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Infinite-dimensional input-to-state stability and Orlicz spaces

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Abstract

In this work, the relation between input-to-state stability and integral input-to-state stability is studied for linear infinite-dimensional systems with an unbounded control operator. Although a special focus is laid on the case L^{∞} , general function spaces are considered for the inputs. We show that integral input-to-state stability can be characterized in terms of input-to-state stability with respect to Orlicz spaces. Since we consider linear systems, the results can also be formulated in terms of admissibility. For parabolic diagonal systems with scalar inputs, both stability notions with respect to L^{∞} are equivalent.

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1 Introduction

In systems and control theory, the question of stability is very natural. Let us consider the situation where the relation between the input (function) u and the state x is governed by the autonomous equation

$$\dot{x} = f(x, u), \quad x(0) = x_0.$$
 (1)

One can then distinguish between stability with respect to the input, external stability and internal stability, that is, when u = 0. For the moment, f is assumed to map from $\mathbb{R}^n \times \mathbb{R}^m$ to \mathbb{R}^n , and to be such that solutions x exist on $[0, \infty)$ for all inputs u in a space Z. Already from this very general view-point, it seems clear that stability notions may strongly depend on the specific choice of Z (and its norm). The concept of input-to-state stability (ISS), combines both external and internal stability in one notion. If Z is chosen to be $L^{\infty}(0, \infty; \mathbb{R}^m)$,

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a system is called ISS (with respect to L^{∞}) if there exist functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that

$$||x(t)|| \le \beta(||x_0||, t) + \gamma(\mathop{\mathrm{ess\,sup}}_{s \in [0, t]} ||u(s)||),$$

for all t > 0 and $u \in Z$. Here \mathcal{KL} and \mathcal{K} denote the classic comparison functions from nonlinear systems theory, see Section 2. Introduced by E. Sontag in 1989 [Son89], ISS has been intensively studied in the past decades, see [Son08] for a survey.

Since ISS cannot be expected in some applications, related stability notions have been studied in the literature. A prominent variant is *integral input-tostate stability* (iISS), [Son98]; this means that for some $\theta \in \mathcal{K}_{\infty}$ and $\mu \in \mathcal{K}$,

$$\|x(t)\| \le \beta(\|x_0\|, t) + \theta\left(\int_0^t \mu(\|u(s)\|) \, ds\right),\tag{2}$$

for all t > 0 and $u \in Z$. For linear systems, i.e., f(x, u) = Ax + Bu with matrices A and B, iISS is equivalent to ISS. To some extent, this observation marks the starting point of this work.

In contrast to the well-established theory for finite-dimensions, a more intensive study of (integral) input-to-state stability for infinite-dimensional systems has only begun recently. We refer to [DM13a, DM13b, JLR08, Log13, Mir16, MI14, MI15, MW15, KK16a, KK16b]. By nature, in the infinite-dimensional setting, the stability notions from finite-dimensions are more subtle. We refer to [MW16] for a listing of failures of equivalences around ISS known from finite-dimensional systems. In most of the mentioned infinite-dimensional references, systems of the form (1) with $f: X \times U \to X$ and Banach spaces X and U are considered. For linear equations, this setting corresponds to evolution equations of the form

$$\dot{x} = Ax + Bu, \quad x(0) = x_0,$$

where B is a bounded control operator. Analogously to finite-dimensions, in this case, ISS and iISS are known to be equivalent, see e.g., Proposition 2.12 below. However, concerning applications the requirement of bounded control operators B is rather restrictive. Typical examples for systems which only allow for a formulation with an unbounded B are boundary control systems. It is clear that such phenomena cannot occur for linear systems in finite-dimensions.

The main point of this paper is to relate and characterize (integral) input-tostate stability for linear, infinite-dimensional systems with unbounded control operators. This is done by using the notion of *admissibility*, [Sal84, Wei89a], which also reveals the connection of the mentioned stability-types with the boundedness of the linear mapping

$$Z \to X, u \mapsto x(t),$$

(for $x_0 = 0$). It is not surprising that the choice of topology for Z, the space of inputs u, is crucial here. However, looking at (2) for $x_0 = 0$, it is not clear how the right-hand-side could define a norm in general. Whereas ISS and iISS are equivalent if the corresponding input space is L^p , $p \in [1, \infty)$, it is shown that L^{∞} -iISS is equivalent to ISS with respect to some Orlicz space. This is one of the main results of this work. Orlicz spaces (or Orlicz–Birnbaum spaces)

| | $B \in \mathcal{L}(U, X)$ | $B \in \mathcal{L}(U, X_{-1})$ | nonlinear |
|-------------------|--|---|---|
| $\dim X < \infty$ | $\mathrm{ISS} \Longleftrightarrow \mathrm{iISS}$ | $\mathrm{ISS}\iff\mathrm{iISS}$ | $\mathrm{ISS} \underset{\Leftarrow}{\Longrightarrow} \mathrm{iISS}$ |
| $\dim X = \infty$ | $ISS \iff iISS$ | ISS $\left(\stackrel{\overleftarrow{?}}{\Longrightarrow}\right)$ iISS | not fully clear |

Table 1: The relation between input-to-state (ISS) and integral input-to-state (iISS) stability in various settings.

appear naturally as generalizations of L^p -spaces and ISS with respect to such spaces can thus be seen as a generalization of classical stability notions.

From the definition, it is clear that iISS stability always implies ISS for linear, infinite-dimensional systems. The converse direction for $Z = L^{\infty}$ remains open in general. It is known that ISS is equivalent to admissibility (together with exponential stability). We will show that L^{∞} -iISS in fact implies zero-class admissibility [JPP09, XLY08], which is slightly stronger than admissibility, see Proposition 2.11. In Table 1, the relation of ISS and iISS in the various abovementioned settings is depicted schematically.

In Section 2, we will discuss the setting and formally introduce the stability notions mentioned above. This includes a general abstract definition of ISS, iISS and admissibility with respect to some function space. Furthermore, we will give some basic facts about their relation.

Section 3 deals with the characterization of ISS and iISS in terms of Orliczspace-admissibility. As a main result, we show that L^{∞} -iISS is equivalent to ISS with respect to some Orlicz space E_{Φ} , where Φ denotes a Young function. Moreover, we show that ISS with respect to an Orlicz space, or equivalently, Orlicz-space-admissibility, is a natural generalization of classic L^p -ISS that "interpolates" the notions of L^1 - and L^{∞} -ISS.

In Section 4, we consider parabolic diagonal systems with scalar input. More precisely, we assume that A possesses a Riesz basis of eigenvectors with eigenvalues lying in a sector in the open left half-plane. For this class of systems we show that L^{∞} -ISS implies ISS with respect to some Orlicz space and thus, by the results of Section 3, the equivalence between iISS and ISS known in finite dimensions holds for this class of systems. Moreover, it turns out that any linear, bounded operator from U to the extrapolation space X_{-1} is L^{∞} admissible which yields a characterization of ISS. The results of this section partially generalize results that were already indicated in [JNPS16].

We illustrate the obtained results by examples in Section 5. In particular, we present a parabolic diagonal system which is L^{∞} -ISS, but not L^{p} -ISS for any $p \in [1, \infty)$. Finally, we conclude by drawing a connection between the question whether L^{∞} -ISS implies L^{∞} -iISS and a problem due to G. Weiss.

2 Stability notions for infinite-dimensional systems

2.1 The setting and definitions

We study systems $\Sigma(A, B)$ of the following form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \quad t \ge 0,$$
(3)

where A generates a C_0 -semigroup $(T(t))_{t\geq 0}$ on a Banach space X and B is a linear and bounded operator from a Banach space U to the extrapolation space X_{-1} . Note that B is possibly unbounded from U to X. Here X_{-1} is the completion of X with respect to the norm

$$||x||_{X_{-1}} = ||(\beta - A)^{-1}x||_X,$$

for some $\beta \in \rho(A)$, the resolvent set of A. It can be shown that the semigroup $(T(t))_{t\geq 0}$ possesses a unique extension to a C_0 -semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} with generator A_{-1} , which is an extension of A. Thus we may consider equation (3) on the Banach space X_{-1} and therefore for $u \in L^1_{loc}(0,\infty;U)$, the *(mild)* solution of (3) is given by the variation of parameters formula

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds, \qquad t \ge 0.$$
(4)

In this paper, we will consider the following type of function spaces. For a Banach space U, let $Z \subseteq L^1_{loc}(0,\infty;U)$ be such that for all t > 0

- (a) $Z(0,t;U) := \{f \in Z : f|_{[t,\infty)} = 0\}$ becomes a Banach space of functions on the interval (0,t) with values in U (in the sense of equivalent classes w.r.t. sets of measure zero).
- (b) Z(0,t;U) is continuously embedded in $L^1(0,t;U)$, that is $Z(0,t;U) \subset L^1(0,t;U)$ and there exists a $\kappa(t) > 0$ such that

$$\|\cdot\|_{L^1(0,t;U)} \le \kappa(t)\|\cdot\|_{Z(0,t;U)}.$$

- (c) For $u \in Z(0, t; U)$ and s > t we have $||u||_{Z(0,t;U)} = ||u||_{Z(0,s;U)}$.
- (d) Z is invariant under the left-shift $S_{\tau}u = u(\cdot + \tau)$ and S_{τ} is contractive on Z(0,t;U) for all $t, \tau > 0$.
- (e) For all $u \in Z$ and 0 < t < s it holds that $u|_{(0,t)} \in Z(0,t;U)$ and

$$||u|_{(0,t)}||_{Z(0,t;U)} \le ||u|_{(0,s)}||_{Z(0,s;U)}.$$

If additionally we have in (b) that

$$\kappa(t) \to 0, \quad t \searrow 0,$$
 (B)

then we say that Z satisfies condition (B).

For example, $Z = L^p$ refers to the spaces $L^p(0,t;U)$, t > 0, for fixed $1 \le p \le \infty$ and U. Other examples can be given by Sobolev spaces and the Orlicz

spaces $L_{\Phi}(0,t;U)$ and $E_{\Phi}(0,t;U)$, see the appendix. If p > 1 (including $p = \infty$) and Φ is a Young function, then L^p , E_{Φ} and L_{Φ} satisfy Condition (B), thanks to Hölder's inequality. Clearly, L^1 does not satisfy condition (B).

In general, the state x(t) given by (4) lies in X_{-1} for $u \in L^1_{loc}$ and t > 0. The notion of *admissibility* ensures that indeed $x(t) \in X$.

Definition 2.1. We call the system $\Sigma(A, B)$ admissible with respect to Z (or Z-admissible), if

$$\forall t > 0, u \in Z(0, t; U) : \quad \int_0^t T_{-1}(s) Bu(s) \ ds \in X.$$
(5)

If $\Sigma(A, B)$ is admissible with respect to Z, then all mild solutions (4) are in X and by the closed graph theorem there exists a constant c(t) (take the infimum over all possible constants) such that

$$\left\| \int_0^t T_{-1}(s) Bu(s) \ ds \right\| \le c(t) \|u\|_{Z(0,t;U)}.$$
(6)

Moreover, it is easy to see that $\Sigma(A, B)$ is admissible if (5) holds for one t > 0.

Definition 2.2. We call the system $\Sigma(A, B)$ infinite-time admissible with respect to Z (or Z-infinite-time admissible), if the system is admissible with $c_{\infty} := \sup_{t>0} c(t) < \infty$. We call the system $\Sigma(A, B)$ zero-class admissible with respect to Z (or Z-zero-class admissible), if it is admissible with respect to Z and $\lim_{t\to 0} c(t) = 0$.

