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Multirate DAE/ODE-Simulation and Model Order Reduction for Coupled Circuit-Field Systems

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Abstract Considering electro-magnetic effects in circuit simulation yields to high dimensional systems of partial-differential-algebraic equations that are time- consuming in simulation and difficult in numerical handling. In this work we present a model order reduction technique for magneto-quasistatic equations where the reduced order model is a system of ordinary differential equations. To exploit the different dynamical behaviour of circuit and field equations we propose multirate time integration schemes which are extended to differential-algebraic equations.

1 Introduction

For the development of modern electrical devices the influence of electromagnetic effects has to be considered in the simulation process very often. In general that leads to a coupled problem where the subsystems provide a quite different behaviour. The electromagnetic effects are described by Maxwell's partial differential equations (PDEs). For numerical simulation a spatial discretisation is necessary which leads to high dimensional subsystems. High dimensional subsystems let increase significantly the computation time for time domain simulation of the coupled system. Therefore a model order reduction is the method of choice to decrease the computational effort. In this paper we use a model order reduction technique for magneto-quasistatic equations (MQS) which starts with a full order differential-algebraic equation (DAE) and ends up with a system of ordinary differential equations (ODE) for the reduced system. This method was developed in [1].

The subsystem which describes the voltages at certain nodes in the device is usually modeled by a system of DAEs. Often it provides a faster dynamic behaviour than the Maxwell's PDEs. Due to the spatial discretisation of the PDEs the number of slow changing components is in general much larger than the number of fast changing components. Such systems can be solved efficiently by multirate time integration schemes:

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Slow changing components are integrated with large macro-step sizes while the fast changing or active components are integrated with small micro-step sizes. For systems of ODEs there are different approaches how the coupling between the subsystems can be realised, see e.g. [2]. Here, we focus on the idea of [3]: The whole system is integrated with macro-step size H. This approximation is accepted for the slow subsystem, the active subsystem is integrated with micro-step size h while the values of the slow subsystem are interpolated. In this work we extend multirate time integration schemes to systems consisting of a fast changing subsystem of ODEs and a slow changing subsystem of DAEs so that the method can be used for an ODE system describing an electrical circuit and a DAE describing the semidiscretised electromagnetic effects in a certain device.

We will simulate magnetic-quasistatic equations for a single-phase 2D transformer embedded in an electrical circuit with a multirate time integration scheme and apply a model order reduction to the Maxwell's PDE.

2 Multirate Time Integration for semi-explicit DAEs

We consider a system of DAEs in the following semi-explicit form

$$\dot{y}_A = f_A(t, y_A, y_L, z_L), \quad \dot{y}_L = f_L(t, y_A, y_L, z_L),\ 0 = g_L(t, y_A, y_L, z_L) \tag{1}$$

with $y_A \in \mathbb{R}^{n_A}$ the fast changing subsystem, $y_L \in \mathbb{R}^{n_{Ly}}$, $z_L \in \mathbb{R}^{n_{Lz}}$ the slow changing subsystem and consistent initial values $y_{A0} \in \mathbb{R}^{n_A}$, $y_{L0} \in \mathbb{R}^{n_{Ly}}$, $z_{L0} \in \mathbb{R}^{n_{Lz}}$. That means the active subsystem is described by a system of ODEs while the algebraic constraints only occur in the slow subsystem. We assume that the slow subsystem is of tractability index one [4]. For semi-explicit DAEs of index one special implicit Runge-Kutta methods for ODEs can be adapted to integrate also DAEs [5]. We are using LobattoIIIC method to construct a multirate integration scheme for DAEs. The Butcher-Tableau and the inverse of the coefficient matrix $\tilde{A}$ of LobattoIIIC are given by

$$\begin{array}{c|cc} 0 & 1/2 & -1/2 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array} \qquad \tilde{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

