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Long Teng, Matthias Ehrhardt, Michael Günther

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L. TENG, M. EHRHARDT, M. GÜNTHER

Lehrstuhl für Angewandte Mathematik und Numerische Analysis,
Fakultät für Mathematik und Naturwissenschaften,
Bergische Universität Wuppertal, Gaußstr. 20, 42119 Wuppertal, Germany,
{teng, ehrhardt, guenther}@math.uni-wuppertal.de

Abstract

Correlation plays an important role in pricing multi-asset option. In this work we incorporate stochastic correlation into pricing Quanto options which is one special and important type of multi-asset option. Motivated by the market observations that the correlations between financial quantities behave like a stochastic process, instead of using a constant correlation, we allow the asset price process and the exchange rate process to be stochastically correlated with stochastic correlation driven by the Ornstein-Uhlenbeck process and the bounded Jacobi process, respectively. We derive an exact Quanto option pricing formula in the stochastic correlation model of the Ornstein-Uhlenbeck process and a highly accurate approximated pricing formula in the stochastic correlation model of the bounded Jacobi process, where correlation risk has been hedged. The comparison between prices using our pricing formula and the Monte Carlo method are provided. We show that the exogenously incorporated stochastic correlation improves the generation of Quanto implied volatility, although keeping constant volatilities for the underlying asset and the exchange rate.

Keywords *Multi-asset option, Quanto Option, Quanto Adjustment, Correlation risk, Stochastic Correlation process, Ornstein-Uhlenbeck process, Bounded Jacobi process, Characteristic function*

1 Introduction

The Quanto option is a cross-currency contract which has a payoff defined with respect to an underlying in one country, but the payoff is converted to another currency for payment. In order to provide investors the possibility not to take a currency conversion risk, Quanto options are settled at a fixed exchange rate which is agreed upon at inception of the contract, e.g., using the exchange rate today. For example, an investor trades a Quanto option on an asset quoted in foreign currency, at maturity, the option's value in foreign currency will be converted at a fixed rate into his home currency (domestic currency).

It is well known that the correlation between the underlying process and the exchange rate process plays a very important role in pricing. We use S_t and R_t to denote the underlying asset price denominated in a foreign currency and the exchange rate between the foreign and domestic currencies, respectively, which are assumed to follow the two-dimensional geometric Brownian motion

$$\begin{cases} dS_t = \mu_s S_t \, dt + \sigma_s S_t \, dW_t^s, \\ dR_t = \mu_r R_t \, dt + \sigma_r R_t \, dW_t^r, \end{cases} \tag{1.1}$$

where the Brownian motions (BMs) W_t^s and W_t^r are correlated by the constant correlation coefficient ρ_{sr}. In [16, 26] it has been shown that the Quanto option price based on the model (1.1) can be given by the extended Black-Scholes formula with a special dividend yield of

$$r_h - r_f + \sigma_s \sigma_r \rho_{sr}, \tag{1.2}$$

where r_h and r_f denote the domestic and foreign interest rate, respectively. This also indicates that pricing Quanto options with the Black-Scholes formula [3] only need to replace the domestic forward F by the quanto adjusted forward

$$F_T^{\text{Quanto}} = F_T \mathrm{e}^{T\sigma_s \sigma_r \rho_{sr}}. \tag{1.3}$$

However, the application of a constant correlation in the model (1.1), and thus in (1.2) and (1.3), is not realistic and might lead to correlation risk, since correlation in the market can not be a constant and should be modelled by a stochastic process like some other quantities, e.g., stochastic volatility or stochastic interest rate. This has been indicated in a couple of papers, e.g., [18, 22, 20, 24]. For this reason, in [22, 20, 24] the Quanto option pricing in stochastic correlation models of different types has been investigated. However, one has to use the Monte-Carlo method to compute the price, which is computationally expensive. A discussion about Quanto option pricing with stochastic volatility and stochastic correlation model of the bounded Jacobi process is provided [14], whereas the computation is rather expensive.

Furthermore, the fact that as a simple forward correlation the quanto adjustment (1.3) can not represent implied volatility skew for the underlying asset and for the exchange rate. For this reason, several models for pricing quanto derivatives with local or stochastic volatility of different types have been proposed, e.g., [8, 12, 13, 17]. Another way for that problem could be to allow a stochastic correlation, as in [1, 15]. Multi-factor models for pricing Quanto option based on a Wishart process [2] are introduced which can capture stochastic correlation between the asset process and the exchange rate process.

In this work, we extend the model (1.1) by imposing a stochastic correlation given by the Ornstein-Uhlenbeck (OU) and the bounded Jacobi (BJ) processes. By applying the OU process which is an affine process, we can derive an exact Quanto option pricing formula of the extended model. By employing the BJ process we have to approximate non-affine terms to bring this extended models in the class of affine diffusion (AD) processes so that the pricing formula can be found in a closed-form. By comparison with the benchmark prices using the Monte-Carlo method we justify the derived pricing formula which takes into account stochastic correlation. Moreover, to recognize the role of stochastic correlation on Quanto pricing, we also compare both Quanto implied volatilities for the model (1.1) and for our models. We find, although the volatilities σ_s and σ_r are still constant in our extended models, that, the embedded stochastic correlation has a significant improvement in generating Quanto implied volatility.

The remainder of the paper is organized as follows. The next section specializes how to impose generally a exogenous stochastic correlation process for pricing Quanto option. In Section 3, in the case of using stochastic correlation driven by the OU process we derive an exact Quanto option pricing formula, as well as a highly accurate approximated Quanto option pricing formula using the BJ process for stochastic correlation. We justify in Section 4 our pricing formulae by comparing prices using the formulae and the Monte Carlo method. Section 5 is devoted to an example of Quanto implied volatility which shows the benefits of imposing a stochastic correlation for Quanto pricing. Finally, Section 6 concludes this work.

2 Quanto pricing with stochastic correlation

We extend the model (1.1) by imposing a stochastic correlation process

$$\begin{cases} dS_t = \mu_s S_t \, dt + \sigma_s S_t \, dW_t^s, \\ dR_t = \mu_r R_t \, dt + \sigma_r R_t \, dW_t^r, \\ d\rho_t = a(t,\rho_t)dt + b(t,\rho_t)dW_t^\rho \qquad \rho_0 \in [-1,1]. \end{cases} \tag{2.1}$$

Different to the simple model (1.1), the BMs in (2.1) have following relationships

$$dW_t^s dW_t^r = \rho_t \, dt, \quad dW_t^s dW_t^\rho = \rho_{s\rho} \, dt, \quad dW_t^r dW_t^\rho = \rho_{r\rho} \, dt, \tag{2.2}$$

i.e. the underlying asset process and the exchange rate process are assumed to be correlated stochastically, driven by the correlation process ρ_t which is by itself correlated with the underlying asset process by $\rho_{s\rho}$ and with the exchange rate by $\rho_{r\rho}$, respectively.

