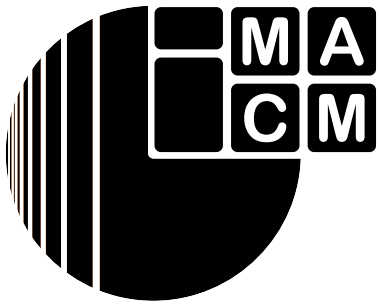

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Multiscale Approach to Parabolic Equations Derivation: Beyond the Linear Theory

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Abstract
The concept of the iterative parabolic approximation based on the multiscale technique is discussed. This approach is compared with the traditional ways to derive the wide-angle parabolic equation. While the latter fail in the nonlinear case, the multiscale derivation technique leading to iterative parabolic equations can be easily adapted to handle it. The nonlinear iterative parabolic approximations for the wave propagation in Kerr media are presented. An example demonstrating the capability of iterative parabolic equations to take nonparaxial propagation effects in Kerr media into account is considered.

Keywords: paraxial approximations, parabolic equation, multiscale technique, Kerr medium, underwater acoustics

1 Introduction

In this work we review our recent results [TPZ13, Pet15, PE16, PME16] in the framework of the history of *the parabolic equation* (PE) theory. This theory emerged from the pioneering work of Leontovich and Fock in 1946 [LF46], and has been being intensively developed since that time. In principle the PEs were designed to describe paraxial propagation, i.e. the propagation at small angles to some axis. In the linear theory this shortcoming has been overcome some three decades ago with the introduction of the wide-angle PEs [PK77, Cla85]. In the case of nonlinear waves however (e.g. waves in a Kerr medium) only the narrow-angle PE was available until now [Fib15].

Recently a new approach to the derivation of the PEs was proposed [TPZ13]. Although in principle it is based on the same parabolic scaling that was first used in the original work of Leontovich and Fock, the breakthrough was achieved when Trofimov implemented it within

the workflow of the method of multiple scales. This approach resulted in the derivation of *the system of iterative PEs* that can very accurately handle wide-angle propagation problems (this capability was validated in [TPZ13, PE16] for the linear case). By contrast to the standard wide-angle PE, the direct multiscale technique can be also used to obtain the wide-angle parabolic approximation in the nonlinear case. It was shown that such approximations can accurately handle nonparaxial propagation effects in nonlinear optics [PME16].

2 A Very Brief History of the Parabolic Equation Theory

2.1 Leontovich and Fock: the Original Derivation

A paraxial approximation for *the linear Helmholtz equation* (HE)

$$\Delta u + k^2 u = 0\,, \tag{1}$$

in the context of the radiowaves propagation along the Earth's surface was first used by Leontovich and Fock in 1946 [LF46]. In (1) $u = u(r, z)$ denotes the vertical component of electric field (referred to as the Hertz function by Leontovich and Fock) in cylindrical coordinates r, z, and k denotes the wavenumber. The derivation of the PE then requires two steps [LF46]:

1. rewrite the HE (1) for the reduced function $W = R\mathrm{e}^{-\mathrm{i}kR}U$, where $R = \sqrt{r^2 + z^2}$;
2. introduce the stretched coordinates (the so-called *parabolic scaling*)

$$\rho = \frac{kr}{2|\eta|}\,, \qquad \zeta = \frac{kz}{\sqrt{|\eta|}}\,, \tag{2}$$

   where η is the relative complex permittivity of the ground which is considered a large parameter.

Retaining only the leading-order terms in $\frac{1}{|\eta|}$, we arrive at the PE

$$\mathrm{i}\frac{\partial W}{\partial \rho} + \frac{\partial^2 W}{\partial \zeta^2} + \mathrm{i}\frac{\zeta}{\rho}\frac{\partial W}{\partial \zeta} = 0\,.$$

Rewriting this equation for another new function W_1 defined by $W = \sqrt{\rho}\mathrm{e}^{-\mathrm{i}\frac{\zeta^2}{4\rho}}W_1$, we obtain the *classical narrow-angle PE* (or *paraxial wave equation*)

$$\mathrm{i}\frac{\partial W_1}{\partial \rho} + \frac{\partial^2 W_1}{\partial \zeta^2} = 0\,. \tag{3}$$

Note that *the parabolic scaling* (2) is the crucial point in obtaining the PE by Leontovich and Fock. As it is usually dropped in the latter works [JKPS11], the entire idea of the derivation becomes somewhat mysterious.