To exploit the special multirate structure of the DAE system (1) with consistent initial values at t_0 we apply the following integration procedure: Integrate the whole system (1) with macro-step size H using LobattoIIIC, the intermediate stages Y_{i1} and Y_{i2} for $i \in \{A, L\}$ are computed by solving the following system

$$\begin{aligned} Y_{ik} &= y_{i0} + \tfrac{H}{2}\left(f_i(t_0, Y_{A1}, Y_{L1}, Z_{L1}) \mp (f_i(t_0 + H, Y_{A2}, Y_{L2}, Z_{L2})\right), \quad k = 1, 2 \\ 0 &= g_L(t_0, Y_{A1}, Y_{L1}, Z_{L1}) = g_L(t_0 + H, Y_{A2}, Y_{L2}, Z_{L2}) \end{aligned} \tag{2}$$

The approximations of the slow subsystem

$$\begin{aligned} y_{LH} &= y_{L0} + \tfrac{H}{2}\left(f_L(t_0, Y_{A1}, Y_{L1}, Z_{L1}) + (f_L(t_0 + H, Y_{A2}, Y_{L2}, Z_{L2})\right) \\ z_{LH} &= Z_{L2} \end{aligned}$$

are accepted. The active subsystem is integrated with smaller micro-step sizes $h = H/n$ for some $n \in \mathbb{N}$. The unknown values of y_L, z_L and $Y_{L1}, Y_{L2}, Z_{L1}, Z_{L2}$ at the intermediate time steps $t_0 + jh$ for $j = 1, \ldots, n$ are achieved by linear interpolation. The first micro-step reads

$$Y_{Ak} = y_{A0} + \tfrac{h}{2}\left(f_A(t_0, Y_{A1}, Y_{L1}, Z_{L1}) \mp (f_A(t_0+h, Y_{A2}, \bar{Y}_{Lh}, \bar{Z}_{Lh})\right), \qquad k = 1,2$$
$$y_{Ah} = y_{A0} + \tfrac{h}{2}\left(f_A(t_0, Y_{A1}, Y_{L1}, Z_{L1}) + (f_A(t_0+h, Y_{A2}, \bar{Y}_{Lh}, \bar{Z}_{Lh})\right)$$

with $\bar{Y}_{Lh}$ denoting the interpolated value of Y_{L1} and Y_{L2} at t_0+h, $\bar{Z}_{Lh}$ respectively. After computing n micro-steps an approximation of the system (1) is given by $[y_{AH}^\top, y_{LH}^\top, z_{LH}^\top]^\top$. This approximation may not fullfil the algebraic constraints. Thus we suggest to define an tolerance ε: if $\|g(t_0+H, y_{AH}, y_{LH}, z_{LH})\| \geq \varepsilon$ resolve system (2) with $i = L$ and fixed Y_{A1}, Y_{A2}. ε has to be chosen according to the strength of the coupling between active and slow subsystem. By using the weights $\tilde{b}_1 = 1$, $\tilde{b}_2 = 0$ in the Runge-Kutta scheme a step size control can be included.

3 Model Order Reduction for Magnetic Quasistatic Equations

We consider Maxwell's equation in the magnetic potential formulation

$$\sigma \tfrac{\partial A}{\partial t} + \nabla \times (\nu \nabla \times A) \;=\; J, \quad \text{ in } \Omega \times (0,T), \tag{3}$$

where Ω is a bounded two- or three-dimensional domain composed of conducting and nonconducting subdomains, A is the magnetic vector potential, ν is the magnetic reluctivity which may depend nonlinearly on A on the conducting subdomain, σ is the electric conductivity vanishing on the nonconducting subdomain and J is the current density applied by external sources. We have the boundary condition $A \times n_o = 0$ on $\partial\Omega \times (0,T)$ with n_o the outer unit normal vector to the boundary $\partial\Omega$ of Ω and the initial condition $A(\cdot,0) = A_0$ in Ω. We assume that $J = \chi\iota$ with a divergence free winding function χ and the electrical current vector ι. The coupling equation connecting Maxwell's equations to the network equations is given by

