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Abstract We consider the computational efficiency of the backward vs. forward approaches and compare these with the respective ones resulting from a parametric reduced order model, whose speed-up can be put to good use in the calibration of the underlying dynamics. We apply a global Proper Orthogonal Decomposition in the time domain to obtain the reduced basis and the Modified Craig-Sneyd (MCS) ADI and Chang-Cooper schemes to numerically solve the partial differential equations. The numerical results are presented for the Black-Scholes and Heston models.

1 Introduction

In Computational Finance option prices can be studied by Stochastic Differential Equations (SDEs) involving Brownian motion [1, 5, 11]. This usually leads to Monte Carlo simulation approaches. Assuming proper hedging and suitable boundary conditions the option prices can also be shown to satisfy a system of time-dependent partial differential equations (PDEs), which are our starting point. In the case of the 1D Black-Scholes-Merton Model, we have the backwards PDE [11]

$$\frac{\partial u}{\partial t}(t,S) = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2}(t,S) - (r-q)\,S\frac{\partial u}{\partial S}(t,S) + ru(t,S)\,, \tag{1}$$

for a call option $u = u(t,S)$,where $0 \le t \le T$, $0 \le S \le S_{\max}$ and whose terminal and boundary conditions depend on the payoff. F.i. for a call option with "strike" K we have $u(T,S) = \max(S-K,0)$, $0 \le K \le S_{\max}$. Furthermore, $\sigma^2 = \sigma_u^2$ is the variance (volatility) of u, r is the risk-free interest and q is the dividend rate.

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Introducing a stochastic square root variance model with a mean reverting process one we obtain again a backwards 2D-PDE, the 2D Heston model [11]

$$\frac{\partial u}{\partial t} = -\frac{1}{2}vS^2\frac{\partial^2 u}{\partial S^2} - \rho\sigma vS\frac{\partial^2 u}{\partial v \partial S} - \frac{1}{2}\sigma^2 v\frac{\partial^2 u}{\partial v^2} - (r-q)S\frac{\partial u}{\partial S} - \kappa(\theta - v)\frac{\partial u}{\partial v} + ru, \tag{2}$$

for $u = u(t,S,v)$, $0 \le S \le S_{\max}$, $0 \le v \le v_{\max}$, where $v = \sigma_u^2$ follows a Cox-Ingersoll-Ross stochastic process with an own variance $\sigma^2 = \sigma_v^2$; ρ is the correlation between the Wiener processes for u and for v. θ is the long-term mean of variance for v. Similar as before, $u(T,S,v) = \max(S-K,0)$, $0 \le K \le S_{\max}$.

For (1) and (2) one needs efficient time integration methods like the Modified Craig-Sneyd (MCS) ADI method [6, 7].

The time-varying density distribution of the option satisfies a dual equation (Fokker-Planck). Then the option can be obtained by a post-processing step by taking a proper integral with the density solution of the dual equation. Clearly for solving the dual PDE one is interested only in non-negative outcomes. This imposes extra conditions to time-integrators. The Chang-Cooper scheme can be guaranteed to generate non-negative solutions [8, 9].

Sensitivity is closely related to dual approaches. In finance the Dupire equation has become famous [1, 4, 5]. It expresses the sensitivity of the option prices w.r.t. the expiration time.

Model Order Reduction (MOR) is a technique used to derive a low-order, highly accurate approximation of the solution of a problem by projecting the original, full order problem onto a subspace of (much) smaller dimension while still capturing the dominant dynamics of the Full Order Model (FOM). We use a Global Proper Orthogonal Decomposition approach with uniform sampling to obtain our Reduced Order Model (ROM).

2 Direct Backward and Adjoint Forward Models

The equations (1) and (2) are both of the form

$$\begin{cases} \partial_t u(t,x) & = Lu(t,x) \\ u(T,x) & = u_T(x) \end{cases} \qquad (t,x) \in \mathscr{Q} \equiv [t_0,T] \times \Omega$$

and equipped with appropriate boundary conditions. Multiplying the above equation by a sufficiently regular test function p and integrating by parts over $\mathscr{Q}$, we obtain

$$\int_{\mathscr{Q}} p(\partial_t - L)u = \int_{\mathscr{Q}} u(-\partial_t + L^*)p + \int_{\Omega} pu\Big|_{t=t_0}^{t=T} \tag{3}$$

where L^* is the adjoint operator of L. Let us choose the test function p as the solution of the adjoint equation $\partial_t p = -L^* p$ with initial condition $p(x,t_0) = p_0(x) = \delta(x-x_0)$. Due to Lagrange's identity and as p satisfies the adjoint equation, we obtain

$$\int_\Omega p(x,T)\,u(x,T)\,dx = \int_\Omega p(x,t_0)\,u(x,t_0)\,dx = u(x_0,t_0)\,. \tag{4}$$

When we scale p such that $\int_\Omega p(T,x)\,dx = 1$, we can interpret p as a probability density function. For (1) and (2) the adjoint equations $\partial_t p = -L^* p$ have the form

$$\frac{\partial p}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}\left(A_i(t,x)\,p\right) + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}\left(\sum_k B_{ik}(t,x)\,B_{jk}(t,x)\,p\right). \tag{5}$$

For (1) we have

$$A = \left[(r-q)\,S\right],\ \ B = \left[\sqrt{\sigma}S\right],$$

resulting in the forward PDE for $p(t,S)$

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma^2 S^2 p\right) - \frac{\partial}{\partial S}\left((r-q)\,Sp\right) \tag{6}$$

with initial condition $p(0,S) = \delta(S-S_0)$ and zero-flux boundary conditions.

