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PERTURBATIONS OF POSITIVE SEMIGROUPS ON AM-SPACES

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Abstract. We consider positive perturbations of positive semigroups on AM-spaces and prove a result which is the dual counterpart of a famous perturbation result of Desch in AL-spaces. As an application we present unbounded perturbations of the shift semigroup.

1. Introduction

Strongly continuous semigroups play a central role in operator theory, partial differential equations, and linear systems theory, as documented in the monographs by Engel and Nagel [8], Pazy [13], Davies [6], Goldstein [9], Tucsnak and Weiss [17], or Jacob and Zwart [10].

One of the central problems of operator semigroup theory is to decide whether a concrete operator is the generator of a semigroup and how this semigroup is represented. Though the famous Hille–Yosida theorem provides a complete characterization of semigroup generators, it is practically never used in applications because of the difficult technical conditions appearing there.

One idea is to write complicated operators as the sum of simple ones and this is why perturbation theory became one of the major topics in semigroup theory. The main question is: Supposing A generates a C_0-semigroup $(T(t))_{t\geqslant 0}$, under which conditions on B does the operator $A+B$ (suitably defined) generate a C_0-semigroup?

There has been enormous development in the perturbation theory of operator semigroups, which is well documented in the monographs by Engel and Nagel [8, Chapter III], Kato [11], Banasiak and Arlotti [4], Tucsnak and Weiss [17, Section 5.4], Bátkai, Kramar Fijavž and Rhandi [2, Chapter 13].

In this note we concentrate on perturbations of positive semigroups in Banach lattices. For Banach lattices and positive operators on them we refer to the monographs by Aliprantis and Burkinshaw [5] or Schaefer [15], and for positive semigroups to Nagel [12] and Bátkai, Kramar Fijavž and Rhandi [2].

Our work is motivated by a well-known perturbation result, originally due to Desch [7] and Voigt [18], which we cite here. To be able to formulate it, we need some notions from Banach lattices and positive operators, which will be explained in the next section.

Theorem 1.1. *Let A be the generator of a positive C_0-semigroup $(T(t))_{t\geqslant 0}$ on a real AL-space E. Let $B\colon \operatorname{dom}(A)\to E$ be a positive operator, and assume that there exists $\lambda > \mathrm{s}(A)$ such that $\operatorname{spr}(B(\lambda-A)^{-1}) < 1$ (where spr denotes the spectral radius). Then $A+B$ generates a positive C_0-semigroup on E.*

We refer to Remark 4.4(b) for this formulation of Desch's result.

Let us formulate here the main theoretical result of our paper.

Theorem 1.2. *Let A be the generator of a positive C_0-semigroup $(T(t))_{t\geqslant 0}$ on a real AM-space E. Let $B\colon E\to E_{-1}$ be a positive operator and suppose that there is a $\lambda > \mathrm{s}(A)$ such that $\operatorname{spr}\big((\lambda - A_{-1})^{-1}B\big) < 1$. Then the part $(A_{-1}+B)_{|E}$ of $A_{-1}+B$ in E generates a positive C_0-semigroup on E.*

We recall that the part of $A_{-1}+B$ in E is the restriction of $A_{-1}+B$ to the domain $\operatorname{dom}\big((A_{-1}+B)_{|E}\big) := \{f \in E : (A_{-1}+B)f \in E\}$, considered as an operator in E. The extrapolation space E_{-1} and the extrapolated operator A_{-1} are explained in Section 2. In Section 3 we discuss a technical tool, the perturbation of so-called resolvent positive operators. The main result is proved in Section 4. Finally, an application is discussed in detail in Section 5.

2. Extrapolation Spaces and Positivity

Let X be a Banach space and $(T(t))_{t\geqslant 0}$ a C_0-semigroup on X, with generator A. For $\lambda \in \rho(A)$ we define the extrapolation space as the completion $X_{-1} := (X, \|\cdot\|_{-1})^{\sim}$, where $\|f\|_{-1} := \|R(\lambda, A)f\|$. Here $R(\lambda, A) = (\lambda - A)^{-1}$ is the resolvent of A. For all $t \geqslant 0$, the operator $T(t)$ has a unique extension $T_{-1}(t) \in \mathcal{L}(X_{-1})$, and $(T_{-1}(t))_{t\geqslant 0}$ is a C_0-semigroup on X_{-1}, with generator A_{-1} satisfying $\operatorname{dom}(A_{-1}) = X$. Moreover, these definitions are independent of the choice of $\lambda \in \rho(A)$, meaning that a different $\lambda \in \rho(A)$ generates the same space with equivalent norms. We refer the reader to Engel and Nagel [8, Chapter II.5] for these properties and for more on this subject.

Definition 2.1. Let E be a Banach lattice and E_{-1} the extrapolation space for the positive C_0-semigroup $(T(t))_{t\geqslant 0}$. We say that $f \in E_{-1}$ is *positive*, if f belongs to the closure of E_+ in E_{-1}. We denote by $E_{-1,+}$ the set of all positive elements in E_{-1}.

From the definition the set of positive elements satisfies $E_+ \subseteq E_{-1,+}$. By

$$\mathrm{s}(A) = \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\}$$

we denote the *spectral bound* of A. We recall that for generators of positive semigroups $R(\lambda, A) \geqslant 0$ holds for $\lambda > \mathrm{s}(A)$.