Remark 2.3. Clearly, zero-class admissibility and infinite-time admissibility imply admissibility.

If the semigroup $(T(t))_{t\geq 0}$ is exponentially stable, that is, there exist constants $M, \omega > 0$ such that

$$||T(t)|| \le M e^{-\omega t}, \quad t \ge 0, \tag{7}$$

then it is easy to see that Z-infinite-time admissibility is equivalent to Z-admissibility.

Since $Z \subseteq L^1_{loc}(0,\infty;U)$, for any $u \in Z$ and any initial value x_0 , the mild solution x of (3) is continuous as function from $[0,\infty)$ to X_{-1} . Next we show that zero-class admissibility guarantees that x even lies in $C(0,\infty;X)$.

Proposition 2.4. If $\Sigma(A, B)$ is Z-zero-class admissible, then for every $x_0 \in X$ and every $u \in Z$ the mild solution of (3), given by (4), satisfies $x \in C([0,\infty); X)$.

Proof. Since x is given by (4), it suffices to consider the case $x_0 = 0$. Let $u \in Z$. We have to show that $t \mapsto \Phi_t u := \int_0^t T_{-1}(s)Bu(s) ds$ is continuous. The proof is divided into two steps.

First, note that $t \mapsto \Phi_t u$ is right-continuous on $[0, \infty)$. In fact, by

$$\Phi_{t+h}u - \Phi_t u = T(t) \int_0^h T_{-1}(s) Bu(s+t) \, ds,$$

h > 0, and Z-zero-class admissibility, it follows that

$$\|\Phi_{t+h}u - \Phi_t u\| \le c(h) \|T(t)\| \|u(\cdot + t)\|_{Z(0,h;U)} \to 0.$$

for $h \searrow 0$ (where we used properties (d), (e) of Z).

Second, we show that $t \mapsto \Phi_t$ is left-continuous on $(0, \infty)$. Since $(\Phi_t - \Phi_{t-h})u = (\Phi_t - \Phi_{t-h})u|_{(0,t)}$, we can assume that $u \in Z(0, t; U)$. Clearly,

$$(\Phi_t - \Phi_{t-h})u = T(t-h)\int_0^h T_{-1}(s)Bu(s+t-h)\,ds.$$

It follows that

$$\left\| \int_{0}^{h} T_{-1}(s) Bu(s+t-h) \, ds \right\| \leq c(h) \|u(\cdot+t-h)\|_{Z(0,h;U)}$$
$$\leq c(h) \|u(\cdot+t-h)\|_{Z(0,t;U)}$$
$$\leq c(h) \|u\|_{Z(0,t;U)} \xrightarrow{h\searrow 0} 0,$$

where the last two inequalities hold by properties (e) and (d) of Z. Since $||T(\cdot)||$ is uniformly bounded on compact intervals, we conclude that $||\Phi_{t+h}u - \Phi_t u|| \to 0$ as $h \to 0$.

Remark 2.5. If $\Sigma(A, B)$ is admissible with respect to L^p , $1 \le p < \infty$, then, by Hölder's inequality, $\Sigma(A, B)$ is L^q -zero-class admissible for any q > p. Thus, Proposition 2.4 implies that the mild solution of (3) lies in $C(0, \infty; X)$ for all $u \in L^q$. Moreover, this continuity even holds for $u \in L^p$, which was already shown by G. Weiss in his seminal paper [Wei89a, Prop. 2.3] on admissible control operators. However, there, a direct, but similar proof is used without using the notion of zero-class admissibility.

As stated in [Wei89a, Problem 2.4], it is an interesting open problem whether the continuity of x is implied by L^{∞} -admissibility. By Proposition 2.4, the answer is 'yes' in the case of L^{∞} -zero-class admissibility. See also Section 6.

To introduce input-to-state stability, we will need the following well-known function classes from Lyapunov theory.

$$\begin{split} \mathcal{K} &= \{ \mu : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid \mu(0) = 0, \ \mu \text{ continuous, strictly increasing} \}, \\ \mathcal{K}_\infty &= \{ \theta \in \mathcal{K} \mid \lim_{x \to \infty} \theta(x) = \infty \}, \\ \mathcal{L} &= \{ \gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid \gamma \text{ continuous, strictly decreasing, } \lim_{t \to \infty} \gamma(t) = 0 \}, \\ \mathcal{K}\mathcal{L} &= \{ \beta : (\mathbb{R}_0^+)^2 \to \mathbb{R}_0^+ \mid \beta(\cdot, t) \in \mathcal{K} \ \forall t \text{ and } \beta(s, \cdot) \in \mathcal{L} \ \forall s \}. \end{split}$$

Definition 2.6. The system $\Sigma(A, B)$ is called input-to-state stable with respect to Z (orZ-ISS), if there exist functions $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_{\infty}$ such that for every $t \geq 0, x_0 \in X$ and $u \in Z(0, t; U)$

- (i) x(t) lies in X and
- (*ii*) $||x(t)|| \le \beta(||x_0||, t) + \mu(||u||_{Z(0,t;U)}).$

The system $\Sigma(A, B)$ is called integral input-to-state stable with respect to Z (or Z-iISS), if there exist functions $\beta \in \mathcal{KL}$, $\theta \in \mathcal{K}_{\infty}$ and $\mu \in \mathcal{K}$ such that for every $t \geq 0$, $x_0 \in X$ and $u \in Z(0, t; U)$

(i) x(t) lies in X and

(*ii*)
$$||x(t)|| \le \beta(||x_0||, t) + \theta\left(\int_0^t \mu(||u(s)||_U)ds\right)$$

The system $\Sigma(A, B)$ is called uniformly bounded energy bounded state with respect to Z (or Z-UBEBS), if there exist functions $\gamma, \theta \in \mathcal{K}_{\infty}, \mu \in \mathcal{K}$ and a constant c > 0 such that for every $t \ge 0, x_0 \in X$ and $u \in Z(0, t; U)$

(i)
$$x(t)$$
 lies in X and

(*ii*)
$$||x(t)|| \le \gamma(||x_0||) + \theta\left(\int_0^t \mu(||u(s)||_U)ds\right) + c$$

Remark 2.7. By the inclusion of L^p spaces on bounded intervals we obtain that L^p -ISS (L^p -iISS, L^p -UBEBS) implies L^q -ISS (L^q -iISS, L^q -UBEBS) for all $1 \leq p < q \leq \infty$. Further the inclusions $L^{\infty} \subseteq E_{\Phi} \subseteq L_{\Phi} \subseteq L^1$ and $Z \subseteq L^1_{loc}$ yield a corresponding chain of implications of ISS, iISS and UBEBS.

2.2 Relations between the stability notions

In the rest of the paper $\Sigma(A, B)$ will always refer to a system of the form introduced in the previous section.

Proposition 2.8. Let $Z \subseteq L^1_{loc}(0,\infty;U)$ be a function space. Then we have:

- (i) The following statements are equivalent
 - (a) $\Sigma(A, B)$ is Z-ISS,
 - (b) $\Sigma(A, B)$ is Z-admissible and $(T(t))_{t\geq 0}$ is exponentially stable,
 - (c) $\Sigma(A, B)$ is Z-infinite-time admissible and $(T(t))_{t\geq 0}$ is exponentially stable.
- (ii) If $\Sigma(A, B)$ is Z-iISS, then the system is Z-admissible and $(T(t))_{t\geq 0}$ is exponentially stable,
- (iii) If $\Sigma(A, B)$ is Z-UBEBS, then $\Sigma(A, B)$ is Z-admissible and $(T(t))_{t\geq 0}$ is bounded, that is, (7) holds for $\omega = 0$.

Proof. Clearly, Z-ISS, Z-iISS and Z-UBEBS imply Z-admissibility, and Z-admissibility and exponential stability of $(T(t))_{t\geq 0}$ show Z-ISS. If, $\Sigma(A, B)$ is Z-ISS or Z-iISS, by setting u = 0, it follows that ||T(t)|| < 1 for sufficiently large t, which shows that $(T(t))_{t\geq 0}$ is exponentially stable. It is easy to see that Z-UBEBS implies boundedness of $(T(t))_{t\geq 0}$. Finally, by Remark 2.3 items (b) and (c) in (i) are equivalent.

Proposition 2.9. If $1 \le p < \infty$, then the following are equivalent

- (i) $\Sigma(A, B)$ is L^p -ISS,
- (ii) $\Sigma(A, B)$ is L^p -iISS,
- (iii) $\Sigma(A, B)$ is L^p -UBEBS and $(T(t))_{t\geq 0}$ is exponentially stable.



Figure 1: Relations between the different stability notions with respect to L^p , $p < \infty$, and L^{∞} for a system $\Sigma(A, B)$, where it is assumed that the semigroup is exponentially stable.

Proof. Clearly, by the definition of iISS and UBEBS, (ii) \Rightarrow (iii). By Proposition 2.8, (iii) \Rightarrow (i). Thus in view of Proposition 2.8 it remains to show that L^{p} -infinite-time admissibility and exponential stability imply L^{p} -iISS. Indeed, L^{p} -infinite-time admissibility and exponential stability show for $x_{0} \in X$ and $u \in L^{p}(0,t;U)$ that

$$||x(t)|| \le M e^{-\omega t} ||x_0|| + c_{\infty} ||u||_{L^p(0,t;U)}$$

= $M e^{-\omega t} ||x_0|| + c_{\infty} \left(\int_0^t ||u(s)||^p \, ds \right)^{1/p}$,

which shows L^p -iISS.

Remark 2.10. Let $1 \leq p < \infty$. If the system $\Sigma(A, B)$ is L^p -admissible and $(T(t))_{t\geq 0}$ is exponentially stable, then the system $\Sigma(A, B)$ is L^p -ISS with the following choices for the functions β and μ :

$$\beta(s,t) := M e^{-\omega t} s \quad and \quad \mu(s) := c_{\infty} s.$$

Here the constants M and ω are given by (7) and $c_{\infty} = \sup_{t>0} c(t)$.

Proposition 2.11. If $\Sigma(A, B)$ is L^{∞} -*iISS*, then $\Sigma(A, B)$ is L^{∞} -zero-class admissible.

Proof. If $\Sigma(A, B)$ is L^{∞} -iISS, then there exist $\theta \in \mathcal{K}_{\infty}$ and $\mu \in \mathcal{K}$ such that for all $t > 0, u \in L^{\infty}(0, t; U), u \neq 0$

$$\frac{1}{\|u\|_{\infty}} \left\| \int_0^t T_{-1}(s) Bu(s) \, ds \right\| \le \theta \left(\int_0^t \mu\left(\frac{\|u(s)\|_U}{\|u\|_{\infty}} \right) \, ds \right). \tag{8}$$

Since the function μ is monotonically increasing and $||u(s)||_U \leq ||u||_{\infty}$ a.e., the right-hand side of (8) is bounded above by $\theta(t\mu(1))$ which converges to zero as $t \searrow 0$.

We illustrate the relations of the different stability notions with respect to L^{∞} discussed above in the diagram depicted in Figure 1.

Proposition 2.12. Suppose B is a bounded operator from U to X and $Z \subseteq L^1_{loc}(0,\infty;U)$ is a function space. Then the following statements are equivalent.

