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Abstract – The method of multiple scales is used to derive parabolic approximations for the nonlinear Helmholtz equation in the Kerr medium. The resulting approximation has the form of a system of iterative parabolic equations. The zero-order approximation coincides with the solution of the standard nonlinear Schrödinger equation. High-order corrections are obtained by solving the linear parabolic equations with the input terms computed from the solutions of the previous equations of the iterative system. The developed approach is applied to the modeling of the nonparaxial optical soliton propagation. It is shown that the wide-angle parabolic approximations allow to take nonparaxiality effects into account and eliminate the significant phase error produced by the nonlinear Schrödinger equation.

Introduction. – The propagation of continuous-wave laser beam in the media featuring Kerr nonlinearity is described by the scalar nonlinear Helmholtz equation (NHE) [1]. The development of various approaches to its solution is very important in many practical contexts. Very few studies report far-reaching results related to the direct numerical solution techniques for the NHE [2,3]. The iterative methods proposed in the latter works are somewhat computationally intense and therefore they can pose certain restrictions on the domain size in the applications.

For many practical purposes the NHE can be approximated by the nonlinear Schrödinger equation (NSE) [1] which is more computationally efficient. Being a one-way paraxial approximation to the NHE, the NSE completely neglects backscattering effects, and is inaccurate for the modeling of nonparaxial propagation. The neglect of nonparaxial effects often manifests is the so-called phase error that becomes considerable for the large propagation distances. Elimination of constraints posed by the paraxial approximation is a challenging task in various fields, for instance, in optical communication [10,11], endoscopic imaging using multimode lasers [12, 13], soliton-soliton scaterring [14], multiplexing of non-dispersive beams [15] and solitonic turbulence [16–19]. Non-paraxial phenomena also manifest in the modelling of intense light propagation through randomly inhomogeneous active media [20].

In the linear theory of wave propagation the nonparaxial effects are usually taken into account by the so-called wide-angle parabolic approximations [4,5]. The traditional approach [4] to the derivation of wide-angle parabolic equations (PEs) cannot be readily generalized to the case of the NHE, as it is based on the formal factorization of the Helmholtz operator and the subsequent approximation of the operator square root. Recently a new approach to the derivation of high-order parabolic approximations for the linear Helmholtz equation was proposed [5, 6]. It is based on the method of multiple scales [7] and results in an iterative system of PEs, where the solution of the n-th equation is used to compute the input term of the $n+1$-th equation.

In this work we show that the approach from [5] can be easily generalized to the nonlinear case. The resulting parabolic approximation also has the form of the system of iterative PEs. The first equation in the system coincides with the NSE (and it is natural to call it narrow-angle, or paraxial nonlinear PE), while the high-order corrections are still computed from the linear PEs (just as in linear case). The study is concluded with an example where the nonparaxial soliton solution [8,9] is compared against its high-order parabolic approximations. It is shown that our approach indeed allows to take the nonparaxial propagation effects into account and eliminate the phase error produced by the NSE. Although our approach also neglects backscattering, it can be used as a compromise between the accurate but costly direct methods of the NHE solution and very efficient NSE that does not take into account many important propagation features (especially nonparaxiality).

Helmholtz equation in the Kerr medium. – Consider the Helmholtz equation in a nonlinear Kerr-type medium:

$$\frac{\partial^2}{\partial z^2}E + \frac{\partial^2}{\partial x^2}E + k_0^2(1+\epsilon|E|^{2\sigma})E = 0\,, \tag{1}$$

where $E = E(x,z)$ denotes the electric field. Hereafter we restrict our attention to the case $\sigma = 1$ (though it will be clear that our approach can be readily used for other possible values of σ. In this section we derive an iterative system of PEs, whose solutions form a wide-angle parabolic approximation for the NHE (1). Note that the derivation here follows closely the case of the linear Helmholtz equation [5], and the only difference arises from the nonlinear term in (1).