By the argument of change of measure and using the Girsanov theorem, the extended model (2.1) can be specified under the risk-neutral measure as

$$\begin{cases} dS_t = (r_f - \sigma_s \sigma_r \rho_t) S_t \, dt + \sigma_s S_t \, d\tilde{W}_t^s, \\ dR_t = (r_h - r_f) R_t \, dt + \sigma_r R_t \, d\tilde{W}_t^r, \\ d\rho_t = \tilde{a}(t, \rho_t) dt + b(t, \rho_t) d\tilde{W}_t^\rho, \qquad \rho_0 \in [-1, 1], \end{cases} \tag{2.3}$$

with

$$d\tilde{W}_t^x d\tilde{W}_t^r = \rho_t \, dt, \quad d\tilde{W}_t^x d\tilde{W}_t^\rho = \rho_{x\rho} \, dt, \quad d\tilde{W}_t^r d\tilde{W}_t^\rho = \rho_{r\rho} \, dt, \tag{2.4}$$

and $\tilde{a}(t, \rho_t) = a(t, \rho_t) - \lambda(S, R, \rho)$, where $\lambda(S_t, R_t, \rho_t, t)$ represents the price of the correlation risk and could be assumed to be constant. Furthermore, under the log-transform for the underlying asset and the exchange rate, i.e. $x_t = \ln(S_t)$ and $y_t = \ln(R_t)$ the model is represented by

$$\begin{cases} dx_t = (r_f - \frac{1}{2}\sigma_s^2 - \sigma_s \sigma_r \rho_t) \, dt + \sigma_s \, d\tilde{W}_t^s, \\ dy_t = (r_h - r_f - \frac{1}{2}\sigma_r^2) \, dt + \sigma_r \, d\tilde{W}_t^r, \\ d\rho_t = \tilde{a}(t, \rho_t) dt + b(t, \rho_t) d\tilde{W}_t^\rho, \qquad \rho_0 \in [-1, 1]. \end{cases} \tag{2.5}$$

We know that the underlying asset S is denominated in the foreign currency (denoted by F). Let the exchange rate R be the number of units of the domestic or home currency (denoted by H) per unit of F, namely $R = H/F$. Let $U(\ln(S_t), \ln(R_t), \rho_t, t)$ denote the value of any contract with the underlying asset in F but paid in H, obviously, based on (2.5), U must satisfy the partial differential equation (PDE)

$$\begin{aligned} &\frac{\partial U}{\partial t} + (r_f - \frac{\sigma_s^2}{2} - \sigma_s \sigma_r \rho_t) \frac{\partial U}{\partial x} + (r_h - r_f - \frac{\sigma_r^2}{2}) \frac{\partial U}{\partial r} + \tilde{a}(t, \rho_t) \frac{\partial U}{\partial \rho} + \frac{\sigma_s^2}{2} \frac{\partial^2 U}{\partial x^2} + \frac{\sigma_r^2}{2} \frac{\partial^2 U}{\partial r^2} \\ &+ \frac{\tilde{b}^2(t, \rho_t)}{2} \frac{\partial^2 U}{\partial \rho^2} + \sigma_s \sigma_r \rho_t \frac{\partial^2 U}{\partial x \partial r} + \sigma_s \tilde{b}(t, \rho_t) \rho_{x\rho} \frac{\partial^2 U}{\partial x \partial \rho} + \sigma_r \tilde{b}(t, \rho_t) \rho_{r\rho} \frac{\partial^2 U}{\partial r \partial \rho} - r_h U = 0. \end{aligned} \tag{2.6}$$

We denote the value of a standard Quanto option by $V(S_t, R_t, \rho_t, t)$ which yields

$$V(S_t, R_t, \rho_t, t) = R_0 \cdot \mathbb{E}^H[\alpha(S_t - K)^+] \tag{2.7}$$

with the terminal condition $R_0 \cdot (\alpha(S_T - K)^+)$, where R_0 is the fixed exchange rate for the payment, e.g., one can take the today's rate, $\mathbb{E}^H[\cdot]$ is the expectation under domestic risk-neutral probability measure and $\alpha = 1$ for Quanto calls and $\alpha = -1$ for Quanto puts. Obviously, as a contract with the underlying asset in a foreign currency but paid in a domestic currency, the value of Quanto option (2.7) must satisfy the pricing PDE (2.6). As an example of Quanto pricing we consider Quanto calls and without loss of generality we assume $R_0 = 1$, we have thus

$$C(S_t, R_t, \rho_t, t) = \mathbb{E}^H[(S_t - K)^+], \tag{2.8}$$

the prices of Quanto puts can be determined straightforwardly from the put-call parity.

By analogy with Quanto pricing formula in [22], we assume a solution of (2.8) has the form

$$\begin{aligned} C(S_t, R_t, \rho_t, t) =& \mathrm{e}^{x_t + (r_f - r_h)\tau}\mathbb{E}[\mathrm{e}^{-\sigma_s\sigma_r \int_t^T \rho_s ds}] P_1(x_T \geq \ln(K)) \\ & - \mathrm{e}^{-r_h\tau} K P_2(x_T \geq \ln(K)) \end{aligned} \tag{2.9}$$

with the time to maturity $\tau = T - t$. Due to the embedded stochastic correlation process, the probabilities P_1 and P_2 are not immediately available in closed form. However, we know that not only P_j, but also their corresponding characteristic functions $\phi_j(x, r, \rho, t; u) = E[\mathrm{e}^{iux_T}|\mathcal{F}_t]$ satisfy the PDE (2.6) subject to the terminal condition

$$\phi_j(x, r, \rho, T; u) = \mathrm{e}^{iux_T}, \quad j = 1, 2, \tag{2.10}$$

see, e.g., [11]. Once we can obtain the characteristic functions $\phi_j(x, r, \rho, t; u)$ by solving the PDE (2.6), from which we can easily and efficiently calculate the probabilities P_j by applying e.g., Fourier techniques [4, 10]. To calculate the term $\mathbb{E}[\mathrm{e}^{-\sigma_s\sigma_r \int_t^T \rho_s ds}]$, we can derive the characteristic function $\mathbb{E}[\mathrm{e}^{iu\mathcal{R}_\tau}]$ of the integrated correlation process $\mathcal{R}_\tau := \int_t^T \rho_s ds$ under risk-neutral probability measure and then set $u = i\sigma_s\sigma_r$.

3 Stochastic correlation processes

In this section, we apply an OU and a BJ process to model the stochastic correlation process $d\rho_t$ in (2.5). We derive the closed-form solution of (2.9) in the stochastic correlation model of the OU process and find its highly accurate approximated solution in a closed form in stochastic correlation model of the BJ process.

3.1 The Ornstein-Uhlenbeck process

Although using an OU process for modelling stochastic correlation has a great drawback that the process is not bounded, this is to say the generated correlations for a small value of κ_ρ and a large value of σ_ρ can be out of the correlation interval $[-1, 1]$, however, due to its analytical tractability we would like to consider an OU process [23] to be a stochastic correlation process which is defined by

$$d\rho_t = \kappa_\rho(\mu_\rho - \rho_t)\, dt + \sigma_\rho\, d\tilde{W}_t^\rho, \tag{3.1}$$

where κ_ρ and σ_ρ are positive, ρ_0, $\mu \in [-1, 1]$. In order to circumvent the above mentioned drawback we could choose a relative large value of κ_ρ and a small value of σ_ρ. It has been indicated by Teng et al. [21] that $P(\rho < 1) = 1$ is valid if and only if

$$\frac{\sqrt{\kappa_\rho}(1 - \mu_\rho)}{\sigma_\rho} \to \infty. \tag{3.2}$$

Analogously, for $P(\rho > -1) = 1$ one obtains

$$\frac{\sqrt{\kappa_\rho}(-1 - \mu_\rho)}{\sigma_\rho} \to -\infty. \tag{3.3}$$

Theoretically, if one limits the mean value μ_ρ to be in $(-1, 1)$, from (3.2) and (3.3) one can conclude that the OU process is bounded in the interval with the condition $\frac{\sqrt{\kappa_\rho}}{\sigma_\rho} \to \infty$. Indeed, in practice, the condition $\frac{\sqrt{\kappa_\rho}}{\sigma_\rho} \geq 3$ is already enough to ensure the generated correlates lying always in $(-1, 1)$.