Also note that this derivation can be easily generalized to the case of *the nonlinear Helmholtz equation* (NHE). For example, the paraxial equation for the Kerr medium can be readily obtained using this technique [Fib15].

A different way to obtain the paraxial wave equation was discovered by Maliuzhinets and Popov. It is based on considering plane waves satisfying the HE (1). The use of the PE in the field of ocean acoustics was pioneered by Tappert in 1974 [Tap77, JKPS11], and his approach is

very similar to that of Leontovich and Fock, although the parabolic scaling was not mentioned in his works.

Another way to derive the PE is based on the so-called *splitting matrix approach* that originates from the work of Corones [Cor75]. This technique involves the rewriting of the original HE in vector form, where the vector components are the forward- and back-propagating waves.

2.2 Formal Factorization and High-Order PEs

The traditional way to obtain high-order (or wide-angle) PEs consists in the formal factorization of the Helmholtz operator. Such factorization results in an equation containing a pseudodifferential operator (PDO), usually an operator square root. A polynomial (Taylor-type) or a rational (Padé-type) approximation of the PDO reduces the latter equation to a wide-angle (or high-order) parabolic equation. The wide-angle PE based on a Taylor approximation was first proposed by Popov and Khozioskii [PK77], while the Padé expansion of the PDO was introduced by Claerbout [Cla85]. Here we outline the standard derivation scheme for the wide-angle PE [JKPS11]. Note that although we used cylindrical coordinates in the previous section (mainly for historical reason, i.e. in order to keep the original PE derivation by Leontovich and Fock intact), in what follows we work in 2D Cartesian coordinates (x, z) for the sake of simplicity, and x always denotes the paraxial direction.

Consider the acoustic Helmholtz equation [JKPS11] describing sound propagation in 2D

$$u_{xx} + u_{zz} + k^2 u = 0\,. \tag{4}$$

Here the function $u = u(x, z)$ is the sound pressure, and k is the wavenumber (the ratio of the angular frequency to the sound speed). Variable z denotes the depth, and x is the range. In ocean acoustics the medium usually varies much slower in range than in depth, and by paraxial propagation we mean the propagation along the x axis.

We start with the formal factorization of the operator on the left-hand side of (4) that results in the product of two PDOs:

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\sqrt{k^2 + \frac{\partial^2}{\partial z^2}}\right)\left(\frac{\partial}{\partial x} - \mathrm{i}\sqrt{k^2 + \frac{\partial^2}{\partial z^2}}\right) u = 0\,. \tag{5}$$

The two factors in (5) correspond to the waves propagating in negative and positive x-directions respectively. Restricting our attention to the case of a right-propagating wave we drop the other factor and consider

$$\left(\partial_x - \mathrm{i}\sqrt{\kappa^2 + \frac{\partial^2}{\partial z^2}}\right) u = 0\,.$$

Introducing the reference wavenumber k_0 and removing the factor $\exp(\mathrm{i}k_0 x)$ (the so-called *principal oscillation*) from u we arrive at *the one-way propagation equation*

$$u_x = \mathrm{i}k_0(-1 + \sqrt{1 + L})u\,, \tag{6}$$

where $L = \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + \frac{k^2 - k_0^2}{k_0^2}$.

Equation (6) is a pseudodifferential PE since it contains the operator square root (which is not a differential operator). The numerical evaluation of pseudodifferential operators is a

very challenging task, and in practice some approximation of $\sqrt{1+L}$ is usually applied. The first-order Taylor approximation

$$\sqrt{1+L} \approx 1 + L/2\,,$$

allows us to obtain *the standard narrow-angle PE* (which is equivalent to that of Leontovich and Fock (3) modulo the notation).

$$i\frac{1}{k_0}\frac{\partial u}{\partial x} + \frac{1}{2k_0^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{2}\frac{k^2 - k_0^2}{k_0^2}u = 0\,.$$

The Taylor approximations of higher order lead us to the wide-angle parabolic equations of Popov and Khozioskii [PK77]. Claerbout [Cla85] proposed to use Padé approximations for the square root in (6). It can be written as

$$(-1 + \sqrt{1+L}) \approx \sum_{j=1}^{n} \frac{\alpha_{j,n} L}{1 + \beta_{j,n} L}\,.$$

Such approximations lead to the high-order wide-angle PEs, which can be solved by the alternating directions method. Another important step in the development of the PE theory was made by Collins [Col15, JKPS11], who rewrote the equation (6) using *the evolution operator*

$$u(x + \Delta x) = \mathrm{e}^{\mathrm{i}k_0(-1+\sqrt{1+L})\Delta x} u(x)\,, \tag{7}$$

and then approximate the latter using the Padé series:

$$\mathrm{e}^{\mathrm{i}k_0(-1+\sqrt{1+L})\Delta x} \approx \sum_{j=1}^{n} \frac{\alpha_{j,n} L}{1 + \beta_{j,n} L}\,.$$

Let us note that the same idea was independently introduced by Avilov [Avi85] to the Soviet acoustics community. The use of (7) allows to perform very large steps Δx when implementing the marching scheme for the PE solution.