Remark 2.2. In the context of Definition 2.1, for $\lambda > \mathrm{s}(A_{-1}) = \mathrm{s}(A)$ it is easy to see that $R(\lambda, A_{-1}) \in \mathcal{L}(E_{-1})$ is a positive operator (i.e., maps positive elements to positive elements). In fact, it will follow from Proposition 2.3 that $R(\lambda, A_{-1})$ is also positive as an operator in $\mathcal{L}(E_{-1}, E)$, see Remark 2.4 below.

Let $B \in \mathcal{L}(E, E_{-1})$. If B is positive, i.e. $Bf \geqslant 0$ for all $f \in E_+$, then $R(\lambda, A_{-1})B$ is positive as an operator in $\mathcal{L}(E)$, for all $\lambda > \mathrm{s}(A)$. Conversely, if there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $(\mathrm{s}(A), \infty)$ tending to ∞, and such that $R(\lambda_n, A_{-1})B \geqslant 0$ for all $n \in \mathbb{N}$, then $B \geqslant 0$. Indeed, if $f \in E_+$, then $R(\lambda_n, A_{-1})Bf \in E_+$ for all $n \in \mathbb{N}$, and the convergence $\lambda_n R(\lambda_n, A_{-1})Bf \to Bf$ in E_{-1} $(n \to \infty)$ implies $Bf \in E_{-1,+}$.

In Example 5.4 we will show that positivity of $R(\lambda, A_{-1})B$ for only a single $\lambda > \mathrm{s}(A_{-1})$ does not imply the positivity of B.

Next we establish some basic properties of the ordering on E_{-1}.

Proposition 2.3. *Let E be a real Banach lattice and $(T(t))_{t\geqslant 0}$ a positive C_0-semigroup on E. The set $E_{-1,+}$ is a closed convex cone in E_{-1}, satisfying*

$$E_+ = E_{-1,+} \cap E.$$

Proof. Taking closures in the inclusions $E_+ + E_+ \subseteq E_+$ and $\alpha E_+ \subseteq E_+$ for $\alpha \geqslant 0$, one obtains the corresponding inclusions for $E_{-1,+}$. Also, $E_{-1,+}$ is closed as the closure of E_+. To show $E_{-1,+} \cap (-E_{-1,+}) = \{0\}$, let $f \in E_{-1,+}$ and assume also that $-f \in E_{-1,+}$. Then there exist sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}$ in E_+ such that $f_n \to f$ and $g_n \to -f$ in E_{-1}, and thereby $f_n + g_n \to 0$ in E_{-1}, as $n \to \infty$. Choose $\lambda > \mathrm{s}(A)$ and let the norm $\|\cdot\|_{-1}$ be defined in terms of this λ. Note that $0 \leqslant f_n \leqslant f_n + g_n$, and hence $0 \leqslant R(\lambda, A)f_n \leqslant R(\lambda, A)(f_n + g_n)$, by the positivity of the semigroup. Therefore

$$\|f_n\|_{-1} = \|R(\lambda, A)f_n\| \leqslant \|R(\lambda, A)(f_n + g_n)\| = \|f_n + g_n\|_{-1} \to 0$$

as $n \to \infty$. This shows that $f = 0$.

Finally, to show that the definition of positivity in the extrapolation space is compatible with the original ordering, we note that $E_+ \subseteq E_{-1,+} \cap E$ is immediate from the definition.

To prove the reverse inclusion let $f \in E_{-1,+} \cap E$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in E_+ such that $\|f - f_n\|_{-1} \to 0$ $(n \to \infty)$. Recalling that the norm $\|\cdot\|_{-1}$ can be defined using any $\lambda \in \rho(A)$ we obtain

$$\|R(\lambda, A)f - R(\lambda, A)f_n\| \to 0 \text{ as } n \to \infty,$$

for all $\lambda > \mathrm{s}(A)$. Because of $R(\lambda, A)f_n \in E_+$ for all $n \in \mathbb{N}$ this implies that $R(\lambda, A)f \in E_+$ for all $\lambda > \mathrm{s}(A)$. From $\lambda R(\lambda, A)f \to f$ (in E) as $\lambda \to \infty$ we therefore obtain $f \in E_+$. □

Remark 2.4. It is important to keep in mind the following simple consequence of the properties shown in Proposition 2.3.

In the context of this proposition, let $C\colon E_{-1} \to E$ be an operator. Then C is positive if and only if C is positive as an operator from E_{-1} to E_{-1}.

Remark 2.5. The extrapolation space for a positive semigroup is not a Banach lattice, in general. This will be shown by Examples 5.1 and 5.3.

If the norm on a Banach lattice E satisfies

$$\|\sup\{f, g\}\| = \sup\{\|f\|, \|g\|\} \tag{2.1}$$

for all $f, g \in E_+$, then the Banach lattice E is called an *abstract M-space* or an *AM-space*. If the norm on a Banach lattice E satisfies

$$\|f + g\| = \|f\| + \|g\| \tag{2.2}$$

for all $f, g \in E_+$, then the Banach lattice E is called an *abstract L-space* or an *AL-space*.

3. Perturbations of Resolvent Positive Operators

In this section we provide a technical tool which will be needed in the proof of the main result.

Let E, F be ordered real Banach spaces. Let E_+ be *generating* ($E = E_+ - E_+$) and *normal* ($E' = E'_+ - E'_+$), and assume that the norm on $\mathcal{L}(E)$ is *monotone* ($A, B \in \mathcal{L}(E)$, $0 \leqslant A \leqslant B$ implies $\|A\| \leqslant \|B\|$). (These conditions on E are satisfied if E is a Banach lattice.)