(i) $(T(t))_{t\geq 0}$ is exponentially stable,

- (ii) $\Sigma(A, B)$ is Z-admissible and $(T(t))_{t\geq 0}$ is exponentially stable,
- (iii) $\Sigma(A, B)$ is Z-infinite-time admissible and $(T(t))_{t\geq 0}$ is exponentially stable,
- (iv) $\Sigma(A, B)$ is Z-ISS,
- (v) $\Sigma(A, B)$ is Z-iISS.
- (vi) $\Sigma(A, B)$ is Z-UBEBS and $(T(t))_{t\geq 0}$ is exponentially stable,
- (vii) $\Sigma(A, B)$ is L^1_{loc} -admissible and $(T(t))_{t\geq 0}$ is exponentially stable.
- If Z satisfies Assumption (B), then the above assertions are equivalent to

(viii) $\Sigma(A, B)$ is Z-zero-class admissible and $(T(t))_{t>0}$ is exponentially stable.

Proof. By Proposition 2.8 we have $(v) \Rightarrow (vi) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (iv) \Rightarrow (i)$, and Proposition 2.9 and Remark 2.7 prove $(vii) \Rightarrow (v)$. The implication $(i) \Rightarrow (vii)$ follows from the fact that by the boundedness of B we have $x(t) \in X$ for all $t \ge 0$ and all $u \in L^1_{loc}(0,t;U)$. Clearly, $(viii) \Rightarrow (ii)$. Thus it remains to show that if Z satisfies Assumption (B), then $(i) \Rightarrow (viii)$. Let $(T(t))_{t\ge 0}$ be exponentially stable, that is, there exist constants $M, \omega > 0$ such that (7) holds. Therefore, for any $u \in L^1_{loc}(0,t;U)$,

$$\begin{aligned} \|x(t)\| &\leq M e^{-\omega t} \|x_0\| + M \|B\| \int_0^t e^{-\omega(t-s)} \|u(s)\|_U \, ds \\ &\leq M e^{-\omega t} \|x_0\| + M \|B\| \int_0^t \|u(s)\|_U \, ds. \end{aligned}$$
(9)

Using that Z(0,t;U) is continuously embedded in $L^1(0,t;U)$, we conclude that

$$\|x(t)\| \le M e^{-\omega t} \|x_0\| + M \|B\| \kappa(t) \|u(s)\|_{Z(0,t;U)}$$
(10)

for all $t \ge 0$. If Assumption (B) holds, then the embedding constants $\kappa(t)$ tend to 0 as $t \searrow 0$. Hence, (10) shows that (i) implies (viii).

Remark 2.13. Note that in Proposition 2.12, the assertions are independent of Z as the assertions only rest on exponential stability. In particular, if one of the equivalent conditions hold, then the system $\Sigma(A, B)$ is L^p -ISS with the following choices for the functions β and μ

$$\beta(s,t) := M e^{-\omega t} s \quad and \quad \mu(s) := \frac{M}{\omega q} \|B\|s,$$

and L^p -iISS with

$$\beta(s,t) := Me^{-\omega t}s, \qquad \mu(s) := s, \quad and \quad \theta(s) := sM\|B\|.$$

Here the constants M and ω are given by (7).

Although, in this case, the notions of ISS and iISS are independent of p, the choices for the functions μ , however, do depend on p. Note, that if B is unbounded, then these notions also depend on p.

Furthermore, note that in the trivial case $X = U = \mathbb{C}$ and A = -1, B = 1, we have that the system $\Sigma(A, B)$ is not L^1 -zero-class admissible.

3 IISS from the viewpoint of Orlicz spaces

In this section we relate L^{∞} -ISS and L^1 -ISS to admissibility with respect to Orlicz spaces E_{Φ} corresponding to a Young function Φ . For the definition and fundamental properties of Orlicz spaces and Young functions, we refer to the Appendix. The main results of this section are summarized in the following three theorems.

Theorem 3.1. Let $(T(t))_{t\geq 0}$ be exponentially stable. Then the following statements are equivalent.

- (i) There is a Young function Φ such that the system $\Sigma(A, B)$ is E_{Φ} -ISS.
- (*ii*) $\Sigma(A, B)$ is L^{∞} -*iISS*.
- (iii) There is a Young function Φ such that the system $\Sigma(A, B)$ is E_{Φ} -UBEBS.

If Φ satisfies the Δ_2 -condition, Definition A.12, more can be said.

Theorem 3.2. If Φ is a Young function that satisfies the Δ_2 -condition, then the following are equivalent.

- (i) $\Sigma(A, B)$ is E_{Φ} -ISS,
- (ii) $\Sigma(A, B)$ is E_{Φ} -iISS,
- (iii) $\Sigma(A, B)$ is E_{Φ} -UBEBS and $(T(t))_{t>0}$ is exponentially stable.

Remark 3.3. Since L^p -spaces are examples of Orlicz spaces where the Δ_2 condition is satisfied, Theorem 3.2 can be seen as a generalization of Proposition
2.9.

Theorem 3.4. Let $(T(t))_{t\geq 0}$ be exponentially stable. Then the following statements are equivalent.

- (i) $\Sigma(A, B)$ is L^1 -ISS.
- (ii) $\Sigma(A, B)$ is L^1 -iISS.
- (iii) $\Sigma(A, B)$ is E_{Φ} -admissible for every Young function Φ .

The proofs of Theorems 3.1, 3.2 and 3.4 are given at the end of this section.



Figure 2: Relations between the different stability notions with respect to Orlicz spaces for a system $\Sigma(A, B)$, where it is assumed that the semigroup is exponentially stable and that Φ satisfies the Δ_2 -condition. **Lemma 3.5.** Let $(T(t))_{t\geq 0}$ be exponentially stable and $\Sigma(A, B)$ be Z-admissible, where Z is either L^{∞} or E_{Φ} for some Young function Φ . Then $\Sigma(A, B)$ is Z-iISS if and only if there exist $\theta \in \mathcal{K}_{\infty}$ and $\mu \in \mathcal{K}$ such that for every $u \in Z(0, 1; U)$,

$$\left\| \int_{0}^{1} T_{-1}(s) Bu(s) \, ds \right\| \le \theta \left(\int_{0}^{1} \mu(\|u(s)\|_{U}) ds \right). \tag{11}$$

Proof. It suffices to show that there exists C > 0 such that for any t > 0 and $u \in Z(0, t; U)$, there exists $\tilde{u} \in Z(0, 1; U)$ with

$$\left\|\int_0^t T_{-1}(s)Bu(s)\,ds\right\| \le C\left\|\int_0^1 T_{-1}(s)B\tilde{u}(s)\,ds\right\|$$

and $\int_0^1 \mu(\|\tilde{u}(s)\|_U) ds \leq \int_0^t \mu(\|u(s)\|_U) ds$ for any $\mu \in \mathcal{K}$. Without loss of generality, we assume that $t \in \mathbb{N}$, otherwise extend u suitably by the zero-function. By splitting the integral, substitution and the fact that $\Sigma(A, B)$ is Z-admissible, we get for $u \in Z(0, t; U)$,

$$\begin{split} \left\| \int_{0}^{t} T_{-1}(s) Bu(s) \, ds \right\| &= \left\| \sum_{k=0}^{t-1} \int_{k}^{k+1} T_{-1}(s) Bu(s) \, ds \right\| \\ &= \left\| \sum_{k=0}^{t-1} T(k) \int_{0}^{1} T_{-1}(s) Bu(s+k) \, ds \right\| \\ &\leq \sum_{k=0}^{t-1} \| T(k) \| \max_{k=0,\dots,n-1} \left\| \int_{0}^{1} T_{-1}(s) Bu(s+k) \, ds \right\| \\ &\leq C \cdot \max_{k=0,\dots,n-1} \left\| \int_{0}^{1} T_{-1}(s) Bu(s+k) \, ds \right\|, \end{split}$$

where $C < \infty$ only depends on the exponentially stable semigroup $(T(t))_{t\geq 0}$. Choose $\tilde{u} = u(\cdot + k_0)|_{(0,1)}$, where k_0 is the argument such that the above maximum is attained. If $Z = L^{\infty}$, it is plain that $\tilde{u} \in Z$. For $Z = E_{\Phi}$, it remains to show that $\tilde{u} \in E_{\Phi}(0,1;U)$. Let $(u_n)_n \subset L^{\infty}(0,t;U)$ such that $u_n \to u$ in $E_{\Phi}(0,t;U)$. By the definition of $\|\cdot\|_{E_{\Phi}} = \|\cdot\|_{L_{\Phi}}$, it is easily seen that $u_n(\cdot + k_0)|_{(0,1)}$ converges to \tilde{u} in $E_{\Phi}(0,1;U)$. Thus, $\tilde{u} \in E_{\Phi}(0,1;U)$.

Lemma 3.6. Let $(T(t))_{t\geq 0}$ be exponentially stable and let $\Sigma(A, B)$ be L^{∞} iISS. Then there exist $\tilde{\theta}, \Phi \in \mathcal{K}_{\infty}$ such that Φ is a Young function which is continuously differentiable on $(0, \infty)$ and

$$\left\|\int_0^t T_{-1}(s)Bu(s)\,ds\right\| \le \tilde{\theta}\left(\int_0^t \Phi(\|u(s)\|)\,ds\right) \tag{12}$$

for all t > 0 and $u \in L^{\infty}(0, t; U)$.

Proof. By assumption, there exist $\theta \in \mathcal{K}_{\infty}$ and $\mu \in \mathcal{K}$ such that (11) holds for $Z = L^{\infty}$. Without loss of generality we can assume that μ belongs to \mathcal{K}_{∞} and that it is continuously differentiable on $(0, \infty)$, see Corollary B.2. Let $\psi_1 \in \mathcal{K}_{\infty}$ such that $\lim_{s \searrow 0} \psi_1(s)\mu'(s) = 0$. Now let us consider the function $\psi_2 := \psi_1 \circ \mu^{-1}$. Since $\psi_2 \in \mathcal{K}_{\infty}$ and hence $\Psi(s) = \int_0^s \psi_2(r) dr$ is a Young function. Thus, by Jensen's inequality we obtain

$$\begin{aligned} \theta\left(\int_0^1 \mu(\|u(s)\|)\,ds\right) &= (\theta\circ\Psi^{-1})\left(\Psi\left(\int_0^1 \mu(\|u(s)\|)\,ds\right)\right)\\ &\leq (\theta\circ\Psi^{-1})\left(\int_0^1 (\Psi\circ\mu)(\|u(s)\|)\,ds\right). \end{aligned}$$

Clearly $\tilde{\theta} := \theta \circ \Psi^{-1}$ belongs to \mathcal{K}_{∞} . We will show that $\tilde{\theta}$ can be majorized by a Young function. To see this, we observe that for s > 0,

$$(\Psi \circ \mu)'(s) = \psi_2(\mu(s))\mu'(s) = \psi_1(s)\mu'(s).$$

Therefore, $\lim_{s\searrow 0} (\Psi \circ \mu)'(s) = 0$ by the choice of ψ_1 . Hence by Lemma B.1 and Corollary B.2 there exists a $\psi_3 \in \mathcal{K}_{\infty}$ which is continuously differentiable on $(0,\infty)$ such that $(\Psi \circ \mu)'(s) \leq \psi_3(s)$ for all s > 0. Define $\Phi(s) = \int_0^s \psi_3(r) dr$. Then Φ is a Young function and the inequality $\Psi \circ \Phi \leq \tilde{\Phi}$ holds on $(0,\infty)$. Altogether, we obtain

$$\theta\left(\int_0^1 \mu(\|u(s)\|)\,ds\right) \le \tilde{\theta}\left(\int_0^1 \Phi(\|u(s)\|)\,ds\right).$$

By Lemma 3.5, the assertion follows.

