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β-admissibility of observation and control operators for hypercontractive semigroups

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Abstract

We prove a Weiss conjecture on β-admissibility of control and observation operators for discrete and continuous γ-hypercontractive semigroups of operators, by representing them in terms of shifts on weighted Bergman spaces and using a reproducing kernel thesis for Hankel operators. Particular attention is paid to the case $\gamma = 2$, which corresponds to the unweighted Bergman shift.

Keywords: Admissibility; semigroup system; dilation theory; Bergman space; hypercontraction; reproducing kernel thesis; Hankel operator
2010 Subject Classification: 30H10, 30H20, 47B32, 47B35, 47D06, 93B28

1 Introduction

We study infinite dimensional observation systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad y(t) = Cx(t), \quad t \geq 0, \\ x(0) &= x_0 \in X, \end{aligned}$$

where A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $\mathcal{H}$ and C is a linear bounded operator from $D(A)$, the domain of A equipped with the graph topology, to another Hilbert space $\mathcal{Y}$. For well-posedness of the system with respect to the output space $L^2_\beta(0,\infty;\mathcal{Y}) := \{f : (0,\infty) \to \mathcal{Y} \mid f \text{ measurable}, \|f\|_\beta^2 := \int_0^\infty \|f(t)\|^2 t^\beta \, dt < \infty\}$ it is required that C is an β-*admissible observation operator* for A, that is, there exists an $M > 0$ such that

$$\|CT(\cdot)x_0\|_{L_\beta(0,\infty;\mathcal{Y})} \leq M\|x_0\|_{\mathcal{H}}, \qquad x_0 \in D(A).$$

It is easy to show that β-admissibility implies the resolvent condition

$$\sup_{\lambda\in\mathbb{C}_+} (\operatorname{Re}\lambda)^{\frac{1+\beta}{2}} \|C(\lambda - A)^{-(1+\beta)}\| < \infty \tag{1}$$

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where $\mathbb{C}_+$ denotes the open right half plane of $\mathbb{C}$. Whether or not the converse implication holds is commonly referred to as a *weighted Weiss conjecture*. For $\beta = 0$ the conjecture was posed by Weiss [23]. In this situation the conjecture is true for contraction semigroups if the output space is finite-dimensional, for right-invertible semigroup and for bounded analytic semigroups if $(-A)^{1/2}$ is 0-admissible. However, in general the conjecture is not true. We illustrate this in Figure 1.