Lemma 3.1. *Let $Q \in \mathcal{L}(E,F)$ be an isomorphism of Banach spaces, $Q^{-1}\colon F \to E$ positive, and let $B\colon E \to F$ be a positive operator. (This implies that $Q^{-1}B \in \mathcal{L}(E)$ by Batty and Robinson* [3, Proposition 1.7.2]*; hence $B \in \mathcal{L}(E,F)$.) Then the following conditions are equivalent:*

(i) $\operatorname{spr}(Q^{-1}B) < 1$,

(ii) $Q - B$ continuously invertible, with $(Q-B)^{-1} \in \mathcal{L}(F,E)$ positive.

If these properties are satisfied, then

$$(Q-B)^{-1} = \Big(\sum_{n=0}^{\infty} (Q^{-1}B)^n\Big) Q^{-1} \geqslant Q^{-1}.$$

Proof. First we show '(i) $\Rightarrow$ (ii)' and the additional assertion. Combining condition (i) and $Q^{-1}B \geqslant 0$ with the Neumann series we obtain

$$\big(I - Q^{-1}B\big)^{-1} = \sum_{n=0}^{\infty} \big(Q^{-1}B\big)^n \geqslant I \geqslant 0,$$

where I denotes the identity operator in E. Using the decomposition

$$Q - B = Q\big(I - Q^{-1}B\big)$$

one sees the continuous invertibility of $Q - B$ and

$$(Q-B)^{-1} = \big(I - Q^{-1}B\big)^{-1} Q^{-1} = \Big(\sum_{n=0}^{\infty} \big(Q^{-1}B\big)^n\Big) Q^{-1} \geqslant Q^{-1}.$$

For the proof of '(ii) $\Rightarrow$ (i)' we first note that the identity

$$\sum_{j=0}^{n} (Q^{-1}B)^j Q^{-1}(Q-B) = I - \big(Q^{-1}B\big)^{n+1}$$

implies

$$\begin{aligned}\sum_{j=1}^{n+1} \big(Q^{-1}B\big)^j &= \Big(I - \big(Q^{-1}B\big)^{n+1}\Big)(Q-B)^{-1}B \\ &= (Q-B)^{-1}B - \big(Q^{-1}B\big)^{n+1}(Q-B)^{-1}B \leqslant (Q-B)^{-1}B.\end{aligned}$$

Now the monotonicity of the norm in $\mathcal{L}(E)$ implies $(1,\infty) \subseteq \rho\big(Q^{-1}B\big)$ and

$$\sup_{\mu>1} \big\| \big(\mu - Q^{-1}B\big)^{-1} \big\| \leqslant \big\| I + (Q-B)^{-1}B \big\|,$$

and this implies that $[1,\infty) \subseteq \rho(Q^{-1}B)$. From the positivity of $Q^{-1}B$ we obtain $\operatorname{spr}(Q^{-1}B) \in \sigma(Q^{-1}B)$, by Pringsheim's theorem (see Schaefer [14, Appendix, 2.2]), and this finally implies $\operatorname{spr}(Q^{-1}B) < 1$. □

Now we specialise the hypotheses to the case that E is a real Banach lattice, $(T(t))_{t\geqslant 0}$ a positive C_0-semigroup on E, with generator A, and that $F := E_{-1}$ is the corresponding extrapolation space. The following result is a version of Voigt [18, Theorem 1.1], with the operator product $B(\lambda - A)^{-1}$ replaced by $(\lambda - A_{-1})^{-1}B$.

Theorem 3.2. *Let $\lambda > \mathrm{s}(A)$, and let $B\colon E \to E_{-1}$ be a positive operator. (Recall from Lemma 3.1 that then $(\lambda - A_{-1})^{-1}B \in \mathcal{L}(E)$ and $B \in \mathcal{L}(E, E_{-1})$.) Then the following conditions are equivalent:*

(i) $\operatorname{spr}\big((\lambda - A_{-1})^{-1}B\big) < 1$,

(ii) $\lambda \in \rho(A_{-1}+B)$ *and* $(\lambda - A_{-1} - B)^{-1} \geqslant 0$.

(In condition (ii) we consider A_{-1} *and* B *as operators in* E_{-1}*, with domain* E*.)*

If these properties are satisfied, then

$$(\lambda - A_{-1} - B)^{-1} = \left(\sum_{n=0}^{\infty}\big((\lambda - A_{-1})^{-1}B\big)^n\right)(\lambda - A_{-1})^{-1} \geqslant (\lambda - A_{-1})^{-1},$$

$\mathrm{s}(A_{-1}+B) < \lambda$, *and* $(\mu - A_{-1} - B)^{-1} \geqslant 0$ *for all* $\mu \geqslant \lambda$ *(i.e.,* $A_{-1}+B$ *is resolvent positive, in the terminology of Arendt* [1]*).*

Proof. With $Q := \lambda - A_{-1}$ and in view of Remark 2.4, all the statements except for the last one are immediate consequences of Lemma 3.1. However, from (i) one immediately obtains $\mathrm{spr}\big((\mu - A_{-1})^{-1}B\big) < 1$ for all $\mu \geqslant \lambda$, therefore $[\lambda,\infty) \subseteq \rho(A_{-1}+B)$ and $(\mu - A_{-1} - B)^{-1} \geqslant 0$ for all $\mu \geqslant \lambda$. □

4. Perturbation Theory with Positive Operators

The cornerstone of the proof of our main result will be the Desch–Schappacher perturbation theorem, which we cite here from Engel and Nagel [8, Chapter III, Corollaries 3.2 and 3.3].