$$\int_\Omega \chi^T \tfrac{\partial}{\partial t} A \, d\xi + R\iota = u, \tag{4}$$

where R is the resistance matrix and u is the voltage vector. Applying the finite element discretization method to (3) and (4) and reordering unknown variables accordingly to the conducting and nonconducting subdomains, we obtain a nonlinear system of differential algebraic equations (DAEs)

$$\mathscr{E}\dot{x} = \mathscr{F}(x)x + \mathscr{B}u, \qquad y = \mathscr{C}x \tag{5}$$

with a singular mass matrix $\mathscr{E}$ and $x = [a_1^T, a_2^T, \iota^T]^T$ while $[a_1^T, a_2^T]^T$ is a semidiscretized vector of magnetic potentials. The properties of the involved matrices guarantee that the system (5) is of index one and can be transformed into a system of ordinary differential equations

$$E\dot{z} = F(z)z + Bu, \qquad y = Cz \tag{6}$$

with $z = [a_1^T, (Z^T a_2)^T]^T$ and $Z = X_2(X_2^T X_2)^{-1}$. X_1 and X_2 are coupling matrices between magnetic vector potential and the electric current. In the transformation into the ODE system there is another projection matrix Y involved. The columns of Y are orthonormal and span $\ker(X_2)$ and is quite costly to compute. Note that the matrix E is nonsingular and system (6) has the same input u and the output y as the DAE system (5) meaning that the input-output relation of (5) is preserved in (6).

If the magnetic reluctivity is constant on the conducting domain, then $F(z)$ in (6) is independent of z, i.e. for constant matrices E, F, B and C. we have

$$E\dot{z} = Fz + Bu, \qquad y = Cz. \tag{7}$$

For model reduction of (7), we use a balanced truncation approach based on the controllability and observability Gramians P and Q which are defined as unique symmetric positive semidefinite solutions of the generalized Lyapunov equations. Since one has the relation $C = -B^T E^{-1} F$ only one equation has to be solved

$$FPE^T + EPF^T = -BB^T. \tag{8}$$

One can show that system (7) is passive and the Gramians satisfy $EQE = FPF$. The Gramians exist if all eigenvalues of the pencil $\lambda E - F$ have negative real part. This condition is guaranteed since the matrices E and $-F$ are symmetric and positive definite. Let $P = SS^T$ be a Cholesky factorization of P. Then the Cholesky factor of Q can be determined from $Q = E^{-1}FSS^TFE^{-1} = LL^T$ as $L = -E^{-1}FS$. Taking into account that the matrix $L^TES = -S^TFS$ is symmetric and positive definite, the Hankel singular values of (7) can be calculated from the eigenvalue decomposition

$$-S^TFS = [U_1,\, U_0]\,\mathrm{diag}(\Lambda_1,\, \Lambda_0)\,[U_1,\, U_0]^T,$$

instead of the more expensive singular value decomposition. For the projection matrices $V = SU_1\Lambda_1^{-1/2}$, $W = LU_1\Lambda_1^{-1/2} = -E^{-1}FSU_1\Lambda_1^{-1/2} = -E^{-1}FV$, we obtain,

$$\tilde{E}\dot{\tilde{z}} = \tilde{F}\tilde{z} + \tilde{B}u, \qquad \tilde{y} = \tilde{C}\tilde{z} \tag{9}$$

the reduced-order model with $\tilde{E} = W^TEV$, $\tilde{F} = W^TFV$, $\tilde{B} = W^TB$, $\tilde{C} = CV$. One can show that the reduced matrices satisfies $\tilde{E} = \tilde{E}^T > 0$, $\tilde{F} = \tilde{F}^T < 0$ and $\tilde{C} = \tilde{B}^T$. Hence, system (9) is passive. We have the $\mathscr{L}_2$-norm error bound for the output

$$\|y - \tilde{y}\|_{\mathscr{L}_2} \leq 2\,\mathrm{trace}(\Lambda_2)\,\|u\|_{\mathscr{L}_2},$$

where Λ_2 contains the truncated Hankel singular values on the diagonal. For solving the generalized Lyapunov equation (8), we can use the low-rank alternating direction implicit method or (rational) Krylov subspace method [6,7]. In both methods, we need to solve linear systems of the form $(\tau E + F)v = b$ for a vector v. Exploiting the structure of the system matrices E and F as in (6), this can be done without computing the basis matrix Y required in F and constructing E and F explicitly [1].