Using the standard multiple-scale approach [7], we introduce the following slow variables (using the so-called 'parabolic scaling')

$$Z = \epsilon z, \qquad X = \epsilon^{1/2}x\,,$$

and the fast variable $\eta = (1/\epsilon)\theta(X,Z)$. Consequently, the derivatives in (1) are replaced according to the chain rule:

$$\frac{\partial}{\partial z} \to \epsilon\left(\frac{\partial}{\partial Z} + \frac{1}{\epsilon}\theta_Z\frac{\partial}{\partial \eta}\right)\,,$$
$$\frac{\partial}{\partial x} \to \epsilon^{1/2}\left(\frac{\partial}{\partial X} + \frac{1}{\epsilon}\theta_X\frac{\partial}{\partial \eta}\right)\,,$$

where the subscripts denote partial derivatives.

Next we postulate the following asymptotic expansion

$$E(x,z) = \mathcal{E}(X,Z,\eta) = \mathcal{E}_0(X,Z,\eta) + \epsilon\mathcal{E}_1(X,Z,\eta) + \epsilon^2\mathcal{E}_2(X,Z,\eta) + \dots\,. \tag{2}$$

Introducing the asymptotic expansion (2) into the equation (1) and collecting the terms of order ϵ^{-1}, ϵ^0, etc, we obtain an infinite sequence of equalities.

There is only one term $\epsilon^{-1}\theta_X$ of the order of ϵ^{-1}, and we have $\theta_X = 0$, hence $\theta = \theta(Z)$, i.e. the fast scale only depends on the range.

Terms of the order ϵ^0 are combined into the following equation

$$(\theta_Z)^2\mathcal{E}_{0\eta\eta} + k_0^2\mathcal{E}_0 = 0\,.$$

In order to satisfy this equality we simply put

$$(\theta_Z)^2 = k_0^2 \tag{3}$$

and readily obtain

$$\mathcal{E}_0 = \exp(\mathrm{i}\eta)\mathcal{A}_0(X,Z)\,.$$

At this point we choose the branch $\theta_Z = k_0$ of the solution of the Hamilton-Jacobi equation (3) and thus retain only the waves propagating in the positive direction of the X-axis (this is exactly the point where the one-way approximation in applied is our approach). From (3) we also find that

$$\theta(Z) = k_0 Z\,.$$

Note that the uniformity of the asymptotic expansion (2) is maintained if and only if

$$\mathcal{E}_j = \exp(\mathrm{i}\eta)\mathcal{A}_j(X,Z)\,, \tag{4}$$

for all $j \geq 1$ (this is a typical result of the application of the multiple-scale expansion method (see [5,7]).

Also note that in the light of representation of $\mathcal{E}_j$ in (4) we immediately rewrite the nonlinear term in (1) as

$$k_0^2\epsilon|\mathcal{E}|^2\mathcal{E} = k_0^2\epsilon\exp(\mathrm{i}\eta)(\mathcal{A}_0+\epsilon\mathcal{A}_1+\dots)^*(\mathcal{A}_0+\epsilon\mathcal{A}_1+\dots)^2\,, \tag{5}$$

where f^* denotes the complex conjugate of f.

Since the exponentials can be cancelled in (1) for the terms of all orders in ϵ, we now rewrite the nonlinear part of the expression (5) in the following way

$$k_0^2\epsilon(|\mathcal{A}_0|^2\mathcal{A}_0) + k_0^2\epsilon^2(2|\mathcal{A}_0|^2\mathcal{A}_1 + \mathcal{A}_0^2\mathcal{A}_1^*) + k_0^2\epsilon^3(2|\mathcal{A}_0|^2\mathcal{A}_2 + \mathcal{A}_0^2\mathcal{A}_2^* + 2\mathcal{A}_0|\mathcal{A}_1|^2 + \mathcal{A}_0^*\mathcal{A}_1^2) + \dots\,. \tag{6}$$

Now we proceed to the terms of the positive orders in ϵ. Collecting the terms containing ϵ^1 we arrive at the following equality

$$2\mathrm{i}k_0\mathcal{A}_{0Z} + \mathcal{A}_{0XX} + k_0^2\mathcal{A}_0|\mathcal{A}_0|^2 = 0\,. \tag{7}$$

Equation (7) is basically the nonlinear (cubic) Schrödinger equation (NSE). The standard derivation of (7) that could be found in many works, e.g. [2,3]. In our view, the derivation presented here is somewhat more clear.

