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Short rate as a sum of two CKLS-type processes

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Abstract. We study the short rate model of interest rates, in which the short rate is defined as a sum of two stochastic factors. Each of these factors is modelled by a stochastic differential equation with a linear drift and the volatility proportional to a power of the factor. We show a calibration methods which - under the assumption of constant volatilities - allows us to estimate the term structure of interest rate as well as the unobserved short rate, although we are not able to recover all the parameters. We apply it to real data and show that it can provide a better fit compared to a one-factor model. A simple simulated example suggests that the method can be also applied to estimate the short rate even if the volatilities have a general form. Therefore we propose an analytical approximation formula for bond prices in such a model and derive the order of its accuracy.

1 Introduction

A discount bond is a security which pays a unit amount of money to its holder at specified time T which is called a maturity of the bond. Its price determines the interest rate for the given maturity. Short rate interest rate models are formulated in terms of a stochastic differential equation (or a system of them in multifactor models) governing the evolution of so called short rate, which is the interest rate for infinitesimal time interval. After specification of the so called market price of risk, the bond prices can be computed as solutions to a parabolic partial differential equation. Alternatively they can be formulate in the equivalent, risk neutral measure, which is sufficient to formulate the partial differential equation problem without any additional input. For more details on short rate models see, e.g. [6], [1].

There are many different specifications of the short rate dynamics available in the literature. A popular model, because of its tractability, is Vasicek model [10], where the short rate follows a mean reversion process $dr = \kappa(\theta - r)dt + \sigma dw$, where w is a Wiener process and $\kappa, \theta, \sigma > 0$ are constants. Its generalization with nonconstant volatility has been proposed in [2] in the form $dr = \kappa(\theta - r)dt + \sigma r^\gamma$ with additional parameter $\gamma > 0$, which we will refer to as a CKLS model. In addition to Vasicek model, it encompasses also other known models as special cases (we particularly note Cox-Ingersoll-Ross model [4], CIR hereafter, with $\gamma = 1/2$). Two factor models include models with stochastic volatility, convergence models modelling interest rates in a country before joining a monetary union or models where the short rate is a sum of certain factors (see [1] for a detailed treatment of different interest rate models). We study the last mentioned class of models. In particular, we are concerned with a model where the short rate r is given by $r = r_1 + r_2$ and the risk neutral dynamics of the factors r_1 and r_2 is as follows:

$$\begin{aligned} dr_1 &= (\alpha_1 + \beta_1 r_1)dt + \sigma_1 r_1^{\gamma_1} dw_1, \\ dr_2 &= (\alpha_2 + \beta_2 r_2)dt + \sigma_2 r_2^{\gamma_2} dw_2, \end{aligned} \quad (1)$$

where the correlation between increments of Wiener processes is ρ, i.e., $\mathbb{E}(dw_1 dw_1) = \rho dt$. In particular we note that by taking $\gamma_1 > 0$ and $\gamma_2 = 0$ we are able to model negative interest rates (both instantaneous short rate and interest rates with other maturities) which were actually a reality recently in Eurozone (see historical data at [11]. This can be accomplished also by a simple one-factor Vasicek model. However, a consequence of Vasicek model is the same variance of short rate, regardless of its level. On the other hand, the real data suggest that volatilities of interest rates decrease as interest rates themselves decrease. The model with $\gamma_1 > 0$ and $\gamma_2 = 0$ has the variance dependent on the level of factor r_1.