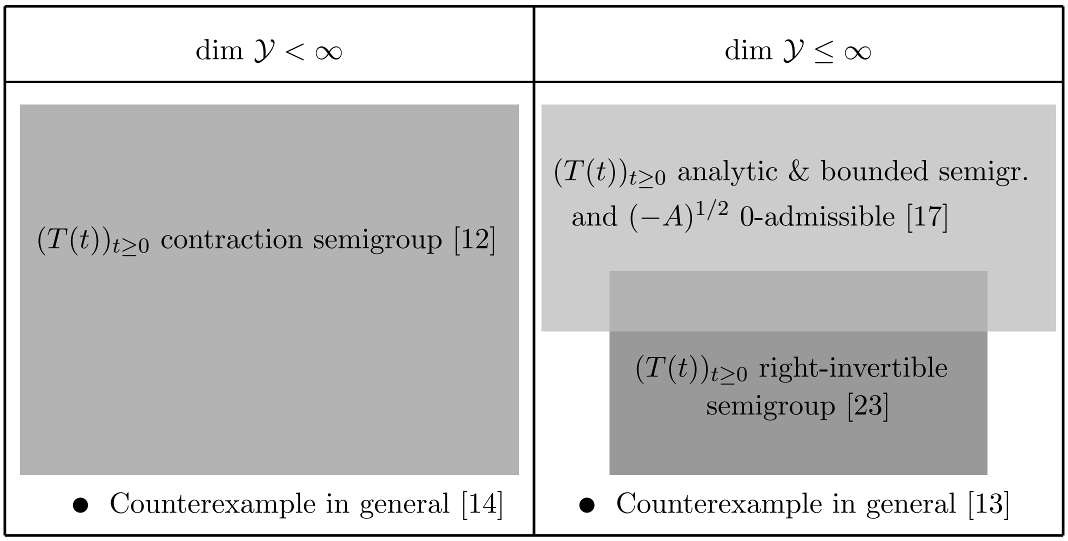

Figure 1: Weighted Weiss conjecture: Case $\beta = 0$

For $\beta \neq 0$, there is much less known. In the situation $\beta < 0$, the weighted Weiss conjecture is true for bounded analytic semigroups if $(-A)^{1/2}$ is 0-admissible [9], but in general the weighted Weiss conjecture does not hold [25]. If $\beta > 0$, then the weighted Weiss conjecture is true for normal contraction semigroups and for the right-shift on $L^2_{-\alpha}(0,\infty)$ for $\alpha > 0$ if the the output space is finite-dimensional, and for bounded analytic semgroups if $(-A)^{1/2}$ is 0-admissible, see Figure 2. Again, in general the conjecture is not true. In Theorem 4.4 we show that the weighted Weiss conjecture holds if the dual of the cogenerator T^* of the semigroup $(T(t))_{t\geq 0}$ is γ-hypercontractive for some $\gamma > 1$. The proof is based on the fact that γ-hypercontractions are unitarily equivalent to the restriction of the backward shift to an invariant subspace of a weighted Bergman space, the Cayley transform between discrete-time and continuous-time systems, and the fact that the weighted Weiss conjecture holds for the backward shift on an invariant subspace of a weighted Bergman space [11]. In order to apply the results of [11] we first have to extend them to the vector-valued Bergman spaces.
Owing to the fact that C is a β-admissible observation operator for $(T(t))_{t\geq 0}$ if and only if C^* is a $(-\beta)$-admissible control operator for $(T^*(t))_{t\geq 0}$, where $\beta \in (-1, 1)$ (cf. Remarks 3.1 and 4.2 below), the resolvent growth conditions for β-admissible control operators can be derived from those of $(-\beta)$-admissible observation operators.
Beside continuous-time systems we also prove a discrete-time version of the Weiss conjecture. For $T \in \mathcal{L}(\mathcal{H})$, $E \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ and $F \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ we consider the discrete time linear systems:

$$x_{n+1} = Tx_n + Eu_{n+1}, \quad y_n = Fx_n \qquad \text{with} \quad x_0 \in \mathcal{H} \tag{2}$$

and $u_n \in \mathcal{U}$, $n \in \mathbb{N}$. Here, $\mathcal{H}$ is the state space, $\mathcal{U}$ the input space and $\mathcal{Y}$ is the output space

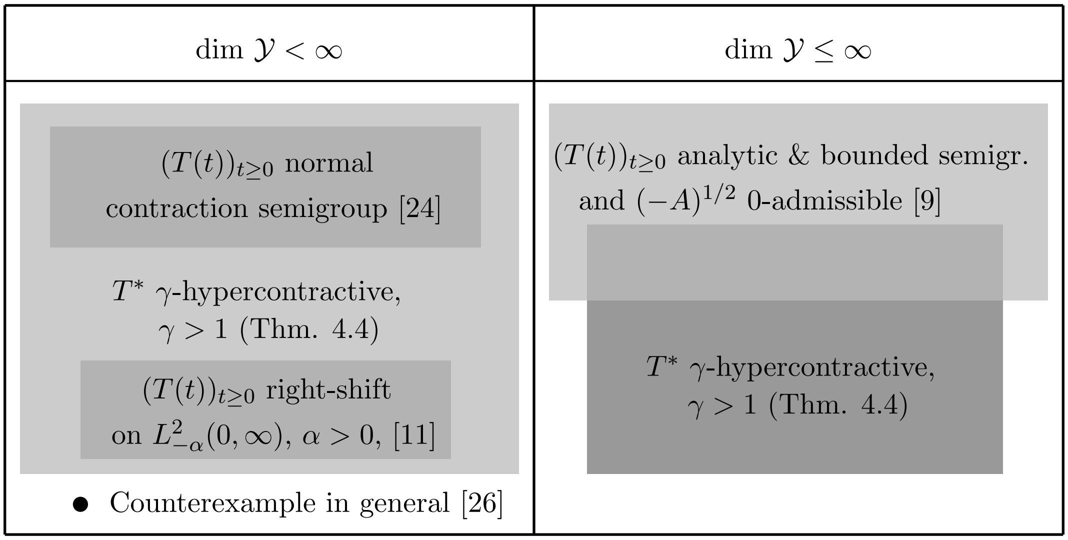

Figure 2: Weighted Weiss conjecture: Case $\beta > 0$

of the system.
Let $\beta > -1$. By $\ell^2_\beta(\mathcal{U})$ we denote the sequence space