We can let the functions $\tilde{a}(t, \rho_t)$ and $\tilde{b}(t, \rho_t)$ in (2.5) be $\kappa_\rho(\mu_\rho - \rho_t)$ and σ_ρ, respectively, Thus, the pricing PDE (2.6) becomes

$$\begin{aligned}
&\frac{\partial U}{\partial t} + (r_f - \frac{\sigma_s^2}{2} - \sigma_s\sigma_r\rho_t)\frac{\partial U}{\partial x} + (r_h - r_f - \frac{\sigma_r^2}{2})\frac{\partial U}{\partial r} + \kappa_\rho(\mu_\rho - \rho_t)\frac{\partial U}{\partial \rho} + \frac{\sigma_s^2}{2}\frac{\partial^2 U}{\partial x^2} \\
&+ \frac{\sigma_r^2}{2}\frac{\partial^2 U}{\partial r^2} + \frac{\sigma_\rho^2}{2}\frac{\partial^2 U}{\partial \rho^2} + \sigma_s\sigma_r\rho_t\frac{\partial^2 U}{\partial x \partial r} + \sigma_s\sigma_\rho\rho_{x\rho}\frac{\partial^2 U}{\partial x \partial \rho} + \sigma_r\sigma_\rho\rho_{r\rho}\frac{\partial^2 U}{\partial r \partial \rho} - r_h U = 0.
\end{aligned} \tag{3.4}$$

By substituting (2.9) into the PDE (3.4) we obtain

$$\frac{\partial P_1}{\partial t} + (r_f + \frac{\sigma_s^2}{2} - \sigma_s\sigma_r\rho_t)\frac{\partial P_1}{\partial x} + (r_h - r_f - \frac{\sigma_r^2}{2} + \sigma_s\sigma_r\rho_t)\frac{\partial P_1}{\partial r}$$
$$+ (\kappa_\rho\mu_\rho - \kappa_\rho\rho_t + \sigma_s\sigma_\rho\rho_{x\rho})\frac{\partial P_1}{\partial \rho} + \frac{\sigma_s^2}{2}\frac{\partial^2 P_1}{\partial x^2} + \frac{\sigma_r^2}{2}\frac{\partial^2 P_1}{\partial r^2} + \frac{\sigma_\rho^2}{2}\frac{\partial^2 P_1}{\partial \rho^2} \tag{3.5}$$
$$+ \sigma_s\sigma_r\rho_t\frac{\partial^2 P_1}{\partial x\partial r} + \sigma_s\sigma_\rho\rho_{x\rho}\frac{\partial^2 P_1}{\partial x\partial \rho} + \sigma_r\sigma_\rho\rho_{r\rho}\frac{\partial^2 P_1}{\partial r\partial \rho} = 0.$$

and

$$\frac{\partial P_2}{\partial t} + (r_f - \frac{\sigma_s^2}{2} - \sigma_s\sigma_r\rho_t)\frac{\partial P_2}{\partial x} + (r_h - r_f - \frac{\sigma_r^2}{2})\frac{\partial P_2}{\partial r} + \kappa_\rho(\mu_\rho - \rho_t)\frac{\partial P_2}{\partial \rho} + \frac{\sigma_s^2}{2}\frac{\partial^2 P_2}{\partial x^2}$$
$$+ \frac{\sigma_r^2}{2}\frac{\partial^2 P_2}{\partial r^2} + \frac{\sigma_\rho^2}{2}\frac{\partial^2 P_2}{\partial \rho^2} + \sigma_s\sigma_r\rho_t\frac{\partial^2 P_2}{\partial x\partial r} + \sigma_s\sigma_\rho\rho_{x\rho}\frac{\partial^2 P_2}{\partial x\partial \rho} + \sigma_r\sigma_\rho\rho_{r\rho}\frac{\partial^2 P_2}{\partial r\partial \rho} = 0, \tag{3.6}$$

as indicated above that the corresponding characteristic functions $\phi_j(x, r, \rho, t; u)$ of P_j, $j = 1, 2$ must also satisfy the PDEs (3.5) and (3.6), respectively. Their solutions can be found in a closed form which are given in the following three lemmas.

Lemma 3.1. *The characteristic function of P_1 in* (2.9)*, with the correlation process driven by an OU process, reads*

$$\phi_1(x, r, \rho, t; u) = e^{D_1(\tau,u) + C_1(\tau,u)\rho_t + iux_t} \tag{3.7}$$

with

$$C_1(u, \tau) = \underbrace{\frac{iu\sigma_s\sigma_r}{\kappa_\rho}}_{:=c_1}(e^{-\kappa_\rho\tau} - 1) \tag{3.8}$$

$$D_1(u, \tau) = d_2(u)\tau - \frac{d_1(u)(e^{-\kappa_\rho\tau} - 1)}{\kappa_\rho} - \frac{\sigma_\rho^2 c_1^2(e^{-2\kappa_\rho\tau} - 1)}{4\kappa_\rho}, \tag{3.9}$$

where

$$d_1(u) = \kappa_\rho\mu_\rho c_1 + \sigma_s\sigma_\rho\rho_{x\rho}c_1(1 + iu) - \sigma_\rho^2 c_1^2, \tag{3.10}$$

$$d_2(u) = r_f iu - \kappa_\rho\mu_\rho c_1 + (\frac{\sigma_s^2 iu}{2} - \sigma_s\sigma_\rho\rho_{x\rho}c_1)(1 + iu) + \frac{\sigma_\rho^2 c_1^2}{2}. \tag{3.11}$$

Proof. According to [5, 6], we assume that the characteristic function is of the following form:

$$\phi_1(x, r, \rho, t; u) = E[\mathrm{e}^{iux_T} | \mathcal{F}_t] = \mathrm{e}^{D_1(\tau,u)+C_1(\tau,u)\rho_t+B_1(\tau,u)r_t+iux_t} \tag{3.12}$$

with terminal conditions $B_1(0, u) = 0$, $C_1(0, u) = 0$, $D_1(0, u) = 0$ and $\tau := T - t$. By substituting (3.12) into (3.5) and collecting the terms for x_t, r_t, ρ_t we obtain the ordinary differential equations (ODEs)

$$B_1'(\tau, u) = 0, \quad B(0, u) = 0, \tag{3.13}$$

$$C_1'(\tau, u) = -\sigma_s\sigma_r iu - \kappa_\rho C_1(\tau, u), \quad C(0, u) = 0, \tag{3.14}$$

$$\begin{aligned} D_1'(\tau, u) = (r_f + \frac{\sigma_s^2}{2})iu - \frac{\sigma_s^2 u^2}{2} + (\kappa_\rho\mu_\rho + \sigma_s\sigma_\rho\rho_{x\rho}(1 + iu))D_1(\tau, u) \\ + \frac{\sigma_\rho^2}{2} D_1^2(\tau, u), \quad D(0, u) = 0. \end{aligned} \tag{3.15}$$