Before using a certain model we need to calibrate it, i.e., estimate its parameters from the available data. One approach to calibration of interest rate models is based on minimizing the weighted squared differences between theoretical yields and the real market ones, see, e.g., [8], [9]. Let R_{ij} be the yield observed at i-th day for j-th maturity τ_j and $R(\tau_j, r_{1j}, r_{2j})$ the yield computed from the two factor model, where r_{1i} and r_{2i} are factors of the short rate at i-th day. We denote by w_{ij} the weight of the i-th day and j-th maturity observation in the objective function. In general, we look for the values of the parameters and the decomposition of the short rate to the factors, which minimize the objective function

$$F(r_{1i}, r_{2i}, \alpha_i, \beta_i, \gamma_i, \sigma_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} \Big(R(\tau_j, r_{1i}, r_{2i}) - R_{ij} \Big)^2. \tag{2}$$

In order to solve this optimization problem, we need to evaluate the yields given by the model which is equivalent to solving the PDE for bond prices $P(\tau, r_1, r_2)$, which reads as

$$\begin{aligned} & -\frac{\partial P}{\partial \tau} + [\alpha_1 + \beta_1 r_1]\frac{\partial P}{\partial r_1} + [\alpha_2 + \beta_2 r_2]\frac{\partial P}{\partial r_2} \\ + \frac{\sigma_1^2 r_1^{2\gamma_1}}{2}\frac{\partial^2 P}{\partial r_1^2} + \frac{\sigma_2^2 r_2^{2\gamma_2}}{2}\frac{\partial^2 P}{\partial r_2^2} & + \rho\sigma_1\sigma_2 r_1^{\gamma_1} r_2^{\gamma_2}\frac{\partial^2 P}{\partial r_1 \partial r_2} - (r_1 + r_2)P = 0 \end{aligned} \tag{3}$$

for any r_1, r_2 from their domain and any time to maturity $\tau \in [0, T)$, with initial condition $P(0, r_1, r_2) = 1$ for any r_1, r_2, see [6]. Closed form solutions are available only in special cases. For the model (1) , cf. [1], it is only the Vasicek case $\gamma_1 = \gamma_2 = 0$ and the CIR case $\gamma_1 = \gamma_2 = 1/2$ but only with zero correlation $\rho = 0$ and a mixed model $\gamma_1 = 0$, $\gamma_2 = 1/2$ again with $\rho = 0$. In the remaining cases we need some approximation, which can be obtained using a certain numerical method, Monte Carlo simulation of an approximate analytical solution.

The paper is formulated as follows: In the following section we consider the uncorrelated case of the two-factor Vasicek model, i.e., the model (1) with $\gamma_1 = \gamma_2 = 0$ and $\rho = 0$, and the possibility to estimate its parameters and the short rate factors using the objective function (2). In Section 3 we apply this algorithm to real data and we note its advantage in fitting the market interest rates, compared to one-factor Vasicek model. Section 4 present a simulated example which shows a performance of this algorithm when estimating the short rate from a general model (1), i.e., a robustness to missspecified volatility. This motivates us to develop an analytical approximation formula for the bond prices for the model (1) and derive the order of its accuracy which we do in Section 5. We end the paper with concluding remarks.

2 Two-factor Vasicek model: singularity and transformation

In this section we consider the model (1) with $\gamma_1 = \gamma_2 = 0$, in which case the formulae for the bond prices are known, see for example [1]. Moreover we assume that $\rho = 0$, so the increments of the Wiener processed determining the factors of the short rate are uncorrelated. We write the bond price P as

$$\log P(\tau, r_1, r_2) = c_{01}(\tau)r_1 + c_{02}(\tau)r_2 + c_{11}(\tau)\alpha_1 + c_{12}(\tau)\alpha_2 + c_{21}(\tau)\sigma_1^2 + c_{22}(\tau)\sigma_2^2,$$

where, for $k = 1$ and $k = 2$,

$$c_{0k} = \frac{1 - e^{\beta_k \tau}}{\beta_k}, c_{1k} = \frac{1}{\beta_k}\left(\frac{1 - e^{\beta_k \tau}}{\beta_k} + \tau\right), c_{2k} = \frac{1}{2\beta_k^2}\left(\frac{1 - e^{\beta_k \tau}}{\beta_k} + \tau + \frac{(1 - e^{\beta_k \tau})^2}{2\beta_k}\right)$$