Contrary to the case of the narrow-angle PE, the derivation of the high-order PEs based on the formal factorization does not admit any generalization to the nonlinear case. Indeed, the crucial factorization (5) is impossible for the nonlinear HE (see below).

3 Iterative Parabolic Approximations

The idea to apply a direct multiscale approach to the derivation of wide-angle PEs was introduced by Trofimov [TPZ13] in the context of 2D propagation in ocean acoustics. It was later generalized to the 3D case by Petrov [Pet15]. Right from the original study of Leontovich and Fock it is clear that the parabolic approximation results from the introduction of two spatial scales (2). The idea of Trofimov [TPZ13] was to use this scaling from the very beginning of the derivation in the framework of the method of multiple scales [Nay73]. Here we briefly outline the idea and the main results of the approach of Trofimov. Note that the detailed derivation of the iterative PEs is presented in the next section for the nonlinear case, and it closely follows the original one [TPZ13] with the nonlinear terms making the only difference.

We start the derivation of the iterative PE with the same equation (4) as in the previous section. First the parabolic scaling of the coordinates is performed, and the equation (4) is

rewritten in the slow variables $X = \epsilon x$ and $Z = \epsilon^{1/2} z$, and the fast variable $\eta = (1/\epsilon)\theta(X, Z)$. We assume that

$$k^2 = k_0^2 + \epsilon\nu(X, Z)\,, \tag{8}$$

$$u = u_0(X, Z, \eta) + \epsilon u_1(X, Z, \eta) + \dots\,. \tag{9}$$

Following the generalized multiple-scale method [Nay73], we replace the derivatives in (4) using the chain rule

$$\frac{\partial}{\partial x} \to \epsilon \left(\frac{\partial}{\partial X} + \frac{1}{\epsilon}\theta_X \frac{\partial}{\partial \eta} \right) \quad \text{and} \quad \frac{\partial}{\partial z} \to \epsilon^{1/2} \left(\frac{\partial}{\partial Z} + \frac{1}{\epsilon}\theta_Z \frac{\partial}{\partial \eta} \right)\,.$$

Substituting (8) and (9) into (4), collecting the terms of like orders in ϵ and solving the resulting equations one by one we eventually obtain the following series for $u(x, z)$:

$$u(x, z) = \exp(\mathrm{i}k_0 z) \sum_{j=0}^{\infty} A_j(x, z)\,. \tag{10}$$

The amplitudes $A_j(x, z)$ in (10) can be obtained from the following hierarchy of *iterative parabolic equations*:

$$2\mathrm{i}k_0 A_{j,x} + A_{j,zz} + \nu A_j + A_{j-1,xx} = 0\,, \quad j = 0, 1, 2 \dots\,, \tag{11}$$

where $A_{-1}(x, z) \equiv 0$. The approximation of $u(x, z)$ obtained by taking into account only the first $N+1$ terms $A_0(x, z), \dots, A_N(x, z)$ of the series (10) is called N-th order iterative parabolic approximation. The iterative parabolic equations (11) should be solved one by one, and the solution of n-th equation is used as the input for $n + 1$-th equation.

Note that at least for the range-independent case (i.e. for the waveguide where the parameters do not vary in range) the series (10) uniformly converges to the exact modal solution of (4) on any finite interval $x \in [0, L]$ cf. [TPZ13].

It is also interesting to note that the system similar to (11) first appeared in the work of Grikurov and Kiselev [GK86]. They studied the accuracy of the solution given by a narrow-angle PE in the ray coordinates, and derived a simplified version of (11) in order to estimate the contribution of the high-order terms.

In a study of Awadallah and Brown [AB98] a narrow-angle PE derivation is also presented within the framework of multiscale approach. This work features a very clear transition from the underlying physical assumptions on the scattering geometry to the introduction of the two spatial scales. Though the parabolic scaling is not explicitly mentioned in [AB98], it clearly equivalent to the steps taken in this work.