Straightforwardly, due to the final condition $B(0, u) = 0$ in (3.13) we have $B(\tau, u) = 0$. The term of exchange rate r_t in (3.12) will thus have no contribution for the characteristic function. This is meaningful corresponding to the concerned pricing formula (2.9), in which we do not care what R_t is, since it can be hedged somehow. Furthermore, we can observe that the term $\rho_{r\rho}$ does not appear in the ODEs above. Now the equations (3.14) and (3.15) can be easily solved to yield the solution in Lemma 3.1. □

Lemma 3.2. *The characteristic function of P_2 in (2.9), with the correlation process driven by an OU process, reads*

$$\phi_2(x, r, \rho, t; u) = e^{D_2(\tau,u)+C_2(\tau,u)\rho_t+iux_t} \tag{3.16}$$

with $C_2(u, \tau) = C_1(u, \tau)$ and

$$\begin{aligned} D_2(u, \tau) =&(d_2(u) + \sigma_s\sigma_\rho\rho_{x\rho}c_1 - \sigma_s^2 iu)\tau - \frac{(d_1(u) - \sigma_s\sigma_\rho\rho_{x\rho}c_1)(e^{-\kappa_\rho\tau} - 1)}{\kappa_\rho} \\ &- \frac{\sigma_\rho^2 c_1^2(e^{-2\kappa_\rho\tau} - 1)}{4\kappa_\rho}, \end{aligned} \tag{3.17}$$

where $d_1(u)$ and $d_2(u)$ are defined in (3.10) and (3.11), respectively.

The proof of this proposition is similar to the proof of Lemma 3.1. We can invert the characteristic functions to receive the desired risk-neutral probabilities P_j in (2.9) by

$$P_j(x_T \geq \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\mathrm{e}^{-iu\ln K}\phi_j(x, r, \rho, t; u)}{iu}\right] du, \quad j = 1, 2. \tag{3.18}$$

Next we turn to the calculation of the term $\mathbb{E}[\mathrm{e}^{-\sigma_s\sigma_r\mathcal{R}_\tau}]$ in (2.9), where $\mathcal{R}_\tau := \int_t^T \rho_s ds$, $\tau = T-t$, is the integrated OU process. Since the OU process is an affine process, we can obtain $\mathbb{E}[\mathrm{e}^{-\sigma_s\sigma_r\mathcal{R}_\tau}]$ in a closed form which has been presented in the following lemma.

Lemma 3.3. *Let $\mathcal{R}_t := \int_0^t \rho_s ds$ be a integrated correlation process defined in* (3.1). *We have*

$$\mathbb{E}[e^{-\sigma_s\sigma_r\mathcal{R}_t}] = e^{-\psi(t)-\rho_0\eta(t)} \tag{3.19}$$

with

$$\eta(t) = \frac{\sigma_s\sigma_r}{\kappa_\rho}(1-e^{-\kappa_\rho t}), \tag{3.20}$$

$$\psi(t) = \frac{\sigma_s^2\sigma_r^2\sigma_\rho^2}{4\kappa_\rho^3}(e^{-2\kappa_\rho t} - 4e^{-\kappa_\rho t} + 3) + \frac{\mu_\rho\sigma_s\sigma_r}{\kappa_\rho}(e^{-\kappa_\rho t} - 1) + (\mu_\rho\sigma_s\sigma_r - \frac{\sigma_s^2\sigma_r^2\sigma_\rho^2}{2\kappa_\rho^2})t. \tag{3.21}$$

Proof. Suppose $0 \le t \le T$ und define

$$f(t,\rho_t) := \mathbb{E}[\mathrm{e}^{-\sigma_s\sigma_r\int_t^T \rho_s ds}|\mathcal{F}_t], \tag{3.22}$$

which is the same as

$$f(t,\rho_t) = \mathbb{E}[\mathrm{e}^{-\sigma_s\sigma_r\int_t^T \rho_s ds}|\rho_t] \tag{3.23}$$

due to the Markov property. Therefore, we have

$$f(t,\rho_t) = \mathrm{e}^{\sigma_s\sigma_r\int_0^t \rho_t ds}\cdot\mathbb{E}[\mathrm{e}^{-\sigma_s\sigma_r\int_0^T \rho_s ds}|\mathcal{F}_t] \tag{3.24}$$

which implies

$$\mathrm{e}^{-\sigma_s\sigma_r\int_0^t \rho_t ds}f(t,\rho_t) = \mathbb{E}[\mathrm{e}^{-\sigma_s\sigma_r\int_0^T \rho_s ds}|\mathcal{F}_t], \tag{3.25}$$

and $\mathrm{e}^{-\sigma_s\sigma_r\int_0^t \rho_s ds}f(t,\rho_t)$ is thus a martingale. Moreover, applying Itô's lemma to $\mathrm{e}^{-\sigma_s\sigma_r\int_0^t \rho_t ds}f(t,\rho_t)$ one obtains

$$\begin{aligned} d\left(\mathrm{e}^{-\sigma_s\sigma_r\int_0^t \rho_s ds}f(t,\rho_t)\right) = \mathrm{e}^{-\sigma_s\sigma_r\int_0^t \rho_s ds}&\left[\frac{\partial f}{\partial t}(t,\rho_t) - \sigma_s\sigma_r\rho_t f(t,\rho_t) + \kappa_\rho(\mu_\rho-\rho_t)\frac{\partial f}{\partial\rho_t}(t,\rho_t)\right.\\ &\left.+\frac{\sigma_\rho^2}{2}\frac{\partial^2 f}{\partial\rho_t^2}(t,\rho_t)\right]dt + \sigma_\rho\mathrm{e}^{-\sigma_s\sigma_r\int_0^t \rho_s ds}\frac{\partial f}{\partial\rho_t}(t,\rho_t)\,dW_t^\rho. \end{aligned}$$

We have seen that $\mathrm{e}^{-\sigma_s\sigma_r\int_0^t \rho_s ds}f(t,\rho_t)$ on the left-hand side is a martingale and the du integral on the right-hand side must be thus zero, therefore

$$\frac{\partial f}{\partial t}(t,\rho_t) - \sigma_s\sigma_r\rho_t f(t,\rho_t) + \kappa_\rho(\mu_\rho-\rho_t)\frac{\partial f}{\partial\rho_t}(t,\rho_t) + \frac{\sigma_\rho^2}{2}\frac{\partial^2 f}{\partial\rho_t^2}(t,\rho_t) = 0 \tag{3.26}$$

where $f(t, \rho_t)$ can be written as

$$\mathbb{E}[\mathrm{e}^{-\sigma_s\sigma_r \int_t^T \rho_s ds}]. \tag{3.27}$$

Furthermore, the process (3.1) is stationary. One thus can write

$$f(t, \rho_t) = \mathbb{E}[\mathrm{e}^{-\sigma_s\sigma_r \int_0^{T-t} \rho_s ds}], \tag{3.28}$$

where the initial value is ρ_t. Now set

$$G(T-t, \rho_t) = \mathbb{E}[\mathrm{e}^{-\sigma_s\sigma_r \int_0^{T-t} \rho_s ds}] \tag{3.29}$$

such that $G(T-t, \rho_t)$ is equal to $f(t, \rho_t)$ in (3.28) and satisfies

$$\frac{\partial G}{\partial t} = -\sigma_s\sigma_r\rho_t G + \kappa_\rho(\mu_\rho - \rho_t)\frac{\partial G}{\partial \rho_t} + \frac{\sigma_\rho^2}{2}\frac{\partial^2 G}{\partial \rho_t^2} = 0 \tag{3.30}$$

with the terminal condition $G(0, \rho_t) = 1$. Following [7, 9] we guess the solution of latter PDE has a form as

$$G(\tau, \rho_t) = \mathrm{e}^{-\psi(t) - \rho_t\eta(t)}, \tag{3.31}$$

which can be substituted into (3.30) to obtain the follow ODEs

$$-\psi'(t) = -\kappa_\rho\mu_\rho\eta(t) + \frac{\sigma_\rho^2}{2}\eta^2(t), \quad \psi(0) = 0, \tag{3.32}$$

$$-\eta'(t) = \kappa_\rho\eta(t) - \sigma_s\sigma_r, \quad \eta(0) = 0. \tag{3.33}$$

