \|g\|^2) = \|A_0^{\frac{1}{2}}\|$, and $|\mathrm{Im}\langle Dg, g\rangle_{H_{-\frac{1}{2}}, H_{\frac{1}{2}}}| \leq \|\mathrm{Im}\, D\|\|g\|^2 \leq \|\mathrm{Im}\, D\|$. □

The following example shows that the numerical range $W(\mathcal{A})$ may indeed fill the entire closed left half-plane.

**Example 3.5.** Let $H = \ell^2(\mathbb{N})$, $\mathbb{N} = \{1, 2, \ldots\}$. The operator

$$\begin{aligned} \mathcal{D}(A_0) &:= \big\{(x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \mid (nx_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})\big\}, \\ A_0(x_n)_{n\in\mathbb{N}} &:= (nx_n)_{n\in\mathbb{N}}, \quad (x_n)_{n\in\mathbb{N}} \in \mathcal{D}(A_0), \end{aligned}$$

satisfies (A1) and $H_{\frac{1}{2}} = \big\{(x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \mid (\sqrt{n}\, x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})\big\}$. Then the operator

$$D(x_n)_{n\in\mathbb{N}} := \big((1 + (-1)^n)nx_n\big)_{n\in\mathbb{N}}, \quad (x_n)_{n\in\mathbb{N}} \in H_{\frac{1}{2}},$$

satisfies (A2) and (A3). As usual, we denote by $e_j := (\delta_{ij})_{i\in\mathbb{N}}$, $j \in \mathbb{N}$, the sequence of unit vectors in $\ell^2(\mathbb{N})$. Then, clearly $W(-D) = (-\infty, 0]$ and hence $(-\infty, 0] \subset W(\mathcal{A})$ by (13). Moreover, for $n \in \mathbb{N}$,

$$\|2^{-\frac{1}{2}}(2n+1)^{-\frac{1}{2}}\, e_{2n+1}\|_{H_{\frac{1}{2}}} = \frac{1}{\sqrt{2}}, \qquad \|\pm \mathrm{i}2^{-\frac{1}{2}}\, e_{2n+1}\| = \frac{1}{\sqrt{2}},$$

and, by (12) since $De_{2n+1} = 0$,

$$\begin{aligned} &\left\langle \mathcal{A}\begin{bmatrix} 2^{-\frac{1}{2}}(2n+1)^{-\frac{1}{2}}\, e_{2n+1} \\ \pm \mathrm{i}2^{-\frac{1}{2}}\, e_{2n+1}\end{bmatrix}, \begin{bmatrix} 2^{-\frac{1}{2}}(2n+1)^{-\frac{1}{2}}\, e_{2n+1} \\ \pm \mathrm{i}2^{-\frac{1}{2}}\, e_{2n+1}\end{bmatrix}\right\rangle_{H_{\frac{1}{2}}\times H} \\ &= \frac{\pm \mathrm{i}}{\sqrt{2n+1}}\big\langle \sqrt{2n+1}\, e_{2n+1}, \sqrt{2n+1}\, e_{2n+1}\big\rangle = (\pm \mathrm{i})\sqrt{2n+1} \longrightarrow \pm \mathrm{i}\infty, \quad n \to \infty. \end{aligned}$$

Altogether, the convexity of $W(\mathcal{A})$, see e.g. [28, Theorem V.3.1], implies that $W(\mathcal{A})$ is the entire closed left half-plane,

$$W(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \leq 0\}.$$

Due to the spectral inclusion property (8), estimates for the numerical range yield estimates for the approximate point spectrum. For the spectrum of $\mathcal{A}$, we obtain the following.

**Corollary 3.6.** *The spectrum of $\mathcal{A}$ satisfies the following inclusions:*

(i) $\sigma(\mathcal{A}) \subset \{z \in \mathbb{C} \setminus \{0\} \mid \mathrm{Re} z \leq 0\}$;

(ii) *if $\gamma_0 < \infty$ and there is $\lambda_0 \in \rho(\mathcal{A})$ with $\mathrm{Re}\lambda_0 < -\gamma_0$, then $\sigma(\mathcal{A}) \subset \big\{z \in \mathbb{C}\setminus\{0\} \mid -\gamma_0 \leq \mathrm{Re} z \leq 0\big\}$;*

(iii) *if $\gamma_0 < \infty$ and $\mu_0 > 0$, then $\sigma(\mathcal{A}) \subset \Big\{z \in \mathbb{C}\setminus\{0\} \mid -\gamma_0 \leq \mathrm{Re} z \leq 0,\ |\mathrm{Im} z| \leq \dfrac{\gamma_0}{\mu_0} + \|\mathrm{Im}\, D\|\Big\}$, and the same inclusion holds if $\gamma_0 < \infty$ and $\delta_0 > 0$.*

*Proof.* By Proposition 2.3, we know $0 \notin \sigma(\mathcal{A})$. Thus, by Proposition 3.4 and Lemma 3.2, in all claims it suffices to prove that $\sigma(\mathcal{A}) \subset \overline{W(\mathcal{A})}$. As $\overline{W(\mathcal{A})}$ is convex, the set $\mathbb{C} \setminus \overline{W(\mathcal{A})}$ consists of one or two components. By [28, Theorem V.3.2], if a component $\Omega$ of $\mathbb{C} \setminus \overline{W(\mathcal{A})}$ contains a point $\lambda_0 \in \rho(\mathcal{A})$, then $\Omega \subset \rho(\mathcal{A})$. Since $0 \in \rho(\mathcal{A})$ by Proposition 2.3 and $\rho(\mathcal{A})$ is open, we always have $\{z \in \mathbb{C} \mid \mathrm{Re} z > 0\} \cap \rho(\mathcal{A}) \neq \emptyset$ and thus (i) follows. The assumption in (ii) ensures that also $\{z \in \mathbb{C} \mid \mathrm{Re} z < -\gamma_0\} \cap \rho(\mathcal{A}) \neq \emptyset$. By Lemma 3.2 the assumptions in (iii) guarantee that $\mathcal{A}$ is bounded which implies that $\sigma(\mathcal{A}) \subset \overline{W(\mathcal{A})}$ and hence the claim follows. □

**Remark 3.7.** Corollary 3.6 provides an alternative proof for the fact that $\sigma(\mathcal{A})$ is contained in the closed left half-plane, see Remark 2.4.

If the operator $D$ has some sectoriality property, then the numerical range of $\mathcal{A}$ is contained in some parabolic region, as the following result shows. We point out that the numerical range cannot lie in a sector with corner 0: recall from Proposition 3.4 (i) that $0 \in W(\mathcal{A})$. Thus 0 being a corner of $W(A)$ would imply $0 \in \sigma(\mathcal{A})$, cf. [35], a contradiction to Proposition 2.3.

**Proposition 3.8.** *Assume there exists $k \geq 0$ such that*

$$|\mathrm{Im}\langle Dz, z\rangle| \leq k \,\mathrm{Re}\langle Dz, z\rangle, \quad z \in H_1. \tag{15}$$

*If $\delta_0 > 0$, then, for every $\delta \in (0, \delta_0]$,*

$$\sigma(\mathcal{A}) \subset \overline{W(\mathcal{A})} \subset \left\{ \lambda \in \mathbb{C} \mid -\gamma \leq \mathrm{Re}\lambda \leq 0,\ |\mathrm{Im}\lambda| \leq k|\mathrm{Re}\lambda| + 2\sqrt{\frac{1}{\delta}|\mathrm{Re}\lambda|} \right\} \tag{16}$$

*where $\delta = \delta_0$ gives the best enclosure.*

*Proof.* Proposition 3.4 (ii) implies $W(\mathcal{A}) \subset \{\lambda \in \mathbb{C} \mid -\gamma \leq \mathrm{Re}\lambda \leq 0\}$. By Remark 3.3, we have $\lambda \in W(\mathcal{A})$ if and only if there exists $(f, g)^\top \in \mathcal{D}(\mathcal{A})$ with $\|f\|^2_{H_{\frac{1}{2}}} + \|g\|^2 = 1$ such that

$$\mathrm{Re}\lambda = -\mathrm{Re}\langle Dg, g\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}, \tag{17}$$

$$\mathrm{Im}\lambda = -2\,\mathrm{Im}\langle f, g\rangle_{H_{\frac{1}{2}}} - \mathrm{Im}\langle Dg, g\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}. \tag{18}$$

If $\delta_0 > 0$ and $\delta \in (0, \delta_0]$, then $\|g\|^2_{H_{\frac{1}{2}}} \leq \frac{1}{\delta}\mathrm{Re}\langle Dg, g\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}$. Using this estimate, $\|f\|_{H_{\frac{1}{2}}} \leq 1$, (17) and (18), we find

$$\begin{aligned} |\mathrm{Im}\lambda| &\leq 2\|f\|_{H_{\frac{1}{2}}}\|g\|_{H_{\frac{1}{2}}} + |\mathrm{Im}\langle Dg, g\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}| \\ &\leq 2\sqrt{\frac{1}{\delta}\mathrm{Re}\langle Dg, g\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}} + k\,\mathrm{Re}\langle Dg, g\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \\ &= 2\sqrt{\frac{1}{\delta}|\mathrm{Re}\lambda|} + k|\mathrm{Re}\lambda|, \end{aligned}$$

which proves the inclusion for $\overline{W(\mathcal{A})}$ in (16). This and the convexity of the set $\overline{W(\mathcal{A})}$ ensures that the complement $\mathbb{C} \setminus \overline{W(\mathcal{A})}$ has only one component, in both cases $\gamma_0 = \infty$ and $\gamma_0 < \infty$. Now the inclusion $\sigma(\mathcal{A}) \subset \overline{W(\mathcal{A})}$ in (16) follows from [28, Theorem V.3.2] in the same way as the inclusion in Corollary 3.6 since we know $0 \in \rho(\mathcal{A})$. □

**Remark 3.9.** Note that the condition $\delta_0 > 0$ implies that assumption (15) is satisfied for some $k \geq 0$ but not vice versa, cf. [26, Lemma 4.1]. In fact, if we denote the norm of $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ by $\|D\|_{\frac{1}{2},-\frac{1}{2}} := \|A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}\|$ and use the definition of $\delta_0$ in (9), we see that

$$\left(\mathrm{Re}\langle Dz,z\rangle_{H_{-\frac12}\times H_{\frac12}}\right)^2+\left(\mathrm{Im}\langle Dz,z\rangle_{H_{-\frac12}\times H_{\frac12}}\right)^2=\left|\langle Dz,z\rangle_{H_{\frac12}\times H_{-\frac12}}\right|^2\le\left(\|D\|_{\frac12,-\frac12}\||z\|^2_{\frac12}\right)^2$$
$$\le\frac{\|D\|^2_{\frac12,-\frac12}}{\delta_0^2}\left(\mathrm{Re}\langle Dz,z\rangle_{H_{-\frac12}\times H_{\frac12}}\right)^2.$$

This shows that $\frac{\|D\|_{\frac12,-\frac12}}{\delta_0}\ge 1$ and

$$|\mathrm{Im}\langle Dz,z\rangle_{H_{-\frac12}\times H_{\frac12}}|\le\left(\frac{\|D\|^2_{\frac12,-\frac12}}{\delta_0^2}-1\right)^{\frac12}\mathrm{Re}\langle Dz,z\rangle_{H_{-\frac12}\times H_{\frac12}}.$$

# 4 The quadratic numerical range (QNR) of $\mathcal{A}$

In this section we establish new spectral enclosures for the operator $\mathcal{A}$ in (6) by means of the so-called quadratic numerical range. The latter is defined for operators in a product Hilbert space $\mathcal{H}_1\times\mathcal{H}_2$ that admit a matrix representation with respect to some decomposition of the space, i.e. that have a domain of the form $\mathcal{D}_1\times\mathcal{D}_2$ with dense subspaces $\mathcal{D}_i$ of $\mathcal{H}_i$, $i=1,2$.

In general, such a decomposition of the domain of the operator $\mathcal{A}$ in (6) requires stronger assumptions on $D$; e.g. if $D$ maps $H_{\frac12}$ even into $H$, then $\mathcal{D}(\mathcal{A})=H_1\times H_{\frac12}$. Under the weaker assumptions (A2), (A3), $H_1\times H_1\subset\mathcal{D}(\mathcal{A})$ is a core of $\mathcal{A}$ by Proposition 2.3 and so the quadratic numerical range of the restriction $\mathcal{A}|_{H_1\times H_1}$ is defined as follows, see [41, Definition 2.5.1].