4 A Generalization: Nonlinear Helmholtz Equation

In this section we show that the iterative parabolic approximations can be also easily derived for the nonlinear case. While in the previous section we worked in the acoustical context, here we switch to nonlinear optics. More precisely, we consider the Helmholtz equation in a nonlinear Kerr-type medium (hereafter we call this equation NHE) [Fib15]:

$$\frac{\partial^2}{\partial z^2} E + \frac{\partial^2}{\partial x^2} E + k_0^2 (1 + \epsilon |E|^{2\sigma}) E = 0\,, \tag{12}$$

where $E = E(x, z)$ denotes the electric field. Hereafter we restrict our attention to the case $\sigma = 1$ (though it will be clear that our approach can be readily used for other possible values of σ).

We use the same slow variables X, Z as in the previous section, and the same fast variable η as well. Now let us rewrite E using these new variables:

$$E(x, z) = \mathcal{E}(X, Z, \eta)\,.$$

According to the chain rule, we have

$$\frac{\partial E}{\partial z} = \epsilon^{1/2}\left(\frac{\partial}{\partial Z} + \frac{1}{\epsilon}\theta_Z\frac{\partial}{\partial \eta}\right)\mathcal{E} \quad \text{and} \quad \frac{\partial E}{\partial x} = \epsilon\left(\frac{\partial}{\partial X} + \frac{1}{\epsilon}\theta_X\frac{\partial}{\partial \eta}\right)\mathcal{E}\,,$$

and the NHE (1) can now be written as

$$\epsilon\left(\frac{\partial}{\partial Z} + \frac{1}{\epsilon}\theta_Z\frac{\partial}{\partial \eta}\right)^2\mathcal{E} + \epsilon^2\left(\frac{\partial}{\partial X} + \frac{1}{\epsilon}\theta_X\frac{\partial}{\partial \eta}\right)^2\mathcal{E} + k_0^2(1 + \epsilon|\mathcal{E}|^2)\mathcal{E} = 0\,.$$

Next we introduce the following asymptotic expansion similar to (9)

$$E(x, z) = \mathcal{E}(X, Z, \eta) = \mathcal{E}_0(X, Z, \eta) + \epsilon\mathcal{E}_1(X, Z, \eta) + \epsilon^2\mathcal{E}_2(X, Z, \eta) + \dots\,. \tag{13}$$

Substituting the series (13) into the equation (12) and collecting the terms of order ϵ^{-1}, ϵ^0, etc, we obtain an infinite sequence of equations. Throughout the remaining of this section we consider them one by one.

There is only one term $\epsilon^{-1}\theta_Z$ of the order of ϵ^{-1}, and the respective equation is $\theta_Z = 0$, hence $\theta = \theta(X)$, i.e. the fast scale only depends on the range.

Terms of the order of ϵ^0 are combined into the following equation

$$(\theta_X)^2\mathcal{E}_{0\eta\eta} + k_0^2\mathcal{E}_0 = 0\,.$$

In order to satisfy this equality we simply put

$$(\theta_X)^2 = k_0^2 \tag{14}$$

and readily obtain

$$\mathcal{E}_0 = \exp(\mathrm{i}\eta)\mathcal{A}_0(X, Z)\,.$$

At this point we choose the branch $\theta_X = k_0$ of the solution of the Hamilton-Jacobi equation (14) and thus retain only the waves propagating in the positive direction of the X-axis (this is exactly the point where the one-way approximation is applied in our approach). From (14) we also find that

$$\theta(X) = k_0 X\,.$$

Note that the uniformity of the asymptotic expansion (13) is maintained if and only if

$$\mathcal{E}_j = \exp(\mathrm{i}\eta)\mathcal{A}_j(X, Z)\,, \tag{15}$$

for all $j \geq 1$ (this is a typical result of the application of the multiple-scale expansion method (see [Nay73, TPZ13]).

In the light of representation of $\mathcal{E}_j$ in (15) we immediately rewrite the nonlinear term in (12) as

$$k_0^2\epsilon|\mathcal{E}|^2\mathcal{E} = k_0^2\epsilon\exp(\mathrm{i}\eta)(\mathcal{A}_0 + \epsilon\mathcal{A}_1 + \dots)^*(\mathcal{A}_0 + \epsilon\mathcal{A}_1 + \dots)^2\,, \tag{16}$$

where f^* denotes the complex conjugate of f.