$$\ell^2_\beta(\mathcal{U}) := \{\{u_n\}_n \mid u_n \in \mathcal{U} \text{ and } \|\{u_n\}_n\|^2_\beta := \sum_{n=0}^{\infty}(1+n)^\beta |u_n|^2 < \infty\}.$$

Clearly, $\ell^2_\beta(\mathcal{U})$ equipped with the norm $\|\cdot\|_\beta$ is a Hilbert space. Following [9] and [24], we say that F is a *β-admissible observation operator for T*, if there exists a constant $M > 0$ such that

$$\sum_{n=0}^{\infty}(1+n)^\beta \|FT^n x\|^2 \leq M\|x\|^2$$

for every $x \in \mathcal{H}$.
To test whether a given observation operator is β-admissible, a frequency-domain characterization is convenient and, to this end, it is not difficult to show that β-admissibility of F for T implies the resolvent growth condition

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\frac{1+\beta}{2}}\|F(I-\bar{z}T)^{-\beta-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{Y})} < \infty, \tag{3}$$

where $\mathbb{D}$ is the open unit disc.
The question of whether the converse statement holds, commonly referred to as a (weighted) Weiss conjecture, is much more subtle. For $\beta = 0$, the conjecture is true if T is a contraction and the output space $\mathcal{Y}$ is finite-dimensional [10]. It was shown by [25, 24] that for T a normal contraction and finite-dimensional output spaces the weighted Weiss conjecture holds for positive β, but not in the case $\beta \in (-1, 0)$. Moreover, the weighted Weiss conjecture holds if T is a Ritt operator and a contraction for $\beta > -1$ [18], but it is not true for general contractions if $\beta > 0$, see [26]. Recently, in [11] it was shown that the Weiss conjecture holds for the forward shift on weighted Bergman spaces. One aim of this paper is to show that the Weiss conjecture holds for adjoint operators of γ-hypercontractions. We obtain a

characterisation of β-admissibility, $\beta > 0$, with respect to γ-hypercontractions ($\gamma > 1$) by characterising β-admissibility with respect to the shift operator on vector-valued weighted Bergman spaces.

It is shown in [11] that in the case of a scalar-valued Bergman space, β-admissibility with respect to the shift operator can be characterised by the resolvent growth bound (3). We extend this analysis to the vector-valued setting.

We proceed as follows. In Section 2 we introduce and study γ-hypercontractive operators and γ-hypercontractive strongly continuous semigroups. In particular, γ-hypercontractions are unitarily equivalent to the restriction of the backward shift to an invariant subspace of a weighted Bergman space. Section 3 is devoted to the weighted Weiss conjecture for discrete-time systems. We first extend the result of [11] concerning the shift operator on a scalar-valued Bergman space to the vector-valued setting and then we prove that the weighted Weiss conjecture holds for $\beta > 0$ if T^* is a γ-hypercontraction for some $\gamma > 1$. Finally, in Section 4 positive results concerning the weighted Weiss conjecture for continuous-time systems are given.

2 γ-hypercontractions

Let $\mathcal{H}$ be a Hilbert space. For $T \in \mathcal{L}(\mathcal{H})$, we define

$$M_T : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}), \quad M_T(X) = T^* X T.$$

Definition 2.1 ([2], [4]). *Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$, $\|T\| \leq 1$. Let $\gamma \geq 1$. We say that T is a* γ-hypercontraction, *if for each $0 < r < 1$,*

$$(\mathbf{1} - M_{rT})^\gamma (I) \geq 0.$$

Note that the left hand side in the definition is well-defined in the sense of the usual holomorphic functional calculus, since $\sigma(\mathbf{1} - M_{rT}) \subset \mathbb{C}_+$. A 1-hypercontraction is of course just an ordinary contraction. If T is a normal contraction, then it is easy to show by the usual continuous functional calculus that T is also a γ-hypercontraction for each $\gamma \geq 1$. Moreover, all strict contractions are γ-hypercontractions, as the next result shows.