**Definition 4.1.** For $(f,g)^\top\in H_1\times H_1\subset\mathcal{D}(\mathcal{A})$, $f,g\neq 0$, let

$$\mathcal{A}_{f,g}:=\begin{bmatrix}0 & \dfrac{\langle g,f\rangle_{H_{\frac12}}}{\|f\|_{H_{\frac12}}\|g\|}\\[2ex] -\dfrac{\langle A_0f,g\rangle}{\|f\|_{H_{\frac12}}\|g\|} & -\dfrac{\langle Dg,g\rangle}{\|g\|^2}\end{bmatrix}\in M_2(\mathbb{C}).$$

The set of all eigenvalues of all these $2\times2$ matrices $\mathcal{A}_{f,g}$,

$$W^2(\mathcal{A}|_{H_1\times H_1}):=\bigcup_{\substack{(f,g)^\top\in H_1\times H_1,\\ f,g\neq0}}\sigma_p(\mathcal{A}_{f,g})=\bigcup_{\substack{(f,g)^\top\in H_1\times H_1,\\ \|f\|_{H_{\frac12}}=\|g\|=1}}\sigma_p(\mathcal{A}_{f,g}),$$

is called the *quadratic numerical range* of the operator matrix $\mathcal{A}|_{H_1\times H_1}$ in $H_{\frac12}\times H$.

**Remark 4.2.** The following equivalent description of $W^2(\mathcal{A}|_{H_1\times H_1})$ is useful, see [41, Proposition 1.1.3]. For $(f,g)^\top\in H_1\times H_1$ with $f,g\neq 0$, set

$$\Delta(f,g;\lambda):=\|f\|^2_{H_{\frac12}}\|g\|^2\det(\mathcal{A}_{f,g}-\lambda)=\|f\|^2_{H_{\frac12}}\|g\|^2\left(\lambda^2+\lambda\frac{\langle Dg,g\rangle}{\|g\|^2}+\frac{|\langle f,g\rangle_{H_{\frac12}}|^2}{\|f\|^2_{H_{\frac12}}\|g\|^2}\right).\tag{19}$$

Then

$$W^2(\mathcal{A}|_{H_1\times H_1})=\left\{\lambda\in\mathbb{C}\mid\exists\,(f,g)^\top\in H_1\times H_1,\ f,g\neq0:\Delta(f,g;\lambda)=0\right\}.\tag{20}$$

The quadratic numerical range is either connected or consists of two components; thus it is in general not convex, and even its components need not be so (see e.g. [34], [41, p. 4/5]).

An important property of the quadratic numerical range is that it is always contained in the numerical range. Together with Proposition 3.4, we obtain:

**Proposition 4.3.**

$$W^2(\mathcal{A}|_{H_1\times H_1}) \subset W(\mathcal{A}|_{H_1\times H_1}) \subset W(\mathcal{A}) \subset \{z \in \mathbb{C} \mid -\gamma \leq \mathrm{Re} z \leq 0\}.$$

*Proof.* The first inclusion was proved in [41, Theorem 2.5.3], the second one is obvious, and the third one was shown in Proposition 3.4 (ii). ☐

In general, the quadratic numerical range may be considerably smaller than the numerical range. The next proposition shows that the extreme points of their real parts are the same.

**Proposition 4.4.** *If* $\dim H > 1$*, then*

$$W(-D) \cup \{0\} \subset W^2(\mathcal{A}|_{H_1\times H_1}) \cap W(\mathcal{A}) \tag{21}$$

*and hence*

$$\inf\bigl(\mathrm{Re}\, W^2(\mathcal{A}|_{H_1\times H_1})\bigr) = -\gamma_0, \quad \max\bigl(\mathrm{Re}\, W^2(\mathcal{A}|_{H_1\times H_1})\bigr) = 0. \tag{22}$$

*Proof.* Since $\dim H_{\frac{1}{2}}, \dim H > 1$, the numerical ranges of the diagonal elements of $\mathcal{A}|_{H_1\times H_1}$, i.e. of the zero operator 0 in $H_{\frac{1}{2}}$ and of $D : H \to H$ with $\mathcal{D}(D) = H_1$, are contained in $W^2(\mathcal{A}|_{H_1\times H_1})$ by [41, Theorem 2.5.4]. This together with Proposition 3.4 (i) proves (21).

The claims in (22) follow from (21), Proposition 4.3 and Proposition 3.4 (ii). ☐

# 5 The spectral inclusion property of the QNR

In this section we establish the spectral inclusion property of $W^2(\mathcal{A}|_{H_1\times H_1})$ under our standard assumptions (A1)–(A3). To obtain inclusions for the spectrum of $\mathcal{A}$, we use that $\mathcal{A} = \overline{\mathcal{A}|_{H_1\times H_1}}$ by Proposition 2.3 and hence, see e.g. [41, Lemma 2.5.16],

$$\sigma_p(\mathcal{A}) \subset \sigma_{ap}(\mathcal{A}|_{H_1\times H_1}), \quad \sigma_{ap}(\mathcal{A}) = \sigma_{ap}(\mathcal{A}|_{H_1\times H_1}). \tag{23}$$

**Theorem 5.1.** *We have*

$$\sigma_p(\mathcal{A}|_{H_1\times H_1}) \subset W^2(\mathcal{A}|_{H_1\times H_1}), \quad \sigma_{ap}(\mathcal{A}|_{H_1\times H_1}) \subset \overline{W^2(\mathcal{A}|_{H_1\times H_1})}, \tag{24}$$

*and hence*

$$\sigma_p(\mathcal{A}) \subset \overline{W^2(\mathcal{A}|_{H_1\times H_1})}, \qquad \sigma_{ap}(\mathcal{A}) \subset \overline{W^2(\mathcal{A}|_{H_1\times H_1})}. \tag{25}$$

*Proof.* It suffices to prove the inclusions (24); the inclusions (25) follow from (24) by means of (23).

The inclusion of the point spectrum in (24) was proved in [41, Theorem 2.5.9]. To prove the inclusion of the approximate point spectrum, let $\lambda \in \sigma_{ap}(\mathcal{A}|_{H_1\times H_1}) = \sigma_{ap}(\mathcal{A})$. Then, by Proposition 2.3, $\lambda \neq 0$, $\mathrm{Re}\lambda \leq 0$, and there exists a sequence $((f_n, g_n)^\top)_{n\in\mathbb{N}}$ in $H_1 \times H_1$ with

$$\left\| \begin{bmatrix} f_n \\ g_n \end{bmatrix} \right\|_{H_{\frac{1}{2}}\times H} = 1, \qquad \lim_{n\to\infty} \left\| (\mathcal{A} - \lambda) \begin{bmatrix} f_n \\ g_n \end{bmatrix} \right\|_{H_{\frac{1}{2}}\times H} = 0.$$

Then we have

$$\|f_n\|^2_{H_{\frac{1}{2}}} + \|g_n\|^2 = 1 \tag{26}$$

and

$$\|g_n - \lambda f_n\|_{H_{\frac{1}{2}}} \to 0, \tag{27}$$

$$\|A_0 f_n + D g_n + \lambda g_n\| \to 0, \quad n \to \infty. \tag{28}$$

Without loss of generality, we may assume that

$$a := \lim_{n\to\infty} \|f_n\|^2_{H_{\frac{1}{2}}}$$

exists. Then $b := \lim_{n\to\infty} \|g_n\|^2 = 1 - a$, by (26). If $a = 0$, then (27) and (10) imply that $b = 0$, a contradiction to $b = 1 - a$. Hence we have $a > 0$.

Now we consider the sequence of polynomials

$$\Delta(f_n, g_n; z) = \det \begin{bmatrix} -z\langle f_n, f_n\rangle_{H_{\frac{1}{2}}} & \langle g_n, f_n\rangle_{H_{\frac{1}{2}}} \\ -\langle A_0 f_n, g_n\rangle & -\langle Dg_n, g_n\rangle - z\langle g_n, g_n\rangle \end{bmatrix}, \quad z \in \mathbb{C},\ n \in \mathbb{N}. \tag{29}$$

By (27) we obtain

$$\lim_{n\to\infty} \langle g_n, f_n\rangle_{H_{\frac{1}{2}}} = \lim_{n\to\infty} \langle \lambda f_n, f_n\rangle_{H_{\frac{1}{2}}} = \lambda a. \tag{30}$$

It follows that $\lim_{n\to\infty}\langle A_0 f_n, g_n\rangle = \overline{\lambda} a$. Note that $(g_n)_{n\in\mathbb{N}}$ is bounded in $H$ by (26). Thus, using (28) and the definitions of $a, b$, we deduce that

$$\lim_{n\to\infty} \langle Dg_n, g_n\rangle = -\lim_{n\to\infty} \langle A_0 f_n + \lambda g_n, g_n\rangle = -\overline{\lambda} a - \lambda b. \tag{31}$$

Then, by (29), (30) and (31), it follows that

$$\Delta(f_n, g_n; z) \to \det \begin{bmatrix} -za & \lambda a \\ -\overline{\lambda} a & \overline{\lambda} a + \lambda b - zb \end{bmatrix} =: \Delta(z), \quad n \to \infty,$$

uniformly for $z$ in compact subsets of $\mathbb{C}$. It is easy to see that $\Delta(\lambda) = 0$ and $\Delta \not\equiv 0$ since $\lambda a \neq 0$. Hence, by Hurwitz' theorem (see e.g. [11, Theorem VII.2.5]), for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ with the property that, for $n \geq N$, the quadratic polynomial $\Delta(f_n, g_n; z)$ has a zero $z_{n,1} \in \mathbb{C}$ with $|z_{n,1} - \lambda| < \varepsilon$. Since $z_{n,1} \in W^2(\mathcal{A}|_{H_1\times H_1})$, it follows that $\lambda \in \overline{W^2(\mathcal{A}|_{H_1\times H_1})}$. □

**Proposition 5.2.** *If, in addition to the assumptions* (A2), (A3)*, the operator $D$ maps the space $H_{\frac{1}{2}}$ into $H$, then*

$$\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}|_{H_1\times H_1}) \subset W^2(\mathcal{A}|_{H_1\times H_1}). \tag{32}$$

*Proof.* By Theorem 5.1, we only have to prove the first identity. If $D$ maps $H_{\frac{1}{2}}$ into $H$, we have $\mathcal{D}(\mathcal{A}) = H_1 \times H_{\frac{1}{2}}$. Since an eigenvector $(f, g)^\top \in \mathcal{D}(\mathcal{A}) = H_1 \times H_{\frac{1}{2}}$ of $\mathcal{A}$ at an eigenvalue $\lambda \in \sigma_p(\mathcal{A})$ satisfies

$$-\lambda f + g = 0,$$

we see that also $g \in H_1$, and $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}|_{H_1\times H_1})$ follows. □

**Remark 5.3.** The stronger assumption in Proposition 5.2 is satisfied if e.g. $D = A_0^\theta$ for some $\theta \in (-\infty, \frac{1}{2}]$.

The following inclusion of the spectrum is immediate from Theorem 5.1.

**Theorem 5.4.** *If a component $\Omega$ of $\mathbb{C} \setminus \overline{W^2(\mathcal{A}|_{H_1\times H_1})}$ contains a point $\lambda_0 \in \rho(\mathcal{A})$, then $\Omega \subset \rho(\mathcal{A})$; in particular, if every component of $\mathbb{C} \setminus \overline{W^2(\mathcal{A}|_{H_1\times H_1})}$ contains a point $\lambda_0 \in \rho(\mathcal{A})$, then*

$$\sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A}|_{H_1\times H_1})}.$$

*Proof.* The claim follows from Theorem 5.1 and the fact that the boundary of the spectrum $\sigma(\mathcal{A})$ belongs to $\sigma_{ap}(\mathcal{A})$, see e.g. [14, IV §1.10]. Alternatively, it follows from Theorem 5.1 and the fact that the mapping $\lambda \mapsto \dim \mathcal{R}(\mathcal{A} - \lambda)^\perp$ is locally constant, see [28, Theorem V.3.2]. □

# 6 Uniformly accretive and sectorial damping: estimates for QNR and spectrum

In this section and the next we show how special properties of the damping operator $D$ such as uniform accretivity and sectoriality are reflected in the quadratic numerical range $W^2(\mathcal{A}|_{H_1\times H_1})$. As a result we obtain new bounds on the spectrum of $\mathcal{A}$ which improve the bounds by the numerical range, see Proposition 3.8, considerably.

In particular, we show that the spectrum is separated into two parts by a spectral free strip if $\beta_0\delta_0 > 4$; in this case, $D$ is uniformly accretive with $\inf(\operatorname{Re} W(D)) = \beta_0 \geq a_0^2\delta_0 > 0$, see (9) and (11). Note that the spectral free strip has to lie between $-\beta_0$ and 0 since $W(-D) \subset W^2(\mathcal{A}|_{H_1\times H_1})$ by (21). We also show that, unlike the numerical range, $W^2(\mathcal{A}|_{H_1\times H_1})$ may lie in a sector with corner 0 even though $0 \notin \sigma(\mathcal{A})$ since the zero operator on the diagonal of $\mathcal{A}$ has 0 in its spectrum, cf. [35, Theorem 3.1].

We begin with a spectral enclosure for the case $\delta_0 > 0$ which may be used if no explicit information on $\operatorname{Im}W(D)$ is available; in this case the only enclosure provided by the numerical range $W(\mathcal{A})$ is the left half-plane.