An important advantage of our multi-scale approach is the possibility to derive the higher order corrections to the NSE (7). These corrections may probably both improve the handling of the nonlinear effects and also describe more accurately wide-angle propagation. The latter capability was already validated in [5] for the linear Helmholtz equation. Collecting the terms of the order ϵ^{s+1} we obtain an equation for A_s:

$$2\mathrm{i}k_0\mathcal{A}_{s,Z} + \mathcal{A}_{s,XX} + k_0^2\left(\sum_{l+n+m=s}\mathcal{A}_l\mathcal{A}_n\mathcal{A}_m^*\right) + \mathcal{A}_{s-1,ZZ} = 0\,. \tag{8}$$

For each $s \geq 1$ equation (8) is a generalization of the linear Schrödinger equation with an input term $\mathcal{A}_{s-1,ZZ}$ that is computed from the solution of previous equation. Note that the coefficients of (8) also contain $\mathcal{A}_{s-1}, \mathcal{A}_{s-2}, \dots$.

Approximate solution of the NHE (1). – Note that the equations (7), (8) are written in slow variables Z, X. Switching back to the physical variables, and recalling that $\eta = \epsilon^{-1}\theta(Z) = \epsilon^{-1}k_0 Z = k_0 z$, we find that the solution of (1) can be approximated by the truncated series of N terms

$$E(x,z) \sim \exp(\mathrm{i}k_0 z)\sum_{j=0}^{N} A_j(x,z)\,, \tag{9}$$

where A_j satisfy the equations

$$\begin{aligned}
&2\mathrm{i}k_0 A_{0z} + A_{0xx} + \epsilon k_0^2 |A_0|^2 A_0 = 0\,,\\
&2\mathrm{i}k_0 A_{1z} + A_{1xx} + \epsilon k_0^2 \left(2|A_0|^2 A_1 + A_0^2 A_1^*\right) + A_{0zz} = 0\,,\\
&2\mathrm{i}k_0 A_{2z} + A_{2xx} + \epsilon k_0^2 \left(2|A_0|^2 A_2 + A_0^2 A_2^*\right) +\\
&\qquad\qquad \epsilon k_0^2 \left(2|A_1|^2 A_0 + A_1^2 A_0^*\right) + A_{1zz} = 0\,,\\
&\qquad \dots\\
&2\mathrm{i}k_0 A_{s,z} + A_{s,xx} + \epsilon k_0^2 \left(2|A_0|^2 A_s + A_0^2 A_s^*\right) +\\
&\qquad\qquad \epsilon k_0^2 \left(\sum_{\substack{l+n+m=s,\\ l,n,m<s}} A_l A_n A_m^*\right) + A_{s-1,zz} = 0\,,\\
&\qquad \dots\,.
\end{aligned} \tag{10}$$

Equations (10) can be solved one by one to obtain the approximation for the solution to (1), and this enables us to call (10) an *iterative system* of equations. Hereafter the right hand side of (9) is called N-th order (wide-angle) parabolic approximation for the solution of the NHE (1). We use the term *iterative parabolic approximation* to emphasize the difference between our approach and classical wide-angle linear PE theory.