Theorem 2.2. *Let $T \in \mathcal{L}(\mathcal{H})$ with $\|T\| < 1$. Then T is a γ-hypercontraction for sufficiently small $\gamma > 1$.*

Proof: Suppose that $\|T\| < 1$. Then $\|M_T\| < 1$, and $\sigma(\mathbf{1} - M_T)$ is bounded away from the negative real axis, so an analytic branch of the logarithm exists on some open set $\Omega \supseteq \sigma(\mathbf{1} - M_T)$. For $\gamma \geq 1$, define $f_\gamma(z) = \exp(\gamma \log z)$, analytic on Ω.

Now $f_\gamma(z) \to z$ uniformly for z in compact subsets of Ω, and therefore $f_\gamma(\mathbf{1} - M_T)$, defined by the analytic functional calculus, converges to $\mathbf{1} - M_T$ in the norm on $\mathcal{L}(\mathcal{L}(\mathcal{H}))$ (see, e.g., [5, Thm. 3.3.3]).

Hence, in particular, $(\mathbf{1} - M_T)^\gamma(I) \to (\mathbf{1} - M_T)(I) = I - T^*T$ in norm in $\mathcal{L}(\mathcal{H})$ as $\gamma \to 1$.

Since $\|T\| < 1$, $\sigma((\mathbf{1} - M_T)(I))$ is strictly contained in the positive real axis, and thus for sufficiently small $\gamma > 1$ the spectrum of $(\mathbf{1} - M_T)^\gamma(I)$ is also strictly contained in the positive real axis, by continuity properties of the spectrum (see, e.g., [5, Thm. 3.4.1]).

Hence $(\mathbf{1} - M_T)^\gamma(I) \geq 0$ for all γ sufficiently close to 1, and so T is a γ-hypercontraction. ∎

If $n \in \mathbb{N}$, then equivalently, $T \in \mathcal{L}(\mathcal{H})$ is an n-hypercontraction if and only if

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k} T^{*k}T^k \geq 0$$

for all $1 \leq m \leq n$.
In particular, a Hilbert space operator T is *2-hypercontractive* if it satisfies

$$I - T^*T \geq 0$$

(that is, it is a contraction), and also

$$I - 2T^*T + T^{*2}T^2 \geq 0. \tag{4}$$

Note, that for $1 < \mu < \gamma$, the γ-hypercontractivity property implies μ-hypercontractivity. We are particularly interested in γ-hypercontractive operators as they are unitarily equivalent to the restriction of the backward shift to an invariant subspace of a weighted Bergman space, which we now define.

Definition 2.3. *Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. For $\alpha > -1$, the* weighted Bergman space $\mathcal{A}^2_\alpha(\mathbb{D}, \mathcal{K})$, *where $\mathcal{K}$ is a Hilbert space, contains of analytic functions $f : \mathbb{D} \to \mathcal{K}$ for which*

$$\|f\|^2_\alpha = \int_{\mathbb{D}} \|f(z)\|^2 dA_\alpha(z) < \infty, \tag{5}$$

where $dA_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha dA(z)$ and $dA(z) := \frac{1}{\pi}dxdy$ is area measure on $\mathbb{D}$ for $z = x + iy$. We note that the norm $\|f\|_\alpha$ is equivalent to

$$\left(\sum_{n=0}^{\infty} \|f_n\|^2 (1+n)^{-(1+\alpha)}\right)^{\frac{1}{2}}, \tag{6}$$

where f_n are the Taylor coefficients of f.

For each $\alpha > -1$, let S_α denote the shift operator on the weighted Bergman space $A^2_\alpha(\mathbb{D}, \mathcal{K})$,

$$S_\alpha f(z) = zf(z) \quad (f \in A^2_\alpha(\mathbb{D}, \mathcal{K}))$$

The following theorem is a special case of Corollary 7 in [4]. For the case of integer γ, this was proved in [2].

Theorem 2.4. *Let $\alpha > -1$. Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$ be an $\alpha + 2$-hypercontraction with $\sigma(T) \subset \mathbb{D}$. Then T is unitarily equivalent to the restriction of S^*_α to an invariant subspace of $A^2_\alpha(\mathbb{D}, \mathcal{K})$, where $\mathcal{K}$ is a Hilbert space.*

Next we introduce the concept of γ-hypercontractive semigroups.