By solving the ODEs above one obtain the solutions for $\psi(t)$ and $\eta(t)$. □

Up to now we can directly compute the Quanto price (2.9) in stochastic correlation model of the OU process with (3.18) and (3.19).

3.2 The bounded Jacobi process

Due to the drawback of using the OU process for stochastic correlations, the process is not bounded to the interval $[-1, 1]$, we investigate now Quanto pricing in stochastic correlation model of the BJ process

$$d\rho_t = \kappa_\rho(\mu_\rho - \rho_t)\,dt + \sigma_\rho\sqrt{1-\rho_t^2}\,d\tilde{W}_t^\rho, \tag{3.34}$$

where κ_ρ and σ_ρ are positive, ρ_0, $\mu \in [-1,1]$. The process is bounded to $(-1,1)$ with the following restriction of the parameter range

$$\kappa_\rho > \frac{\sigma_\rho^2}{1 \pm \mu_\rho}, \tag{3.35}$$

see [20] and [24]. Compared to the OU process we only have a different diffusion coefficient $\sigma_\rho\sqrt{1-\rho_t^2}$. Similar to (2.9), we write the Quanto pricing formula in stochastic correlation model of the BJ process as

$$\begin{aligned} C(S_t, R_t, \rho_t, t) =& \mathrm{e}^{x_t + (r_f - r_h)\tau}\mathbb{E}[\mathrm{e}^{-\sigma_s\sigma_r\int_t^T \rho_s ds}]\tilde{P}_1(x_T \geq \ln(K)) \\ &- \mathrm{e}^{-r_h\tau}K\tilde{P}_2(x_T \geq \ln(K)) \end{aligned} \tag{3.36}$$

where $\tilde{P}_1$ and $\tilde{P}_2$ can be straightforwardly obtained by updating (3.5) and (3.6), and thus given by

$$\begin{aligned} &\frac{\partial \tilde{P}_1}{\partial t} + (r_f + \frac{\sigma_s^2}{2} - \sigma_s\sigma_r\rho_t)\frac{\partial \tilde{P}_1}{\partial x} + (r_h - r_f - \frac{\sigma_r^2}{2} + \sigma_s\sigma_r\rho_t)\frac{\partial \tilde{P}_1}{\partial r} \\ &\quad + (\kappa_\rho\mu_\rho - \kappa_\rho\rho_t + \sigma_s\sigma_\rho\rho_{x\rho})\frac{\partial \tilde{P}_1}{\partial \rho} + \frac{\sigma_s^2}{2}\frac{\partial^2 \tilde{P}_1}{\partial x^2} + \frac{\sigma_r^2}{2}(1 - \underline{\rho_t^2})\frac{\partial^2 \tilde{P}_1}{\partial r^2} + \frac{\sigma_\rho^2}{2}\frac{\partial^2 \tilde{P}_1}{\partial \rho^2} \\ &\quad + \sigma_s\sigma_r\rho_t\frac{\partial^2 \tilde{P}_1}{\partial x \partial r} + \sigma_s\sigma_\rho\underline{\sqrt{1-\rho_t^2}}\rho_{x\rho}\frac{\partial^2 \tilde{P}_1}{\partial x \partial \rho} + \sigma_r\sigma_\rho\underline{\sqrt{1-\rho_t^2}}\rho_{r\rho}\frac{\partial^2 \tilde{P}_1}{\partial r \partial \rho} = 0 \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} &\frac{\partial \tilde{P}_2}{\partial t} + (r_f - \frac{\sigma_s^2}{2} - \sigma_s\sigma_r\rho_t)\frac{\partial \tilde{P}_2}{\partial x} + (r_h - r_f - \frac{\sigma_r^2}{2})\frac{\partial \tilde{P}_2}{\partial r} + \kappa_\rho(\mu_\rho - \rho_t)\frac{\partial \tilde{P}_2}{\partial \rho} \\ &\quad + \frac{\sigma_s^2}{2}\frac{\partial^2 \tilde{P}_2}{\partial x^2} + \frac{\sigma_r^2}{2}\frac{\partial^2 \tilde{P}_2}{\partial r^2} + \frac{\sigma_\rho^2}{2}(1 - \underline{\rho_t^2})\frac{\partial^2 \tilde{P}_2}{\partial \rho^2} + \sigma_s\sigma_r\rho_t\frac{\partial^2 \tilde{P}_2}{\partial x \partial r} \\ &\quad + \sigma_s\sigma_\rho\underline{\sqrt{1-\rho_t^2}}\rho_{x\rho}\frac{\partial^2 \tilde{P}_2}{\partial x \partial \rho} + \sigma_r\sigma_\rho\underline{\sqrt{1-\rho_t^2}}\rho_{r\rho}\frac{\partial^2 \tilde{P}_2}{\partial r \partial \rho} = 0. \end{aligned} \tag{3.38}$$

Because of the nonlinear coefficients ρ_t^2 and $\sqrt{1-\rho_t^2}$, the corresponding characteristic functions $\tilde{\phi}_j(x, r, \rho_t, t; u)$ of $\tilde{P}_j$, $j = 1, 2$ can not be derived in a closed form. However, as indicated by Teng et al. in [21], such nonlinear coefficient could be linearized by its expectation which further can be well approximated by a linear combination of exponential functions.

$\mathbb{E}[\rho_t^2]$ can be calculated and given in a closed-form by

$$\begin{aligned}\mathbb{E}[\rho_t^2] =& \frac{1}{\sigma_\rho^4 + 3\kappa_\rho\sigma_\rho^2 + 2\kappa_\rho^2} \mathrm{e}^{-t(\sigma_\rho^3+2\kappa_\rho)}\Bigg((\sigma_\rho^4 + 3\kappa_\rho\sigma_\rho^2 + 2\kappa_\rho^2)\rho_0^2 \\ &+ 2\mu_\rho\kappa_\rho\rho_0(\sigma_\rho^2 + 2\kappa_\rho)(\mathrm{e}^{t(\sigma_\rho^2+\kappa_\rho)} - 1) + \sigma_\rho^2(\sigma_\rho^2 + \kappa_\rho)(\mathrm{e}^{t(\sigma_\rho^2+2\kappa_\rho)} - 1) \\ &- 2\mu_\rho^2\kappa_\rho\big(\kappa_\rho(2\mathrm{e}^{t(\sigma_\rho^2+\kappa_\rho)} - \mathrm{e}^{t(\sigma_\rho^2+2\kappa_\rho)} - 1) - \sigma_\rho^2\mathrm{e}^{t(\sigma_\rho^2+\kappa_\rho)}(\mathrm{e}^{t\kappa_\rho} - 1)\big)\Bigg),\end{aligned} \tag{3.39}$$

for the detailed calculation we refer to [25]. However, the equation (3.39) is rather long and thus not convenient for the further calculation. For this we apply the result in [19] or [21]:

Proposition 3.1. *Let ρ_t be a BJ process defined in* (3.34) *and denote the original solution of $\mathbb{E}[\rho_t^2]$ by $f_\rho(t)$. The solution can be approximated by*

$$e^{-mt} + be^{-nt} + a, \tag{3.40}$$

where

$$a = \frac{(\sigma_\rho^2 + \kappa_\rho)(\sigma_\rho^2 + 2\kappa_\rho\mu_\rho^2)}{\sigma_\rho^4 + 3\kappa_\rho\sigma_\rho^2 + 2\kappa_\rho^2}, \quad b = \rho_0^2 - a - 1, \tag{3.41}$$

$$m = -2\log\left(\gamma_1 - be^{-\frac{n}{2}}\right), \quad n = -2\log\left(\frac{b\gamma_1 - \sqrt{b^2\gamma_1^2 - \gamma_2\gamma_3}}{\gamma_2}\right), \tag{3.42}$$

with

$$\gamma_1 := f_\rho(0.5) - a, \quad \gamma_2 := b + b^2, \quad \gamma_3 := \gamma_1^2 + a - f_\rho(1). \tag{3.43}$$

The approximation quality has been measured in [21].

Now we consider the other term $\sqrt{1-\rho_t^2}$ which needs approximation. It has been shown in [21] that

$$g_\rho(t) := \mathbb{E}\left[\sqrt{1-\rho_t^2}\right] = \sqrt{\mathbb{E}[1-\rho_t^2] - \frac{\mathbb{E}[\rho_t]^2}{1-\mathbb{E}[\rho_t]^2}\mathbb{V}[\rho_t]} = \sqrt{1 - \frac{\mathbb{E}[\rho_t^2] - \mathbb{E}[\rho_t]^4}{1-\mathbb{E}[\rho_t]^2}} \tag{3.44}$$

with $\mathbb{E}[\rho_t] = \mu_\rho + (\rho_0 - \mu_\rho)\mathrm{e}^{-\kappa_\rho t}$.

Proposition 3.2. *Let ρ_t be a BJ process given in* (3.34), *$\mathbb{E}[\sqrt{1-\rho_t^2}]$ can be approximated by*

$$g_\rho(t) := \mathbb{E}[\sqrt{1-\rho_t^2}] \approx e^{-\hat{m}t} + \hat{b}e^{-\hat{n}t} + \hat{a}, \tag{3.45}$$

where

$$\hat{a} = \sqrt{1 - \frac{(\sigma_\rho^2 + \kappa_\rho)(\sigma_\rho^2 + 2\kappa_\rho\mu_\rho^2) - \mu_\rho^4(\sigma_\rho^4 + 3\kappa_\rho\sigma_\rho^2 + 2\kappa_\rho^2)}{(1-\mu_\rho^2)(\sigma_\rho^4 + 3\kappa_\rho\sigma_\rho^2 + 2\kappa_\rho^2)}}, \quad \hat{b} = \sqrt{1-\rho_0^2} - \hat{a} - 1 \tag{3.46}$$

$$\hat{m} = -2\log\left(\zeta_1 - \hat{b}e^{-\frac{\hat{n}}{2}}\right), \quad \hat{n} = -2\log\left(\frac{\hat{b}\zeta_1 - \sqrt{\hat{b}^2\zeta_1^2 - \zeta_2\zeta_3}}{\zeta_2}\right), \tag{3.47}$$

with

$$\zeta_1 := f_\rho(0.5) - \hat{a}, \quad \zeta_2 := \hat{b} + \hat{b}^2, \quad \zeta_3 := \zeta_1^2 + \hat{a} - f_\rho(1). \tag{3.48}$$

For the proof and measure quality of the approximation we refer to [21]. Whilst one substitutes the nonlinear coefficient $\sqrt{1-\rho^2}$ in the PDEs (3.37) and (3.38) by its approximation (3.45), the corresponding approximated characteristic functions $\tilde{\phi}_j(x, r, \rho_t, t; u)$ of $\tilde{P}_j$, $j = 1, 2$ can thus be found by solving PDEs.

Lemma 3.4. *The characteristic function of $\tilde{P}_1$ in* (3.36)*, with the correlation process driven by an BJ process, reads*

$$\tilde{\phi}_1(x, r, \rho, t; u) = e^{\tilde{D}_1(\tau,u) + \tilde{C}_1(\tau,u)\rho_t + iux_t}, \tag{3.49}$$

with $\tilde{C}_1(u,\tau) = C_1(u,\tau)$ given in (3.8) *and*

$$\begin{aligned}
\tilde{D}_1(u,\tau) =& \frac{d_1(u)(1-e^{-\kappa_\rho\tau})}{\kappa_\rho} + \frac{\sigma_\rho^2 c_1^2(1-a)(1-e^{-2\kappa_\rho\tau})}{4\kappa_\rho} + \frac{\sigma_\rho^2 c_1^2 e^{-mT+(m-\kappa_\rho)\tau}}{m-\kappa_\rho} \\
&+ \frac{\sigma_\rho^2 c_1^2 e^{-mT+(m-2\kappa_\rho)\tau}}{2(m-2\kappa_\rho)} + \frac{b\sigma_\rho^2 c_1^2 e^{-nT+(n-\kappa_\rho)\tau}}{n-\kappa_\rho} + \frac{b\sigma_\rho^2 c_1^2 e^{-nT+(n-2\kappa_\rho)\tau}}{2(n-2\kappa_\rho)} \\
&+ \frac{\sigma_s\sigma_\rho\rho_{x\rho}c_1(1+iu)e^{-\hat{m}T+(\hat{m}-\kappa_\rho)\tau}}{\hat{m}-\kappa_\rho} + \frac{\sigma_s\sigma_\rho\rho_{x\rho}c_1\hat{b}(1+iu)e^{-\hat{n}T+(\hat{n}-\kappa_\rho)\tau}}{\hat{n}-\kappa_\rho} \\
&- \frac{\sigma_\rho^2 c_1^2 e^{-(T-\tau)m}}{2m} - \frac{b\sigma_\rho^2 c_1^2 e^{-(T-\tau)n}}{2n} - \frac{\sigma_s\sigma_\rho\rho_{x\rho}c_1(1+iu)e^{-(T-\tau)\hat{m}}}{\hat{m}} \\
&- \frac{\hat{b}\sigma_s\sigma_\rho\rho_{x\rho}c_1(1+iu)e^{-(T-\tau)\hat{n}}}{\hat{n}} + \tilde{d}_2(u)\tau + \tilde{d}_3(u),
\end{aligned} \tag{3.50}$$

where

$$\tilde{d}_1(u) = \kappa_\rho \mu_\rho c_1 + \sigma_s \sigma_\rho \rho_{x\rho} \hat{a} c_1 (1 + iu) + \sigma_\rho^2 c_1^2 (a - 1), \tag{3.51}$$