An example: non-paraxial soliton propagation. – One of very few exact solutions of NHE was proposed in [8, 9]. It describes a non-paraxial soliton propagating in the Kerr media. The solution [8] is described by its envelope function $A(x,z) = E(x,z)\exp(-\mathrm{i}k_0 z)$, which writes as

$$\begin{aligned}
A(x,z) = \frac{\sigma}{k_0}\sqrt{\frac{2}{\epsilon}}\mathrm{sech}\left(\frac{\sigma k_0 x + \sigma v z}{\sqrt{k_0^2+v^2}}\right) \times\\
\exp\left(-\mathrm{i}vx\sqrt{\frac{k_0^2+\sigma^2}{k_0^2+v^2}} + \mathrm{i}k_0 z\left(-1+\sqrt{\frac{k_0^2+\sigma^2}{k_0^2+v^2}}\right)\right)\,.
\end{aligned} \tag{11}$$

In fact the equation (11) contains two independent parameters v and σ. In the notation from [8] they are expressed as

$$\sigma = \sqrt{2}\frac{\eta}{w_0}\,,\qquad v = \sqrt{2}\frac{V}{w_0}\,,$$

where V is the transverse velocity of the soliton, η is the amplitude parameter, and w_0 is related to the width of the nonparaxial soliton [8]. Note that the expression $E(x,y) = A(x,z)\exp(\mathrm{i}kz)$ satisfies the NHE (1) exactly. We now want to reproduce the solution (11) using the iterative PE system (10). Note that for the small paraxial parameter

$$\kappa = \frac{1}{4\pi^2 n_0^2}\left(\frac{\lambda}{w_0}\right)^2$$

(i.e. when $\kappa \to 0$, and the wavelength is much smaller than the soliton width w_0) the solution (11) turns into the solution of the NSE that coincides with the equation for A_0:

$$2\mathrm{i}k_0 A_{0z} + A_{0xx} + \epsilon k_0^2 |A_0|^2 A_0 = 0\,.$$

The latter solution writes as

$$\begin{aligned}
A_0(x,z) = \frac{\sigma}{k_0}\sqrt{\frac{2}{\epsilon}}\mathrm{sech}\left(\sigma x + \frac{\sigma v}{k_0}z\right) \times\\
\exp\left(-\mathrm{i}vx + \frac{\mathrm{i}}{2k_0}\left(\sigma^2 - v^2\right)z\right)\,.
\end{aligned} \tag{12}$$

It is easy to see the difference between the nonparaxial soliton solution (11) and its paraxial approximation (12). Even this relatively simple example clearly demonstrates that the accuracy of the NSE approximation (7) can be insufficient. By contrast, the use of iterative PEs (8) results in an approximation of substantially higher quality.

For simplicity we consider the propagation of the nonparaxial soliton with zero transverse velocity $V = 0$ using the exact solution (11) of the NHE as a reference. We solve the Cauchy problems for the equations (8) on the unbounded domain $\Omega = \{(x,z) \in \mathbb{R}\times\mathbb{R}^+\}$. The initial condition at $z = 0$ for the equation (7) is obtained from the formula (11):

$$A_0|_{z=0} = A(x,0) = \frac{\sigma}{k_0}\sqrt{\frac{2}{\epsilon}}\mathrm{sech}\left(\sigma x\right)\,, \tag{13}$$

while zero Cauchy initial data $A_s|_{z=0} = 0$ at $z = 0$ for $s > 0$ is supplied to the high-order equations (8). Obviously, the solution of the Cauchy problem for (7) with the initial conditions (13) will therefore coincide with (12). Now our goal is to resolve the higher-order equations in (10). In this particular case this can be accomplished analytically. We introduce the following ansatz into the equations for $A_1, A_2, \dots$:

$$A_s(x,z) = P_s(z)\mathrm{sech}(\sigma x)\exp\left(\frac{\mathrm{i}\sigma^2}{2k_0}z\right)\,. \tag{14}$$