Definition 2.5. *Let $(T(t))_{t\geq 0}$ be a strongly continuous contraction semigroup on a Hilbert space $\mathcal{H}$, with infinitesimal generator A. We call a C_0-semigroup $(T(t))_{t\geq 0}$* γ-hypercontractive *if each operator $T(t)$ is a γ-hypercontraction.*

In the following we assume that $(T(t))_{t\geq 0}$ is a strongly continuous contraction semigroup on a Hilbert space $\mathcal{H}$, with infinitesimal generator A. As in [22], the *cogenerator* $T := (A+I)(A-I)^{-1}$ exists, and is itself a contraction. Rydhe [21] studied the relation between γ-hypercontractivity of a strongly continuous contraction semigroup and its cogenerator. He proved that T is γ-hypercontractive if every operator $T(t)$, $t \geq 0$, is γ-hypercontractive. Conversely, if every operator $T(t)$, $t \geq 0$, is N-hypercontractive for some $N \in \mathbb{N}$, then T is N-hypercontractive. However, by means of an example, Rydhe [21] showed that for general γ-hypercontractivity this reverse implication is false. Clearly, if A generates a contraction semigroup of normal operators, then the cogenerator of $(T(t))_{t\geq 0}$ is γ-hypercontractive for each $\gamma \geq 1$.
In particular 2-hypercontractivity can be characterized as follows, see [21]. For completeness we include a more elementary proof, which also yields additional information.

Proposition 2.6. *Let $(T(t))_{t\geq 0}$ be a strongly continuous contraction semigroup acting on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent.*

1. *$(T(t))_{t\geq 0}$ is 2-hypercontractive.*
2. *The function $t \mapsto \|T(t)x\|^2$ is convex for all $x \in H$.*
3. $$\operatorname{Re}\langle A^2 y, y\rangle + \|Ay\|^2 \geq 0 \qquad (y \in \mathcal{D}(A^2)). \tag{7}$$
   or equivalently,
   $$\|(A+A^*)x\|^2 + \|Ax\|^2 \geq \|A^*x\|^2 \qquad (y \in \mathcal{D}(A)\cap\mathcal{D}(A^*)).$$
4. *The cogenerator T is a 2-hypercontraction.*

Proof We first prove that Part 1 and Part 2 are equivalent. Take $t \geq 0$ and $\tau > 0$. If $T(\tau)$ is a 2-hypercontraction, then, by (4) we have

$$\langle T(t)x, T(t)x\rangle - 2\langle T(t+\tau)x, T(t+\tau)x\rangle + \langle T(t+2\tau)x, T(t+2\tau)x\rangle \geq 0,$$

or

$$\|T(t+\tau)x\|^2 \leq \frac{1}{2}\left(\|T(t)x\|^2 + \|T(t+2\tau)x\|^2\right), \tag{8}$$

which is the required convexity condition.
Conversely, the convexity condition (8) implies that $T(\tau)$ is a 2-hypercontraction (take $t = 0$).
Next we show that Part 2 are Part 3 equivalent. For $t > 0$ and $y \in \mathcal{D}(A^2)$ we calculate the second derivative of the function $g : t \mapsto \|T(t)y\|^2$.

$$g'(t) = \frac{d}{dt}\langle T(t)y, T(t)y\rangle = \langle AT(t)y, T(t)y\rangle + \langle T(t)y, AT(t)y\rangle.$$

Similarly,

$$g''(t) = \langle A^2T(t)y, T(t)y\rangle + 2\langle AT(t)y, AT(t)y\rangle + \langle T(t)y, A^2T(t)y\rangle.$$

If g is convex, then letting $t \to 0$ gives the condition (7).
Conversely, the condition (7) gives the convexity of $t \to \|T(t)y\|^2$ for $y \in \mathcal{D}(A^2)$, and by density this holds for all y.