Theorem 4.1. *Let* A *be the generator of a* C_0*-semigroup* $(T(t))_{t\geqslant 0}$ *on a Banach space* X *and let* $B \in \mathcal{L}(X, X_{-1})$*. Moreover, assume that there exist* $\tau > 0$ *and* $K \in [0,1)$ *such that*

(i) $\displaystyle\int_0^\tau T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s \in X$,

(ii) $\displaystyle\left\|\int_0^\tau T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s\right\| \leqslant K\|u\|_\infty$

for all continuous functions $u \in \mathrm{C}([0,\tau], X)$*. Then the operator* $(A_{-1}+B)_{|X}$ *generates a* C_0*-semigroup* $(S(t))_{t\geqslant 0}$ *on* X*. Furthermore this semigroup is given by the Dyson–Phillips series*

$$S(t) = \sum_{n=0}^{\infty} S_n(t), \text{ for all } t \geqslant 0, \tag{4.1}$$

where $S_0(t) := T(t)$ *and*

$$S_n(t)f := \int_0^t T_{-1}(t-s)BS_{n-1}(s)f\,\mathrm{d}s \text{ for all } f \in X. \tag{4.2}$$

In this case B *is said to be a Desch–Schappacher perturbation of* A*.*

Let us state and prove our result for AM-spaces and positive semigroups in a special case, using the above theorem.

Proposition 4.2. *Let* E *be a real AM-Space,* $(T(t))_{t\geqslant 0}$ *a positive* C_0*-semigroup on* E *with generator* A*. Let* $B \in \mathcal{L}(E, E_{-1})$ *be a positive operator and suppose further that there exists* $\lambda > \mathrm{s}(A)$ *such that* $K := \|R(\lambda, A_{-1})B\| < 1$*. Then* $(A_{-1}+B)_{|E}$ *is the generator of a positive* C_0*-semigroup* $(S(t))_{t\geqslant 0}$*, and the extrapolation space* E_{-1} *for this semigroup is the same as for* $(T(t))_{t\geqslant 0}$*.*

The key technical tool in verifying the conditions of Theorem 4.1 will be the following lemma, which we state separately.

Lemma 4.3. *Let E be a real Banach lattice, E_{-1} the extrapolation space for a positive C_0-semigroup $(T(t))_{t\geqslant 0}$ on E, let $B \in \mathcal{L}(E, E_{-1})$ be a positive operator, and let $\tau > 0$. Then we have:*

(i) $(T_{-1}(t))_{t\geqslant 0}$ *is positive.*

(ii) For each step function $u \in \mathrm{L}^\infty([0,\tau]; E)$ *we have*

$$\int_0^\tau T_{-1}(s)Bu(s)\,\mathrm{d}s \in E.$$

(iii) For all $f \in E_+$ *we have* $\int_0^\tau T_{-1}(s)Bf\,\mathrm{d}s \in E_+$.

(iv) If in addition $(T(t))_{t\geqslant 0}$ *is exponentially stable, then we have*

$$\int_0^\tau T_{-1}(s)Bf\,\mathrm{d}s \leqslant \int_0^\infty T_{-1}(s)Bf\,\mathrm{d}s$$

in E, for all $f \in E_+$.

Proof. (i) This follows because $T_{-1}(t)$ is the continuous extension of $T(t)$, for all $t \geqslant 0$.

(ii) Let $u \in \mathrm{L}^\infty([0,\tau]; E)$ be a step function, i.e., $u(t) = \sum\limits_{n=1}^N u_n\chi_{I_n}(t)$ where $u_1, \dots, u_N \in E$, $I_1, \dots, I_N \subseteq [0,\tau]$ are pairwise disjoint intervals with $\bigcup_{n=1}^N I_n = [0,\tau]$, and where χ_{I_n} denotes the indicator function of I_n. It suffices to show

$$\int_{I_n} T_{-1}(s)Bu_n\,\mathrm{d}s = \int_{t_{n-1}}^{t_n} T_{-1}(s)Bu_n\,\mathrm{d}s \in E,$$

where $(t_{n-1}, t_n) \subseteq I_n \subseteq [t_{n-1}, t_n]$. With the substitution $s' = s - t_{n-1}$ we get

$$\begin{aligned}\int_{t_{n-1}}^{t_n} T_{-1}(s)Bu_n\,\mathrm{d}s &= \int_0^{t_n - t_{n-1}} T_{-1}(s + t_{n-1})Bu_n\,\mathrm{d}s \\ &= T_{-1}(t_{n-1})\int_0^{t_n - t_{n-1}} T_{-1}(s)Bu_n\,\mathrm{d}s.\end{aligned}$$

Because $(T_{-1}(t))_{t\geqslant 0}$ is a C_0-semigroup on E_{-1} with generator A_{-1}, we have that $\int_0^{t_n - t_{n-1}} T_{-1}(s)Bu_n\,\mathrm{d}s$ belongs to $\mathrm{dom}(A_{-1}) = E$, and the assertion follows.