**Theorem 6.1.** *Suppose that $\delta_0 > 0$ and hence $\beta_0 > 0$, so that $D$ is uniformly accretive. Then, for every $\beta \in (0, \beta_0]$, $\delta \in (0, \delta_0]$,*

$$\sigma(\mathcal{A}) \subset \big\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0,\ |\operatorname{Re}\lambda| \notin I_0,\ |\operatorname{Im}\lambda| \leq h_0(|\operatorname{Re}\lambda|)\big\}$$

*where $I_0$ is a (possibly empty) interval centred at $\frac{\beta}{2}$, given by*

$$I_0 := \begin{cases} \emptyset & \text{if } \beta\delta \leq 4, \\ \Big(\frac{\beta}{2}\Big(1-\sqrt{1-\frac{4}{\beta\delta}}\Big), \frac{\beta}{2}\Big(1+\sqrt{1-\frac{4}{\beta\delta}}\Big)\Big) & \text{if } \beta\delta > 4, \end{cases} \tag{33}$$

*and*

$$h_0(t) := \begin{cases} \sqrt{\dfrac{\beta}{\delta}\dfrac{t}{\beta-t} - t^2}, & 0 \leq t < \beta,\ t \notin I_0, \\ \infty, & \beta \leq t < \infty; \end{cases}$$

*in particular, if $\beta\delta > 4$, then $\mathcal{A}$ has a spectral free strip around $\operatorname{Re}\lambda = -\frac{\beta}{2}$. The choice $\beta = \beta_0$, $\delta = \delta_0$ gives the best enclosure and, if $\beta_0\delta_0 > 4$, the widest spectral free strip around $\operatorname{Re}\lambda = -\frac{\beta_0}{2}$. If $\gamma_0 < \infty$ and there is a $\lambda_0 \in \rho(\mathcal{A})$ with $\operatorname{Re}\lambda_0 < -\gamma_0$, then $\sigma(\mathcal{A}) \cap \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < -\gamma_0\} = \emptyset$.*

*Proof.* If we show that $W^2(\mathcal{A}|_{H_1\times H_1}) \setminus \{0\}$ satisfies the asserted inclusion, then so does $\sigma(\mathcal{A})$ due to Theorem 5.4, the fact that $0 \in \rho(\mathcal{A})$ by Proposition 2.3 and that $h_0$ is bounded on the subinterval in $[0, \frac{\beta}{2}]$ where it is defined with $h_0(0) = 0$.

Let $\beta \in (0, \beta_0]$, $\delta \in (0, \delta_0]$. Since $\operatorname{Re} W^2(\mathcal{A}|_{H_1\times H_1}) \leq 0$ it suffices to consider $\lambda \in W^2(\mathcal{A}|_{H_1\times H_1})$ with $-\beta < \operatorname{Re}\lambda \leq 0$. By Definition 4.1, there exists $(f,g)^\top \in H_1 \times H_1$, $\|f\|_{H_{\frac{1}{2}}} = \|g\| = 1$, with

$$0 = \det(\mathcal{A}_{f,g} - \lambda) = \lambda\big(\lambda + \langle Dg, g\rangle\big) + |\langle f, g\rangle_{H_{\frac{1}{2}}}|^2. \tag{34}$$

Together with $|\langle f, g\rangle_{H_{\frac{1}{2}}}|^2 \leq \|g\|^2_{H_{\frac{1}{2}}} \leq \dfrac{\operatorname{Re}\langle Dg, g\rangle}{\delta}$, this implies that

$$\operatorname{Re}\langle Dg, g\rangle = -|\langle f, g\rangle_{H_{\frac{1}{2}}}|^2 \operatorname{Re}\frac{1}{\lambda} - \operatorname{Re}\lambda \leq \Big(\frac{\operatorname{Re}\langle Dg, g\rangle}{\delta}\frac{1}{|\lambda|^2} + 1\Big)|\operatorname{Re}\lambda|$$

and hence
$$\frac{1}{|\mathrm{Re}\lambda|} \le \frac{1}{\delta|\lambda|^2} + \frac{1}{\mathrm{Re}\langle Dg, g\rangle}.$$
Using this estimate and $\mathrm{Re}\langle Dg, g\rangle \ge \beta > |\mathrm{Re}\lambda| > 0$, we obtain
$$|\mathrm{Re}\lambda|^2 + |\mathrm{Im}\lambda|^2 = |\lambda|^2 \le \frac{1}{\delta}\Big(\frac{1}{|\mathrm{Re}\lambda|} - \frac{1}{\mathrm{Re}\langle Dg, g\rangle}\Big)^{-1} \le \frac{1}{\delta}\Big(\frac{1}{|\mathrm{Re}\lambda|} - \frac{1}{\beta}\Big)^{-1} = \frac{1}{\delta}\frac{\beta|\mathrm{Re}\lambda|}{\beta - |\mathrm{Re}\lambda|},$$
which proves the claimed spectral inclusion. Note that, if $\beta\delta > 4$, then estimating the left hand side above further by $|\mathrm{Re}\lambda|^2 \le |\mathrm{Re}\lambda|^2 + |\mathrm{Im}\lambda|^2$ yields that $|\mathrm{Re}\lambda|$ must satisfy the inequality $|\mathrm{Re}\lambda|(\beta - |\mathrm{Re}\lambda|) \le \frac{\beta}{\delta}$ or, equivalently, $\big||\mathrm{Re}\lambda| - \frac{\beta}{2}\big|^2 \ge \big(\frac{\beta}{2}\big)^2\big(1 - \frac{4}{\beta\delta}\big) > 0$.

It is not difficult to see that, for increasing $\delta \in (0, \delta_0]$ and $\beta \in (0, \beta_0]$, the enclosure gets tighter since the right endpoint of $I_0$ is increasing, the left end-point is decreasing and $h_0(t)$ is decreasing. The last assertion follows from Proposition 4.3. □

**Theorem 6.2.** *Assume there exists $k \ge 0$ such that*
$$|\mathrm{Im}\langle Dz, z\rangle| \le k\,\mathrm{Re}\langle Dz, z\rangle, \quad z \in H_1. \tag{35}$$

(i) *If $\beta_0 > 0$ and $\beta \in (0, \beta_0]$, then*
$$\sigma(\mathcal{A}) \subset \big\{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda < 0,\ |\mathrm{Im}\lambda| \le h_{\mathrm{i}}(|\mathrm{Re}\lambda|)\big\} \tag{36}$$
*where $h_{\mathrm{i}} : [0, \infty) \to [0, \infty]$ is given by*
$$h_{\mathrm{i}}(t) := \begin{cases} \frac{1}{1-\frac{2}{\beta}t}kt, & 0 \le t < \dfrac{\beta}{2}, \\ \infty, & \dfrac{\beta}{2} \le t < \infty; \end{cases} \tag{37}$$
*if $\gamma_0 < \infty$ and there is a $\lambda_0 \in \rho(\mathcal{A})$ with $\mathrm{Re}\lambda_0 < -\gamma_0$, then*
$$\sigma(\mathcal{A}) \subset \big\{\lambda \in \mathbb{C} \mid -\gamma_0 \le \mathrm{Re}\lambda < 0,\ |\mathrm{Im}\lambda| \le h_{\mathrm{i}}(|\mathrm{Re}\lambda|)\big\}.$$

(ii) *If $\mu_0 > 0$ and $\mu \in (0, \mu_0]$, then*
$$\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C} \mid -\gamma_0 \le \mathrm{Re}\lambda < 0,\ |\mathrm{Im}\lambda| \le h_{\mathrm{ii}}(|\mathrm{Re}\lambda|)\} \tag{38}$$
*where $h_{\mathrm{ii}} : [0, \infty) \to [0, \infty)$ is given by*
$$h_{\mathrm{ii}}(t) := k_\mu t, \quad k_\mu^2 := \frac{2}{\mu^2} + \frac{k^2-1}{2} + \sqrt{\Big(\frac{2}{\mu^2} + \frac{k^2-1}{2}\Big)^2 + k^2}, \tag{39}$$
*with $k_\mu \in [0, \infty)$ satisfying $k \le k_\mu \le \sqrt{k^2 + \frac{4}{\mu^2}}$.*

(iii) *If $\delta_0 > 0$ and $\delta \in (0, \delta_0]$, then*
$$\sigma(\mathcal{A}) \subset \big\{\lambda \in \mathbb{C} \mid -\gamma \le \mathrm{Re}\lambda < 0,\ |\mathrm{Im}\lambda| \le h_{\mathrm{iii}}(|\mathrm{Re}\lambda|)\big\} \tag{40}$$
*where $h_{\mathrm{iii}} : [0, \infty) \to [0, \infty)$ is defined by $h_{\mathrm{iii}}(t)$ being the largest non-negative solution $y$ of*
$$(y^2 + t^2)(y - kt) = \frac{2}{\delta}ty, \tag{41}$$
*which satisfies the estimates*
$$kt \le h_{\mathrm{iii}}(t) \le \min\Big\{kt + \frac{1}{\delta}, \frac{kt}{2} + \sqrt{\Big(\frac{kt}{2}\Big)^2 + \frac{2t}{\delta}}\Big\} \le kt + \min\Big\{\frac{1}{\delta}, \sqrt{\frac{2t}{\delta}}\Big\}, \quad t \in [0, \infty). \tag{42}$$

*The choice $\beta = \beta_0$, $\mu = \mu_0$ and $\delta = \delta_0$, respectively, yields the best enclosures.*

**Remark 6.3.** (a) If $k > 0$, then the function $\mu \mapsto k_\mu$ is strictly decreasing on $(0, \infty)$ from a pole at $\mu = 0$ to $\lim_{\mu\to\infty} k_\mu = k$; for $k = 0$, it is strictly decreasing on $(0, 2)$ and equal to 0 for $\mu \geq 2$,

$$k_\mu^2 = \frac{2}{\mu^2} - \frac{1}{2} + \left|\frac{2}{\mu^2} - \frac{1}{2}\right| = \begin{cases} \frac{4}{\mu^2} - 1, & 0 < \mu < 2, \\ 0, & \mu \geq 2, \end{cases} \qquad \text{if } k = 0. \tag{43}$$

Note that, in general, $k_\mu = k$ if and only if $k = 0$ and $\mu \geq 2$.

(b) The spectral enclosure (40) by the quadratic numerical range in Theorem 6.2 (iii) is better than the one by the numerical range in Proposition 3.8; indeed, the term $\sqrt{\frac{2t}{\delta}}$ in the last upper bound for $h_{\mathrm{iii}}$ in (42) is better than the corresponding term there by a factor of $\sqrt{2}$.

*Proof of Theorem* 6.2. If we show that $W^2(\mathcal{A}|_{H_1\times H_1}) \setminus \{0\}$ satisfies the asserted inclusions, then so does $\sigma(\mathcal{A})$ due to Theorem 5.4, the fact that $0 \in \rho(\mathcal{A})$ by Proposition 2.3, and $h_k(0) = 0$ for $k \in \{\mathrm{i}, \mathrm{ii}, \mathrm{iii}\}$.

Let $\lambda \in W^2(\mathcal{A}|_{H_1\times H_1}) \setminus \{0\}$. Proposition 4.3 implies that $-\gamma_0 \leq \mathrm{Re}\lambda \leq 0$, so we only have to show the estimates for $\mathrm{Im}\lambda$. Further, we can assume that $\mathrm{Im}\lambda \neq 0$ since all enclosing sets contain $\{t \in \mathbb{R} \mid -\gamma \leq t < 0\}$. By Definition 4.1, there exists $(f, g)^\top \in H_1 \times H_1$ with $\|f\|_{H_{1/2}} = \|g\| = 1$ such that (34) holds. Dividing by $\lambda$ and taking real and imaginary parts, we obtain

$$\mathrm{Re}\langle Dg, g\rangle = \left(\frac{|\langle f, g\rangle_{H_{1/2}}|^2}{|\lambda|^2} + 1\right)|\mathrm{Re}\lambda|, \tag{44}$$

$$\mathrm{Im}\langle Dg, g\rangle = \left(\frac{|\langle f, g\rangle_{H_{1/2}}|^2}{|\lambda|^2} - 1\right)\mathrm{Im}\lambda. \tag{45}$$

Since in all cases $\beta_0 > 0$, either by assumption or because of (11), we have $\mathrm{Re}\langle Dg, g\rangle \neq 0$. Thus (44) implies $\mathrm{Re}\lambda \neq 0$ and we conclude

$$\frac{\mathrm{Im}\langle Dg, g\rangle}{\mathrm{Im}\lambda} = \frac{\mathrm{Re}\langle Dg, g\rangle}{|\mathrm{Re}\lambda|} - 2 = \frac{\mathrm{Re}\langle Dg, g\rangle - 2\,|\mathrm{Re}\lambda|}{|\mathrm{Re}\lambda|}, \tag{46}$$

$$\frac{\mathrm{Re}\langle Dg, g\rangle}{|\mathrm{Re}\lambda|} + \frac{\mathrm{Im}\langle Dg, g\rangle}{\mathrm{Im}\lambda} = 2\frac{|\langle f, g\rangle_{H_{1/2}}|^2}{|\lambda|^2}. \tag{47}$$