Finally we show the equivalence of Part 3 and Part 4. We start with the condition (7) and calculate

$$\langle (I - 2T^*T + T^{*2}T^2)x, x\rangle$$

for $x = (A-I)^2 y$ (note that $(A-I)^{-2} : H \to H$ is defined everywhere and has dense range). We obtain

$$\begin{aligned}\langle (A-I)^2 y, (A-I)^2 y\rangle - 2\langle (A^2-I)y, (A^2-I)y\rangle + \langle (A+I)^2 y, (A+I)^2 y\rangle \\ = 4\langle A^2 y, y\rangle + 8\langle Ay, Ay\rangle + 4\langle y, A^2 y\rangle \geq 0.\end{aligned}$$

Thus condition (7) holds if and only if the cogenerator T is 2-hypercontractive. ■

Thus every normal contraction semigroup is 2-hypercontractive. Moreover, even every hyponormal contraction semigroup is 2-hypercontractive. Note, that a semigroup is hyponormal if the generator A satisfies $D(A) \subset D(A^*)$ and $\|A^*x\| \leq \|Ax\|$ for all $x \in D(A)$, see [15, 19]. Clearly, a C_0-semigroup $(T(t))_{t\geq 0}$ is contractive if and only if the adjoint semigroup $(T^*(t))_{t\geq 0}$ is contractive. Unfortunately, a similar statement does not hold for 2-hypercontractions: The right shift semigroup on $L^2(0,\infty)$ is 2-hypercontractive, but the adjoint semigroup, the left shift semigroup on $L^2(0,\infty)$ is not.

3 Discrete-time β-admissibility

Let $\mathcal{H}$, $\mathcal{U}$, $\mathcal{Y}$ be Hilbert spaces, $T \in \mathcal{L}(\mathcal{H})$, $E \in \mathcal{L}(\mathcal{U},\mathcal{H})$ and $F \in \mathcal{L}(\mathcal{H},\mathcal{Y})$. Consider the discrete time linear system:

$$x_{n+1} = Tx_n + Eu_{n+1}, \quad y_n = Fx_n \qquad \text{with} \quad x_0 \in \mathcal{H} \tag{9}$$

and $u_n \in \mathcal{U}$, $n \in \mathbb{N}$.

Following [9] and [24], we say that F is a β-admissible observation operator for T, if there exists a constant $M > 0$ such that

$$\sum_{n=0}^{\infty} (1+n)^\beta \|FT^n x\|^2 \leq M\|x\|^2$$

for every $x \in \mathcal{H}$. Moreover, we say that E is a β-admissible control operator for T, if there exists a constant $M > 0$ such that

$$\left\| \sum_{n=1}^{\infty} T^n E u_n \right\|_{\mathcal{H}} \leq M \|\{u_n\}_n\|_\beta$$

for every $\{u_n\}_n \in \ell^2_\beta(\mathcal{U})$.

Remark 3.1. *Let $x \in \mathcal{H}$ and $\{y_n\}_n \in \ell^2_{-\beta}(\mathcal{Y})$. Then the calculation*

$$\begin{aligned}|\langle \{FT^n x\}_n, \{y_n\}_n\rangle_{\beta\times -\beta}| &= \left| \sum_{n=0}^{\infty} \langle FT^n x, y_n\rangle_{\mathcal{Y}} \right| \\ &= |\langle x, \sum_{n=0}^{\infty} (T^*)^n F^* y_n\rangle_{\mathcal{H}}|\end{aligned}$$

implies that F is a β-admissible observation operator for T if and only if F^* is a $(-\beta)$-admissible control operator for T^*.

A characterisation of β-admissibility with respect to γ-hypercontractions ($\gamma > 1$) may be obtained by characterising β-admissibility with respect to the shift operator on vector-valued weighted Bergman spaces, as defined just after Definition 2.3.
It is shown in [11] that in the case of a scalar-valued Bergman spaces, β-admissibility with respect to S_α can be characterised by the resolvent growth bound (3). This result was obtained by noting that β-admissibility is equivalent to boundedness of an appropriate little Hankel operator, while (3) is equivalent to boundedness of the same Hankel operator on a set of reproducing kernels. That such Hankel operators satisfy a Reproducing Kernel Thesis (boundness on the reproducing kernels is equivalent to operator boundedness) is equivalent to the characterisation of β-admissibility by the growth condition (3).