We fix the values of β_1 and β_2. Then the objective function (2) can be written as

$$
\begin{aligned}
F &= \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{w_{ij}}{\tau_j^2} \left(\log P(\tau_j, r_{1i}, r_{2i}) + R_{ij}\tau_j\right)^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{w_{ij}}{\tau_j^2} \left(c_{01}(\tau_j)r_{1i} + c_{02}(\tau)r_{2i} + c_{11}(\tau_j)\alpha_1 + c_{12}(\tau_j)\alpha_2 + \right. \\
&\left. c_{21}(\tau_j)\sigma_1^2 + c_{22}(\tau_j)\sigma_2^2 + R_{ij}\tau_j\right)^2,
\end{aligned}
$$

which can be represented as a weighted linear regression problem without intercept, with parameters $r_{1i}, r_{2i}, \alpha_1, \alpha_2, \sigma_1^2, \sigma_2^2$ to be estimated. However, the regressors are linearly dependent and hence the estimates minimizing the objective function are not uniquely determined. In the context of calibrating the yield curves, this means that different sets of parameter values and factor evolutions lead to the same optimal fit of the term structures. In particular, we have

$$
-\frac{1}{\beta_2}c_{01}(\tau) + \frac{1}{\beta_2}c_{02}(\tau) + \frac{\beta_1}{\beta_2}c_{11}(\tau) = c_{12}(\tau).
$$

Substituting this into the formula for the logarithm of the bond price we get

$$
\begin{aligned}
\log P(\tau, r_1, r_2) &= c_{01}(\tau)r_1 + c_{02}(\tau)r_2 + c_{11}(\tau)\alpha_1 + c_{12}(\tau)\alpha_2 + c_{21}(\tau)\sigma_1^2 + c_{22}(\tau)\sigma_2^2 \\
&= \left(r_{1i} - \frac{\alpha_2}{\beta_2}\right) c_{01}(\tau_j) + \left(r_{2i} + \frac{\alpha_2}{\beta_2}\right) c_{02}(\tau_j) \left(\alpha_1 + \frac{\alpha_2\beta_1}{\beta_2}\right) c_{11}(\tau_j) \\
&\quad + c_{21}(\tau_j)\sigma_1^2 + c_{22}(\tau_j)\sigma_2^2.
\end{aligned}
$$

The objective function of the regression problem then reads as

$$
\begin{aligned}
F = \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{w_{ij}}{\tau_j^2} &\left(\left(r_{1i} - \frac{\alpha_2}{\beta_2}\right) c_{01}(\tau_j) + \left(r_{2i} + \frac{\alpha_2}{\beta_2}\right) c_{02}(\tau_j)\right. \\
&\left. + \left(\alpha_1 + \frac{\alpha_2\beta_1}{\beta_2}\right) c_{11}(\tau_j) + c_{21}(\tau_j)\sigma_1^2 + c_{22}(\tau_j)\sigma_2^2 + R_{ij}\tau_j\right)^2, \qquad (4)
\end{aligned}
$$

which is already regular. Note that we are not able to estimate all the parameters, nor the separate factors r_1 and r_2. However, the sum of the parameters corresponding to c_{01} and c_{02} is the sum of r_1 and r_2, i.e., the short rate r.

Thus, for a given pair (β_1, β_2) we find the optimal values of the regression problem above and note the attained value of the objective function. Then, we optimize for the values of β_1, β_2. For these optimal β_1, β_2 we note the coefficients corresponding to c_{01} and c_{02}. These are estimated shifted factors and their sum is the estimate of the short rate.

3 Application to real data

We use this algorithm to the two data sets considered in paper [5] dealing with estimating the short rate using one-factor Vasicek model: Euribor data from last quarter of 2008 and last quarter of 2011. We note that in the first case, the fit of the one-factor Vasicek was much better then in the second case.

It can be expected that in the case when already a one-factor model provides a good fit, estimating a two-factor model does not bring much change into the results. However, if the fit of a one-factor model is not satisfactory, the estimates from the two-factor model can be more substantially different. From Figure 1 we can see that the fit of the term structures has significantly improved by adding the second factor in the last quarter of 2011.