Proof of Theorem 3.1. (i) \Rightarrow (ii): Since $\Lambda(s) = s^2$ defines a Young function with $\Lambda(1) = 1$, it can be easily seen that

$$\Phi_1(s) = \begin{cases} \Phi(s), & s < 1, \\ \Phi(\Lambda(s)), & s \ge 1, \end{cases}$$

defines another Young function such that $\Phi \leq \Phi_1$. Furthermore, Φ_1 increases essentially more rapidly than Φ (see Def. A.13), since the composition $\Phi \circ \Lambda$ of two Young functions Φ, Λ is known to be increasing essentially more rapidly than Φ (see page 114 of [KR61]). We define $\theta: [0, \infty) \to [0, \infty)$ by

$$\theta(\alpha) = \sup\left\{ \left\| \int_0^1 T_{-1}(s) Bu(s) \, ds \right\| \, \Big| \, u \in L^\infty(0,1;U), \int_0^1 \Phi_1(\|u(s)\|) \, ds \le \alpha \right\},$$

for $\alpha > 0$ and $\theta(0) = 0$. Clearly, θ is non-decreasing. Admissibility and Remark A.10 yield that for $u \in L^{\infty}(0, 1; U)$,

$$\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| \le c(1)\|u\|_{E_{\Phi}(0,1;U)} \le c(1)\left(1+\int_0^1 \Phi_1(\|u(s)\|)\,ds\right).$$

Hence, $\theta(\alpha) < \infty$ for all $\alpha \ge 0$.

If we can show that $\lim_{t\searrow 0} \theta(t) = 0$, then, by Corollary B.2, there exists $\tilde{\theta} \in \mathcal{K}_{\infty}$ such that $\theta \leq \tilde{\theta}$ pointwise. Therefore, let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to 0. By the definition of θ , for any $n \in \mathbb{N}$ there exists $u_n \in L^{\infty}(0, 1; U)$ such that

$$\int_0^1 \Phi_1(\|u_n(s)\|) \, ds < \alpha_n$$

and

$$\left\|\theta(\alpha_n) - \left\|\int_0^1 T_{-1}(s)Bu_n(s)\,ds\right\|\right\| < \frac{1}{n}.\tag{13}$$

Hence the sequence $(||u_n||_U)_{n \in \mathbb{N}}$ is Φ_1 -mean convergent to zero (see Def. A.11). By Theorem 13.4 in [KR61], the sequence even converges to zero with respect to the norm of the space $L_{\Phi}(0, 1)$, and thus also in $E_{\Phi}(0, 1)$. Hence

$$\lim_{n \to \infty} \|u_n\|_{E_{\Phi}(0,1;U)} = \lim_{n \to \infty} \|\|u_n(\cdot)\|_U\|_{E_{\Phi}(0,1)} = 0.$$

Hence, by admissibility,

$$\left\|\int_0^1 T_{-1}Bu_n(s)\,ds\right\| \le c(1)\|u_n\|_{E_{\Phi}(0,1;U)} \to 0,$$

as $n \to \infty$. Altogether we obtain that

$$\begin{aligned} \theta(\alpha_n) &\leq \left| \theta(\alpha_n) - \left\| \int_0^1 T_{-1}(s) B u_n(s) \, ds \right\| \right\| + \left\| \int_0^1 T_{-1}(s) B u_n(s) \, ds \right\| \\ &\leq \frac{1}{n} + c(1) \| u_n \|_{E_{\Phi}(0,1;U)}, \end{aligned}$$

and thus, $\lim_{n\to\infty} \theta(\alpha_n) = 0$.

Therefore, there exists $\tilde{\theta} \in \mathcal{K}_{\infty}$ such that $\theta \leq \tilde{\theta}$ pointwise. The function $\Phi_1: [0,\infty) \to [0,\infty)$ is a Young function, in particular we have $\Phi_1 \in \mathcal{K}_{\infty}$. The definition of θ yields that

$$\left\| \int_0^1 T_{-1}(s) Bu(s) \, ds \right\| \le \theta \left(\int_0^1 \Phi_1(\|u(s)\|) \, ds \right) \le \tilde{\theta} \left(\int_0^1 \Phi_1(\|u(s)\|) \, ds \right)$$

for all $u \in L^{\infty}(0,1;U)$. By Lemma 3.5, we conclude that $\Sigma(A, B)$ is L^{∞} -iISS.

(ii) \Rightarrow (i): Now assume that $\Sigma(A, B)$ is L^{∞} -iISS. It suffices to show that there is a Young function Φ such that $\int_0^t T_{-1}(s)Bu(s)ds \in X$ for all $u \in E_{\Phi}(0,t)$. Note that since $E_{\Phi}(0,t;U) \subset L^1(0,t;U)$ for any Young function Φ the integral always exists in X_{-1} . By assumption, $\int_0^t T_{-1}(s)Bu(s)ds \in X$ for all $u \in L^{\infty}(0,t)$. By Lemma 3.6, there exist $\tilde{\theta} \in \mathcal{K}_{\infty}$ and a Young function Φ such that (12) holds. Let $u \in E_{\Phi}$. By definition, there is a sequence $(u_n)_{n \in \mathbb{N}} \subset L^{\infty}(0,t;U)$ such that $\lim_{n\to\infty} \|u_n - u\|_{E_{\Phi}(0,t;U)} = 0$. Since $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $E_{\Phi}(0,t;U)$, we can assume without loss of generality that $\|u_n - u_m\|_{E_{\Phi}(0,t;U)} < 1$ for all $m, n \in \mathbb{N}$. Lemma 3.8.4 (i) in $[\text{KJF77}]^1$ now yield

$$\left\| \int_{0}^{t} T_{-1}(s) B(u_{n}(s) - u_{m}(s)) \, ds \right\| \leq \tilde{\theta} \left(\int_{0}^{t} \Phi(\|u_{n}(s) - u_{m}(s)\|_{U}) \, ds \right)$$
$$\leq \tilde{\theta} \left(\|\|(u_{n} - u_{m})(\cdot)\|_{U}\|_{E_{\Phi}(0,t)} \right)$$
$$= \tilde{\theta} \left(\|u_{n} - u_{m}\|_{E_{\Phi}(0,t;U)} \right).$$

 $^{^1\}mathrm{note}$ that this theorem is stated for scalar-valued functions only. However, this suffices here.

Hence $(\int_0^t T_{-1}(s)Bu_n(s) ds)_{n \in \mathbb{N}}$ is a Cauchy sequence in X and thus converges. Let y denote its limit. Since $E_{\Phi}(0, t; U)$ is continuously embedded in $L^1(0, t; U)$, see Remark A.10, it follows that

$$\lim_{n \to \infty} \int_0^t T_{-1}(s) B u_n(s) \, ds = \int_0^t T_{-1}(s) B u(s) \, ds$$

in X_{-1} . Since X is continuously embedded in X_{-1} , we conclude that

$$y = \int_0^t T_{-1}(s) Bu(s) \, ds.$$

Thus, we have shown that $\int_0^t T_{-1}(s)Bu(s) ds \in X$ for all $u \in E_{\Phi}$ and hence, by the closed graph theorem, $\Sigma(A, B)$ is admissible with respect to E_{Φ} .

(i) \Rightarrow (iii): This follows since for all $u \in E_{\Phi}(0,t;U)$ it holds that $u \in \tilde{L}_{\Phi}(0,t;U)$ and

$$||u||_{E_{\Phi}} \le 1 + \int_0^t \Phi(||u(s)||_U) \, ds,$$

see Remark A.6.

(iii) \Rightarrow (i): This is clear by the closed graph theorem.

Proof of Theorem 3.2. The implications (ii) \Rightarrow (iii) \Rightarrow (i) follow, analogously as for the L^p -case by Proposition 2.8.

(i) \Rightarrow (ii): Similarly to the proof of Theorem 3.1, we can define a non-decreasing function θ by

$$\theta(\alpha) = \sup\left\{ \left\| \int_0^1 T_{-1}(s) Bu(s) \, ds \right\| \, \Big| \, E_{\Phi}(0,1;U), \int_0^1 \Phi(\|u(s)\|) \, ds \le \alpha \right\},\$$

for $\alpha > 0$ and $\theta(0) := 0$. By admissibility and Remark A.10, we have that

$$\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| \le c(1)\|u\|_{E_{\Phi}(0,1;U)} \le c(1)\left(1+\int_0^1 \Phi(\|u(s)\|)\,ds\right),$$

for $u \in E_{\Phi}(0, 1; U) \subset \tilde{L}_{\Phi}(0, t; U)$. Hence, θ is well-defined. In analogy to the proof of Theorem 3.1, it remains to show that θ is right-continuous at 0. This follows because Φ satisfies the Δ_2 -condition. In fact, if the latter is true, it is known that a sequence $(u_n)_{n\in\mathbb{N}}$ in E_{Φ} converges to 0 if and only if the sequence is Φ -mean convergent to zero (see Def. A.11). Therefore, $\alpha_n \searrow 0$ implies that there exists a sequence $u_n \in E_{\Phi}(0,1;U)$ that converges to 0 in E_{Φ} and such that

$$\left|\theta(\alpha_n) - \left\|\int_0^1 T_{-1}Bu_n(s)\,ds\right\|\right| \le \frac{1}{n}, \quad n \in \mathbb{N}.$$

By E_{Φ} -admissibility, we conclude that $\theta(\alpha_n) \to 0$ as $n \to \infty$. Hence, by Lemma B.2, we find $\tilde{\theta} \in \mathcal{K}_{\infty}$ such that $\theta \leq \tilde{\theta}$ pointwise. By definition of θ , this implies

$$\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| \le \tilde{\theta}\left(\int_0^1 \Phi(\|u(s)\|)\,ds\right)$$

for all $u \in E_{\Phi}(0,1;U)$. Finally, Lemma 3.5 yields that $\Sigma(A,B)$ is E_{Φ} -iISS. \Box

Proof of Theorem 3.4. By Propositions 2.8 and 2.9, we only need to show the equivalence of (i) and (iii). That (i) implies (iii) follows immediately since E_{Φ} is continuously embedded in L^1 .

Conversely, let $\Sigma(A, B)$ be E_{Φ} -admissible for every Young function Φ . According to Proposition 2.8 (a), we have to show that $\Sigma(A, B)$ is L^1 -admissible. Let t > 0 and $u \in L^1(0, t; U)$. It remains to prove that $\int_0^t T_{-1}(s)Bu(s) ds \in X$. By [KR61, p. 61], there exists a Young function Φ satisfying the Δ_2 -condition such that $||u|| \in L_{\Phi}^{-2}$. The Δ_2 -condition implies that $E_{\Phi} = L_{\Phi}$ and $E_{\Phi}(0, t; U) =$ $L_{\Phi}(0, t; U)$, see [RR91, p. 303] or [Sch05, Thm. 5.2]. Thus $\int_0^t T_{-1}(s)Bu(s) ds \in$ X by assumption.