Statements (iii) and (iv) follow directly from Proposition 2.3. □

Proof of Proposition 4.2. In the first (main) part of the proof we will assume that the given semigroup is exponentially stable, and that $\lambda = 0$.

Let $\tau > 0$. Let us denote by $\mathrm{T}([0,\tau]; E)$ the vector space of E-valued step functions. In fact, $\mathrm{T}([0,\tau]; E)$ is a normed vector lattice, a sublattice of $\mathrm{L}^\infty([0,\tau]; E)$. We define a linear operator $R\colon \mathrm{T}([0,\tau]; E) \to E$ by

$$Ru := \int_0^\tau T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s.$$

Note that Proposition 2.3 implies that R is a positive operator. We show that

$$\|Ru\|_E \leqslant K\|u\|_\infty, \tag{4.3}$$

for all $u \in \mathrm{T}([0,\tau]; E)$.

First, let u be a positive step function, $u = \sum_{n=1}^N u_n\chi_{I_n}$ as above, with $u_1, u_2, \dots, u_N \geqslant 0$. Then $0 \leqslant u \leqslant z\chi_{[0,\tau]}$, where $z := \sup_n u_n$. We conclude, with the help of Proposition 2.3 and Lemma 4.3, that

$$\begin{aligned}\|Ru\| &\leqslant \left\| \int_0^\tau T_{-1}(\tau - s) Bz \,\mathrm{d}s \right\| \\ &\leqslant \left\| \int_0^\infty T_{-1}(\tau) Bz \,\mathrm{d}s \right\| \leqslant \|A_{-1}^{-1}B\| \|z\| \\ &= K\|z\| = K\| \sup_n u_n \| = K \sup_n \|u_n\| = K\|u\|_\infty,\end{aligned}$$

where we have used the AM-property of E in the last line. If u is an arbitrary E-valued step function, then $u = u^+ - u^-$, $|Ru| = |Ru^+ - Ru^-| \leqslant Ru^+ + Ru^- = R|u|$, hence $\|Ru\| \leqslant \|R|u|\| \leqslant K\||u|\|_\infty = K\|u\|_\infty$.

The estimate (4.3) implies that R possesses a (unique linear) continuous extension – still denoted by R – to the closure of $\mathrm{T}([0,\tau];E)$ in $\mathrm{L}^\infty([0,\tau];E)$. This closure contains $\mathrm{C}([0,\tau];E)$, and the estimate (4.3) carries over to all u in the closure.

If $u \in \mathrm{C}([0,\tau];E)$, and $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathrm{T}([0,\tau];E)$ converging to u uniformly on $[0,\tau]$, then $Ru_n \to Ru$ in E. But also

$$Ru_n = \int_0^\tau T_{-1}(\tau - s) Bu_n(s) \,\mathrm{d}s \to \int_0^\tau T_{-1}(\tau - s) u(s) \,\mathrm{d}s$$

in E_{-1}, because $B\colon E \to E_{-1}$ is continuous and $(T_{-1}(t))_{t\geqslant 0}$ is bounded on $[0,\tau]$. This implies that $\int_0^\tau T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s = Ru \in E$, and that

$$\left\| \int_0^\tau T_{-1}(\tau - s) Bu(s) \,\mathrm{d}s \right\| \leqslant K\|u\|_\infty.$$

Therefore both conditions in Theorem 4.1 are satisfied; hence $(A_{-1} + B)_{|E}$ generates a C_0-semigroup $(S(t))_{t\geqslant 0}$ which is given by the Dyson–Phillips series (see Equations (4.1) and (4.2)). Using Lemma 4.3(i) and Remark 2.4 we conclude that the iterates $S_n(t)$ as well as the semigroup operators $S(t)$ are positive. This shows all the statements for the present case, except for the assertion concerning the extrapolation spaces.

For the general case we note that from a basic rescaling procedure in semigroup theory we know that A is the generator of the (positive) C_0-semigroup $(T(t))_{t\geqslant 0}$ if and only if $A - \lambda$ is the generator of the (positive) C_0-semigroup $(\mathrm{e}^{-\lambda t}T(t))_{t\geqslant 0}$. Observe that the function $(\mathrm{s}(A), \infty) \ni \lambda \mapsto \|R(\lambda, A_{-1})B\|$ is decreasing. Now choose $\lambda > \mathrm{s}(A)$ such that $A - \lambda$ generates a positive exponentially stable C_0-semigroup and such that $\|R(\lambda, A_{-1})B\| < 1$. Then the case treated so far implies that $(A_{-1} - \lambda + B)_{|E}$ generates a positive C_0-semigroup.

Now we show the equality of the extrapolation spaces. We choose

$$\lambda > \max\bigl\{\mathrm{s}(A), \mathrm{s}\bigl((A_{-1} + B)_{|E}\bigr)\bigr\}$$

and such that $\|R(\lambda, A_{-1}))B\| < 1$. From the identity

$$(\lambda - A_{-1} - B) = (\lambda - A_{-1})\bigl(I - (\lambda - A_{-1})^{-1}B\bigr)$$

we obtain

$$(\lambda - A_{-1} - B)^{-1} = \bigl(I - (\lambda - A_{-1})^{-1}B\bigr)^{-1}(\lambda - A_{-1})^{-1}.$$

Restricting this equality to E we conclude that

$$(\lambda - (A_{-1} - B)_{|E})^{-1} = \bigl(I - (\lambda - A_{-1})^{-1}B\bigr)^{-1}(\lambda - A)^{-1}.$$