(i) Let $\beta \in (0, \beta_0]$. Assume that $|\mathrm{Re}\lambda| < \frac{\beta}{2}$. By (44) and the definition of $\beta_0$ in (9), we have

$$\frac{|\langle f, g\rangle_{H_{1/2}}|^2}{|\lambda|^2} + 1 = \frac{\mathrm{Re}\langle Dg, g\rangle}{|\mathrm{Re}\lambda|} \geq \frac{\beta_0}{|\mathrm{Re}\lambda|} \geq \frac{\beta}{|\mathrm{Re}\lambda|} \;(> 2).$$

Then, from (44), (45), (35) and the above estimate it follows that

$$\frac{|\mathrm{Im}\lambda|}{|\mathrm{Re}\lambda|} = \frac{|\mathrm{Im}\langle Dg, g\rangle|}{|\mathrm{Re}\langle Dg, g\rangle|}\,\frac{\frac{|\langle f,g\rangle_{H_{1/2}}|^2}{|\lambda|^2} + 1}{\left|\frac{|\langle f,g\rangle_{H_{1/2}}|^2}{|\lambda|^2} - 1\right|} \leq k\,\frac{1}{1 - 2\left(\frac{|\langle f,g\rangle_{H_{1/2}}|^2}{|\lambda|^2} + 1\right)^{-1}} \leq k\,\frac{1}{1 - \frac{2}{\beta}|\mathrm{Re}\lambda|}.$$

(ii) Let $\mu \in (0, \mu_0]$. By (44) and (45) we obtain

$$\left(\frac{\mathrm{Re}\langle Dg, g\rangle}{\mathrm{Re}\lambda}\right)^2 - \left(\frac{\mathrm{Im}\langle Dg, g\rangle}{\mathrm{Im}\lambda}\right)^2 = 4\,\frac{|\langle f, g\rangle_{H_{1/2}}|^2}{|\lambda|^2}.$$

Multiplying this identity by $\dfrac{|\lambda|^2}{(\mathrm{Re}\langle Dg,g\rangle)^2}$, we infer that

$$\frac{|\lambda|^2}{|\mathrm{Re}\lambda|^2}-\frac{|\lambda|^2}{|\mathrm{Im}\lambda|^2}\left(\frac{\mathrm{Im}\langle Dg,g\rangle}{\mathrm{Re}\langle Dg,g\rangle}\right)^2=4\left(\frac{|\langle f,g\rangle_{H_{1/2}}|}{\mathrm{Re}\langle Dg,g\rangle}\right)^2.$$

Using $\|f\|_{H_{1/2}}=\|g\|=1$ and the definition of $\mu_0$ in (9), we estimate $|\langle f,g\rangle_{H_{1/2}}|\le\|g\|_{H_{1/2}}$ $\le\frac{1}{\mu_0}\mathrm{Re}\langle Dg,g\rangle\le\frac{1}{\mu}\mathrm{Re}\langle Dg,g\rangle$. Thus from the sectoriality of $D$, i.e. from (35), it follows that

$$1-k^2+\frac{|\mathrm{Im}\lambda|^2}{|\mathrm{Re}\lambda|^2}-\frac{|\mathrm{Re}\lambda|^2}{|\mathrm{Im}\lambda|^2}k^2=\frac{|\lambda|^2}{|\mathrm{Re}\lambda|^2}-\frac{|\lambda|^2}{|\mathrm{Im}\lambda|^2}k^2\le\frac{4}{\mu^2}.$$

Hence

$$\frac{|\mathrm{Im}\lambda|^2}{|\mathrm{Re}\lambda|^2}\left(\frac{|\mathrm{Im}\lambda|^2}{|\mathrm{Re}\lambda|^2}+1-k^2-\frac{4}{\mu^2}\right)-k^2\le 0.$$

The latter is a quadratic inequality for $\frac{|\mathrm{Im}\lambda|^2}{|\mathrm{Re}\lambda|^2}$. If we note that $1-k^2-\frac{4}{\mu^2}=-2\big(\frac{2}{\mu^2}+\frac{k^2-1}{2}\big)$, we see that this inequality is satisfied if and only if $\frac{|\mathrm{Im}\lambda|^2}{|\mathrm{Re}\lambda|^2}\le k_\mu^2$, due to the definition of $k_\mu^2$.

The inequalities for $k_\mu$ are not difficult to check: for the lower bound we note that $k_\mu^2$ is strictly decreasing in $\mu$ and $\lim_{\mu\to\infty}k_\mu^2=k^2$; for the upper bound we use the inequality $\big(\frac{2}{\mu^2}+\frac{k^2-1}{2}\big)^2+k^2\le\big(\frac{2}{\mu^2}+\frac{k^2+1}{2}\big)^2$.

(iii) Let $\delta\in(0,\delta_0]$. Multiplying (47) by $\dfrac{|\mathrm{Re}\lambda|\,\mathrm{Im}\lambda\,|\lambda|^2}{\mathrm{Re}\langle Dg,g\rangle}$, we conclude that

$$|\lambda|^2\Big(\mathrm{Im}\lambda+\frac{\mathrm{Im}\langle Dg,g\rangle}{\mathrm{Re}\langle Dg,g\rangle}|\mathrm{Re}\lambda|\Big)=2\frac{|\langle f,g\rangle_{H_{1/2}}|^2}{\mathrm{Re}\langle Dg,g\rangle}|\mathrm{Re}\lambda|\,\mathrm{Im}\lambda.$$

From the sectoriality of $D$, i.e. from (35), the inequality $|\langle f,g\rangle_{H_{1/2}}|^2\le\|g\|_{H_{1/2}}^2$, and the definition of $\delta_0$ in (9), it follows that

$$|\lambda|^2\big(|\mathrm{Im}\lambda|-k|\mathrm{Re}\lambda|\big)\le\frac{2}{\delta_0}|\mathrm{Re}\lambda||\mathrm{Im}\lambda|\le\frac{2}{\delta}|\mathrm{Re}\lambda||\mathrm{Im}\lambda|,$$

which is satisfied if and only if $|\mathrm{Im}\lambda|\le h_{\mathrm{iii}}(|\mathrm{Re}\lambda|)$ by definition of $h_{\mathrm{iii}}$.

The three upper bounds for $h_{\mathrm{iii}}$ in (42) are not difficult to check: for the first bound in the first inequality we use the estimate $2ty\le t^2+y^2$ on the right hand side of (41), while for the second bound in the first inequality we use $y^2\le(y^2+t^2)$ on the left hand side of (41); the very last bound is obvious. □

**Remark 6.4.** By means of a different method, the spectral inclusion of Theorem 6.2 (i) was also shown in [26, Theorem 4.2], while Theorem 6.2 (iii) improves the corresponding statement of [26, Theorem 4.2].

Note that due to (11), $\mu_0>0$ implies $\beta_0>0$, and $\delta_0>0$ implies $\mu_0>0$ and thus $\beta_0>0$. Therefore if, in Theorem 6.2, (ii) applies then so does (i) and if (iii) applies, then so do (i) and (ii).

In the following Proposition 6.5 we work out the precise form of the corresponding intersections of the bounding sets in Theorem 6.1 and Theorem 6.2 (i), (ii), and (iii).

Figures 1–4 below illustrate how the spectral enclosures by means of the quadratic numerical range (red for colour online/pdf version, dark grey for black and white print) compare to those obtained by means of the numerical range (in light grey) and how the enclosures improve successively for the cases $\beta>0$, $\mu>0$, $\delta>0$, and $\beta\delta>4$.

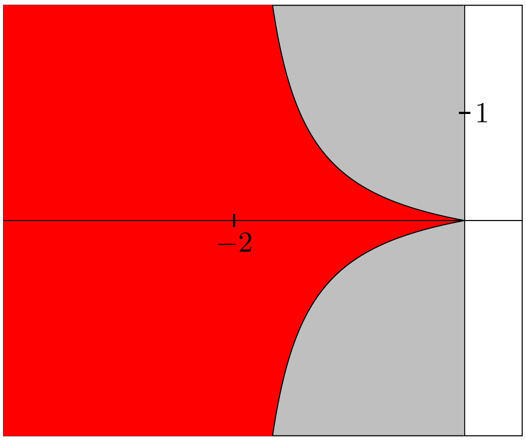


Figure 1:
Theorem 6.2 (i) with $k=0.2$, $\beta=4$, $\mu=0$, $\delta=0$.

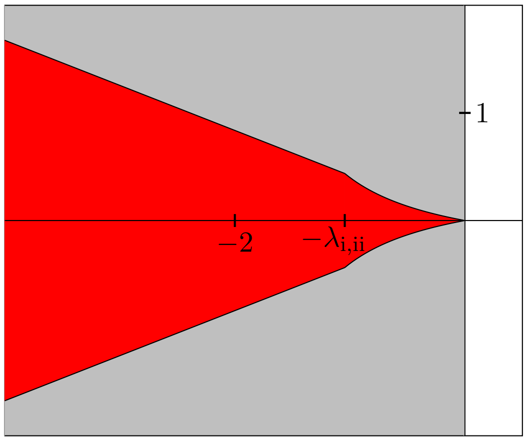


Figure 2:
Theorem 6.2 (i), (ii) with $k=0.2$, $\beta=4$, $\mu=2.1$, $\delta=0$; here $\lambda_{\mathrm{i,ii}}\approx 1.04$.

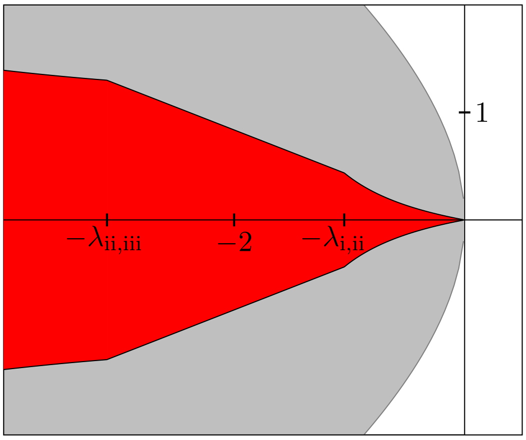


Figure 3:
Theorem 6.2 (i), (ii), (iii) without Theorem 6.1 with $k=0.2$, $\beta=4$, $\mu=2.1$, $\delta=1.05$; here $\lambda_{\mathrm{i,ii}}\approx 1.04$, $\lambda_{\mathrm{ii,iii}}\approx 3.10$, see Remark 6.6.

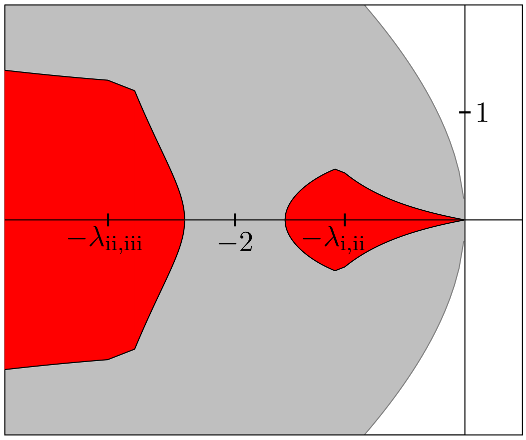


Figure 4:
Theorem 6.2 (i), (ii), (iii) and Theorem 6.1 with $k=0.2$, $\beta=4$, $\mu=2.1$, $\delta=1.05$; here $\beta\delta>4$, $k>\frac{4}{\beta\delta}-1$, $\lambda_{\mathrm{i,ii}}\approx 1.04$, $\lambda_{\mathrm{ii,iii}}\approx 3.10$, $I_{0,\mu}\approx(1.12,2.87)$, $I_0\approx(1.56,2.44)$, see Remark 6.6.

Figures 1–4: Spectral enclosures obtained from $W(\mathcal{A})$ (light grey) and from $W^2(\mathcal{A})$ (red for colour online/pdf version, dark grey for black and white print).