After some lengthy computation we find that the functions $P_s(z)$ are merely polynomials which can be obtained from a simple ordinary differential equation

$$\frac{dP_s}{dz} = \frac{\mathrm{i}}{2k}\frac{d^2 P_{s-1}}{dz^2} - \frac{\sigma^2}{2k^2}\frac{dP_{s-1}}{dz} - \frac{\mathrm{i}\sigma^4}{8k^3}P_{s-1}\,,\quad z>0\,, \tag{15}$$

subject to the obvious initial condition $P_s(0) = 0$ at $z = 0$. The first few polynomials computed using (15) are presented here

$$P_0(z) = \sqrt{\frac{2}{\epsilon}}\frac{\sigma}{k}\,,$$
$$P_1(z) = -\sqrt{\frac{2}{\epsilon}}\frac{\mathrm{i}\sigma^5}{8k^4}z\,,$$
$$P_2(z) = -\sqrt{\frac{2}{\epsilon}}\frac{\sigma^9}{128k^7}z^2 + \sqrt{\frac{2}{\epsilon}}\frac{\mathrm{i}\sigma^7}{16k^6}z\,,$$
$$P_3(z) = \sqrt{\frac{2}{\epsilon}}\frac{\mathrm{i}\sigma^{13}}{3072k^{10}}z^3 + \sqrt{\frac{2}{\epsilon}}\frac{\sigma^{11}}{128k^9}z^2 - \sqrt{\frac{2}{\epsilon}}\frac{5\mathrm{i}\sigma^9}{128k^8}z^3\,. \quad (16)$$

Here we consider the evolution of the soliton with the transverse velocity $V = 0$, energy parameter $\eta = 1$, and width $w_0 = 1.005 \cdot 10^{-6}$ m, the light wavelength in vacuum is $\lambda = 1\,\mu$m. The soliton is evolving in the Kerr medium with refractive index $n_0 = 2.0$, and Kerr nonlinearity parameter $n_2 = 0.2 \cdot 10^{-16}$ m^2/W (in our notation $\epsilon = 2n_2/n_0$). We compute the solution for $z \in [0, 6 \cdot 10^{-3}$ m]. The comparison of the solution of the NHE (1) and the respective parabolic approximations (9) is presented in Fig. 1. Note that the improvement due to the high-order terms A_s is noticeable even for $z \in [1$ mm$, 1.5$ mm] (see top subplot in Fig. 1), and the phase error of the paraxial solution A_0 in clearly visible. This phase error grows further for $z \in [3$ mm$, 3.5$ mm] (see middle subplot in Fig. 1), and we have to include more terms of the iterative parabolic approximation in order to obtain more accurate solution amplitude at greater ranges. For $z \in [5$ mm$, 5.5$ mm] (see bottom subplot in Fig. 1) we find that the third-order iterative parabolic approximation is still highly accurate, while lower-order approximations fail to reproduce either correct phase or amplitude. Overall, the advantage of the high-order parabolic approximations (9),(8) over the standard paraxial equation (7) is clearly demonstrated in our example.

Conclusion. – In this study we presented the construction of the high-order parabolic approximations for the NHE. On one hand, the derived system of the iterative PEs is a direct generalization of the parabolic approximations obtained in the previous work for the linear Helmholtz equation [5]. On the other hand, our theory provides corrections to the NSE that allow to take the nonparaxiality effects into account and eliminate the phase errors produced by the NSE. There are still many questions related to the parabolic approximations proposed here, that should be resolved. In particular, it is necessary to develop efficient numerical methods, derive some artificial boundary conditions (e.g. a generalization of the boundary conditions from [6]) and obtain some energy estimates similar to the asymptotic energy flux conservation theorem [5]. We also stress that our approach does not compete with the direct numerical methods for the NHE solution, e.g. [2, 3], as it is not capable of handling the backscattering. Yet it presents a more computationally efficient alternative which can be useful for applications, where the nonparaxial propagation is dominant.