For model reduction of the nonlinear system (6), we can use proper orthogonal decomposition combined with the discrete empirical interpolation method (DEIM) for efficient evaluation of the nonlinearity $f(z) = F(z)z$ and matrix DEIM for fast computation of the Jacobi matrix $J_f(z)$, see [1] for details.

4 Simulation of a Coupled Electric Circuit/Field System

We simulate the electromagnetic effects of a single-phase 2D transformer in a coupled field/circuit system. Since the electromagnetic effects in our transformer do not react immediately on fast changes in the input voltage this system suits for integration by a multirate scheme. The active subsystem describes

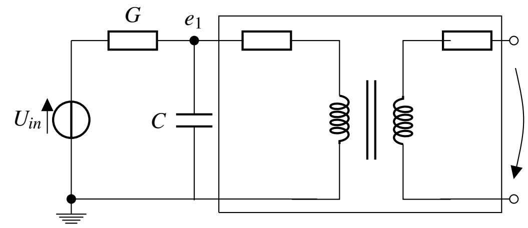


Fig. 1 Circuit diagram for no load test of the coupled systems with lumped elements for the electromagnetic effects (box).

the circuit while the slow subsystem the electromagnetic effects of the transformer. Figure 1 shows a circuit diagram of the coupled system, the electromagnetic effects are represented by the lumped devices of a transformer in the box. The input voltage is given by a superposed sinus function $U_{in}(t) = 45.5 \cdot 10^3 \sin(900\pi t) + 10^3 \sin(45000\pi t)$.

MQS-Device Modeling. We consider the linear magneto-quasistatic equations for a single-phase 2D transformer with an iron core and two coils in the form of (5). The dimension of the spatially discretised MQS system is $n_L = 7823$. The material parameters are $\sigma = 5 \cdot 10^5 \Omega^{-1}\mathrm{m}^{-1}$, $\nu_1 = 14872\mathrm{Am/(Vs)}$ on the conducting and $\nu_2 = 1\mathrm{Am/(Vs)}$ on the non-conducting subdomain. The FEM discretisation is done by the free available software FEniCS[1]. To apply a time domain simulation the model was exported to Matlab 2015a. The input of the subsystem is given by the voltage at the primary coil. The output is the current through the primary coil. The reduced model was computed by balanced truncation method as it is described in section 3. The size of the reduced model is $r = 4$.

Circuit Modeling and Coupling. Electric circuit and transformer are coupled by a source coupling approach [8]: We add an additional controlled current source to the circuit subsystem and an additional voltage source to transformer. Than, the circuit can be described by an ODE for the potential of node e_1

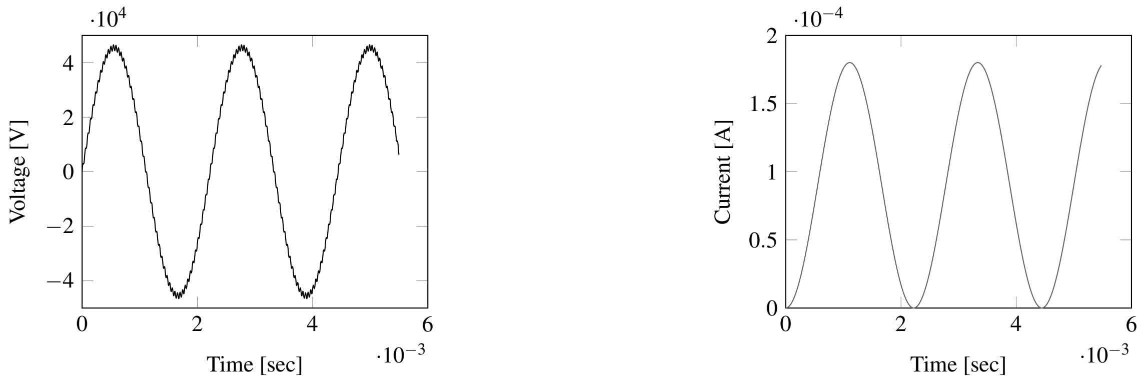
$$\tfrac{d}{dt} e_1(t) C = G(e_1(t) - U_{in}(t)) - I_{CO}(t)$$