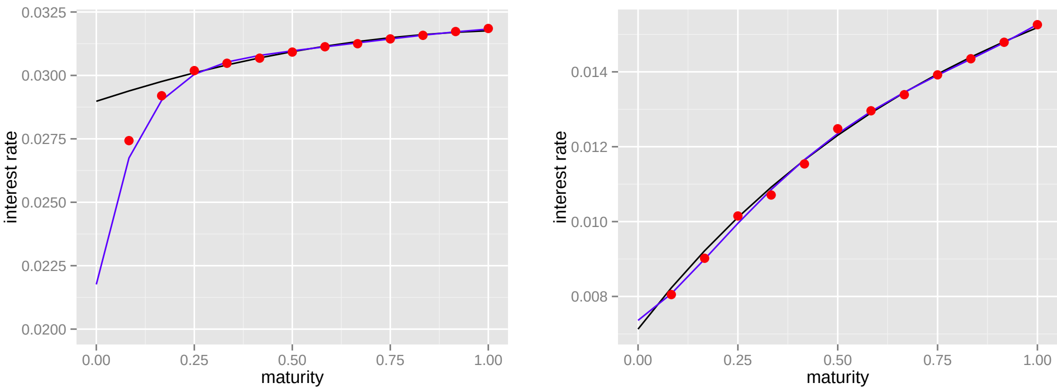

Fig. 1. Fitted yield curves using real data - a selected day in 2008 (left) and 2010 (right): blue lines show the fit from the 2-factor model, black lines from the 1-factor model, red circles are market data

4 Robustness of the short rate estimates

Naturally, the algorithm described in the previous section works well in case of data simulated from the two-factor Vasicek model. However, we noted the estimate of the short rate is remarkable accurate even when the volatility is misspecified. In particular, since we are able to compute exact bond prices from the two-factor CIR model with uncorrelated factors and test the algorithm on these data.

We simulate two factor CIR model with the parameters taken from [3]: $\kappa_1 = 1.8341, \theta_1 = 0.05148, \sigma_1 = 0.1543$, $\kappa_2 = 0.005212, \theta_2 = 0.03083, \sigma_2 = 0.06689$. We simulate daily data from one quarter (assuming 252 trading days in a year). Then, we consider market prices of risk $\lambda)1 = -0.1253, \lambda_2 = -0.06650$ from [3] and compute the term structures for maturities $1, 2, , \ldots, 12$ months for each day using the exact formulae. These data are used as inputs to estimation of the two-factor Vasicek model. A sample result, comparing the simulated short rate and its estimate is presented in Figure 2.

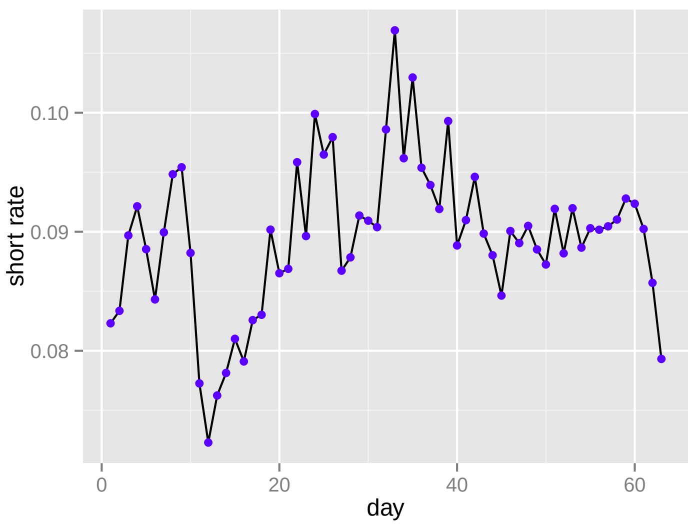

Fig. 2. Estimating short rate using data simulated from the two-factor CIR model: simulated (points) and estimated (line) short rate.