4 Stability of parabolic diagonal systems

In the previous section we have proved that for infinite-dimensional systems L^{∞} iISS implies L^{∞} -ISS. It is an open question whether the converse implication holds. Here, we give a positive answer for parabolic diagonal systems.

In this section we assume that $U = \mathbb{C}$ and that the operator A possesses a q-Riesz basis of eigenvectors $(e_n)_{n \in \mathbb{N}}$ with eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ lying in a sector in the open left half-plane \mathbb{C}_- . More precisely, $(e_n)_{n \in \mathbb{N}}$ is a q-Riesz basis of X, if $(e_n)_{n \in \mathbb{N}}$ is a Schauder basis and for some constants $c_1, c_2 > 0$ we have

$$c_1 \sum_k |a_k|^q \le \left\| \sum_k a_k e_k \right\|^q \le c_2 \sum_k |a_k|^q$$

шa

for all sequences (a_k) in ℓ^q . Thus without loss of generality we can assume that $X = \ell^q$ and that $(e_n)_{n \in \mathbb{N}}$ is the canonical basis of ℓ^q . We further assume that the sequence $(\lambda_n)_{n \in \mathbb{N}}$ lies in \mathbb{C} with $\sup_n \operatorname{Re}(\lambda_n) < 0$ and that there exists a constant k > 0 such that $|\operatorname{Im} \lambda_n| \leq k |\operatorname{Re} \lambda_n|, n \in \mathbb{N}$, i.e., $(\lambda_n)_n \subset S_\theta$ for some $\theta \in (0, \pi/2)$, where

$$S_{\theta} = \{ z \in \mathbb{C} : |z| > 0, |\arg z| < \theta \}.$$

Then the linear operator $A: D(A) \subset \ell^q \to \ell^q$ is given by

$$Ae_n = \lambda_n e_n, \qquad n \in \mathbb{N},$$

and $D(A) = \{(x_n) \in \ell^q \mid \sum |x_n\lambda_n|^q < \infty\}$. A generates an analytic exponentially stable C_0 -semigroup $(T(t))_{t\geq 0}$ on ℓ^q , which is given by $T(t)e_n = e^{t\lambda_n}e_n$. An easy calculation shows that the extrapolation space $(\ell^q)_{-1}$ is given by

$$(\ell^q)_{-1} = \left\{ x = (x_n)_{n \in \mathbb{N}} \mid \sum_n \frac{|x_n|^q}{|\lambda_n|^q} < \infty \right\},\$$
$$\|x\|_{X_{-1}} = \|A^{-1}x\|_{\ell^q}.$$

$$\exists c, u_0 > 0 \ \forall u, v \ge u_0: \quad Q(uv) \le cQ(u)Q(v).$$

In fact, it is easy to see that this property implies that $Q \circ Q$ satisfies

$$u \ge u_0: \quad (Q \circ Q)(\ell u) \le k(\ell)(Q \circ Q)(u),$$

for some $\ell > 1$ and $k(\ell) > 0$, which is known to be equivalent to $Q \circ Q$ satisfying the Δ_2 condition, see [KR61, p. 23].

²In [KR61, p. 61] is a ctually shown that for given $f \in L^1(0,t)$, there exists an Young function Q such that $f \in L_{Q \circ Q}(0,t)$ and such that Q satisfies the Δ' -condition, i.e.,

Thus the linear bounded operator B from \mathbb{C} to $(\ell^q)_{-1}$ can be identified with a sequence $(b_n)_{n\in\mathbb{N}}$ in \mathbb{C} satisfying

$$\sum_{n \in \mathbb{N}} \frac{|b_n|^q}{|\lambda_n|^q} < \infty$$

Thanks to the sectoriality condition for $(\lambda_n)_{n \in \mathbb{N}}$ this equivalent to

$$\sum_{n \in \mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re} \lambda_n|^q} < \infty.$$

Parabolic diagonal systems are a well-studied class of systems in the literature, see e.g. [TW09].

The following result shows that, under the above assumptions, the system $\Sigma(A, B)$ is L^{∞} -iISS. Thus for this class of systems L^{∞} -iISS is equivalent to L^{∞} -ISS, and both notions are implied by $B \in (\ell^q)_{-1}$, that is, $\sum_n \frac{|b_n|^q}{|\lambda_n|^q} < \infty$. The following theorem generalizes the main result in [JNPS16], where the case q = 2 is studied.

Proposition 4.1. Let A possess a q-Riesz basis of X consisting of eigenvectors $(e_n)_{n \in \mathbb{N}}$ with eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ lying in a sector in the open left half-plane \mathbb{C}_- with $\sup_n \operatorname{Re}(\lambda_n) < 0$ and $B \in \mathcal{L}(\mathbb{C}, X_{-1})$. Then, for each $\alpha \in (0, 1)$, $\Sigma(A, A_{-1}^{-\alpha}B)$ is L^p -admissible for $p > \max(1/\alpha, q)$, where³

$$A_{-1}^{-\alpha}B = \left(b_n\lambda_n^{-\alpha}\right)_{n\in\mathbb{N}} \in (\ell^q)_{-1}.$$

Proof. The proof follows from the characterization of L^p -admissibility for parabolic diagonal systems with scalar input given in [JPP14, Thm. 3.5]. In fact, for $p \in (q, \infty), \Sigma(A, \tilde{B})$ is infinite-time L^p -admissible if and only if

$$\left(2^{-\frac{nq(p-1)}{p}}\mu(S_n)\right)_{n\in\mathbb{Z}}\in\ell^{\frac{p}{p-q}}(\mathbb{Z}),$$

where $\mu = \sum_{n \in \mathbb{Z}} |\tilde{b}_n|^q \delta_{\lambda_n}$ and $S_n = \{z \in \mathbb{C} : \operatorname{Re} z \in (2^{n-1}, 2^n]\}, n \in \mathbb{Z}^4$. Since the considered semigroup is exponentially stable, \mathbb{Z} can be replaced by \mathbb{N} in the above line. Hence, it remains to show that

$$\left(2^{-\frac{nq(p-1)}{p-q}}\mu(S_n)^{\frac{p}{p-q}}\right)_{n\in\mathbb{N}}\in\ell^1(\mathbb{N}),$$

for $\mu = \sum_{n \in \mathbb{N}} |b_n^q \lambda_n^{-\alpha q}| \delta_{\lambda_n}$. By definition of S_n , it follows that

$$\mu(S_n) = \sum_{k:\lambda_k \in S_n} |b_k^q \lambda_k|^{-\alpha q}$$
$$= \sum_{k:\lambda_k \in S_n} \frac{|b_k|^q}{|\lambda_k|^q} |\lambda_k|^{q-\alpha q}$$
$$\leq C \cdot 2^{nq(1-\alpha)} \sum_{k \in \mathbb{N}} \frac{|b_k|^q}{|\lambda_k|^q}$$
$$= C \cdot 2^{nq(1-\alpha)} \cdot ||B||_{X_{-1}}^q,$$

³up to identification

⁴Here, δ_{λ} denotes the Dirac measure at λ .

where C is a constant only depending on the sector in which the λ_n 's lie. Thus

$$2^{-\frac{nq(p-1)}{p-q}}\mu(S_n)^{\frac{p}{p-q}} \le (k\|B\|_{X_{-1}}^q)^{\frac{p}{p-q}} \cdot 2^{\frac{nq}{p-q}(1-\alpha p)}, \quad n \in \mathbb{N}.$$

Theorem 4.2. Let $U = \mathbb{C}$, and assume that the operator A possesses a q-Riesz basis of X consist of eigenvectors $(e_n)_{n \in \mathbb{N}}$ with eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ lying in a sector in the open left half-plane \mathbb{C}_- with $\sup_n \operatorname{Re}(\lambda_n) < 0$ and $B \in \mathcal{L}(\mathbb{C}, X_{-1})$. Then the system $\Sigma(A, B)$ is L^{∞} -iISS, and hence also L^{∞} -ISS and L^{∞} -zero-class admissible.

Remark 4.3. In the situation of Theorem 4.2, $\Sigma(A, B)$ is L^{∞} -iISS if and only if $\Sigma(A, B)$ is L^{∞} -ISS.

Proof of Theorem 4.2. Without loss of generality we may assume $X = \ell^q$ and that $(e_n)_{n \in \mathbb{N}}$ is the canonical basis of ℓ^q . Let $f: (0, \infty) \to [0, \infty)$ be defined by

$$f(s) = \sum_{n \in \mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re} \lambda_n|^{q-1}} e^{\operatorname{Re} \lambda_n s}.$$

Then it is easy to see that f is smooth, strictly decreasing, belongs to $L^1(0,\infty)$, and satisfies $\lim_{s \to 0} f(s) = \infty$ and $\lim_{s \to \infty} f(s) = 0$.

We remark that boundedness of $(\operatorname{Re} \lambda_n)_{n \in \mathbb{N}}$ implies boundedness of $(\lambda_n)_{n \in \mathbb{N}}$. Thus if the sequences $(\operatorname{Re} \lambda_n)_{n \in \mathbb{N}}$ is bounded or $b_n = 0$ for all but finitely many $n \in \mathbb{N}$, then B is a bounded operator from \mathbb{C} to ℓ^q and therefore $\Sigma(A, B)$ is L^{∞} -iISS by Proposition 2.12. Moreover, the series defining the function f is absolutely convergent and

$$\frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^{q-1}}e^{\operatorname{Re}\lambda_n s} + \frac{|b_m|^q}{|\operatorname{Re}\lambda_m|^{q-1}}e^{\operatorname{Re}\lambda_m s} = \frac{|b_n|^q + |b_m|^q}{|\operatorname{Re}\lambda_n|^{q-1}}e^{\operatorname{Re}\lambda_n s}$$

if $\operatorname{Re} \lambda_n = \operatorname{Re} \lambda_m$.