**Proposition 6.5.** *Suppose that condition* (35) *holds and define*

$$\lambda_{\mathrm{i,ii}} := \frac{\beta}{2}\Big(1-\frac{k}{k_\mu}\Big) \in \Big[0,\frac{\beta}{2}\Big], \qquad \text{if } \mu_0>0 \ (\text{which implies } \beta_0>0),$$

$$\lambda_{\mathrm{ii,iii}} := \begin{cases} \dfrac{\mu^2}{2\delta}\Big(1+\dfrac{k}{k_\mu}\Big) \in \Big[\dfrac{\beta}{2},\dfrac{\mu^2}{\delta}\Big), & k_\mu > k, \\ \infty, & k_\mu = k = 0, \end{cases} \qquad \text{if } \delta_0>0 \ (\text{which implies } \mu_0>0 \text{ and } \beta_0>0)$$

*where* $\beta \in (0,\beta_0]$, $\mu \in (0,\mu_0]$, $\delta \in (0,\delta_0]$, *respectively, and* $\beta\delta \le \mu^2$ *if* $\delta_0 > 0$. *Then the spectrum of* $\mathcal{A}$ *satisfies the following inclusions:*

(a) *if* $\mu_0 > 0$ (*and hence* $\beta_0 > 0$), *then*

$$\sigma(\mathcal{A}) \subset \left\{ \lambda \in \mathbb{C} \mid -\gamma_0 \le \mathrm{Re}\lambda < 0,\ |\mathrm{Im}\lambda| \le \begin{cases} \dfrac{1}{1-\frac{2}{\beta}|\mathrm{Re}\lambda|}\, k|\mathrm{Re}\lambda|, & 0<|\mathrm{Re}\lambda|\le\lambda_{\mathrm{i,ii}} \\ k_\mu|\mathrm{Re}\lambda|, & \lambda_{\mathrm{i,ii}}<|\mathrm{Re}\lambda|\le\gamma_0 \end{cases} \right\};$$

(b) *if* $\delta_0 > 0$ (*and hence* $\mu_0 > 0$, $\beta_0 > 0$) *and* $\beta\delta \le \mu^2$, *then* $\lambda_{\mathrm{i,ii}} \le \lambda_{\mathrm{ii,iii}}$ *and*

$$\sigma(\mathcal{A}) \subset \left\{ \lambda\in\mathbb{C} \mid -\gamma_0\le\mathrm{Re}\lambda<0,\ |\mathrm{Re}\lambda|\notin I_0,\ |\mathrm{Im}\lambda|\le \begin{cases} \dfrac{1}{1-\frac{2}{\beta}|\mathrm{Re}\lambda|}\, k|\mathrm{Re}\lambda|, & |\mathrm{Re}\lambda|\in[0,\lambda_{\mathrm{i,ii}}] \\ k_\mu|\mathrm{Re}\lambda|, & |\mathrm{Re}\lambda|\in(\lambda_{\mathrm{i,ii}},\lambda_{\mathrm{ii,iii}})\setminus I_{0,\mu} \\ h_0(|\mathrm{Re}\lambda|), & |\mathrm{Re}\lambda|\in I_{0,\mu}\setminus I_0, \\ h_{\mathrm{iii}}(|\mathrm{Re}\lambda|), & |\mathrm{Re}\lambda|\in[\lambda_{\mathrm{ii,iii}},\gamma_0) \end{cases} \right\}$$

*where* $I_0$, $h_0$ *are as defined in Theorem* 6.1, $k_\mu$, $h_{\mathrm{iii}}$ *are as defined in Theorem* 6.2, *and* $I_{0,\mu}$ *is a* (*possibly empty*) *interval centred at* $\frac{\beta}{2}$, $I_0 \subset I_{0,\mu} \subset \big(\lambda_{\mathrm{i,ii}},\lambda_{\mathrm{ii,iii}}\big)$, *given by*

$$I_{0,\mu} := \begin{cases} \emptyset & \text{if } k_\mu^2 \le \frac{4}{\beta\delta}-1, \\ \Big(\frac{\beta}{2}\Big(1-\sqrt{1-\frac{4}{\beta\delta}\frac{1}{k_\mu^2+1}}\Big), \frac{\beta}{2}\Big(1+\sqrt{1-\frac{4}{\beta\delta}\frac{1}{k_\mu^2+1}}\Big)\Big) & \text{if } k_\mu^2 > \frac{4}{\beta\delta}-1, \end{cases}$$

*which satisfies* $I_0 = I_{0,\mu}$ *if and only if* $k_\mu = 0$ *and* $I_{0,\mu} = \big(\lambda_{\mathrm{i,ii}},\lambda_{\mathrm{ii,iii}}\big)$ *if and only if* $\mu^2 = \beta\delta$. *The choice* $\beta = \beta_0$, $\mu = \mu_0$ *and* $\delta = \delta_0$, *respectively, gives the best enclosures; in this case, the assumption* $\beta_0\delta_0 \le \mu_0^2$ *in* (b) *is automatically satisfied by* (11).

**Remark 6.6.** If the interval $I_{0,\mu}$ is non-empty, then Theorem 6.1 gives an improvement of Theorem 6.2. This improvement is most substantial if even $I_0$ is non-empty.

In fact, $I_0 \ne \emptyset$ if and only if $\beta\delta > 4$, see (33); in this case Theorem 6.1 yields a spectral free strip for $|\mathrm{Re}\lambda| \in I_0$ which is not provided by Theorem 6.2 (ii). Further, $I_{0,\mu} \ne \emptyset$ if and only if $k_\mu^2 > \frac{4}{\beta\delta} - 1$; in this case Theorem 6.1 yields a better estimate than Theorem 6.2 (ii) for $|\mathrm{Re}\lambda| \in I_{0,\mu} \subset \big(\lambda_{\mathrm{i,ii}},\lambda_{\mathrm{ii,iii}}\big)$.

Independently of $\mu$, there is always an improvement if $\beta\delta > 4$ since then $I_0 \ne \emptyset$. Similarly, if $\beta\delta < 4$ and $k^2 \ge \frac{4}{\beta\delta} - 1$, then $I_{0,\mu} \ne \emptyset$ since $k_\mu > k$ due to Remark 6.3 (a); the same applies if $\beta\delta = 4$ and $k > 0$. This is illustrated in Figure 4.

Depending on $\mu$, for $\beta$, $\delta$ fixed, the interval $I_{0,\mu}$ is decreasing for increasing $\mu$ since $k_\mu$ decreases, see Remark 6.3 (a). More precisely, since $\mu^2 \ge \beta\delta$ by assumption, starting from

$I_{0,\sqrt{\beta\delta}} = \big(\lambda_{\mathrm{i,ii}}, \lambda_{\mathrm{ii,iii}}\big)$ for $\mu^2 = \beta\delta$, the interval $I_{0,\mu}$ shrinks down to a (possibly empty) limiting interval $I_{0,\infty}$ obtained from $I_{0,\mu}$ by replacing $k_\mu$ by its limit $\lim_{\mu\to\infty} k_\mu = k$. For $k > 0$, we have

$$\begin{cases} I_{0,\mu} \supsetneq I_{0,\infty} \supsetneq I_0 & \text{if } \; k^2 \geq \frac{4}{\beta\delta} - 1, \\ I_{0,\mu} \supsetneq I_0 = \emptyset, \; \mu \in (0,\mu_0), \qquad I_{0,\mu} = I_{0,\infty} = I_0 = \emptyset, \; \mu \in [\mu_0, \infty), & \text{if } \; k^2 < \frac{4}{\beta\delta} - 1, \end{cases}$$

where $\nu_0 \in (0,\infty)$ is the threshold where $k_{\nu_0}^2 = \frac{4}{\beta\delta} - 1$. For $k = 0$, we always have $I_{0,\mu} = I_0$, and this interval is non-empty if and only if $\beta\delta > 4$; see also Theorem 7.2 (iii) and Figures 7, 8.

*Proof of Proposition* 6.5. A straightforward computation shows that, for $t \in (0, \frac{\beta}{2})$, we have $h_{\mathrm{i}}(t) \leq h_{\mathrm{ii}}(t)$ if and only if $t \leq \frac{\beta}{2}\big(1 - \frac{k}{k_\mu}\big) = \lambda_{\mathrm{i,ii}}$.

To compare the functions $h_{\mathrm{ii}}$ and $h_{\mathrm{iii}}$ we consider the equation (41) defining $h_{\mathrm{iii}}(t)$ with $y$ replaced by $h_{\mathrm{ii}}(t) = k_\mu t$, which leads to the equation

$$(k_\mu^2 + 1)(k_\mu - k)t = \frac{2}{\delta} k_\mu. \tag{48}$$

By definition (39), $k_\mu$ satisfies

$$0 = k_\mu^2\Big(k_\mu^2 + 1 - k^2 - \frac{4}{\mu^2}\Big) - k^2 = k_\mu^4 + k_\mu^2 - k_\mu^2 k^2 - \frac{4}{\mu^2}k_\mu^2 - k^2 = (k_\mu^2+1)(k_\mu^2 - k^2) - \frac{4}{\mu^2}k_\mu^2. \tag{49}$$

Therefore, if $k_\mu > k$, we obtain a unique solution of (48),

$$\lambda_{\mathrm{ii,iii}} = \frac{2}{\delta}\frac{k_\mu}{(k_\mu^2+1)(k_\mu - k)} = \frac{2}{\delta}\frac{k_\mu(k_\mu + k)}{\frac{4}{\mu^2}k_\mu^2} = \frac{\mu^2}{2\delta}\frac{k_\mu + k}{k_\mu} = \frac{\mu^2}{2\delta}\Big(1 + \frac{k}{k_\mu}\Big) \leq \frac{\mu^2}{\delta},$$

for which $h_{\mathrm{ii}}(t) \leq h_{\mathrm{iii}}(t)$ if and only if $t \leq \lambda_{\mathrm{ii,iii}}$. If $k_\mu = k$, then $k_\mu = k = 0$ due to Remark 6.3 (a) and thus, in this case, $h_{\mathrm{ii}}(t) = 0$ for all $t \in [0,\infty)$ and $\lambda_{\mathrm{ii,iii}} = \infty$.

Since $\beta\delta \leq \mu^2$ by assumption or by (11), it is easy to see that $\lambda_{\mathrm{i,ii}} \leq \frac{\beta}{2} \leq \frac{\mu^2}{2\delta} \leq \lambda_{\mathrm{ii,iii}}$ and hence

$$\begin{aligned} &h_{\mathrm{i}}(t) \leq h_{\mathrm{ii}}(t) \leq h_{\mathrm{iii}}(t), && t \in [0, \lambda_{\mathrm{i,ii}}], \\ &h_{\mathrm{ii}}(t) \leq \min\{h_{\mathrm{i}}(t), h_{\mathrm{iii}}(t)\}, && t \in [\lambda_{\mathrm{i,ii}}, \lambda_{\mathrm{ii,iii}}], \\ &h_{\mathrm{iii}}(t) \leq h_{\mathrm{ii}}(t) \leq h_{\mathrm{i}}(t), && t \in [\lambda_{\mathrm{ii,iii}}, \infty). \end{aligned}$$

Finally, if $\delta > 0$, we compare the enclosures of Theorem 6.2 with Theorem 6.1. It is not difficult to see that, for $t \in [0,\beta)$,

$$h_0(t) \leq h_{\mathrm{ii}}(t) = k_\mu t \iff t \in I_{0,\mu}.$$

Since $\frac{1}{k_\mu^2+1} \leq 1$, it is obvious that $I_0 \subset I_{0,\mu}$ and $I_{0,\mu} = I_0$ if and only if $k_\mu = 0$. By (49) one obtains $\frac{k}{k_\mu} = \sqrt{1 - \frac{4}{\mu^2}\frac{1}{k_\mu^2+1}}$; since $\mu^2 \geq \beta\delta$, it follows that $I_{0,\mu} = \big(\lambda_{\mathrm{i,ii}}, \lambda_{\mathrm{ii,iii}}\big)$ if and only if $\mu^2 = \beta\delta$. Now the inclusion $I_{0,\mu} \subset \big(\lambda_{\mathrm{i,ii}}, \lambda_{\mathrm{ii,iii}}\big)$ for $\mu^2 > \beta\delta$ follows if we recall that $I_{0,\mu}$ is decreasing for increasing $\mu$, see Remark 6.6. ☐

# 7 Self-adjoint damping: estimates for QNR and spectrum

In this section we assume that the damping operator is not only sectorial but even self-adjoint, i.e. $A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}$ is self-adjoint. In this case, it is known, see [42, Proof of Lemma 4.5], that the operator $\mathcal{A}$ is $\mathcal{J}$-self-adjoint, i.e. $\mathcal{A}^* = \mathcal{J}\mathcal{A}\mathcal{J}$ with

$$\mathcal{J} = \begin{bmatrix} I_{H_{\frac{1}{2}}} & 0 \\ 0 & -I_H \end{bmatrix} \quad \text{in } H_{\frac{1}{2}} \times H.$$

Hence the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is symmetric with respect to the real line, see [33, Satz I.2]. This property is reflected by both the numerical range and the quadratic numerical range.

**Lemma 7.1.** *Assume $A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}$ is a bounded self-adjoint operator in $H$. Then $W(\mathcal{A}|_{H_1\times H_1})$ and $W^2(\mathcal{A}|_{H_1\times H_1})$ are symmetric with respect to the real line, and hence so are $\overline{W(\mathcal{A})} = \overline{W(\mathcal{A}|_{H_1\times H_1})}$ and $\overline{W^2(\mathcal{A}|_{H_1\times H_1})}$. Moreover, $\sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A}|_{H_1\times H_1})}$.*

*Proof.* We have $\lambda \in W(\mathcal{A}|_{H_1\times H_1})$ if and only if there is $(f,g)^\top \in H_1 \times H_1$ with $\|f\|^2_{H_{\frac{1}{2}}} + \|g\|^2 = 1$ so that (12) holds. Clearly, $(f,-g)^\top \in H_1 \times H_1$ and (12) shows that $\overline{\lambda}$ is in $W(\mathcal{A}|_{H_1\times H_1})$.