In spite of misspecification of the model, the terms corresponding to $\left(r_{1i} - \frac{\alpha_2}{\beta_2}\right)$ and $\left(r_{2i} + \frac{\alpha_2}{\beta_2}\right)$ indeed estimate the factors up to a constant shift. This is displayed in Figure 3; note the vertical axis for each pair of the graphs.

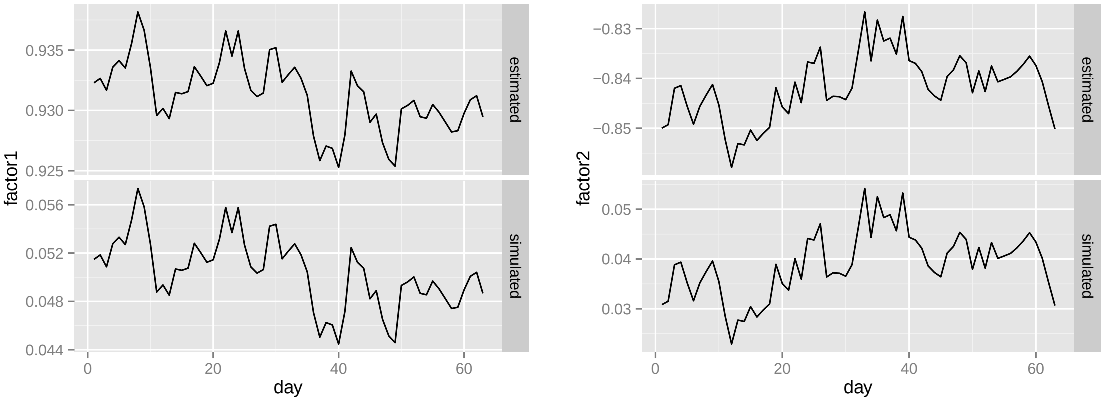

Fig. 3. Estimating factors up to an additive constant using data simulated from the two-factor CIR model.

5 Approximation of the bond prices in the CKLS model

Based on the example in the previous section, we might want to estimate the short rate by application of the algorithms for the two-factor Vasicek model, even though we expect the volatility to have a more general form. Estimates of the short rate factors, up to an additive constant, might be a valuable results, since their knowledge greatly reduced the dimension of the optimization problem (2). However, we need to compute the bond prices in a CKLS general model - either their exact values or a sufficiently accurate approximation. Since they are going to be used in a calibration of a certain kind, they should be calculated quickly and without numerical problems. The aim of this section is to provide an analytical approximation formula for these bond prices and to derive order of its accuracy.

The motivation comes from the paper [7] where an approximation of bond prices for a one-factor CKLS model was proposed. Note that if the correlation in the two-factor CKLS model is zero, the bond price is equal to the sum of two terms corresponding to solutions to bond pricing PDE originating from one factor CKLS models, with factors r_1 and r_2 taking the role of a short rate. Therefore, the bond price could be approximated as a sum of the approximations corresponding to these one-factor models. They are obtained from the Vasicek bond price formula, by substituting its constant volatility by instantaneous volatility from the CKLS model. It is shown in [7] that the error of logarithm of the bond price is then $O(\tau^4)$ as $\tau \to 0^+$. We generalize this idea to the two-factor case and suggest the following approximation.

Theorem 1. *Let P^{ap} be the approximative and P^{ex} be the exact price of the bond in CKLS model.Then for $\tau \to 0^+$*

$$ln P^{ap}(\tau, r_1, r_2) - ln P^{ex}(\tau, r_1, r_2) = c_4(r_1, r_2)\tau^4 + o(\tau^4) \quad (5)$$

where coefficient c_4 is given by

$$c_4(r_1,r_2) = -\frac{1}{24 r_1^2 r_2^2}\Big((2\gamma_1^2-\gamma_1)(r_1^{4\gamma_1} r_2^2\sigma_1^4) + (2\gamma_2^2-\gamma_2)(r_1^2 r_2^{4\gamma_2}\sigma_2^4) \tag{6}$$