In view of the continuous invertibility of the first operator on the right hand side, this equality shows that the $\|\cdot\|_{-1}$-norms corresponding to $(A_{-1}-B)_{|E}$ and A are equivalent on E, and therefore the completions are the same. $\square$

Proof of Theorem 1.2. The following is an adaptation of the proof given in Voigt [18, Proof of Theorem 0.1]. It is assumed that there exists $\lambda > \mathrm{s}(A)$ such that $\mathrm{spr}\big((\lambda - A_{-1})^{-1}B\big) < 1$, which by Theorem 3.2 is equivalent to the requirement $\lambda \in \rho(A_{-1}+B)$, $(\lambda - A_{-1} - B)^{-1} \geqslant 0$. From Theorem 3.2 we then conclude that $\lambda \in \rho(A_{-1}+sB)$,

$$(\lambda - A_{-1})^{-1} \leqslant (\lambda - A_{-1} - sB)^{-1} \leqslant (\lambda - A_{-1} - B)^{-1}$$

for all $s \in (0,1)$. We choose $n \in \mathbb{N}$ such that $\|(\lambda - A_{-1} - B)^{-1}B\| < n$. This implies $\|(\lambda - A_{-1} - (j/n)B)^{-1}(1/n)B\| < 1$ for all $j = 0, \dots, n-1$.

Applying Proposition 4.2 successively to the operators

$$A, (A_{-1} + (1/n)B)_{|E}, \dots, (A_{-1} + ((n-1)/n)B))_{|E},$$

with the perturbation $(1/n)B$, the desired result is obtained. An important point in this sequence of steps is that the extrapolation space E_{-1} does not change; this issue is taken care of by the last statement of Proposition 4.2. $\square$

Remark 4.4. (a) In our main theorem, Theorem 1.2, the hypothesis that 'there exists $\lambda > \mathrm{s}(A)$ such that $\mathrm{spr}\big((\lambda - A_{-1})^{-1}B\big) < 1$' could have been formulated equivalently as 'there exists $\lambda > \mathrm{s}(A)$ such that $\lambda \in \rho(A_{-1}+B)$ and $(\lambda - A_{-1} - B)^{-1} \geqslant 0$', or else as '$A_{-1}+B$ is resolvent positive'. This is a consequence of Theorem 3.2.

(b) Similarly, in Theorem 1.1, as it is stated in Voigt [18, Theorem 0.1], the condition that 'there exists $\lambda > \mathrm{s}(A)$ such that $\mathrm{spr}(B(\lambda - A)^{-1}) < 1$' appears as '$A+B$ is resolvent positive'. The equivalence of these conditions is a consequence of Voigt [18, Theorem 1.1].

5. Examples

We start this section with an application of Theorem 1.2.

Example 5.1. Let $h \in \mathrm{L}^1(0,1)_+$. Consider the partial differential equation

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) = \frac{\partial}{\partial x}u(t,x) + \int_0^1 u(t,y)\,\mathrm{d}y \cdot h(x), \qquad x \in [0,1], t \geqslant 0,\\ u(0,x) = u_0(x), \quad u(t,1) = 0, \qquad x \in [0,1], t \geqslant 0.\end{aligned}$$

Trying to interpret this equation as an abstract Cauchy problem on the AM-space $E := \{f \in \mathrm{C}([0,1]) : f(1) = 0\}$ with norm $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$,

$$\begin{aligned}\dot{u}(t) &= Au(t) + Bu(t)\\ u(0) &= u_0,\end{aligned}$$

where the operator A is defined by

$$Af = f', \qquad \mathrm{dom}(A) = \{f \in \mathrm{C}^1[0,1] : f(1) = f'(1) = 0\}, \tag{5.1}$$

one realises that it is not evident how to associate the right hand side of the equation with a linear operator in E. In Engel and Nagel [8, Chapter II, Example 3.19(i)], it is shown that A is the generator of the nilpotent positive left-shift semigroup $(T(t))_{t\geqslant 0}$ with $\mathrm{s}(A) = -\infty$, given by

$$(T(t)f)(x) = \begin{cases} f(s+t) & \text{if } x+t \leqslant 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

We want to calculate the extrapolation space of E for the generator A. Our aim is to show the equality

$$E_{-1} = \{g \in \mathscr{D}(0,1)' : g = \partial f \text{ for some } f \in E\}, \tag{5.3}$$

where $\mathscr{D}(0,1) = C_{\mathrm{c}}^{\infty}(0,1)$ denotes the usual space of 'test functions', with the inductive limit topology, $\mathscr{D}(0,1)'$ its dual space, and ∂ is differentiation on distributions.