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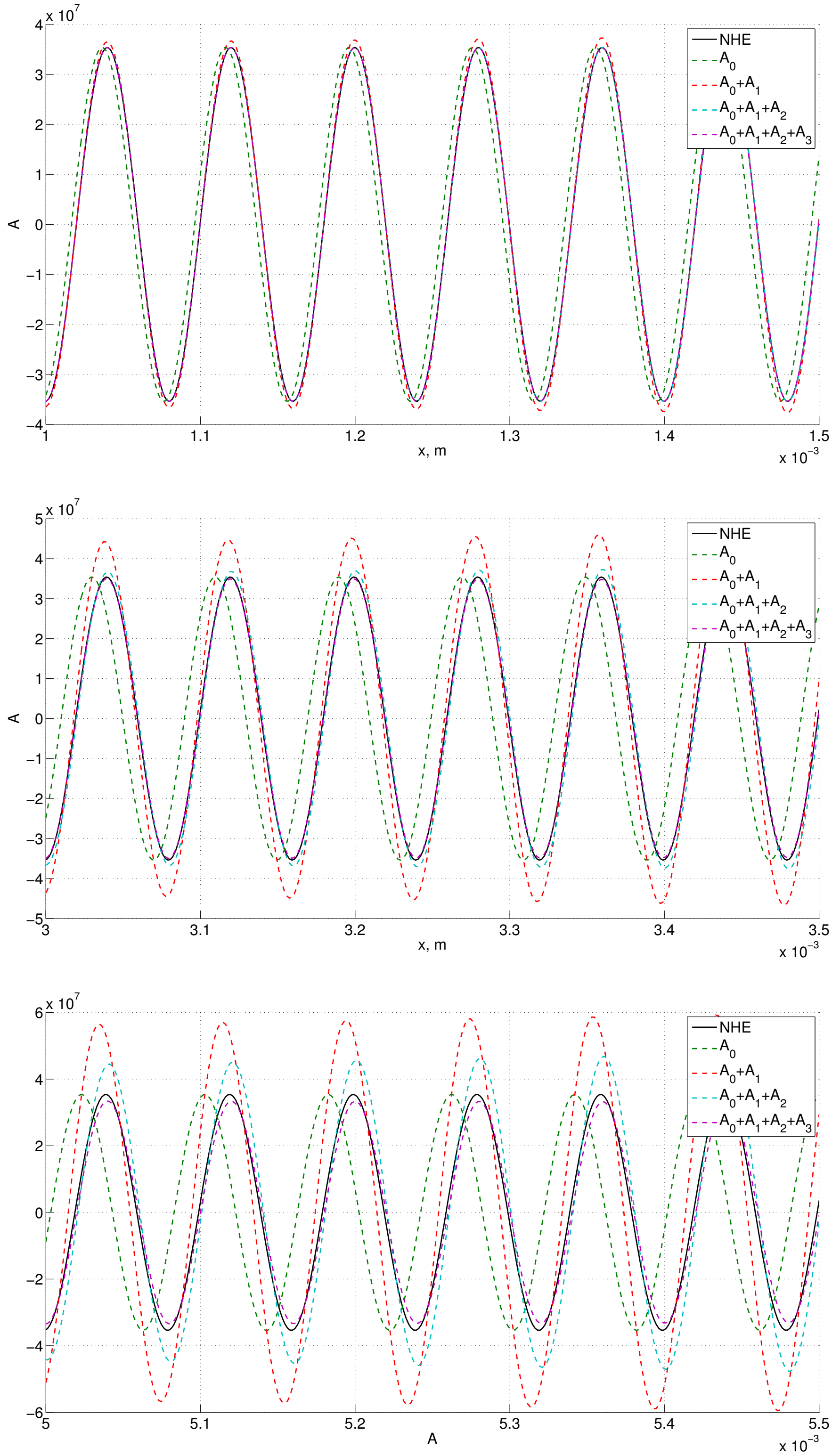

Fig. 1: The real part of the exact solution $A(x, z)$ of the NHE (solid line) and the parabolic approximations of the order 0 to 4 (dashed lines) for different intervals of z: $z \in [1\text{ mm}, 1.5\text{ mm}]$ (top subplot), $z \in [3\text{ mm}, 3.5\text{ mm}]$ (middle subplot), $z \in [5\text{ mm}, 5.5\text{ mm}]$ (bottom subplot) .