$$+ \rho\gamma_1(\gamma_1-1) r_1^{3\gamma_1} r_2^{\gamma_2+2}\sigma_1^3\sigma_2 + \rho\gamma_2(\gamma_2-1) r_1^{\gamma_1+2} r_2^{3\gamma_2}\sigma_1\sigma_2^3 \tag{7}$$

$$+ 2\gamma_2(\alpha_2+\beta_2 r_2)(\rho\sigma_1\sigma_2 r_1^{2+\gamma_1} r_2^{1+\gamma_2} + \sigma_2^2 r_1^2 r_2^{1+2\gamma_2}) + 2\gamma_1\gamma_2\rho^2\sigma_1^2\sigma_2^2 r_1^{2\gamma_1+1} r_2^{2\gamma_2+1} \tag{8}$$

$$+ 2\gamma_1 r_1 r_2^2 \sigma_1(\alpha_1+\beta_1 r_1)\big(r_1^{2\gamma_1}\sigma_1 + \rho\sigma_2 r_1^{\gamma_1} r_2^{\gamma_2}\big)\Big). \tag{9}$$

Remark 1. From the above considerations it follows that $\log P^{ap} - \log P^{ex}$ is $O(\tau^4)$ in the case of zero correlation ρ. What needs to be done is showing that the same order of accuracy is achieved also in the case of general ρ.

Proof. Let us define function $f^{ex}(\tau, r_1, r_2) = \ln P^{ex}(\tau, r_1, r_2)$, where P^{ex} is the exact solution of the equation (3) Then the partial differential equation (3) for f^{ex} is given by:

$$\begin{aligned}
&-\frac{\partial f^{ex}}{\partial \tau} + [\alpha_1+\beta_1 r_1]\frac{\partial f^{ex}}{\partial r_1} + [\alpha_2+\beta_2 r_2]\frac{\partial f^{ex}}{\partial r_2} \\
&+ \frac{\sigma_1^2 r_1^{2\gamma_1}}{2}\left[\left(\frac{\partial f^{ex}}{\partial r_i}\right)^2 + \frac{\partial^2 f^{ex}}{\partial r_i^2}\right] + \frac{\sigma_2^2 r_2^{2\gamma_2}}{2}\left[\left(\frac{\partial f^{ex}}{\partial r_i}\right)^2 + \frac{\partial^2 f^{ex}}{\partial r_i^2}\right] \\
&+ \rho\sigma_1\sigma_2 r_1^{\gamma_1} r_2^{\gamma_2}\left[\frac{\partial f^{ex}}{\partial r_1}\frac{\partial f^{ex}}{\partial r_2} + \frac{\partial^2 f^{ex}}{\partial r_1 \partial r_2}\right] - (r_1+r_2) = 0.
\end{aligned}$$

For the approximation $f^{ap}(\tau, r_1, r_2) = \ln P^{ap}(\tau, r_1, r_2)$ we obtain from the former PDE equation with nontrivial right-hand side $h(\tau, r_1, r_2)$:

$$\begin{aligned}
&-\frac{\partial f^{ap}}{\partial \tau} + [\alpha_1+\beta_1 r_1]\frac{\partial f^{ap}}{\partial r_1} + [\alpha_2+\beta_2 r_2]\frac{\partial f^{ap}}{\partial r_2} \\
&+ \frac{\sigma_1^2 r_1^{2\gamma_1}}{2}\left[\left(\frac{\partial f^{ap}}{\partial r_i}\right)^2 + \frac{\partial^2 f^{ap}}{\partial r_i^2}\right] + \frac{\sigma_2^2 r_2^{2\gamma_2}}{2}\left[\left(\frac{\partial f^{ap}}{\partial r_i}\right)^2 + \frac{\partial^2 f^{ap}}{\partial r_i^2}\right] \\
&+ \rho\sigma_1\sigma_2 r_1^{\gamma_1} r_2^{\gamma_2}\left[\frac{\partial f^{ap}}{\partial r_1}\frac{\partial f^{ap}}{\partial r_2} + \frac{\partial^2 f^{ap}}{\partial r_1 \partial r_2}\right] - (r_1+r_2) = h(\tau, r_1, r_2).
\end{aligned}$$