To extend this analysis to the vector-valued setting, let $\mathcal{K}, \mathcal{Y}$ be Hilbert spaces and consider an analytic function $C : \mathbb{D} \to \mathcal{L}(\mathcal{K}, \mathcal{Y})$ given by

$$C(z) = \sum_{n=0}^{\infty} C_n z^n, \qquad z \in \mathbb{D},$$

where $C_n \in \mathcal{L}(\mathcal{K}, \mathcal{Y})$, for each n. We write $L^2_\alpha(\mathbb{D}, \mathcal{K})$ for the space of measurable functions $f : \mathbb{D} \to \mathcal{K}$ satisfying (5). We also write

$$\overline{A^2_\alpha}(\mathbb{D}, \mathcal{K}) = \{z \mapsto g(\overline{z}) : g \in A^2_\alpha(\mathbb{D}, \mathcal{K})\}.$$

The little Hankel operator $h_C : A^2_{\beta-1}(\mathbb{D}, \mathcal{K}) \to \overline{A^2_\alpha}(\mathbb{D}, \mathcal{Y})$ acting between weighted Bergman spaces is defined by

$$h_C(f) := \overline{P_\alpha}(C(\overline{\iota})f(\iota)), \qquad f \in A^2_{\beta-1}(\mathbb{D}, \mathcal{K}), \tag{10}$$

where $\overline{P_\alpha} : L^2_\alpha(\mathbb{D}, \mathcal{K}) \to \overline{A^2_\alpha}(\mathbb{D}, \mathcal{K})$ is the orthogonal projection onto the anti-analytic functions and $\iota(z) = z, z \in \mathbb{D}$. The following result links β-admissibility with little Hankel operators of the form (10).

Proposition 3.2. *Let $\alpha > -1$ and $\beta > 0$. Let $\mathcal{K}$, $\mathcal{Y}$ be Hilbert spaces. Given $F \in \mathcal{L}(A^2_\alpha(\mathbb{D}, \mathcal{K}), \mathcal{Y})$, define bounded linear operators $F_n \in \mathcal{L}(\mathcal{K}, \mathcal{Y})$ by*

$$F_n x = F(x\iota^n), \qquad x \in \mathcal{K}, n \in \mathbb{N},$$

and symbols $C : \mathbb{D} \to \mathcal{L}(\mathcal{K}, \mathcal{Y})$, $\tilde{C} : \mathbb{D} \to \mathcal{L}(\mathcal{Y}, \mathcal{K})$ by

$$C(z) = \sum_{n=0}^{\infty} (1+n)^\alpha F_n z^n, \qquad \tilde{C}(z) = \sum_{n=0}^{\infty} (1+n)^\alpha F_n^* z^n.$$

The following conditions are equivalent:

(i) The resolvent condition (3) *holds with $T = S_\alpha$ and $\mathcal{H} = A^2_\alpha(\mathbb{D}, \mathcal{K})$;*

(ii) The Hankel operator $h_{\tilde{C}} : A^2_{\beta-1}(\mathbb{D}, \mathcal{Y}) \to \overline{A^2_\alpha}(\mathbb{D}, \mathcal{K})$ satisfies

$$\sup_{\omega \in \mathbb{D}, \|y\|_{\mathcal{Y}}=1} \|h_{\tilde{C}} k^{\beta-1}_{\omega,y}\|_{\overline{A^2_\alpha}(\mathbb{D},\mathcal{K})} < \infty,$$

where

$$k^{\beta-1}_{\omega,y}(z) := y \frac{(1-|\omega|^2)^{\frac{1+\beta}{2}}}{(1-\bar{\omega}z)^{1+\beta}}, \qquad z, \omega \in \mathbb{D},\ y \in \mathcal{Y},$$

are the normalized reproducing kernels for $A^2_{\beta-1}(\mathbb{D}, \mathcal{Y})$;

(iii) The Hankel operator $h_C : A^2_{\beta-1}(\mathbb{D},\mathcal{K}) \to \overline{A^2_\alpha}(\mathbb{D},\mathcal{Y})$ satisfies

$$h_C \in \mathcal{L}(A^2_{\beta-1}(\mathbb{D},\mathcal{K}), \overline{A^2_\alpha}(\mathbb{D},\mathcal{Y}));$$

(iv) F is β-admissible for S_α on $A^2_\alpha(\mathbb{D},\mathcal{K})$.

Proof $(i) \Leftrightarrow (ii)$ follows directly from a vectorial analogue of [11, Proposition 2.3 (ii)].