For (2), with $x = (S,\nu)$, we obtain

$$A = \begin{bmatrix}(r-q)\,S\\ \kappa(\theta-\nu)\end{bmatrix},\ \ B = \begin{bmatrix}\sqrt{\nu}S & 0\\ \sigma\sqrt{\nu}\rho & \sigma\sqrt{\nu}\sqrt{1-\rho^2}\end{bmatrix}$$

yielding

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\nu S^2 p\right) + \frac{\partial^2}{\partial S \partial \nu}\left(\rho\sigma\nu S p\right) + \frac{1}{2}\frac{\partial^2}{\partial \nu^2}\left(\sigma^2 \nu p\right) - \frac{\partial}{\partial S}\left((r-q)\,Sp\right) - \frac{\partial}{\partial \nu}\left(\kappa(\theta-\nu)\,p\right), \tag{7}$$

with initial condition

$$p(0,x) = p(0,S,\nu) = \delta(S-S_0)\,\delta(\nu-\nu_0)$$

and zero-flux (reflecting) boundary conditions. The adjoint PDEs (6) and (7) are also called Fokker-Planck equations.

Summarizing, we can calculate $u(t,x)$ directly by solving first the PDE (1) or (2) backward in time, but also by solving first the adjoint, forward PDE (6) or (7), respectively, for $p(t,x)$, and, next, applying a postprocessing step (4). We note that in the last step quadrature is needed. The integrand is a product of a smooth function $p(T,x)$ with a piecewise linear terminal function $u(T,x)$ – the last one can be interpolated exactly.

The adjoint solution $p(t,x)$ can be interpreted as a conditional probability $p(t,x|t_0,x_0)$. Hence, in solving the dual PDE one is interested only in non-negative outcomes.

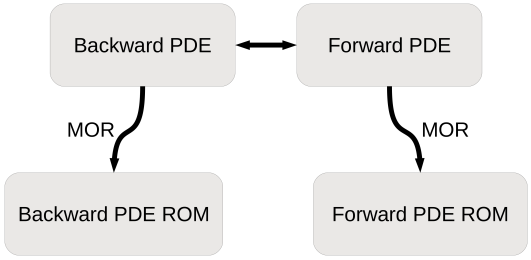
This imposes extra conditions to time-integrators [8]. The Chang-Cooper scheme can be guaranteed to generate non-negative solutions [8, 9]. Additionally, one is interested in keeping the probability constant in time:

$$\int_{\Omega} p(t,x)\,dx = 1. \tag{8}$$

If one likes to determine $u(t,x)$ for several K or T, or to determine the sensitivities of U with respect to K and T, the adjoint integration becomes favourable over the direct backwards time integration for $u(t,x)$.

Both, to the backward models as well as to the adjoint forward models MOR can be applied. We use a global Proper Orthogonal Decomposition (GPOD) approach [10] together with a uniform sampling to obtain our Reduced Order Model (ROM) [12] than in [2].

We have then the following setup



3 Numerical Results

Due to the difference in the behaviour of the solutions of the backward and forward equations, different methods were selected for each of the two cases. In the backward equation case, plenty of methods have been proposed (see [13] and [7]) and from these, we selected the class of ADI schemes for its speed, especially in the high-dimensional cases. In particular, the ADI MCS scheme, which offers a second-order consistency in time and whose stability has been well established [6]. In the forward (adjoint) case, the conservation of probability imposes an extra condition and, accordingly, we choose the Chang & Cooper scheme with BDF2 [9, 3] , which is also second-order consistent in time. Performance-wise, the Chang-Cooper scheme will still suffer from the curse of dimensionality, as the whole system matrix is used to obtain the solution at the following time step.

To demonstrate the speed-up improvement, we choose to price an European Put Option with the 1D Black-Scholes model and with the Heston Model, choosing the parameters in a range of $\pm 20\%$ around the reference values in Table 1.

r	σ
0.03	0.3

(a) Black-Scholes

r	θ	κ	σ
0.03	0.4^2	4	0.3

(b) Heston Model

Table 1: Center of parameters' range.