The symmetry of $W^2(\mathcal{A}|_{H_1\times H_1})$ follows from the fact that, for self-adjoint $D$ and $(f,g)^\top \in H_1 \times H_1$ with $\|f\|_{H_{\frac{1}{2}}} = \|g\| = 1$, the polynomial $\det(\mathcal{A}_{f,g} - \lambda) = \lambda^2 + \lambda\langle Dg, g\rangle + |\langle f, g\rangle_{H_{\frac{1}{2}}}|^2$ has real coefficients and so its zeros are symmetric with respect to the real line.

For the next claim it remains to be noted that $H_1 \times H_1$ is a core (cf. Proposition 2.3) and thus $\overline{W(\mathcal{A})} = \overline{W(\mathcal{A}|_{H_1\times H_1})}$ by [28, Problem V.3.7].

Finally, let $\lambda \in \sigma(\mathcal{A})$. Then either $\lambda \in \sigma_{ap}(\mathcal{A})$ and hence $\lambda \in \overline{W^2(\mathcal{A}|_{H_1\times H_1})}$ by Theorem 5.1, or $\lambda \in \sigma_r(\mathcal{A})$. In the latter case we obtain $\overline{\lambda} \in \sigma_p(\mathcal{A}) \subset \overline{W^2(\mathcal{A}|_{H_1\times H_1})}$ by [5, Theorem VI.6.1] since $\mathcal{A}$ is $\mathcal{J}$-selfadjoint. Hence $\lambda \in \overline{W^2(\mathcal{A}|_{H_1\times H_1})}$ by the symmetry shown before. $\square$

**Theorem 7.2.** *Assume $A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}$ is a bounded self-adjoint operator in $H$.*

(i) *If $\beta_0 > 0$ and $\beta \in (0, \beta_0]$, then*

$$\sigma(\mathcal{A}) \subset \Big\{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \le -\frac{\beta}{2}\Big\} \cup \Big[-\frac{\beta}{2}, 0\Big). \tag{50}$$

(ii) *If $0 < \mu_0 < 2$, then $\beta_0 \ge \mu_0 a_0 > 0$ and, for all $\beta \in (0, \beta_0]$, $\mu \in (0, \mu_0]$,*

$$\sigma(\mathcal{A}) \subset \Big\{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \le -\frac{\beta}{2},\ |\mathrm{Im}\lambda| \le \frac{\sqrt{4-\mu^2}}{\mu}|\mathrm{Re}\lambda|\Big\} \cup \Big[-\frac{\beta}{2}, 0\Big); \tag{51}$$

*if $\mu_0 \ge 2$, then*

$$\sigma(\mathcal{A}) \subset (-\infty, 0).$$

(iii) *If $\delta_0 > 0$, then $\beta_0 \ge \delta_0 a_0^2 > 0$, $\sigma(\mathcal{A}) \setminus \mathbb{R}$ is bounded and confined to a part of a disk, and, for all $\beta \in (0, \beta_0]$, $\delta \in (0, \delta_0]$,*

$$\sigma(\mathcal{A}) \subset \Big(-\infty, -\frac{2}{\delta}\Big] \cup \Big\{\lambda \in \mathbb{C} \mid -\frac{2}{\delta} \le \mathrm{Re}\lambda \le -\frac{\beta}{2},\ \Big|\lambda + \frac{1}{\delta}\Big| \le \frac{1}{\delta}\Big\} \cup \Big[-\frac{\beta}{2}, 0\Big); \tag{52}$$

*if $\beta\delta > 4$, then*

$$\sigma(\mathcal{A}) \subset \Big(-\infty, -\frac{\beta}{2}\Big(1 + \sqrt{1 - \frac{4}{\beta\delta}}\Big)\Big] \cup \Big[-\frac{\beta}{2}\Big(1 - \sqrt{1 - \frac{4}{\beta\delta}}\Big), 0\Big).$$

*The choice $\beta = \beta_0$, $\mu = \mu_0$ and $\delta = \delta_0$, respectively, yields the best enclosures. If, in any of the above cases, in addition $\gamma_0 < \infty$, then also*

$$\sigma(\mathcal{A}) \subset \Big[-\gamma_0, -\frac{\gamma_0}{2}\Big] \cup \Big\{\lambda \in \mathbb{C} \mid -\frac{\gamma_0}{2} \le \mathrm{Re}\lambda < 0\Big\}. \tag{53}$$

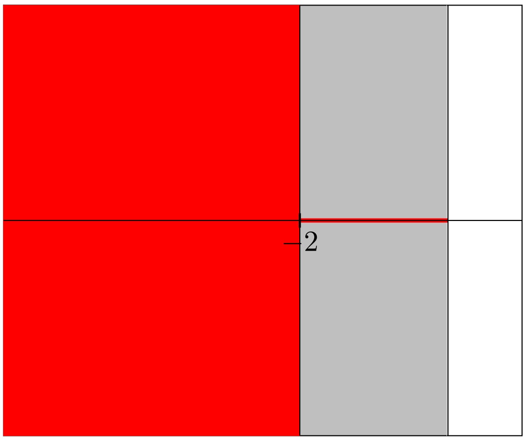


Figure 5:
Theorem 7.2 (i) with $\beta = 4$, $\mu = 0$, $\delta = 0$.

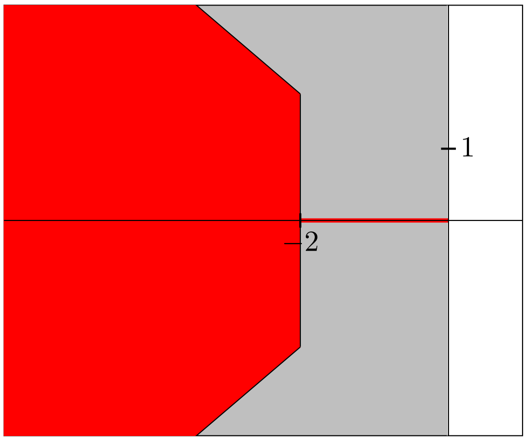


Figure 6:
Theorem 7.2 (i), (ii) with $\beta = 4$, $\mu = 1.5$, $\delta = 0$.

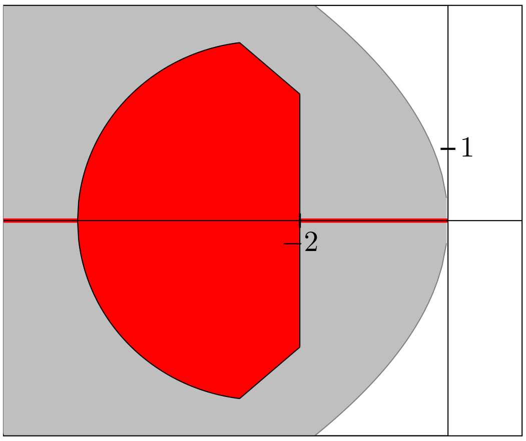


Figure 7:
Theorem 7.2 (i), (ii), (iii) with $\beta = 4$, $\mu = 1.5$, $\delta = 0.4$; here $k_\mu^2 = \frac{7}{9} < \frac{3}{2} = \frac{4}{\beta\delta} - 1$, $I_0 = I_{0,\mu} = \emptyset$, so no improvement by Theorem 6.1, see Remark 6.6.

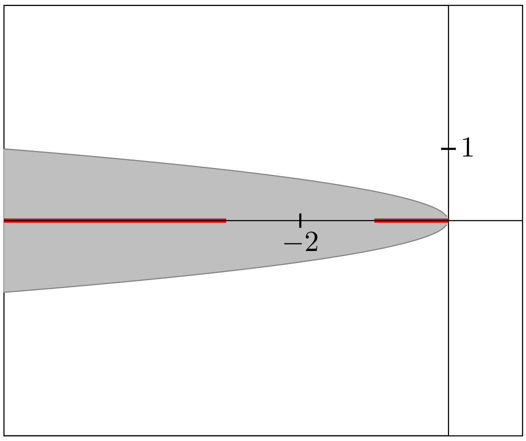


Figure 8:
Theorem 7.2 (iii) with $\beta = 4$, $\delta = \frac{4}{3}$; here spectral gap in $-I_0 = -I_{0,\mu} = (-3, -1)$ by Theorem 6.1, see Remark 6.6.

Figures 5–8: Spectral enclosures obtained from $W(\mathcal{A})$ (light grey) and from $W^2(\mathcal{A})$ (red for colour online/pdf version, dark grey for black and white print).

*Proof.* The self-adjointness of $A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}$ implies that $\mathrm{Im}\langle Dg, g\rangle = 0$ for $g \in H_1$ and hence (35) holds with $k = 0$.

(i) The inclusion (50) follows from Theorem 6.2 (i) since $k = 0$ implies $h_{\mathrm{i}}(t) = 0$ for all $t \in [0, \frac{\beta}{2})$.

(ii) If we use $\beta_0 \geq \mu_0 a_0$, see (11), and formula (43) which describes $k_\mu$ in the case $k = 0$, both inclusions follow from part (i) and Theorem 6.2 (ii).

(iii) By (11) we have $\beta_0 \geq \delta_0 a_0^2$. Further, for $k = 0$, the equation (41) defining $h_{\mathrm{iii}}$ reads $(y^2 + t^2)y = \frac{2}{\delta} t y$. Thus, $h_{\mathrm{iii}}(t) = \sqrt{\frac{2}{\delta}t - t^2}$ for $t \in [0, \frac{2}{\delta}]$ and $h_{\mathrm{iii}}(t) = 0$ for $t > \frac{2}{\delta}$. Now both assertions in (iii) follow from part (i), Theorem 6.2 (iii) and Theorem 6.1.

By Lemma 7.1 it suffices to prove the inclusion in (53) for $\overline{W^2(\mathcal{A}|_{H_1 \times H_1})} \setminus \{0\}$ in place of $\sigma(\mathcal{A})$. Let $\lambda \in W^2(\mathcal{A}|_{H_1 \times H_1}) \setminus \mathbb{R}$. Then there exists $(f, g)^\top \in H_1 \times H_1$ with $\|f\|_{H_{\frac{1}{2}}} = \|g\| = 1$ such that (34) and hence (44), (45) hold. Using $\mathrm{Im}\lambda \neq 0$ and $\mathrm{Im}\langle Dg, g\rangle = 0$ in (46) we find

$$|\mathrm{Re}\lambda| = \frac{1}{2}\langle Dg, g\rangle \leq \frac{\gamma_0}{2}\,.$$

Then, by Proposition 4.3, we conclude that

$$W^2(\mathcal{A}|_{H_1 \times H_1}) \subset \Big[-\gamma_0, -\frac{\gamma_0}{2}\Big] \cup \Big\{\lambda \in \mathbb{C} \mid -\frac{\gamma_0}{2} \leq \mathrm{Re}\lambda < 0\Big\}. \qquad \square$$

**Remark 7.3.** We mention that, by means of a different method, the inclusions in (i), the second inclusion in (ii), and the first inclusion in (iii) were shown in [25, Theorem 3.3], while (53) is an improvement of a corresponding inclusion therein.

As in the previous section, due to (11), $\mu_0 > 0$ implies $\beta_0 > 0$, and $\delta_0 > 0$ implies $\mu_0 > 0$ and thus $\beta_0 > 0$. Therefore if, in Theorem 7.2, (ii) applies then so does (i) and if (iii) applies, then so do (i) and (ii). The precise form of the combination of all inclusions is given in the next proposition.

Figures 5–8 illustrate how the spectral enclosures by means of the quadratic numerical range (red for colour online/pdf version, dark grey for black and white print) compare to those obtained by means of the numerical range (light grey) and how the enclosures improve successively for the cases $\beta_0 > 0$, $0 < \mu_0 < 2$, $\delta_0 > 0$ and $\beta\delta > 4$.