$$\tilde{d}_2(u) = r_f iu - \kappa_\rho \mu_\rho c_1 + (\frac{\sigma_s^2 iu}{2} - \sigma_s \sigma_\rho \rho_{x\rho} \hat{a} c_1)(1 + iu) + \sigma_\rho^2 c_1^2 (a - 1) \tag{3.52}$$

$$\begin{aligned}\tilde{d}_3(u) = &-\frac{\sigma_\rho^2 c_1^2 e^{-mT}}{m - \kappa_\rho} + \frac{\sigma_\rho^2 c_1^2 e^{-mT}}{2(m - 2\kappa_\rho)} - \frac{b \sigma_\rho^2 c_1^2 e^{-nT}}{n - \kappa_\rho} + \frac{\sigma_\rho^2 c_1^2 e^{-nT}}{2(n - 2\kappa_\rho)} \\ &- \frac{\sigma_s \sigma_\rho \rho_{x\rho} c_1 (1 + iu) e^{-\hat{m}T}}{\hat{m} - \kappa_\rho} - \frac{\sigma_s \sigma_\rho \rho_{x\rho} c_1 \hat{b} (1 + iu) e^{-\hat{n}T}}{\hat{n} - \kappa_\rho} + \frac{\sigma_\rho^2 c_1^2 e^{-mT}}{2m} \\ &+ \frac{b \sigma_\rho^2 c_1^2 e^{-nT}}{2n} + \frac{\sigma_s \sigma_\rho \rho_{x\rho} c_1 (1 + iu) e^{-\hat{m}T}}{\hat{m}} + \frac{\hat{b} \sigma_s \sigma_\rho \rho_{x\rho} c_1 (1 + iu) e^{-\hat{n}T}}{\hat{n}}.\end{aligned} \tag{3.53}$$

The proof can be finished quite similarly as the proof of Lemma 3.1, the only differences are the coefficients in that ODEs, see (3.13)-(3.15).. Analogously, $\tilde{\phi}_2(x, r, \rho_t, t; u)$ can also be found.

Lemma 3.5. *The characteristic function of $\tilde{P}_2$ in* (3.36)*, with the correlation process driven by an BJ process, reads*

$$\tilde{\phi}_2(x, r, \rho, t; u) = e^{\tilde{D}_2(\tau, u) + \tilde{C}_2(\tau, u)\rho_t + iux_t} \tag{3.54}$$

with $\tilde{C}_2(u, \tau) = C_1(u, \tau)$ given in (3.8) *and*

$$\begin{aligned}\tilde{D}_2(u, \tau) = &\frac{(\tilde{d}_1(u) - \rho_{\rho x} \sigma_s \sigma_r \hat{a} c_1)(1 - e^{-\kappa_\rho \tau})}{\kappa_\rho} + \frac{\sigma_\rho^2 c_1^2 (1 - a)(1 - e^{-2\kappa_\rho \tau})}{4\kappa_\rho} \\ &+ \frac{\sigma_\rho^2 c_1^2 e^{-mT + (m - \kappa_\rho)\tau}}{m - \kappa_\rho} + \frac{\sigma_\rho^2 c_1^2 e^{-mT + (m - 2\kappa_\rho)\tau}}{2(m - 2\kappa_\rho)} + \frac{b \sigma_\rho^2 c_1^2 e^{-nT + (n - \kappa_\rho)\tau}}{n - \kappa_\rho} \\ &+ \frac{b \sigma_\rho^2 c_1^2 e^{-nT + (n - 2\kappa_\rho)\tau}}{2(n - 2\kappa_\rho)} + \frac{\sigma_s \sigma_\rho \rho_{x\rho} c_1 iu e^{-\hat{m}T + (\hat{m} - \kappa_\rho)\tau}}{\hat{m} - \kappa_\rho} \\ &+ \frac{\sigma_s \sigma_\rho \rho_{x\rho} c_1 \hat{b} iu e^{-\hat{n}T + (\hat{n} - \kappa_\rho)\tau}}{\hat{n} - \kappa_\rho} - \frac{\sigma_\rho^2 c_1^2 e^{-(T - \tau)m}}{2m} - \frac{b \sigma_\rho^2 c_1^2 e^{-(T - \tau)n}}{2n} \\ &- \frac{\sigma_s \sigma_\rho \rho_{x\rho} c_1 iu e^{-(T - \tau)\hat{m}}}{\hat{m}} - \frac{\hat{b} \sigma_s \sigma_\rho \rho_{x\rho} c_1 iu e^{-(T - \tau)\hat{n}}}{\hat{n}} \\ &+ (\tilde{d}_2(u) + \rho_{\rho x} \sigma_s \sigma_r \hat{a} c_1 - \sigma_s^2 iu)\tau + \tilde{d}_3(u) + \frac{\sigma_s \sigma_\rho \rho_{x\rho} c_1 e^{-\hat{m}T}}{\hat{m} - \kappa_\rho} \\ &+ \frac{\sigma_s \sigma_\rho \rho_{x\rho} c_1 \hat{b} 1 e^{-\hat{n}T}}{\hat{n} - \kappa_\rho} - \frac{\sigma_s \sigma_\rho \rho_{x\rho} c_1 e^{-\hat{m}T}}{\hat{m}} - \frac{\hat{b} \sigma_s \sigma_\rho \rho_{x\rho} c_1 e^{-\hat{n}T}}{\hat{n}},\end{aligned} \tag{3.55}$$

where $\tilde{d}_1(u), \tilde{d}_2(u)$ and $\tilde{d}_3(u)$ are defined in the last Proposition.

Clearly,

$$\tilde{P}_j(x_T \geq \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\mathrm{e}^{-iu\ln K}\tilde{\phi}_j(x,r,\rho,t;u)}{iu}\right]du, \quad j = 1,2. \tag{3.56}$$

For calculating the Quanto price in (3.36), we then only need $\mathbb{E}[\mathrm{e}^{-\sigma_s\sigma_r\mathcal{R}_\tau}]$ which is adressed in the following lemma.

Lemma 3.6. *Let $\mathcal{R}_t := \int_0^t \rho_s ds$ be a integrated correlation process defined in (3.34). We have*

$$\mathbb{E}[e^{-\sigma_s\sigma_r\mathcal{R}_t}] = e^{-\tilde{\psi}(t)-\rho_0\tilde{\eta}(t)} \tag{3.57}$$

with

$$\tilde{\eta}(t) = \frac{\sigma_s\sigma_r}{\kappa_\rho}(1-e^{-\kappa_\rho t}), \tag{3.58}$$

$$\begin{aligned}\tilde{\psi}(t) = {} & \frac{\sigma_s^2\sigma_r^2\sigma_\rho^2}{2\kappa_\rho^2}\left(\frac{2(e^{-(\kappa_\rho+m)t}-1)}{\kappa_\rho+m} + \frac{2b(e^{-(\kappa_\rho+n)t}-1)}{\kappa_\rho+n} - \frac{e^{-(2\kappa_\rho+m)t}-1}{2\kappa_\rho+m}\right.\\ & - \frac{b(e^{-(2\kappa_\rho+n)t}-1)}{2\kappa_\rho+n} + \frac{2(a-1)(e^{-\kappa_\rho t}-1)}{\kappa_\rho} - \frac{(a-1)(e^{-2\kappa_\rho t}-1)}{2\kappa_\rho}\\ & \left. -\frac{e^{-mt}-1}{m} - \frac{b(e^{-nt}-1)}{n} + (a-1)t\right) + \frac{\sigma_s\sigma_r t}{\kappa_\rho} + \frac{\sigma_s\sigma_r(e^{-\kappa_\rho t}-1)}{\kappa_\rho^2},\end{aligned} \tag{3.59}$$

where a, b, m and n have been defined in Proposition 3.1.