while I_{CO} describes the coupling interface which is the current through the primary coil of the transformer. The circuit parameters are given by $C = 1\mathrm{nF}$, $G = 10^{-3}\mathrm{S}$. The input voltage to the MQS subsystem is given by the node potential e_1.

Simulation Results. We integrate the system by the multirate LobattoIIIC scheme (section 2) over the time interval $[0\mathrm{s},\ 0.0055\mathrm{s}]$. Since we are interested in the influence of the multirate approach we consider a reference solution that is computed by the LobattoIIIC method with constant, global step size using 2500 time steps, this gives a solution with a moderate accuracy. For the full order DAE the computation time was 728.2s. Fig. 2 shows the outputs of the two subsystem: (a) the superposed sinusoidal oscillations in e_1, which belongs to the fast changing subsystem and (b) the current through the primary coil of the transformer, which belongs to the slow subsystem.

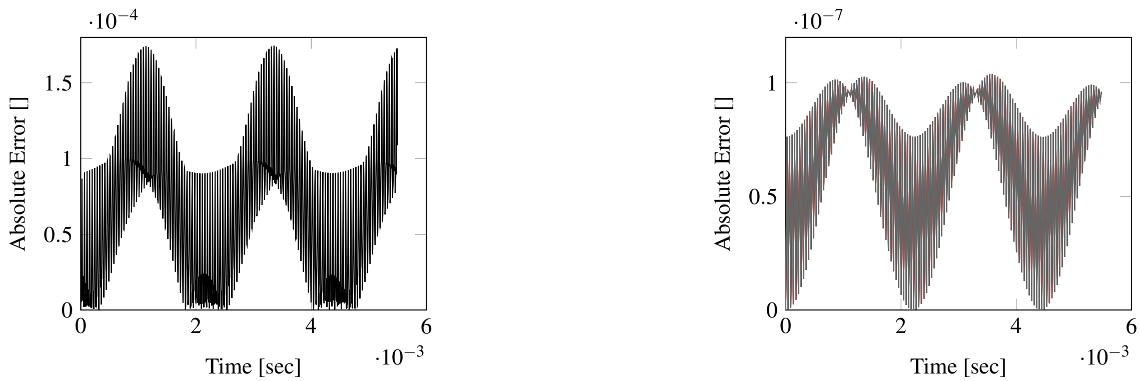
To investigate the influence of the multirate approach on the full order DAE system the time interval is discretised into 250 macro-step and each macro-step is refined into 10 micro-steps. Here the computation ended after 77.4s. Fig. 3 shows the absolute error in the outputs of the two subsystems. The error in the active subsystem is a typical multirate error: At the macro-steps the subsystems are integrated together so the error at this time points is usually a bit smaller. But during the intermediate micro-steps the error in the active subsystem increases. In the slow subsystem every second approximation gives better results while the intermediate approximation is worse. Until now this phenomena is not yet understood completely. Since the