**Proposition 7.4.** *Let $A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}$ be a bounded self-adjoint operator in $H$.*

(a) *If $\mu_0 \geq 2$, then*

$$\sigma(\mathcal{A}) \subset \begin{cases} (-\infty, 0) & \text{if } \gamma_0 = \infty, \\ [-\gamma, 0) & \text{if } \gamma_0 < \infty; \end{cases}$$

*if, in addition, $\delta_0 > 0$ (and hence $\beta_0 > 0$), then, for $\beta \in (0, \beta_0]$, $\delta \in (0, \delta_0]$ with $\beta\delta > 4$ and $\gamma \in [\gamma_0, \infty)$ if $\gamma_0 < \infty$,*

$$\sigma(\mathcal{A}) \subset \begin{cases} \Big(-\infty, -\frac{\beta}{2}\Big(1 + \sqrt{1 - \frac{4}{\beta\delta}}\Big)\Big] \cup \Big[-\frac{\beta}{2}\Big(1 - \sqrt{1 - \frac{4}{\beta\delta}}\Big), 0\Big) & \text{if } \gamma_0 = \infty, \\ \Big[-\gamma, -\frac{\beta}{2}\Big(1 + \sqrt{1 - \frac{4}{\beta\delta}}\Big)\Big] \cup \Big[-\frac{\beta}{2}\Big(1 - \sqrt{1 - \frac{4}{\beta\delta}}\Big), 0\Big) & \text{if } \gamma_0 < \infty. \end{cases}$$

(b) *If* $0 < \mu_0 < 2$ (*and hence* $\beta_0 > 0$), *then, for* $\beta \in (0, \beta_0]$, $\mu \in (0, \mu_0]$ *and* $\gamma \in [\gamma_0, \infty)$ *if* $\gamma_0 < \infty$,

$$\sigma(\mathcal{A}) \subset \begin{cases} \Big\{\lambda \in \mathbb{C} \,|\, -\infty \le \mathrm{Re}\lambda \le -\dfrac{\beta}{2},\ |\mathrm{Im}\lambda| \le \dfrac{\sqrt{4-\mu^2}}{\mu}|\mathrm{Re}\lambda|\Big\} \cup \Big[-\dfrac{\beta}{2}, 0\Big) & \text{if } \gamma_0 = \infty, \\ \Big[-\gamma, -\dfrac{\gamma}{2}\Big] \cup \Big\{\lambda \in \mathbb{C} \,|\, -\dfrac{\gamma}{2} \le \mathrm{Re}\lambda \le -\dfrac{\beta}{2},\ |\mathrm{Im}\lambda| \le \dfrac{\sqrt{4-\mu^2}}{\mu}|\mathrm{Re}\lambda|\Big\} \cup \Big[-\dfrac{\beta}{2}, 0\Big) & \text{if } \gamma_0 < \infty; \end{cases}$$

*if, in addition to* $0 < \mu_0 < 2$, *also* $\delta_0 > 0$, *then, for* $\beta \in (0, \beta_0]$, $\mu \in (0, \mu_0]$, $\delta \in (0, \delta_0]$ *with* $\mu^2 \ge \beta\delta$ *and* $\gamma \in [\gamma_0, \infty)$ *if* $\gamma_0 < \infty$,

$$\sigma(\mathcal{A}) \subset \begin{cases} \Big(-\infty, -\dfrac{2}{\delta}\Big) \cup \Big\{\lambda \in \mathbb{C} \,|\, -\dfrac{2}{\delta} \le \mathrm{Re}\lambda \le -\dfrac{\mu^2}{2\delta},\ \Big|\lambda + \dfrac{1}{\delta}\Big| \le \dfrac{1}{\delta}\Big\} & \\ \cup \Big\{\lambda \in \mathbb{C} \,|\, -\dfrac{\mu^2}{2\delta} \le \mathrm{Re}\lambda \le -\dfrac{\beta}{2},\ |\mathrm{Im}\lambda| \le \dfrac{\sqrt{4-\mu^2}}{\mu}|\mathrm{Re}\lambda|\Big\} \cup \Big[-\dfrac{\beta}{2}, 0\Big) & \text{if } \gamma_0 = \infty, \\ \Big[-\gamma, -\min\Big\{\dfrac{2}{\delta}, \dfrac{\gamma}{2}\Big\}\Big) \cup \Big\{\lambda \in \mathbb{C} \,|\, -\min\Big\{\dfrac{2}{\delta}, \dfrac{\gamma}{2}\Big\} \le \mathrm{Re}\lambda \le -\dfrac{\mu^2}{2\delta},\ \Big|\lambda + \dfrac{1}{\delta}\Big| \le \dfrac{1}{\delta}\Big\} & \\ \cup \Big\{\lambda \in \mathbb{C} \,|\, -\dfrac{\mu^2}{2\delta} \le \mathrm{Re}\lambda \le -\dfrac{\beta}{2},\ |\mathrm{Im}\lambda| \le \dfrac{\sqrt{4-\mu^2}}{\mu}|\mathrm{Re}\lambda|\Big\} \cup \Big[-\dfrac{\beta}{2}, 0\Big) & \text{if } \gamma_0 < \infty,\ \mu^2 < \gamma\delta, \\ \Big[-\gamma, -\dfrac{\gamma}{2}\Big) & \\ \cup \Big\{\lambda \in \mathbb{C} \,|\, -\dfrac{\gamma}{2} \le \mathrm{Re}\lambda \le -\dfrac{\beta}{2},\ |\mathrm{Im}\lambda| \le \dfrac{\sqrt{4-\mu^2}}{\mu}|\mathrm{Re}\lambda|\Big\} \cup \Big[-\dfrac{\beta}{2}, 0\Big) & \text{if } \gamma_0 < \infty,\ \mu^2 \ge \gamma\delta. \end{cases}$$

*The choice* $\beta = \beta_0$, $\mu = \mu_0$ *and* $\delta = \delta_0$, *respectively, gives the best enclosures; in this case, the assumption* $\beta_0\delta_0 \le \mu_0^2$ *in the second part of* (b) *is automatically satisfied by* (11).

*Proof.* (a) The first claim for $\mu_0 \ge 2$ is immediate from Theorem 7.2 (ii) and (53); the second claim follows if we additionally use Theorem 6.1 and observe that $\gamma \ge \gamma_0 \ge \beta_0 \ge \beta$ by (11) and hence $\gamma > \frac{\beta}{2}\Big(1 + \sqrt{1 - \frac{4}{\beta\delta}}\Big)$.

(b) The first claim for $0 < \mu_0 < 2$ follows from Theorem 7.2 (i), (ii) and (53). It remains to consider the case $0 < \mu_0 < 2$ and $\delta_0 > 0$. First we determine if, for $\mathrm{Re}\lambda \in \big(-\frac{2}{\delta}, -\frac{\beta}{2}\big]$, the boundary of the sector in (51) intersects the circle $(\mathrm{Re}\lambda + \frac{1}{\delta})^2 + (\mathrm{Im}\lambda)^2 = \frac{1}{\delta^2}$ in (52). The imaginary part of boundary points of the sector equals $\pm\frac{\sqrt{4-\mu^2}}{\mu}|\mathrm{Re}\lambda|$ and so points $\lambda$ of the intersection satisfy

$$\Big(\mathrm{Re}\lambda + \frac{1}{\delta}\Big)^2 + \frac{4 - \mu^2}{\mu^2}(\mathrm{Re}\lambda)^2 = \frac{1}{\delta^2}\,.$$

A simple calculation yields $\mathrm{Re}\lambda = -\frac{\mu^2}{2\delta}$. Observe that $-\frac{2}{\delta} < -\frac{\mu^2}{2\delta} \le -\frac{\beta}{2}$ since $\mu < 2$ and $\mu^2 \ge \beta\delta$ by assumption. Now Theorem 7.2 (i), (ii), and (iii) implies all the claims for $\gamma_0 = \infty$. For $\gamma_0 < \infty$, we additionally use (53) and recall that $\gamma \ge \gamma_0 \ge \beta_0 \ge \beta$ by (9); then $-\frac{\gamma}{2} \le -\frac{\beta}{2}$, and it remains to note that $-\min\big\{\frac{2}{\delta}, \frac{\gamma}{2}\big\} < -\frac{\mu^2}{2\delta}$ if and only if $\mu^2 < \gamma\delta$. □

# 8 Applications: wave equation with strong damping and small transverse oscillations of an ideal incompressible fluid in a pipe

**Example 8.1.** We begin with an application of our first result, Theorem 6.1, for the case that the damping $D$ is uniformly accretive with respect to $A_0$, but no information on $\mathrm{Im}\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}$, and hence on the numerical range of $\mathcal{A}$, is available. To this end, we consider the wave equation in a bounded domain $\Omega \subset \mathbb{R}^n$ with $C^2$-boundary with viscoelastic and frictional damping subject to Dirichlet conditions on $\partial\Omega$, see Example 2.1 with $b \equiv 0$. Here

$$A_0 = -\Delta, \quad a_0^2 = \min \sigma(-\Delta) = \lambda_1(\Omega) > 0,$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian on $\Omega$ and

$$D = -d\Delta + V, \quad d > 0, \quad V_{\mathrm{Re,inf}} := \operatorname{ess\,inf} \operatorname{Re} V \geq 0,$$

where $V = V_1 + V_2$ and $V_1$, $V_2$ satisfy the dimension dependent assumptions specified in Example 2.1, e.g. $V_1 \in L^2(\Omega, \mathbb{C})$, $V_2 \in L^\infty(\Omega, \mathbb{C})$ if $n \leq 3$. The constants $\beta_0$, $\delta_0$ in (9) can be estimated by

$$\begin{aligned} \beta_0 &= \inf_{z \in H_0^1(\Omega,\mathbb{C})\setminus\{0\}} \frac{d\langle -\Delta z, z\rangle + \langle \operatorname{Re} V z, z\rangle}{\|z\|^2} \geq d\lambda_1(\Omega) + V_{\mathrm{Re,inf}} =: \beta, \\ \delta_0 &= \inf_{z \in H_0^1(\Omega,\mathbb{C})\setminus\{0\}} \frac{d\langle -\Delta z, z\rangle + \langle \operatorname{Re} V z, z\rangle}{\langle -\Delta z, z\rangle} \geq d =: \delta. \end{aligned}$$

Otherwise we do not impose any further conditions on $V$, in particular, $V$ is not assumed to be symmetric and we suppose that no explicit information on $\mathrm{Im} V$ is available.

In this case the only enclosure obtained from the numerical range of the corresponding operator $\mathcal{A}$ is that $W(\mathcal{A})$ is contained in the left half-plane, see Proposition 3.4, while the quadratic numerical range provides not only a non-trivial, but interesting spectral enclosure in the strip

$$S := \big\{ z \in \mathbb{C} : \operatorname{Re} z \in \big( -(d\lambda_1(\Omega) + V_{\mathrm{Re,inf}}), 0 \big] \big\}$$

by means of Theorem 6.1. In terms of the viscoelastic damping constant $d$, the condition $\beta\delta > 4$, which implies $\beta_0\delta_0 > 4$, is equivalent to $d > d_{\mathrm{crit}}$ where

$$d_{\mathrm{crit}} := -\frac{1}{2}\frac{V_{\mathrm{Re,inf}}}{\lambda_1(\Omega)} + \sqrt{\Big(\frac{1}{2}\frac{V_{\mathrm{Re,inf}}}{\lambda_1(\Omega)}\Big)^2 + \frac{4}{\lambda_1(\Omega)}},$$

so the corresponding operator $\mathcal{A}$ has a spectral free strip centred at $-\frac{\beta}{2} = -\frac{1}{2}\big(d\lambda_1(\Omega) + V_{\mathrm{Re,inf}}\big)$ if $d > d_{\mathrm{crit}}$. More precisely, if $d \leq d_{\mathrm{crit}}$, then $\sigma(\mathcal{A}) \cap S$ is contained in the connected set

$$\left\{ z \in S : \operatorname{Re} z \neq 0,\ |\operatorname{Im} z| \leq \sqrt{\Big(\lambda_1(\Omega) + \frac{V_{\mathrm{Re,inf}}}{d}\Big)\Big(\frac{|\operatorname{Re} z|}{d\lambda_1(\Omega) + V_{\mathrm{Re,inf}} - |\operatorname{Re} z|}\Big) - (\operatorname{Re} z)^2} \right\},$$

and if $d > d_{\mathrm{crit}}$, then $\sigma(\mathcal{A}) \cap S$ is contained in the set

$$\left\{ z \in S : |\operatorname{Re} z| \notin I_0 \cup \{0\},\ |\operatorname{Im} z| \leq \sqrt{\Big(\lambda_1(\Omega) + \frac{V_{\mathrm{Re,inf}}}{d}\Big)\Big(\frac{|\operatorname{Re} z|}{d\lambda_1(\Omega) + V_{\mathrm{Re,inf}} - |\operatorname{Re} z|}\Big) - (\operatorname{Re} z)^2} \right\}$$

which consists of two disjoint sets separated by a spectral free strip $-I_0 + \mathrm{i}\mathbb{R}$ where $I_0 \subset \mathbb{R}$ is the interval

$$I_0 = \Big(\frac{\beta}{2}(1 - i_0), \frac{\beta}{2}(1 + i_0)\Big), \quad i_0 := \sqrt{1 - \frac{4}{d(d\lambda_1(\Omega) + V_{\mathrm{Re,inf}})}}.$$

Figures 9, 10 show the difference of the spectral enclosures by means of the numerical range (light grey) and by means of the quadratic numerical range (red for colour online/pdf version, dark grey for black and white print) for the case that $\Omega$ is the unit disk in $\mathbb{R}^2$ and $V_{\mathrm{Re,inf}} = \operatorname{ess\,inf} \operatorname{Re} V = 0$.