Since the exponentials can be cancelled in (12) for the terms of all orders in ϵ, we now rewrite the nonlinear part of the expression (16) in the following way

$$k_0^2\epsilon(|\mathcal{A}_0|^2\mathcal{A}_0)+k_0^2\epsilon^2(2|\mathcal{A}_0|^2\mathcal{A}_1+\mathcal{A}_0^2\mathcal{A}_1^*)+k_0^2\epsilon^3(2|\mathcal{A}_0|^2\mathcal{A}_2+\mathcal{A}_0^2\mathcal{A}_2^*+2\mathcal{A}_0|\mathcal{A}_1|^2+\mathcal{A}_0^*\mathcal{A}_1^2)+\dots\,. \quad (17)$$

Now we proceed to the terms with positive orders in ϵ. Collecting the terms containing ϵ^1 we arrive at the following equality

$$2\mathrm{i}k_0\mathcal{A}_{0X} + \mathcal{A}_{0ZZ} + k_0^2\mathcal{A}_0|\mathcal{A}_0|^2 = 0\,. \quad (18)$$

Equation (18) is basically the nonlinear (cubic) Schrödinger equation (NSE). The standard derivation of (18) could be found in many works, see e.g. [Fib15] and references therein. In our view, the derivation by the method of multiple scales presented here is somewhat more clear.

An important advantage of our multiscale approach is the possibility to derive higher-order corrections to the NSE (18). These corrections may probably both improve the handling of the nonlinear effects and also describe more accurately wide-angle propagation.

Collecting the terms of the order of ϵ^{s+1} we obtain an equation for A_s:

$$2\mathrm{i}k_0\mathcal{A}_{s,X} + \mathcal{A}_{s,ZZ} + k_0^2\left(\sum_{l+n+m=s}\mathcal{A}_l\mathcal{A}_n\mathcal{A}_m^*\right) + \mathcal{A}_{s-1,XX} = 0\,. \quad (19)$$

For each $s \geq 1$ equation (19) is a generalization of the linear Schrödinger equation with an input term $\mathcal{A}_{s-1,XX}$ that is computed from the solution of the previous equation. Note that the coefficients of (19) also contain $\mathcal{A}_{s-1}, \mathcal{A}_{s-2}, \dots$.

4.1 Approximate solution of the NHE: a Recipe

Note that the equations (18), (19) are written in the slow variables Z, X. Switching back to the physical variables, and recalling that $\eta = \epsilon^{-1}\theta(X) = \epsilon^{-1}k_0X = k_0x$, we find that the solution of the NHE (12) can be approximated by the truncated series of $N+1$ terms

$$E(x,z) \sim \exp(\mathrm{i}k_0z)\sum_{j=0}^{N}A_j(x,z)\,, \quad (20)$$

where A_j satisfy the equations

$$\begin{aligned}
2\mathrm{i}k_0A_{0x} + A_{0zz} + \epsilon k_0^2|A_0|^2A_0 &= 0\,,\\
2\mathrm{i}k_0A_{1x} + A_{1zz} + \epsilon k_0^2\left(2|A_0|^2A_1 + A_0^2A_1^*\right) + A_{0xx} &= 0\,,\\
2\mathrm{i}k_0A_{2x} + A_{2zz} + \epsilon k_0^2\left(2|A_0|^2A_2 + A_0^2A_2^*\right) + \epsilon k_0^2\left(2|A_1|^2A_0 + A_1^2A_0^*\right) + A_{1xx} &= 0\,,\\
&\dots\\
2\mathrm{i}k_0A_{s,x} + A_{s,zz} + \epsilon k_0^2\left(2|A_0|^2A_s + A_0^2A_s^*\right) + \epsilon k_0^2\left(\sum_{\substack{l+n+m=s,\\ l,n,m<s}}A_lA_nA_m^*\right) + A_{s-1,xx} &= 0\,,\\
&\dots\,.
\end{aligned} \quad (21)$$

Just like (11), equations (21) can be solved one by one to obtain the approximation for the solution to (12). Hereafter the right hand side of (20) is called N-th order (wide-angle) parabolic approximation for the solution of the NHE (12).

5 Conclusion and Future Work

In this study we attempted to overview various techniques that can be used for the derivation of the PEs starting with the pioneering work of Leontovich and Fock. Our goal was also to show the place of our recent results within the entire PE theory and to emphasize the links connecting them to the previous work. While many features of iterative PEs look quite attractive, its development still requires additional efforts. The most obvious gaps to fill include the general initial conditions construction (especially in the nonlinear case), and the development of efficient numerical techniques for the solution of practical problems. The rational iterative parabolic approximation is also an interesting subject for the future work that can combine the advantages of the Padé-type wide-angle PEs and the iterative PEs.

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