In the next step we substitute to the previous equation approximation of the bond price and make a Taylor expansion of all the terms with respect to τ:

$$h(\tau, r_1, r_2) = k_3(r_1, r_2)\tau^3 + o(\tau^3),$$

where k_3 reads as

$$\begin{aligned}
k_3(r_1,r_2) = \frac{1}{6 r_1^2 r_2^2}\Big(&(2\gamma_1^2-\gamma_1)(r_1^{4\gamma_1} r_2^2\sigma_1^4) + (2\gamma_2^2-\gamma_2)(r_1^2 r_2^{4\gamma_2}\sigma_2^4) \\
&+ \rho\gamma_1(\gamma_1-1) r_1^{3\gamma_1} r_2^{\gamma_2+2}\sigma_1^3\sigma_2 + \rho\gamma_2(\gamma_2-1) r_1^{\gamma_1+2} r_2^{3\gamma_2}\sigma_1\sigma_2^3 \\
&+ 2\gamma_2(\alpha_2+\beta_2 r_2)(\rho\sigma_1\sigma_2 r_1^{2+\gamma_1} r_2^{1+\gamma_2} + \sigma_2^2 r_1^2 r_2^{1+2\gamma_2}) + 2\gamma_1\gamma_2\rho^2\sigma_1^2\sigma_2^2 r_1^{2\gamma_1+1} r_2^{2\gamma_2+1} \\
&+ 2\gamma_1 r_1 r_2^2 \sigma_1(\alpha_1+\beta_1 r_1)\big(r_1^{2\gamma_1}\sigma_1 + \rho\sigma_2 r_1^{\gamma_1} r_2^{\gamma_2}\big)\Big).
\end{aligned}$$

Let us consider function $g(\tau, r_1, r_2) = f^{ap} - f^{ex}$. It satisfies the equation

$$\begin{aligned}
&-\frac{\partial g}{\partial \tau} + [\alpha_1+\beta_1 r_1]\frac{\partial g}{\partial r_1} + [\alpha_2+\beta_2 r_2]\frac{\partial g}{\partial r_2} + \frac{\sigma_1^2 r_1^{2\gamma_1}}{2}\left[\left(\frac{\partial^2 g}{\partial r_1^2}\right)^2 + \frac{\partial^2 g}{\partial r_1}\right] \\
&+ \quad \frac{\sigma_2^2 r_2^{2\gamma_2}}{2}\left[\left(\frac{\partial^2 g}{\partial r_2^2}\right)^2 + \frac{\partial^2 g}{\partial r_2}\right] + \rho\sigma_1\sigma_2 r_1^{\gamma_1} r_2^{\gamma_2}\left[\frac{\partial g}{\partial r_1}\frac{\partial g}{\partial r_2} + \frac{\partial^2 g}{\partial r_1 \partial r_2}\right] \\
&= h(\tau, r_1, r_2) - \sigma_1^2 r_1^{2\gamma_1}\frac{\partial f^{ex}}{\partial r_1}\frac{\partial g}{\partial r_1} - \sigma_2^2 r_2^{2\gamma_2}\frac{\partial f^{ex}}{\partial r_2}\frac{\partial g}{\partial r_2} - \rho\sigma_1\sigma_2 r_1^{\gamma_1} r_2^{\gamma_2}\left[\frac{\partial g}{\partial r_1}\frac{\partial f^{ex}}{\partial r_2} - \frac{\partial g}{\partial r_2}\frac{\partial f^{ex}}{\partial r_1}\right].
\end{aligned} \tag{10}$$