Thus without loss of generality we may assume that $\operatorname{Re} \lambda_n < \operatorname{Re} \lambda_m$ for m < n, $\lim_{n\to\infty} \operatorname{Re} \lambda_n = -\infty$, $b_n \neq 0$ for $n \in \mathbb{N}$ and B is unbounded. By Remark 178 in [Kno28] there is a strictly increasing unbounded sequence $(h_n)_{n\in\mathbb{N}}$ of positive numbers such that

$$\sum_{n \in \mathbb{N}} \frac{h_n |b_n|^q}{|\operatorname{Re} \lambda_n|^q} < \infty.$$

We define the smooth, strictly decreasing function $g: (0, \infty) \to [0, \infty)$ by

$$g(s) = \sum_{n \in \mathbb{N}} \frac{h_n |b_n|^q}{|\operatorname{Re} \lambda_n|^{q-1}} e^{\operatorname{Re} \lambda_n s},$$

for s > 0. Clearly, $g \in L^1(0, \infty)$. The function $\eta: [0, \infty) \to [0, \infty)$, $\eta(s) = g'(s)/f'(s)$, is strictly decreasing and positive. Indeed, for s > t > 0, we have to show that g'(s)f'(t) < g'(t)f'(s) holds which is equivalent to

$$\sum_{n\in\mathbb{N}} \frac{h_n |b_n|^q}{|\operatorname{Re}\lambda_n|^{q-2}} e^{\operatorname{Re}\lambda_n s} \cdot \sum_{n\in\mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^{q-2}} e^{\operatorname{Re}\lambda_n t} < \sum_{n\in\mathbb{N}} \frac{h_n |b_n|^q}{|\operatorname{Re}\lambda_n|^{q-2}} e^{\operatorname{Re}\lambda_n t} \cdot \sum_{n\in\mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^{q-2}} e^{\operatorname{Re}\lambda_n s}.$$

Since all series appearing are absolutely convergent, it is sufficient to show that the following inequality holds for all $m < n \in \mathbb{N}$:

$$\frac{h_{n}|b_{n}|^{q}e^{\operatorname{Re}\lambda_{n}s}}{|\operatorname{Re}\lambda_{n}|^{q-2}} \cdot \frac{|b_{m}|^{q}e^{\operatorname{Re}\lambda_{m}t}}{|\operatorname{Re}\lambda_{m}|^{q-2}} + \frac{h_{m}|b_{m}|^{q}e^{\operatorname{Re}\lambda_{m}s}}{|\operatorname{Re}\lambda_{m}|^{q-2}} \cdot \frac{|b_{n}|^{q}e^{\operatorname{Re}\lambda_{n}t}}{|\operatorname{Re}\lambda_{n}|^{q-2}} \\
< \frac{h_{n}|b_{n}|^{q}e^{\operatorname{Re}\lambda_{n}t}}{|\operatorname{Re}\lambda_{n}|^{q-2}} \cdot \frac{|b_{m}|^{q}e^{\operatorname{Re}\lambda_{m}s}}{|\operatorname{Re}\lambda_{m}|^{q-2}} + \frac{h_{m}|b_{m}|^{q}e^{\operatorname{Re}\lambda_{m}t}}{|\operatorname{Re}\lambda_{m}|^{q-2}} \cdot \frac{|b_{n}|^{q}e^{\operatorname{Re}\lambda_{n}s}}{|\operatorname{Re}\lambda_{n}|^{q-2}}. \quad (14)$$

As $b_n \neq 0$ for all $n \in \mathbb{N}$, inequality (14) is equivalent to

$$h_n e^{\operatorname{Re}\lambda_n(s-t)} + h_m e^{\operatorname{Re}\lambda_m(s-t)} < h_n e^{\operatorname{Re}\lambda_m(s-t)} + h_m e^{\operatorname{Re}\lambda_n(s-t)},$$

which is equivalent to

$$(h_n - h_m)(e^{\operatorname{Re}\lambda_m(s-t)} - e^{\operatorname{Re}\lambda_n(s-t)}) > 0.$$

The latter inequality is true as $\operatorname{Re} \lambda_n < \operatorname{Re} \lambda_m$ and $h_n > h_m$. In particular the following limit exists

$$a := \lim_{s \to \infty} \frac{g'(s)}{f'(s)} \ge 0$$

Define $\Phi: [0,\infty) \to [0,\infty)$ by $\Phi(f(s)) = g(s) - af(s)$. Then Φ is a smooth Young function. Indeed:

$$\Phi'(f(s)) = \frac{g'(s)}{f'(s)} - a$$

and hence

$$\Phi'(0) = \lim_{s \to \infty} \Phi'(f(s)) = \lim_{s \to \infty} \frac{g'(s)}{f'(s)} - a = 0.$$

Moreover, that Φ' is positive and nondecreasing on $(0, \infty)$ follows directly from the fact that both functions η and f are strictly decreasing and η bounded below by a.

We are left to verify that $\lim_{s\to\infty} \Phi'(s) = \infty$. Since Φ' is strictly increasing, it is sufficient to show that Φ' is unbounded. Assume there is some constant $C \ge 0$ such that $\Phi'(s) \le C - a$ for all $s \ge 0$. Then $g'(s)/f'(s) \le C$ for all s > 0 and hence

$$\sum_{n\in\mathbb{N}} \frac{h_n |b_n|^q}{|\operatorname{Re}\lambda_n|^{q-2}} e^{\operatorname{Re}\lambda_n s} \le C \sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|\operatorname{Re}\lambda_n|^{q-2}} e^{\operatorname{Re}\lambda_n s}.$$
(15)

Since the sequence $(h_n)_{n \in \mathbb{N}}$ is strictly increasing and unbounded there is an integer $n_0 \in \mathbb{N}$ such that $h_n \geq C$ if $n \geq n_0$ and $h_n < C$ if $n < n_0$. We can rewrite (15) as

$$\sum_{n \ge n_0} \frac{(h_n - C)|b_n|^q}{|\operatorname{Re}\lambda_n|^{q-2}} e^{\operatorname{Re}\lambda_n s} \le \sum_{n < n_0} \frac{(C - h_n)|b_n|^q}{|\operatorname{Re}\lambda_n|^{q-2}} e^{\operatorname{Re}\lambda_n s}$$
(16)

By passing to the limit as $s \searrow 0$ in (16), we obtain that

$$\sum_{n \in \mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re} \lambda_n|^{q-2}} < \infty.$$

If q > 2, then this shows that $B = A_{-1}^{-2/q} B_0$ for some $B_0 \in \mathcal{L}(\mathbb{C}, X_{-1})$. Therefore, by Proposition 4.1, $\Sigma(A, B)$ is L^p -admissible for p > q and thus L^{∞} -iISS. If $q \leq 2$, then the monotonicity of $\operatorname{Re} \lambda_n$ implies that B is bounded. Combining these, we can conclude that if Φ' is bounded, the proof is finished. Hence, it remains to consider the case where Φ indeed defines a Young function. Let Ψ be the complementary function to Φ . We define $\Theta \colon [0, \infty) \to [0, \infty)$ as $\Theta(s) = \Psi(s^q)$. Then Θ is a Young function and we obtain for all $u \in E_{\Theta}$ using the Hölder's inequality with respect to the measure given by $|\operatorname{Re} \lambda_n| e^{\operatorname{Re} \lambda_n s} ds$

and the Young inequality (we denote by q' the Hölder conjugate of q)

$$\begin{split} \left\| \int_{0}^{t} T_{-1}(s) Bu(s) \, ds \right\|_{\ell^{q}}^{q} &= \sum_{n \in \mathbb{N}} |b_{n}|^{q} \left| \int_{0}^{t} e^{\lambda_{n} s} u(s) \, ds \right|^{q} \\ &\leq \sum_{n \in \mathbb{N}} |b_{n}|^{q} \left(\int_{0}^{t} e^{\operatorname{Re} \lambda_{n} s} |u(s)| \, ds \right)^{q} \\ &= \sum_{n \in \mathbb{N}} \frac{|b_{n}|^{q}}{(\operatorname{Re} \lambda_{n})^{q}} \left(\int_{0}^{t} |\operatorname{Re} \lambda_{n}| e^{\operatorname{Re} \lambda_{n} s} |u(s)| \, ds \right)^{q} \\ &\leq \sum_{n \in \mathbb{N}} \frac{|b_{n}|^{q}}{(\operatorname{Re} \lambda_{n})^{q}} \left(\int_{0}^{t} |\operatorname{Re} \lambda_{n}| e^{\operatorname{Re} \lambda_{n} s} |u(s)|^{q} \, ds \right) \left(\int_{0}^{t} |\operatorname{Re} \lambda_{n}| e^{\operatorname{Re} \lambda_{n} s} \, ds \right)^{1/q'} \\ &\leq \sum_{n \in \mathbb{N}} \frac{|b_{n}|^{q}}{|\operatorname{Re} \lambda_{n}|^{q}} \left(\int_{0}^{t} |\operatorname{Re} \lambda_{n}| e^{\operatorname{Re} \lambda_{n} s} |u(s)|^{q} \, ds \right) \\ &= \int_{0}^{t} \sum_{n \in \mathbb{N}} \frac{|b_{n}|^{q}}{|\operatorname{Re} \lambda_{n}|^{q-1}} e^{\operatorname{Re} \lambda_{n} s} |u(s)|^{q} \, ds \\ &= \int_{0}^{t} f(s) |u(s)|^{q} \, ds \\ &\leq \int_{0}^{t} \left(\int_{0}^{f(s)} \Phi'(r) \, dr + \int_{0}^{|u(s)|^{q}} \Psi'(r) \, dr \right) \, ds \\ &= \int_{0}^{t} \Phi(f(s)) \, ds + \int_{0}^{t} \Theta(|u(s)|) \, ds \\ &< \infty, \end{split}$$

since $E_{\Theta} \subset \tilde{L}_{\Theta}$. This shows that for all $u \in E_{\Theta}(U)$ we have

$$\int_0^t T_{-1}(s)Bu(s)\,ds \in X,$$

that is, $\Sigma(A, B)$ is E_{Θ} -admissible or equivalently E_{Θ} -ISS. The claim now follows from Theorem 3.1.

Lemma 4.4. Let μ be a measure supported on a sector S_{ϕ} with $\phi \in (0, \frac{\pi}{2})$, and let $1 \leq q < \infty$. Then the following are equivalent:

- (i) The Laplace transform $\mathcal{L}: L^{\infty}(0,\infty) \to L^q(\mathbb{C}_+,\mu)$ is bounded.
- (ii) The function $s \mapsto \frac{1}{s}$ lies in $L^q(\mathbb{C}_+, \mu)$.

Proof. (i) \implies (ii): Taking f(t) = 1 for $t \ge 0$ we have that $\mathcal{L}f(s) = 1/s$ and the result follows.

(ii) \implies (i): For $f \in L^{\infty}(0, \infty)$ and $s \in \mathbb{C}_+$ we have

$$\left| \int_0^\infty f(t) e^{-st} \, dt \right| \le \|f\|_\infty \int_0^\infty |e^{-st}| \, dt \le \|f\|_\infty / (\operatorname{Re} s) \le M \|f\|_\infty / |s|,$$

where M is a constant depending only on ϕ . Now condition (*ii*) implies that \mathcal{L} is bounded.

Theorem 4.5. Let A possess a q-Riesz basis of X consisting of eigenvectors $(e_n)_{n \in \mathbb{N}}$ with eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ lying in a sector in the open left half-plane \mathbb{C}_- and $B \in X_{-1}$. Then the following assertions are equivalent.

- (i) B is infinite-time admissible for L^{∞} .
- (*ii*) $\sup_{\lambda \in \mathbb{C}_+} \| (\lambda A)^{-1} B \| < \infty.$
- (iii) The function $s \mapsto 1/s$ lies in $L^q(\mathbb{C}_+, \mu)$, where μ is the measure $\sum |b_k|^2 \delta_{-\lambda_k}$.

Proof. By [JPP14, Thm 2.1], admissibility is equivalent to the boundedness of the Laplace transform $\mathcal{L} : L^{\infty}(0,\infty) \to L^{q}(\mathbb{C}_{+},\mu)$, and hence (i) and (iii) are equivalent by Lemma 4.4.

Note that

$$\|(\lambda - A)^{-1}B\|^q = \sum_k \frac{|b_k|^q}{|\lambda - \lambda_k|^q}.$$

Now if *(ii)* holds, then *(iii)* also holds, letting $\lambda \to 0$. Conversely, if *(iii)* holds, then by sectoriality we have that

$$\sum_k \frac{|b_k|^q}{|\operatorname{Re} \lambda_k|^q} < \infty,$$

and hence $\sum_{k} |b_k|^q / |\lambda - \lambda_k|^q$ is bounded independently of $\lambda \in \mathbb{C}_+$; that is, (ii) holds.