It is important to note that this range of parameters always *guarantees that the Feller condition is preserved*, i.e. $\frac{2\kappa\theta}{\sigma^2} > 1$. In the Black-Scholes case we choose $n_x = 1280$ and in the Heston Model one we choose $n_x = 64$ points per dimension for the spatial discretization, with $n_t = 40$ for the temporal discretization in both cases. This choice of temporal discretization points gives us enough snapshots to build a sufficiently large basis for the ROM. The spatial and temporal steps, h_i and k, are defined as $h_i = \frac{x_{i,max} - x_{i,min}}{n_x}$ and $k = \frac{T}{n_t}$. We evaluate the price of the option at-the-money, i.e. at $S = K$ and at the long-term mean value, $v = \theta$, with $K = 120$. For the Black-Scholes case we choose the domain $S \in [0, 4K]$ and for the Heston Model $(S, v) \in [0, 4K] \times [0, 1]$.

We collect the snapshots at every time step and at a discretized grid for the parametric domain. We discretize the parametric domain $\mathscr{P} = \prod_{i=1}^{n} [\theta_{i,min}, \theta_{i,max}]$ with a uniform grid p_k containing 5 equally distanced points per parameter, including the extreme values. We will have then 2^5 and 4^5 parameter vectors to generate our snapshots for the Black-Scholes and Heston Model, respectively. For the evaluation of the ROM accuracy, we compare the reduced and the full models at the midpoints of the parametric grid, p_l, which results for the Black-Scholes and for the Heston Model in 2^4 and 4^4 evaluations, respectively. The computational time for both models taking the FOM as reference for the speed-up, is presented in Table 2.

	Black-Scholes	Heston Model
Backward FOM	1	1
Backward ROM	3.6×	3×
Forward FOM	1	1
Forward ROM	10×	1.5×

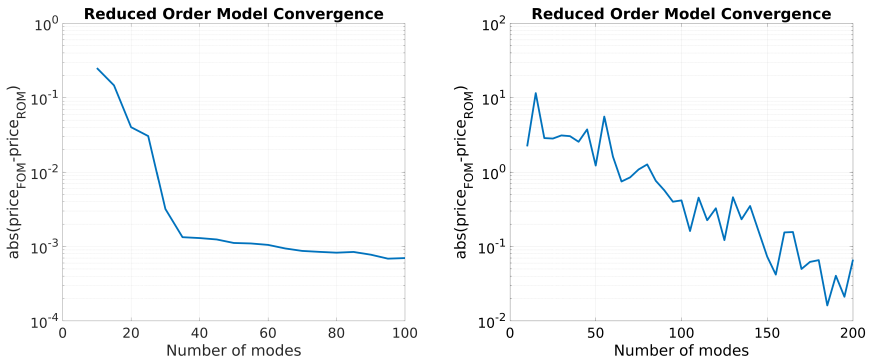
Table 2: Numerical Speed-up of the ROM Model.

In Fig.1 we can see the maximum absolute price error between the ROM and FOM forward models over all the parameter vectors corresponding to the comparison points.

$$error = \max_{\mu \in p_l} |price_{FOM}(\mu) - price_{ROM}(\mu)|$$

For the Heston case, we need around 150 modes to span a subspace which approximates the FOM in the selected range of the parameters. In Fig.2 we can see that the model shows a uniform error across the 256 vectors of parameters, proving its

applicability in the whole parametric domain and not only for the parameters which were used for the snapshots.



(a) Black-Scholes (b) Heston

Fig. 1: Error between FOM and ROM.

4 Conclusion

The proposed MOR approach shows an improvement in speeding-up the time integration to solve both the backward and the forward (adjoint) PDE, which, in practice, would be cumulatively reflected in the effective calibration time. For a least-squares minimization run in Matlab® for the backward and comparable forward equations, the authors found a speed-up of $3.3\times, 5.1\times$, and $12\times$ for the backward ROM, forward FOM and forward ROM, with 20 option prices to calibrate. Note that those are speed-ups in comparison with the backward FOM, which is already a very efficient one to solve numerically with the operator splitting schemes. Despite the already reasonable improvement, we expect an even greater improvement by tuning the model in different areas such as the location of the temporal snapshots, the addition of the mixed derivative term, selection of training points based on a greedy approach and the use of non-uniform grids.

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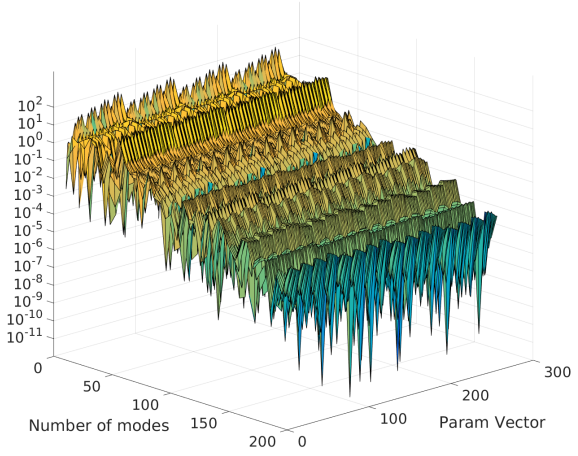

Fig. 2: Validity of the ROM across comparison points.

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