First we note that the 'standard embedding' $j\colon E \hookrightarrow \mathscr{D}(0,1)'$ can be extended to a mapping

$$j_{-1}\colon E_{-1} \to \mathscr{D}(0,1)',$$

defined by

$$\langle j_{-1}(g), \varphi\rangle := \big\langle A_{-1}^{-1}g, -\varphi'\big\rangle = -\int_0^1 \big(A_{-1}^{-1}g\big)(x)\varphi'(x)\,\mathrm{d}x.$$

Indeed, if $g \in E$, then

$$\langle j_{-1}(g), \varphi\rangle = -\int_0^1 \big(A^{-1}g\big)(x)\varphi'(x)\,\mathrm{d}x = \int_0^1 \big(A^{-1}g\big)'(x)\varphi(x)\,\mathrm{d}x = \int_0^1 g(x)\varphi(x)\,\mathrm{d}x,$$

which shows that j_{-1} is an extension of j. In fact, the definition of j_{-1} shows that $j_{-1} = \partial \circ j \circ A_{-1}^{-1}$, and this formula shows that j_{-1} maps E_{-1} continuously to $\mathscr{D}(0,1)'$. Finally we note that j_{-1} is injective. Indeed, if $g \in E_{-1}$ is such that $j_{-1}(g) = 0$, then $\int_0^1 \big(A_{-1}^{-1}g\big)(x)\varphi'(x)\,\mathrm{d}x = 0$ for all $\varphi \in \mathscr{D}(0,1)$, which implies that the continuous function $A_{-1}^{-1}g$ is constant, and this constant is zero because $\big(A_{-1}^{-1}g\big)(1) = 0$. The injectivity of A_{-1}^{-1} then implies $g = 0$.

Rewriting the above formula for j_{-1} as

$$j_{-1} \circ A_{-1} = \partial \circ j,$$

valid on E, we see the validity of (5.3) as well as the property that in the image of E in $\mathscr{D}(0,1)'$ the operator A_{-1} acts as differentiation ∂.

Next we are going to show that, in the image of E_{-1} in $\mathscr{D}(0,1)'$, one has

$$\begin{aligned} E_{-1,+} &= \{\mu : \mu \text{ a finite continuous positive Borel measure on } (0,1)\} \\ &= \{g \in \mathscr{D}(0,1)' : g = \partial f \text{ for some increasing function } f \in E\}, \end{aligned} \tag{5.4}$$

where the second equality is standard and will not be discussed further.

So, let $g \in E_{-1,+}$. Then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in E_+ such that $g_n \to g$ in E_{-1}, and thus in $\mathscr{D}(0,1)'$. Hence

$$\int_0^1 g_n(x)\varphi(x)\,\mathrm{d}x \to \langle g, \varphi\rangle \qquad (n \to \infty),$$

for all $\varphi \in \mathscr{D}(0,1)$, and therefore $\langle g, \varphi\rangle \geqslant 0$ for all $0 \leqslant \varphi \in \mathscr{D}(0,1)$, i.e., g is a 'positive distribution'. It is known that this implies that g is a positive Borel measure; see Schwartz [16, Chap. I, Théorème V]. As g is also the distributional derivative of a function $f \in E$, it follows that f is increasing and that μ is finite and continuous (i.e., does not have a discrete part).

For the reverse inclusion, let $f \in E$ be an increasing function. Then one shows by standard methods of Analysis that f can be approximated in E by a sequence $(f_n)_{n\in\mathbb{N}}$ in $E \cap \mathrm{C}^1[0,1]$, all f_n increasing and vanishing in a neighbourhood of 1. This implies that $f_n' \in E_+$ for all $n \in \mathbb{N}$, and that $f_n' \to A_{-1}f = \partial f$ $(n \to \infty)$ in E_{-1}; hence $g := \partial f \in E_{-1,+}$.

As a consequence of (5.4) we obtain that $E_{-1,+} - E_{-1,+}$, the set of finite continuous signed measures, is a proper subset of E_{-1} (because the distributional derivative of a function in E that is not of bounded variation is not a measure). This implies that $E_{-1,+}$ is not generating in E_{-1}, and therefore E_{-1} is not a lattice – let alone a Banach lattice.

Now we come back to treating the initial value problem stated at the beginning. There exists an increasing function $f \in E$ such that $h = \partial f$, and therefore $h \in E_{-1,+}$. Hence the operator $B \in \mathcal{L}(E, E_{-1})$, defined by $g \mapsto \int_0^1 g(x)\,\mathrm{d}x \cdot h$, is a positive operator. We calculate

$$\begin{aligned}
\|(\lambda - A_{-1})^{-1}B\| &= \sup_{\|g\|=1} \|(\lambda - A_{-1})^{-1}Bg\|_\infty \\
&= \sup_{\|g\|=1} \left\| \left(\int_0^1 g(x)\,\mathrm{d}x \right) \cdot (\lambda - A_{-1})^{-1}h \right\|_\infty \\
&= \|(\lambda - A_{-1})^{-1}h\|_E.
\end{aligned}$$

It is a standard fact from semigroup theory that $\|(\lambda - A_{-1})^{-1}h\|_E \to 0$ as $\lambda \to \infty$, for all $h \in E_{-1}$; hence $\|(\lambda - A_{-1})^{-1}B\| < 1$ for large λ. Therefore Theorem 1.2 implies that $(A_{-1} + B)_{|E}$ is the generator of a positive semigroup.

Remark 5.2. (a) Clearly, the function h in Example 5.1 could also be replaced by measures in $E_{-1,+}$ which are not absolutely continuous with respect to the Lebegue measure.

(b) If, more strongly, $h \in \mathrm{L}_\infty(0,1)$ (but not necessarily positive), then h belongs to the extrapolated Favard class $F_0 = \mathrm{L}_\infty(0,1)$ of the semigroup $(T(t))_{t\geqslant 0}$, and the generator property of $(A_{-1} + B)_{|E}$ follows from Engel and Nagel [8, Chapter III, Corollary 3.6].

Example 5.1 shows that E_{-1} need not be a Banach lattice if E is an AM-space without order unit. Next we present a counterexample of an AM-Space with order unit.