Remark 4.6. Let $\mathfrak{b}_p(X)$ denote the set of L^p -admissible control operators from \mathbb{C} to X for a given A. By Theorem 4.2, we have that $\mathfrak{b}_{\infty}(X) = X_{-1}$ for exponentially stable, parabolic diagonal systems. Using [Wei89b, Theorem 6.9], and the inclusion of the L^p -spaces, we obtain the following chain of inclusions for $X = \ell^q$ with $q > 1^5$.

$$X = \mathfrak{b}_1(X) \subset \mathfrak{b}_p(X) \subset \mathfrak{b}_\infty(X) = X_{-1}.$$
(17)

It is not so hard to show that the equality $\mathfrak{b}_{\infty}(X) = X_{-1}$ does not hold in general if the exponential stability is dropped. In fact, a counterexample on $X = \ell^2$ with the standard basis is given by $\lambda_n = 2^n$, $n \in \mathbb{Z}$, $b_n = 2^n/n$ for n > 0 and $b_n = 2^n$ for n < 0.

The relations of the different stability notions with respect to L^{∞} for parabolic diagonal systems are summarized in the diagram shown in Figure 3.

⁵here, q = 1 is also allowed if $(T^*(t))_{t \ge 0}$ is strongly continuous.



Figure 3: Relations between the different stability notions for parabolic diagonal system (assuming that the semigroup is exponentially stable).

5 Some Examples

Example 5.1. Let us consider the following boundary control system given by the one-dimensional heat equation on the spatial domain [0,1] with Neumann boundary control at the point 1,

$$\begin{split} &\frac{\partial}{\partial t}x(\xi,t) = \frac{\partial^2}{\partial \xi^2}x(\xi,t), \quad \xi \in (0,1), t > 0, \\ &\frac{\partial}{\partial \xi}x(0,t) = 0, \quad \frac{\partial}{\partial \xi}x(1,t) = u(t), \quad t > 0, \\ &x(\xi,0) = x_0(\xi), \end{split}$$

see e.g., [JPP14, Example 3.6]. It can be shown that this system can be written in the form $\Sigma(A, B)$ in (3). Here $X = L^2(0, 1)$ and

$$\begin{split} Af &= \frac{\partial^2}{\partial \xi^2} f, \quad f \in D(A), \\ D(A) &= \left\{ f \in L^2(0,1) : f, \frac{\partial}{\partial \xi} f \text{ is absolutely continuous,} \right. \\ &\left. \frac{\partial^2}{\partial \xi^2} f \in L^2(0,1), \frac{\partial}{\partial \xi} f(0) = \frac{\partial}{\partial \xi} f(1) = 0 \right\}. \end{split}$$

Moreover, with $\lambda_n = -\pi^2 n^2$,

$$Ae_n = \lambda_n e_n, \quad n \in \mathbb{N},$$

where the functions $e_0 = 1$ and $e_n = \sqrt{2} \cos(n\pi \cdot)$, $n \ge 1$, form an orthonormal basis of X. With respect to this basis, the operator B = b can be identified with $(b_n)_{n \in \mathbb{N}}$ for $b_n = 1$, $n \in \mathbb{N}$. Therefore,

$$\sum_{n\in\mathbb{N}}\frac{|b_n|^2}{|\lambda_n|^2}<\infty,$$

which shows that $b \in X_{-1}$. By Theorem 4.2, we conclude that the system is L^{∞} -iISS. A choice of functions β, μ, θ is given by

$$\beta(s,t) := e^{-\pi^2 t} s, \quad \mu(s) := s^p, \quad and \quad \theta(s) := c \cdot s^{\frac{1}{p}},$$

for $p \geq \frac{4}{3}$ and some constant c = c(p) > 0. This follows from the fact that $\Sigma(A, B)$ is even L^p -admissible for $p \geq \frac{4}{3}$, see [JPP14, Example 3.6].

Example 5.2. As remarked, Example 5.1 provides a system $\Sigma(A, B)$ which is even L^p -admissible for $p > \frac{4}{3}$. In the following we present a system which is L^{∞} -admissible, but not L^p -admissible for any $p < \infty$. In order to find such an example, we use the characterization of L^p -admissibility from [JPP14, Thm. 3.5], which we have already used in the proof of Proposition 4.1.

Let $X = \ell^2$ and let $(\lambda_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ define a parabolic diagonal system $\Sigma(A, B)$ as in Section 4. Furthermore, let $p \in (2, \infty)$. Then $\Sigma(A, B)$ is infinite-time L^p admissible if and only if

$$\left(2^{-\frac{n2(p-1)}{p}}\mu(S_n)\right)_{n\in\mathbb{Z}}\in\ell^{\frac{p}{p-2}}(\mathbb{Z}),$$

where $\mu = \sum_{n \in \mathbb{Z}} |b_n|^q \delta_{\lambda_n}$ and $S_n = \{z \in \mathbb{C} : \operatorname{Re} z \in (2^{n-1}, 2^n]\}, n \in \mathbb{Z}.$ We choose $\lambda_n = -2^n$ and $b_n = \frac{2^n}{n}$ for $n \in \mathbb{N}$. Clearly, $B = (b_n) \in X_{-1}$. Then we have that

$$2^{-\frac{n2(p-1)}{p}}\mu(S_n) = 2^{-\frac{2n(p-1)}{p}}\frac{2^{2n}}{n^2} = \frac{2^{\frac{2n}{p}}}{n^2},$$

and thus for p > 2,

$$\left(\left(2^{-\frac{n2(p-1)}{p}}\mu(S_n)\right)^{\frac{p}{p-2}}\right)_{n\in\mathbb{Z}} = \left(\frac{2^{\frac{2n}{p-2}}}{n^{\frac{2p}{p-2}}}\right)_{n\in\mathbb{Z}} \notin \ell^1.$$

Hence, $\Sigma(A, B)$ is not L^p -admissible for any p > 2, and therefore also not for any $p \ge 1$. However, since $\sum_{n \in \mathbb{N}} |b_n|^2 / |\operatorname{Re} \lambda_n|^2 = \sum_{n \in \mathbb{N}} 1/n^2 < \infty$, Theorem 4.2 shows that $\Sigma(A, B)$ is L^{∞} -iISS and, in particular infinite-time L^{∞} admissible.

We observe that by Theorem 3.1, there exists a Young function Φ such that $\Sigma(A, B)$ is E_{Φ} -admissible. However, as the system is not L^p -admissible, such a Φ cannot be of the form $\Phi(s) = s^p$ for any $p \in [1, \infty)$.

6 Conclusions and Outlook

In this paper, we have studied the relation between input-to-state stability and integral input-to-state stability for linear infinite-dimensional systems with a (possibly) unbounded control operator and inputs in function spaces. We have related the notions of iISS with respect to L^1 and L^{∞} to ISS with respect to Orlicz spaces. The known result that these stability concepts are equivalent for L^p -inputs with $p < \infty$, was generalized to Orlicz spaces that satisfy the Δ_2 condition. Moreover, we have shown that for parabolic diagonal systems and scalar input, the notions of L^{∞} -iISS and L^{∞} -ISS coincide.

Among possible directions for future research are the investigation of the non-analytic diagonal case, general analytic systems and the relation of zeroclass admissibility and input-to-state stability.

Finally, we mention that the existence of a counterexample for one of the unknown (converse) implications in Figure 1 can be related to the following open question posed by G. Weiss in [Wei89a, Problem 2.4].

Question A: Does the mild solution x belong to $C([0,\infty), X)$ for any $x_0 \in X$ and $u \in Z = L^{\infty}(0,\infty; U)$ provided that $\Sigma(A, B)$ is L^{∞} -admissible?

Although we do not provide an answer to this question, we relate it to

Proposition 6.1. At least one of the following assertions is true.

- 1. The answer to Question A is positive for every system $\Sigma(A, B)$.
- There exists a system Σ(A₀, B₀), with A₀ generating an exponentially stable semigroup and Σ(A₀, B₀) is L[∞]-admissible, but not L[∞]-zero-class admissible.

Proof. This follows directly from Proposition 2.4.

A Orlicz Spaces

In this section we recall some basic definitions and facts about Orlicz spaces. More details can be found in [KR61], [KJF77] and [Ada75]. For the generalization to vector-valued functions see [RR91, VII, Section 7.5]. In the following the Lebesgue measure will be denoted by $\lambda, I \subset \mathbb{R}$ is an open bounded interval, U a Banach space and $\Phi: [0, \infty) \to [0, \infty)$ a function.

Definition A.1. The Orlicz class $\tilde{L}_{\Phi}(I, U)$ is the set of all equivalence classes (with respect to equality almost everywhere) of Bochner-measurable functions $u: I \to U$ such that

$$\rho(u;\Phi) := \int_I \Phi(\|u(x)\|) \, dx < \infty.$$

In general, $\tilde{L}_{\Phi}(I, U)$ is not a vector space. Of particular interest are Orlicz classes generated by Young functions.

Definition A.2. A function $\Phi : [0, \infty) \to \mathbb{R}$ is called a Young function (or Young function generated by φ) if

$$\Phi(t) = \int_0^t \varphi(s) \, ds, \qquad t \ge 0,$$

where the function $\varphi : [0, \infty) \to \mathbb{R}$ has the following properties: $\varphi(0) = 0$, $\varphi(s) > 0$ for s > 0, φ is right continuous at any point $s \ge 0$, φ is nondecreasing on $(0, \infty)$ and $\lim_{s\to\infty} \varphi(s) = \infty$.

Theorem A.3. Let Φ be a Young function. Then $\tilde{L}_{\Phi}(I, U)$ is a convex set and $\tilde{L}_{\Phi}(I, U) \subset L^1(I, U)$. Conversely, for $u \in L^1(I, U)$ there is a Young function Φ such that $u \in \tilde{L}_{\Phi}(I, U)$.

Definition A.4. Let Φ be a Young function generated by the function φ . We set for $t \geq 0$

$$\psi(t) = \sup_{\varphi(s) \le t} s \quad and \quad \Psi(t) = \int_0^t \psi(s) \, ds.$$

The function Ψ is called the complementary function to Φ .

The complementary function of a Young function is again as Young function. If φ is continuous and strictly increasing in $[0, \infty)$ then ψ is the inverse function φ^{-1} and vice versa. We call Φ and Ψ a *pair of complementary Young functions*. **Theorem A.5** (Young's inequality). Let Φ , Ψ be a pair of complementary Young functions and φ , ψ their generating functions. Then for all $u, v \in [0, \infty)$ we have that

$$uv \le \Phi(u) + \Psi(v).$$

Equality holds if and only if $v = \varphi(u)$ or $u = \psi(v)$.

Remark A.6. Let Φ , Ψ be a pair of complementary Young functions, $u \in \hat{L}_{\Phi}(I)$ and $v \in \tilde{L}_{\Psi}(I)$. By integrating Young's inequality we get

$$\int_{I} |u(x)v(x)| \, dx \leq \rho(u; \Phi) + \rho(v; \Psi)$$

We are now in the position to define the Orlicz spaces. There are equivalent definitions of Orlicz spaces available. Here we use the so-called *Luxemburg norm*.

Definition A.7. The set $L_{\Phi}(I, U)$ of all equivalence classes (with respect to equality almost everywhere) of Bochner measurable functions $u: I \to U$ for which there is a k > 0 such that

$$\int_I \Phi(k^{-1} \| u(x) \|_U) \, dx < \infty$$

is called the Orlicz space. The Luxemburg norm of $u \in L_{\Phi}(I, U)$ is defined as

$$\|u\|_{\Phi} := \|u\|_{L_{\Phi}(I,U)} := \inf\left\{k > 0 \mid \int_{I} \Phi(k^{-1}\|u(x)\|) \, ds \le 1\right\}$$

For the choice $\Phi(t) := t^p$, $1 \le p < \infty$, the Orlicz space $L_{\Phi}(I, U)$ equals the vector-valued L^p -spaces with equivalent norms.