[1] http://fenicsproject.org



(a) Node Potential of e_1 (b) Current through the Primary Coil

Fig. 2: Numerical Solution of the Subsystems



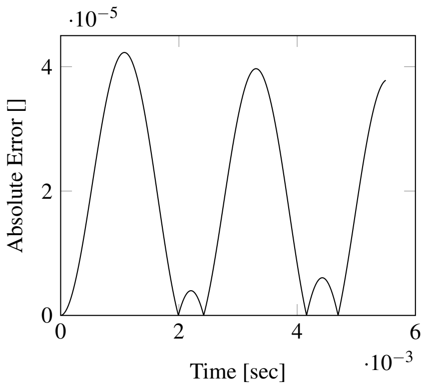
(a) Circuit Subsystem (b) Electromagnetic Subsystem

Fig. 3: Absolute Error between Multirate and Singlerate Approximation

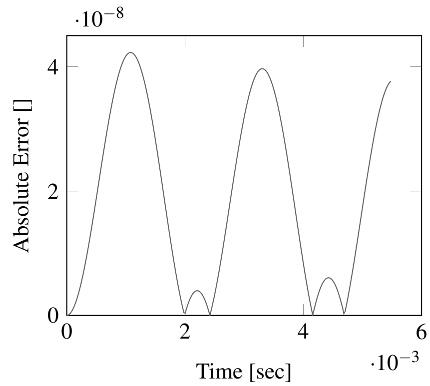
size of the error is in total small the improvement in computation time motivates and justifies the usage of multirate time integration schemes for these DAEs.

The reduced order ODE system is integrated by the same multirate method with the same integration parameters as for the full order DAE system. The simulation needed 0.20s to compute. Figure 4 shows the absolute error between both multirate approximations. The error here is very small and fits to the error bound results of [1].

Finally we integrated the coupled system with reduced ODE subsystem without multirating, so we used the same integration parameters as for the DAE reference solution. The computation time was 0.13s, so it was a bit faster than with multirating. This phenomena can be explained by the ratio between the number of fast and slow changing variables, for the full order system we have a ration of 1 : 7821 and for the reduced system 1 : 4.



(a) Circuit Subsystem



(b) Electromagnetic Subsystem

Fig. 4: Absolute Error between Full-Order DAE and Reduced ODE Subsystem

5 Conclusions

We combined two approaches for an efficient simulation of coupled circuit/field systems. By extending multirate time integration to DAE systems these schemes can be applied to a larger class of problems to reduce the computation time significantly. Model order reduction for magneto-quasistatic equations minimises the computational effort and the numerical handling is much easier since we only have to deal with a system of ODEs. Both approaches and their combination provide reliable approximations with small errors. We pointed out that the efficiency of multirate time integration schemes strongly depends on the ratio between the number of fast and slow changing variables. The combination of model order reduction and multirate time integration is advisible for systems where the dimension of the reduced order subsystem remains high compared to the dimension of the fast changing subsystem.

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References

1. Kerler-Back, J., Stykel, T.: Model order reduction for magneto-quasistatic equations. Preprint 14/2015, Universität Augsburg (2015)
2. Kværnø, A., Rentrop, P.: Low order multirate runge-kutta methods in electric circuit simulation (1999). Preprint No. 2/99
3. Savcenco, V., Hundsdorfer, W., Verwer, G.J.: A multirate time stepping strategy for stiff ordinary differential equations. BIT Numerical Mathematics **47**(1), 137–155 (2007)
4. Lamour, R., März, R., Tischendorf, C.: Differential-algebraic Equations: A Projector Based Analysis. Differential-algebraic equations forum. Springer Berlin Heidelberg (2013)
5. Brenan, K., Campbell, S., Petzold, L.: Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations. Society for Industrial and Applied Mathematics (1995)

6. Benner, P., Saak, J.: Numerical solution of large and sparse continuous time algebraic matrix Riccati and Lyapunov equations: a state of the art survey. GAMM-Mitteil. **36**(1), 32–52 (2013)
7. Simoncini, V.: Computational methods for linear matrix equations. SIAM Review **58**(3), 377–441 (2016)
8. Bartel, A., Brunk, M., Günther, M., Schöps, S.: Dynamic iteration for coupled problems of electric circuits and distributed devices. SIAM J. Sci. Comput. **35**(2), B315–B335 (2013)