Example 5.3. Consider the space $E = \{f \in \mathrm{C}[0,1] : f(0) = f(1)\}$ and the operator

$$Ah = h', \qquad \mathrm{dom}(A) = \{f \in \mathrm{C}^1[0,1] \cap E : \, f'(0) = f'(1)\}.$$

It is shown in Nagel [12, Chapter A-I, page 11], that A is the generator of the positive periodic bounded semigroup $(T(t))_{t\geqslant 0}$ given by

$$(T(t)f)(s) = f(y) \qquad \text{for } y \in [0,1], y = s + t \bmod 1.$$

The description of E_{-1} and $E_{-1,+}$ will be analogous to the description in Example 5.1, but slightly more involved because in the present case the operator A is not invertible. Our first aim is to obtain the representation

$$E_{-1} = \{h \in \mathscr{D}(0,1)' : \, h = f - \partial f \text{ for some } f \in E\}. \tag{5.5}$$

We refer to Engel and Nagel [8, Chapter II, Example 5.8(ii)] for this expression in a similar context. As above, $j\colon E \to \mathscr{D}(0,1)'$ will be the standard injection. We define its extension to E_{-1} by

$$\langle j_{-1}(h), \varphi\rangle := \int_0^1 \big((1-A_{-1})^{-1}h\big)(x)(\varphi+\varphi')(x)\,\mathrm{d}x.$$

Then $j_{-1} = (1-\partial)\circ j\circ(1-A_{-1})^{-1}$. To show the injectivity of j_{-1}, assume that $h \in E_{-1}$ is such that $j_{-1}(h) = 0$. Then $f := (1-A_{-1})^{-1}h \in E$ satisfies

$$0 = \int_0^1 f(x)(\varphi+\varphi')(x)\,\mathrm{d}x = \int_0^1 f(x)\frac{1}{\exp(x)}(\exp\cdot\varphi)'(x)\,\mathrm{d}x$$

for all $\varphi \in \mathscr{D}(0,1)$. This implies that $f\frac{1}{\exp}$ is constant, and therefore $f(0) = f(1)$ shows that $f = 0$, $h = 0$. Rewriting the formula for j_{-1} as

$$j_{-1}\circ(1-A_{-1}) = (1-\partial)\circ j$$

we obtain (5.5). For the following, it will be important to keep in mind the identity $f - \partial f = \exp\partial\big(\frac{f}{\exp}\big)$, for $f \in E$.

For the present context we are going to show that

(5.6) $\qquad E_{-1,+} = \{\mu :\ \mu \text{ a finite continuous positive Borel measure on } (0,1)\}.$

The inclusion '$\subseteq$' is shown as in Example 5.1. For the reverse inclusion let μ be as on the right hand side of (5.6). Then there exists $g \in \mathrm{C}[0,1]$ with $\frac{g}{\exp}$ increasing and such that $\frac{1}{\exp}\mu = \partial\big(\frac{g}{\exp}\big)$. Note that then also

$$\frac{1}{\exp}\mu = \partial\Big(\frac{1}{\exp}(g + c\exp)\Big)$$

for all $c \in \mathbb{R}$. It is clear that there exists a unique $c \in \mathbb{R}$ such that $f := g + c\exp \in E$, and this implies $\mu = \exp\partial\big(\frac{f}{\exp}\big) = f - \partial f$. In order to show that $\mu \in E_{-1,+}$ we still have to approximate f suitably. To do so we first extend f to $\mathbb{R}$ as a continuous periodic function. Then we define $f_k := \rho_k * f$, where $(\rho_k)_{k\in\mathbb{N}}$ is a δ-sequence in $C_c^\infty(\mathbb{R})$. Then $f_k \in \mathrm{C}^\infty(\mathbb{R})\cap E$ for all $k \in \mathbb{N}$, $f_k \to f$ in E as $k \to \infty$, and it is not too difficult to show that $\frac{f_k}{\exp}$ is increasing for all k. Then $f_k - f_k' \in E_+$ for all $k \in \mathbb{N}$, and $f_k - f_k' \to f - \partial f$ $(k\to\infty)$ in E_{-1} implies $\mu = f - \partial f \in E_{-1,+}$.

From (5.5) and (5.6) it follows as in Example 5.1 that E_{-1} is not a lattice and a fortiori not a Banach lattice.

Our final example shows that, for an operator $B \in \mathcal{L}(E, E_{-1})$ to be positive it is not sufficient that $R(\lambda, A_{-1})B$ is positive in $\mathcal{L}(E)$ for some $\lambda > \mathrm{s}(A)$.

Example 5.4. Let E and A be as in Example 5.1. Define

$$h := -\chi_{[0,1/2)} + \chi_{[1/2,1]}.$$

Then the description of $E_{-1,+}$ in Example 5.1 shows that h is not positive in E_{-1}, because $h = \partial g$, for the function $g \in E$ given by

$$g(x) = \begin{cases} -x & \text{if } x \in [0,\frac12[,\\ x-1 & \text{if } x \in [\frac12, 1],\end{cases}$$

which is not increasing. However, $(-A_{-1})^{-1}h = -g$ belongs to E_+.

Defining the operator $B \in \mathcal{L}(E, E_{-1})$ by

$$Bf := \int_0^1 f(x)\,\mathrm{d}x \cdot h$$

we see that $(0 - A_{-1})^{-1}B \in \mathcal{L}(E)$ is positive, but B is not positive.

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