Theorem A.8. $(L_{\Phi}(I,U), \|\cdot\|_{\Phi})$ is a Banach space.

Clearly, $L^{\infty}(I, U)$ is a linear subspace of $L_{\Phi}(I, U)$.

Definition A.9. The space $E_{\Phi}(I, U)$ is defined as

$$E_{\Phi}(I,U) = \overline{L^{\infty}(I,U)}^{\|\cdot\|_{L_{\Phi}(I,U)}}.$$

The norm $\|\cdot\|_{E_{\Phi}(I;U)}$ refers to $\|\cdot\|_{L_{\Phi}(I;U)}$.

If $U = \mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then we write $L_{\Phi}(I) := L_{\Phi}(I, \mathbb{K})$ and $E_{\Phi}(I) := E_{\Phi}(I, \mathbb{K})$ for short. We remark the following properties of the Banach spaces $E_{\Phi}(I, U)$ and $L_{\Phi}(I, U)$.

Remark A.10. 1. $E_{\Phi}(I, U)$ is separable, see e.g. [Sch05, Thm. 6.3].

2. For a measurable $u: I \to U$, $u \in L_{\Phi}(I, U)$ if and only if $f = ||u(\cdot)||_U \in L_{\Phi}(I, \mathbb{R})$. This follows from the fact that

$$\|u\|_{\Phi} = \|f\|_{\Phi}.$$

Thus, $(u_n)_{n\in\mathbb{N}} \subset L_{\Phi}(I,U)$ converges to 0 if and only if $(||u_n(\cdot)||_U)_{n\in\mathbb{N}}$ converges to 0 in $L_{\Phi}(I,\mathbb{R})$. 3. Let Φ , Ψ be a pair of complementary Young functions. By an extension of Hölder's inequality [KJF77, Thm. 3.7.5 and Remark 3.8.6] we have for all $u \in L_{\Phi}(I, U)$ that

$$\|u\|_{L^1(0,t;U)} = \int_0^t \|u(s)\|_U \, ds \le 2\|\chi_{(0,t)}\|_{\Psi} \|u\|_{\Phi},$$

i.e., $L_{\Phi}(I, U)$ is continuously embedded in $L^{1}(I; U)$. Moreover, $\|\chi_{(0,t)}\|_{\Psi} \to 0$ as $t \searrow 0$. Here $\chi_{(0,t)}$ denotes the characteristic function of the interval (0,t).

4. $E_{\Phi}(I,U) \subset \tilde{L}_{\Phi}(I,U) \subset L_{\Phi}(I,U)$, see e.g. [Sch05, Thm. 5.1]. For $u \in \tilde{L}_{\Phi}(I,U)$, it also holds that

$$||u||_{\Phi} \le \rho(||u(\cdot)||_{U}; \Phi) + 1 < \infty.$$

Definition A.11 (Φ -mean convergence). A sequence $(u_n)_{n \in \mathbb{N}}$ in $L_{\Phi}(I)$ is said to converge in Φ -mean to $u \in L_{\Phi}(I)$ if

$$\lim_{n \to \infty} \rho(u_n - u; \Phi) = \lim_{n \to \infty} \int_I \Phi(|u_n(x) - u(x)|) \, dx = 0.$$

Definition A.12. We say that a Young function Φ satisfies the Δ_2 -condition if

$$\exists k > 0, u_0 \ge 0 \ \forall u \ge u_0: \quad \Phi(2u) \le k \Phi(u).$$

We remark, that $E_{\Phi}(I,U) = \tilde{L}_{\Phi}(I,U) = L_{\Phi}(I,U)$ if Φ satisfies the Δ_2 condition.

Definition A.13. Let Φ and Φ_1 be two Young functions. We say that the function Φ_1 converges essentially more rapidly than the function Φ if, for arbitrary s > 0,

$$\lim_{t \to \infty} \frac{\Phi(st)}{\Phi_1(t)} = 0$$

B Some technical results

Lemma B.1. Let $f: [0, \infty) \to [0, \infty)$ be non-decreasing and such that

$$\lim_{t \searrow 0} f(t) = 0.$$

Then there exists a continuous, non-decreasing function $g: [0, \infty) \to [0, \infty)$, which is continuously differentiable on $(0, \infty)$ and such that g(0) = 0 and $f \leq g$ pointwise on $[0, \infty)$.

Proof. We define

$$a_n := \max_{s \in [2^n, 2^{n+1}]} f(s) = f(2^{n+1}), \qquad n \in \mathbb{Z}.$$

Then $(a_n)_{n\in\mathbb{Z}}$ is a non-decreasing sequence with $a_n \ge 0$ and $\lim_{n\to\infty} a_n = 0$. We define g_n as the unique polynomial of degree 3 which solves the Hermite interpolation problem on $[2^{n-1}, 2^n]$ with

$$g_n(2^{n-1}) = a_{n-1}, g_n(2^n) = a_n, g'_n(2^{n-1}) = g'_n(2^n) = 0.$$

Since g'_n has zeroes at 2^n and 2^{n-1} , and is of degree 2, g_n has to be monotonic. Hence, the function g_n is non-decreasing since the sequence (a_n) is non-decreasing. We define $g : [0, \infty) \to [0, \infty)$ by setting $g(s) = g_n(s)$ for $s \in (2^{n-1}, 2^n]$ for $n \in \mathbb{Z}$, and g(0) = 0. The function g is continuous, continuously differentiable on $(0, \infty)$, and non-decreasing. By $g(2^n) = a_n = f(2^{n+1})$ and monotonicity we also have that $f \leq g$ pointwise.

Clearly, one can achieve any C^k by the appropriate spline. The method above is known as *monotone cubic spline interpolation*.

Corollary B.2. The function g from the above theorem can be chosen to be strictly increasing and such that $\lim_{t\to\infty} g(t) = \infty$. In particular, for every function $f \in \mathcal{K}_{\infty}$, there exists a $g \in \mathcal{K}_{\infty}$, which is continuously differentiable on $(0, \infty)$, such that $f \leq g$.

Proof. Simply multiply the function derived in Lemma B.1 by the function $s \mapsto s + 1$.

References

- [Ada75] R.A. Adams. Sobolev spaces. Academic Press, New York-London, 1975. Pure and Applied Mathematics, Vol. 65.
- [DM13a] S. Dashkovskiy and A. Mironchenko. Input-to-state stability of infinitedimensional control systems. *Mathematics of Control, Signals, and Systems*, 25(1):1–35, August 2013.
- [DM13b] S. Dashkovskiy and A. Mironchenko. Input-to-State Stability of Nonlinear Impulsive Systems. SIAM Journal on Control and Optimization, 51(3):1962–1987, May 2013.
- [JLR08] B. Jayawardhana, H. Logemann, and E.P. Ryan. Infinite-dimensional feedback systems: the circle criterion and input-to-state stability. *Commun. Inf. Syst.*, 8(4):413–414, 2008.
- [JNPS16] B. Jacob, R. Nabiullin, J.R. Partington, and F. Schwenninger. On input-tostate-stability and Integral input-to-state-stability for parabolic boundary control systems. *To appear in Proceedings of the MTNS 2016*, 2016.
- [JPP09] B. Jacob, J.R. Partington, and S. Pott. Zero-class admissibility of observation operators. Systems Control Lett., 58(6):406–412, 2009.
- [JPP14] B. Jacob, J.R. Partington, and S. Pott. Applications of Laplace-Carleson embeddings to admissibility and controllability. SIAM J. Control Optim., 52(2):1299–1313, 2014.
- [KJF77] A. Kufner, O. John, and S. Fučík. Function spaces. Noordhoff International Publishing, Leyden; Academia, Prague, 1977. Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis.
- [KK16a] I. Karafyllis and M. Krstic. ISS in different norms for 1-D parabolic PDEs with boundary disturbances. Preprint available at arXiv: 1605.01364, 2016.
- [KK16b] I. Karafyllis and M. Krstic. ISS with respect to boundary disturbances for 1-D parabolic PDEs. To appear in IEEE Trans. Automat. Control, 2016.
- [Kno28] K. Knopp. Theory and application of infinite series. London: Blackie & Sons. XII, 571 pp., 1928.

- [KR61] M. A. Krasnosel'skiĭ and Ya. B. Rutickiĭ. Convex functions and Orlicz spaces. Translated from the first Russian edition by Leo F. Boron. P. Noordhoff Ltd., Groningen, 1961.
- [Log13] H. Logemann. Stabilization of well-posed infinite-dimensional systems by dynamic sampled-data feedback. SIAM J. Control Optim., 51(2):1203–1231, 2013.
- [MI14] A. Mironchenko and H. Ito. Integral input-to-state stability of bilinear infinite-dimensional systems. In Proc. of the 53th IEEE Conference on Decision and Control, pages 3155–3160, 2014.
- [MI15] A. Mironchenko and H. Ito. Construction of Lyapunov Functions for Interconnected Parabolic Systems: An iISS Approach. SIAM Journal on Control and Optimization, 53(6):3364–3382, 2015.
- [Mir16] A. Mironchenko. Local input-to-state stability: Characterizations and counterexamples. Systems & Control Letters, 87:23–28, 2016.
- [MW15] A. Mironchenko and F. Wirth. A note on input-to-state stability of linear and bilinear infinite-dimensional systems. In Proc. of the 54th IEEE Conference on Decision and Control, pages 495–500, 2015.
- [MW16] A. Mironchenko and F. Wirth. Restatements of input-to-state stability in infinite dimensions: what goes wrong. In Proc. of 22th International Symposium on Mathematical Theory of Systems and Networks (MTNS 2016), 2016.
- [RR91] M. M. Rao and Z. D. Ren. Theory of Orlicz spaces, volume 146 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1991.
- [Sal84] D. Salamon. Control and Observation of Neutral Systems, volume 91 of Research Notes in Math. Pitman, Boston, London, 1984.
- [Sch05] G. Schappacher. A notion of Orlicz spaces for vector valued functions. Appl. Math., 50(4):355–386, 2005.
- [Son89] E.D. Sontag. Smooth stabilization implies coprime factorization. IEEE Trans. Automat. Control, 34(4):435–443, 1989.
- [Son98] E.D. Sontag. Comments on integral variants of ISS. Systems Control Lett., 34(1-2):93–100, 1998.
- [Son08] E.D. Sontag. Input to state stability: basic concepts and results. In Nonlinear and optimal control theory, volume 1932 of Lecture Notes in Math., pages 163–220. Springer Berlin, 2008.
- [TW09] M. Tucsnak and G. Weiss. Observation and control for operator semigroups. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2009.
- [Wei89a] G. Weiss. Admissibility of unbounded control operators. SIAM J. Control Optim., 27(3):527–545, 1989.
- [Wei89b] G. Weiss. Admissible observation operators for linear semigroups. Israel J. Math., 65(1):17–43, 1989.
- [XLY08] G.Q. Xu, C. Liu, and S.P. Yung. Necessary conditions for the exact observability of systems on Hilbert spaces. Systems Control Lett., 57(3):222–227, 2008.