Taylor expansion of this equation with respect to τ is given by:

$$g(\tau, r_1, r_2) = \sum_{i=0}^{\infty} c_i(r_1, r_2)\tau^i = \sum_{i=\omega}^{\infty} c_i(r_1, r_2)\tau^i,$$

where coefficient $c_\omega(r_1, r_2)\tau^\omega$ is the first non-zero term. Thus we have $\partial_\tau g = \omega c_\omega(r_1, r_2)\tau^{\omega-1} + o(\tau^{\omega-1})$. Note that $\omega \neq 0$. Coefficient c_0 can not be the first non-zero term in the expansion, because it represents value of the function g in the maturity time of the bond and hence it equals zero (since both f^{ap} and f^{ex} are equal to 1 at maturity). Except for function $h(\tau, r_1, r_2) = k_3(r_1, r_2)\tau^3 + o(\tau^3)$, all the terms in the equation (10) are multiplied by at least one of the derivatives $\partial_{r_1} g$, $\partial_{r_2} g$, which are of order $O(\tau)$. Hence all the terms, except $h(\tau, r_1, r_2)$, are of the order $o(\tau^{\omega-1})$ for $\tau \to 0^+$. Equation (10) then implies

$$-\omega c_\omega(r_1, r_2)\tau^{\omega-1} = k_3(r_1, r_2)\tau^3.$$

We get $\omega = 4$, which means that

$$g(\tau, r_1, r_2) = \ln P^{ap}(\tau, r_1, r_2) - \ln P^{ex}(\tau, r_1, r_2) = -\frac{1}{4}k_3(r_1, r_2)\tau^4 + o(\tau^4).$$

Note that considering a difference of the logarithms of the bond prices is convenient because of calculation of the relative error and the differences in the term structures.

6 Conclusions

In this paper we studied a particular class of two-factor models of interest rates, in which the short rate is defined as a sum of two CKLS-type processes. We developed a method of estimating the short rate and fitting the term structures for a special Vasicek case model and showed its usefulness by applying it to fitting Euribor interest rates. An example from the simulated data, where the procedure gave a very precise estimate of the short rate even if applied to a data generated from a model with nonconstant volatilities, motivated us to propose an approximation of bond prices in such a model and prove its order of accuracy. We note that besides a precise estimate of the short rate, we have also its decomposition into the factors, but these are shifted by a constant. Still, it provides a lot of information about the process and hence our future work will be concerned with using this information together with the approximation of the bonds which we derived to obtain estimates for all the parameters of the model.

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References

1. D. Brigo, F. Mercurio: *Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit*. Springer, 2007.
2. K. C. Chan, G. A. Karolyi, F. A. Longstaff, A. B. Sanders: *An empirical comparison of alternative models of the short-term interest rate*. Journal of Finance, 47(3), 1992, 1209-1227.
3. R. R. Chen, L. Scott: *Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates*. The Journal of Fixed Income 3.3, 1993, 14-31.
4. J. C. Cox, J. E. Ingersoll, S. A. Ross. *A theory of the term structure of interest rates*. Econometrica, 53(2), 1985, 385-407
5. J. Halgašová, B. Stehlíková, Z. Bučková: *Estimating the short rate from the term structures in the Vasicek model*. Tatra Mountains Mathematical Publications 61, 2014, 1-17.
6. Y. - K. Kwok: *Mathematical Models of Financial Derivatives*. Springer, 2008.
7. B. Stehlíková: *A simple analytic approximation formula for the bond price in the Chan-Karolyi-Longstaff-Sanders model*. International Journal of Numerical Analysis and Modeling - Series B, Volume 4, Number 3, 2013, pp. 224-234
8. D. Ševčovič, A. Urbánová Csajková: *Calibration of one factor interest rate models*. Journal of Electrical Engineering, 55, No. 12/s (2004), 46-50
9. D. Ševčovič, A. Urbánová Csajková: *On a two-phase minmax method for parameter estimation of the Cox, Ingersoll, and Ross interest rate model*. Central European Journal of Operation Research, 13, 2005, pp. 169-188.
10. O. Vasicek: *An equilibrium characterization of the term structure*. Journal of financial economics 5.2, 1977, 177-188.
11. EMMI - European Money Markets Institute. Euribor Rates. `http://www.emmi-benchmarks.eu/euribor-org/euribor-rates.html`