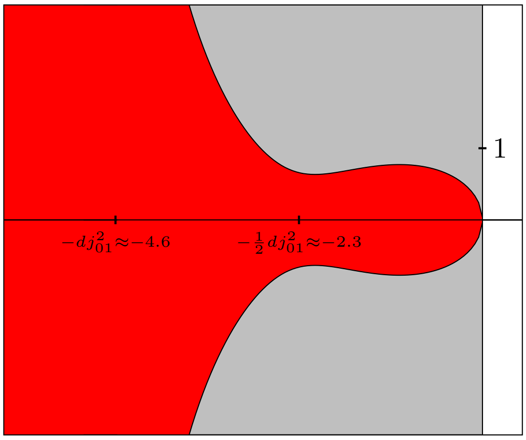


Figure 9:
Example 8.1 with $d=0.8$, $\operatorname{ess\,inf}\operatorname{Re}V=0$: $\beta=4.62$, $\delta=d$; here $\beta\delta\approx3.7<4$, $I_0=\emptyset$.

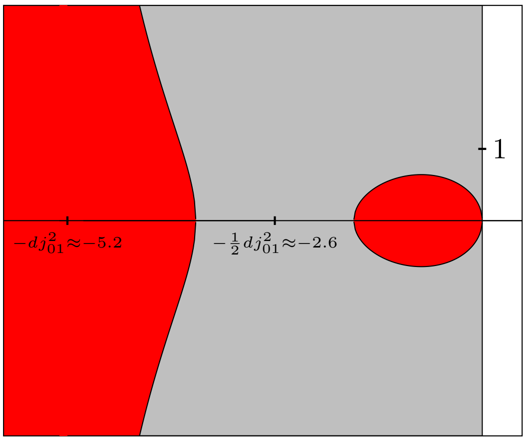


Figure 10:
Example 8.1 with $d=0.9$, $\operatorname{ess\,inf}\operatorname{Re}V=0$: $\beta=5.2$, $\delta=d$; here $\beta\delta\approx 4.68>4$, $I_0\approx(-3.6,1.6)$.

Figures 9, 10: Spectral enclosures obtained from $W(\mathcal{A})$ (light grey) and from $W^2(\mathcal{A})$ (red for colour online/pdf version, dark grey for black and white print) for the wave equation with strong damping on the unit disk in $\mathbb{R}^2$.

Here $\lambda_1(\Omega)=j_{0,1}^2$ where $j_{0,1}\approx 2.4048$ is the first positive zero of the Bessel function $J_0$ and hence $d_{\rm crit}=\frac{2}{j_{0,1}}\approx 0.83$; the viscoelastic damping constants are $d=0.8<d_{\rm crit}$ in Figure 9 and $d=0.9>d_{\rm crit}$ in Figure 10.

**Example 8.2.** The small transverse oscillations of a horizontal pipe of length normalized to 1 carrying a steady-state flow of an ideal incompressible fluid are described by

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2}{\partial r^2}\left[E\frac{\partial^2 u}{\partial r^2}+C\frac{\partial^3 u}{\partial r^2\partial t}\right]+K\frac{\partial^2 u}{\partial t\partial r}=0, \qquad r\in(0,1),\ t>0, \tag{54}$$

see e.g. [39]. Here $u(r,t)$ denotes the transverse displacement at time $t$ and position $r$, and $E$, $C$, $K$ are positive physical constants. The last term on the left hand side of (54) is called the gyroscopic term. If the pipe is pinned at both endpoints, the boundary conditions

$$u\big|_{r=0}=0,\quad \frac{\partial^2 u}{\partial r^2}\bigg|_{r=0}=0,\quad u\big|_{r=1}=0,\quad \frac{\partial^2 u}{\partial r^2}\bigg|_{r=1}=0 \tag{55}$$

have to be imposed at any time $t>0$.

The partial differential equation (54) with boundary conditions (55) is a second order problem (1) in the Hilbert space $H=L^2(0,1)$. Here the operator $A_0$ in $H$ is given by

$$A_0=E\frac{\mathrm{d}^4}{\mathrm{d}r^4},\quad \mathcal{D}(A_0)=\left\{z\in H^4(0,1)\mid z(0)=z(1)=z''(0)=z''(1)=0\right\},$$

where $H^4(0,1)$ is the fourth order Sobolev space associated with $L^2(0,1)$. Clearly, $A_0$ satisfies assumption (A1), $A_0^{-1}$ is a compact operator, and

$$A_0^{\frac12}=-\sqrt{E}\frac{\mathrm{d}^2}{\mathrm{d}r^2},\quad H_{\frac12}=\mathcal{D}(A_0^{\frac12})=\left\{z\in H^2(0,1)\mid z(0)=z(1)=0\right\},$$

with inner product and norm on $H_{\frac{1}{2}}$ given by

$$\langle z, v\rangle_{H_{\frac{1}{2}}} = E\langle z'', v''\rangle, \quad \|z\|^2_{H_{\frac{1}{2}}} \geq E\pi^4\|z\|^2, \quad z, v \in H_{\frac{1}{2}}, \tag{56}$$

i.e. $a_0 = \sqrt{E}\pi^2$. The damping operator $D$ defined as

$$D = C\frac{\mathrm{d}^4}{\mathrm{d}r^4} + K\frac{\mathrm{d}}{\mathrm{d}r} = \frac{C}{E}A_0 + K\frac{\mathrm{d}}{\mathrm{d}r} : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$$

is bounded and maps $\mathcal{D}(A_0)$ into $H$. Moreover, for $z \in H_{\frac{1}{2}}$,

$$\mathrm{Re}\langle Dz, z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = C\langle z'', z''\rangle = \frac{C}{E}\|z\|^2_{H_{\frac{1}{2}}} \geq \frac{C}{\sqrt{E}}\pi^2\|z\|_{H_{\frac{1}{2}}}\|z\| \geq C\pi^4\|z\|^2. \tag{57}$$

Thus assumptions (A2) and (A3) hold as well. However, $D$ is not self-adjoint due to the first order derivative coming from the gyroscopic term in (54).

From (57) we obtain the following information on the constants in the spectral enclosures in Theorem 6.2 which were defined at the beginning of Section 3.

**Proposition 8.3.** *For the operator $D$, we have*

$$\beta_0 = C\pi^4, \quad \gamma_0 = \infty, \quad \delta_0 = \frac{C}{E}, \quad \mu_0 = \frac{C}{\sqrt{E}}\pi^2,$$

*and one can choose*

$$k = \frac{K}{C\pi^3}.$$

*Proof.* From (57) we obtain $\beta_0 \geq C\pi^4$, $\gamma_0 = \infty$, $\delta_0 = \frac{C}{E}$ and $\mu_0 \geq \frac{C}{\sqrt{E}}\pi^2$. Since in (57) equality holds everywhere if we choose $z = z_0$ where $z_0(t) = \sin(\pi t)$, $t \in [0, 1]$, is the eigenfunction of $A_0^{\frac{1}{2}}$ corresponding to its smallest eigenvalue $\pi^2\sqrt{E}$, the equalities $\beta_0 = C\pi^4$, $\mu_0 = \frac{C}{\sqrt{E}}\pi^2$ follow.

To prove the last claim, we let $z \in H_{\frac{1}{2}}$ and estimate

$$\|z'\|^2 = \langle z', z'\rangle = -\langle z'', z\rangle \leq \|z''\|\|z\|.$$

Using this estimate, $\|z\| \leq \frac{1}{\pi^2}\|z''\|$ and (57), we conclude that

$$\left|\mathrm{Im}\langle Dz, z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}\right| = K\left|\langle z', z\rangle\right| \leq K\|z''\|^{1/2}\|z\|^{3/2} \leq \frac{K}{\pi^3}\|z''\|^2 = \frac{K}{C\pi^3}\mathrm{Re}\langle Dz, z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}. \quad \square$$

**Theorem 8.4.** *The spectrum of the operator $\mathcal{A}$ given by* (6) *associated with the boundary value problem* (54), (55) *satisfies the inclusion*

$$\sigma(\mathcal{A}) \subset \left\{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda < 0,\ \left|\mathrm{Re}\lambda - \frac{\beta_0}{2}\right|^2 \geq \left(\frac{\beta_0}{2}\right)^2\left(1 - \frac{4}{\beta_0\delta_0}\right),\ |\mathrm{Im}\lambda| \leq h(|\mathrm{Re}\lambda|)\right\},$$

*where*

$$h(t) = \begin{cases} \dfrac{kt}{1 - \frac{2}{\beta_0}t}, & 0 \leq t < \lambda_{\mathrm{i,ii}}, \\[2ex] \sqrt{\dfrac{\beta_0}{\delta_0}\dfrac{t}{\beta_0 - t} - t^2}, & \lambda_{\mathrm{i,ii}} \leq t \leq \lambda_{\mathrm{ii,iii}}, \\[2ex] h_{\mathrm{iii}}(t), & \lambda_{\mathrm{ii,iii}} < t < \infty, \end{cases}$$

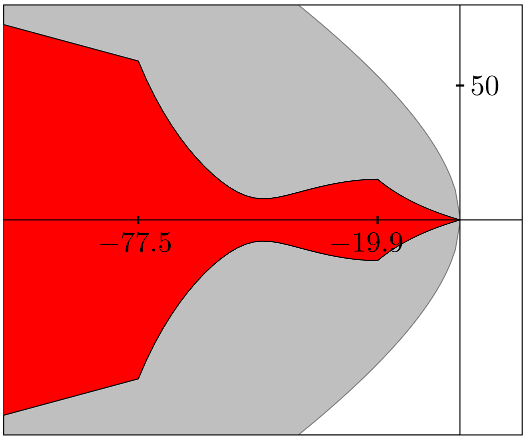


Figure 11:
Example 8.2 for physical parameters $E=25$, $C=1$, $K=14$, see Remark 8.5.

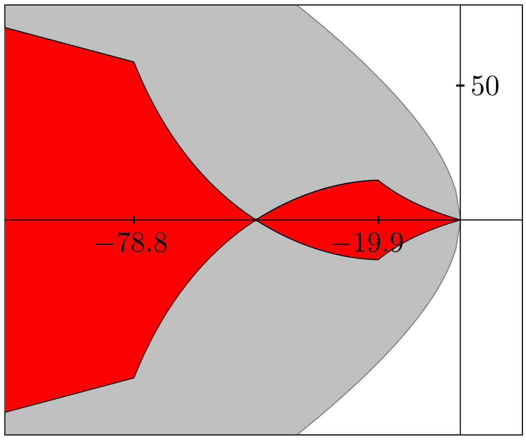


Figure 12:
Example 8.2 for physical parameters $E=25$, $C=\frac{10}{\pi^2}$, $K=14$, see Remark 8.5.

Figures 11, 12: Spectral enclosures obtained from $W(\mathcal{A})$ (light grey) and from $W^2(\mathcal{A})$ (red for colour online/pdf version, dark grey for black and white print) for small oscillations of pipe flow.

*with $k_\mu$, $h_{\rm iii}$ as defined in Theorem 6.2, $\lambda_{\rm i,ii} = \frac{\beta_0}{2}\big(1-\frac{k}{k_{\mu_0}}\big)$, $\lambda_{\rm ii,iii} = \frac{\beta_0}{2}\big(1+\frac{k}{k_{\mu_0}}\big)$, and the constants $\beta_0$, $\delta_0$, $\mu_0$, $k$ as defined in Proposition 8.3; in particular, there is a spectral free strip if $\beta_0\delta_0 > 4$,*

$$\operatorname{Re}\sigma(\mathcal{A}) \cap \left(-\frac{C\pi^4}{2}\left(1+\sqrt{1-\frac{4E}{C^2\pi^4}}\right), -\frac{C\pi^4}{2}\left(1-\sqrt{1-\frac{4E}{C^2\pi^4}}\right)\right) = \emptyset \quad \textit{if} \quad C > \frac{2\sqrt{E}}{\pi^2}.$$

*Proof.* All claims follow from Proposition 6.5 and Remark 6.6 if we note that here $\beta\delta = \mu^2$ whence $\lambda_{\rm ii,iii}$ has the claimed form and $I_{0,\mu} = \big(\lambda_{\rm i,ii}, \lambda_{\rm ii,iii}\big)$. The form of the spectral free strip $|{\rm Re}\lambda| \notin I_0$ is obtained by inserting the constants from Proposition 8.3 into (33). □

**Remark 8.5.** For the physical constants

$$E = 25, \quad C = 1, \quad K = 14$$

one can compute that

$$\lambda_{\rm i,ii} \approx 19.859, \qquad \lambda_{\rm ii,iii} \approx 77.550;$$

the corresponding spectral inclusion in Theorem 8.4 is displayed in Figure 11. Note that here Theorem 8.4 does not yield a spectral gap since $C = 1 < 10/\pi^2 = 2\sqrt{E}/\pi^2$. If we increase $C$ to the critical value $2\sqrt{E}/\pi^2$, i.e. if we choose

$$E = 25, \quad C = \frac{10}{\pi^2}, \quad K = 14,$$

then

$$\lambda_{\rm i,ii} \approx 19.852, \qquad \lambda_{\rm ii,iii} \approx 78.844;$$

Figure 12 shows the corresponding spectral inclusion in Theorem 8.4 right before the opening of the spectral free strip.

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