Following the train of thoughts in the proof of the Lemma of 3.1, similar to (3.30), one can obtain

$$\frac{\partial\tilde{G}}{\partial t} = -\sigma_s\sigma_r\rho_t\tilde{G} + \kappa_\rho(\mu_\rho - \rho_t)\frac{\partial\tilde{G}}{\partial\rho_t} + \frac{\sigma_\rho^2(1-\underline{\rho_t^2})}{2}\frac{\partial^2\tilde{G}}{\partial\rho_t^2} = 0 \tag{3.60}$$

for the case of the BJ process. The only difference to (3.30) is the nonlinear coefficient which has been underlined. We substitute the nonlinear coefficient by its approximation (3.40). Following the same way as in the proof of the Lemma 3.1, one can thus obtain ODEs which can be analytically solved to get the $\tilde{\psi}(t)$ and $\tilde{\eta}(t)$.

4 Numerical experiments

In this section we compare the Quanto option prices using the pricing formulas (2.9) and (3.36) to the prices computed by performing a Monte-Carlo simulation. For Quanto pricing using the Monte-Carlo method we refer to [22]. Furthermore, with an example of calibration to market data we show that using a stochastic correlation is helpful to create a Quanto implied volatility smile as market requires.

4.1 Comparison with Monte-Carlo Valuation

We consider Quanto calls on a foreign stock and assume $S_0 = 100$, $r_h = 0.03$, $r_f = 0.05$, $T = 5$, $\sigma_s = 0.3$, $\sigma_r = 0.4$, $\rho_{x\rho} = 0$ and $R_0 = 1$. Note that one should choose a large value of κ_ρ and a small value of σ_ρ in order to ensure that the generated correlations by the OU process lie in the interval $(-1, 1)$. For using the BJ process we only need to take care of the condition (3.35). Both correlation processes share the same parameter values. In Table 1 we report the prices using different models for different strikes. For the case of the OU process, the pricing formula (2.9) is exact. However, for the case of the BJ process a few approximations have been used, therefore, we call the price using the formula (3.36) *approximated price*. The absolute price differences to the Monte-Carlo prices are also provided for both models, where $20T$ steps and 10^5 paths are used for the Monte-Carlo simulation. Let us recall that

Model	Stoch. Corr. by the OU process			Stoch. Corr. by the BJ process		
Strike	MC price	Exact price	Diff.	MC price	Approx. price	Diff.
40	45.5769 (0.73)	46.5751	0.0019	46.5742 (0.6)	46.5740	0.0002
80	25.0777 (0.55)	25.0733	0.0040	25.0735 (0.20)	25.0731	0.0004
100	18.4768 (0.43)	18.4730	0.0038	18.4760 (0.37)	18.4730	0.0030
120	13.7334 (0.37)	13.7302	0.0032	13.7330 (0.30)	13.7304	0.0026
160	7.8226 (0.24)	7.8175	0.0051	7.8202 (0.20)	7.8178	0.0025

Table 1: The parameters of correlation process: $\rho_0 = 0$, $\kappa_\rho = 2.6$, $\mu_\rho = 0.6$, $\sigma_\rho = 0.1$, the numbers in round brackets represent the standard deviations.

the Quanto prices in the stochastic correlation model of the OU process in Table 1, computed by the model (2.9) are exact. Furthermore, by comparison to the Monte-Carlo prices we observe that the model (3.36) can also provide highly accurate results, although approximations have been used. Note that the OU and BJ process possess the same structure for the drift, if one chooses a sufficiently small value for σ_ρ, then both model prices could be identical.

4.2 Quanto implied Volatility

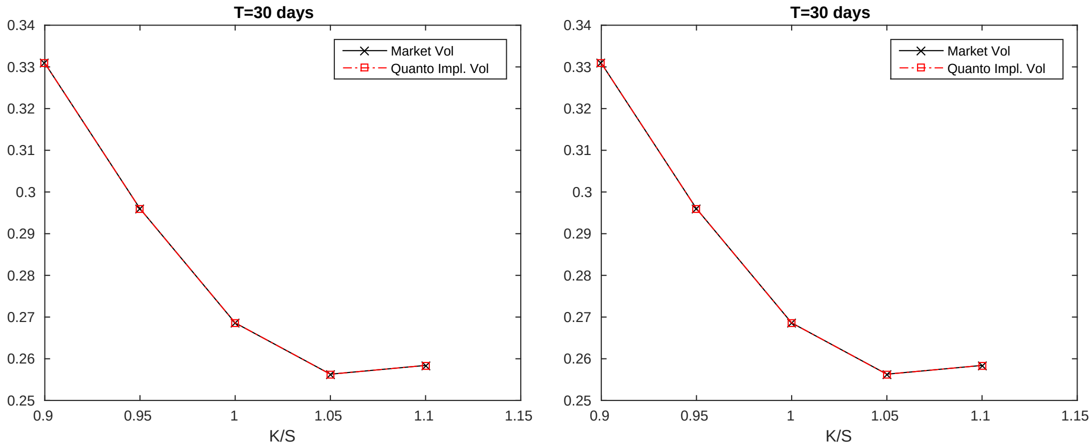
Note that the model (1.1) can not take Quanto implied volatility into account, i.e. one can not find appropriate values for the constant volatilities and correlation σ_s, σ_r, ρ_{sr} by fitting the model to the market data. In our models, both volatilities are still constant, however, a stochastic correlation has been included. Therefore, in order to check how well the incorporated stochastic correlations instead of local or stochastic volatility can help to represent market Quanto volatility, we could fit model prices to market prices only by adjusting the values of exogenously incorporated stochastic correlations. The volatilities are constant so that the obtained parameter values might be not meaningful for the market; however, our aim is only to show that exogenously embedded stochastic correlation is of great help for the Quanto implied volatility.

For the market data, we choose Put-options on the Nikk225 index on July 31, 2009, $S_0 = 10165.2, r_h = r_f = 5\%$. Since our aim is to check the benefit of incorporating stochastic correlation to the Quanto implied volatility, we set the fixed exchange rate for the payoff to be 1, so that the standard implied volatility can be seen as the Quanto implied volatility. We consider several different strikes and a short maturity $T = 30$ days which is representative for the skew and patterns observed. A comparison of implied volatilities for different models is provided in Figure 1.

5 Conclusion

In this paper we have presented how to incorporate stochastic correlations driven by appropriate stochastic processes for Quanto pricing. We have investigated to use the OU and BJ process to model correlations und derived an exact Quanto option pricing formula in stochastic correlation model of the OU process and a highly accurate approximated pricing formula in stochastic correlation model of the BJ process. The comparison of both model prices to the Monte-Carlo prices has also been given. Finally, we conducted an experiment of the calibration in order to show the benefits of the embedded stochastic correlation to the Quanto implied volatility.

However, beyond stochastic correlations, we should also consider a local or stochastic volatility for a better description of financial market phenomena. We leave this to



(a) stochastic correlation by OU process (b) stochastic correlation by BJ process

Figure 1

future work.

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