$(ii) \Rightarrow (iii)$ Note first that [11, Theorem 2.7] extends to the vector-valued setting to imply that $h_{\tilde{C}} : A^2_{\beta-1}(\mathbb{D},\mathcal{Y}) \to \overline{A^2_\alpha}(\mathbb{D},\mathcal{K})$ is bounded. An alternative characterisation of boundedness of little Hankel operators can be given in terms of generalized Hankel matrices of the form

$$\Gamma^{a,b}_\Phi := \Big((1+m)^a(1+n)^b\Phi_{n+m}\Big)_{m,n\geq 0}$$

where $a,b>0$ and $\Phi : \mathbb{D} \to \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ is given by $\Phi(z) = \sum_{n\geq 0}\Phi_n z^n$, for some Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. In particular, the vectorial analogue of [11, Proposition 2.3 (i)] implies that

$$h_{\tilde{C}} \in \mathcal{L}\left(A^2_{\beta-1}(\mathbb{D},\mathcal{Y}), \overline{A^2_\alpha}(\mathbb{D},\mathcal{K})\right) \Longleftrightarrow \Gamma^{\frac{\beta}{2},\frac{1+\alpha}{2}}_{\tilde{C}} \in \mathcal{L}(\ell^2(\mathcal{Y}),\ell^2(\mathcal{K})). \tag{11}$$

Now, it is shown in [20, Theorem 9.1] that

$$\Gamma^{\frac{\beta}{2},\frac{1+\alpha}{2}}_{\tilde{C}} \in \mathcal{L}(\ell^2(\mathcal{Y}),\ell^2(\mathcal{K})) \Leftrightarrow \tilde{C} \in \Lambda_{\frac{1+\alpha+\beta}{2}}\left(\mathcal{L}(\mathcal{Y},\mathcal{K})\right). \tag{12}$$

Here, for $s>0$ and a Banach space X, $\Lambda_s(X)$ is the Besov space containing functions $f \in L^\infty(\mathbb{D},X)$ for which

$$\sup_{\tau\in\mathbb{T},\tau\neq 1}\frac{\|\Delta^n_\tau f\|_{L^\infty(\mathbb{D},X)}}{|1-\tau|^s} < \infty, \qquad \Big((\Delta_\tau f)(\xi) := f(\xi\tau) - f(\tau),\ \Delta^n_\tau := \Delta_\tau\Delta^{n-1}_\tau\Big),$$

for some integer $n>s$. It follows immediately that $C \in \Lambda_{\frac{1+\alpha+\beta}{2}}\left(\mathcal{L}(\mathcal{K},\mathcal{Y})\right)$ and hence, by (11) and (12), that

$$h_C \in \mathcal{L}(A^2_{\beta-1}(\mathbb{D},\mathcal{K}), \overline{A^2_\alpha}(\mathbb{D},\mathcal{Y})).$$

$(iii) \Leftrightarrow (iv)$: The vectorial analogue of [11, Proposition 2.1] implies that (iv) holds if and only if $\Gamma^{\frac{1+\alpha}{2},\frac{\beta}{2}}_C \in \mathcal{L}(\ell^2(\mathcal{K}),\ell^2(\mathcal{Y}))$. By (11), boundedness (iii) of the little Hankel operator h_C is equivalent to $\Gamma^{\frac{\beta}{2},\frac{1+\alpha}{2}}_C \in \mathcal{L}(\ell^2(\mathcal{K}),\ell^2(\mathcal{Y}))$. That (iii) and (iv) are equivalent then follows from [20, Theorem 9.1] and the fact that $\alpha > -1, \beta > 0$.

$(iv) \Rightarrow (i)$ is well known. See, for example, [26]. ■

Theorem 3.3. *Let $\beta > 0$. Let $\mathcal{H}$, $\mathcal{Y}$ be Hilbert spaces and let $T^* \in \mathcal{L}(\mathcal{H})$ be a γ-hypercontraction for some $\gamma > 1$. Let $F \in \mathcal{L}(\mathcal{H},\mathcal{Y})$. Then the following are equivalent:*

1. *F is a β-admissible observation operator for T.*

2. $$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\frac{1+\beta}{2}}\|F(I-\bar{z}T)^{-\beta-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{Y})} < \infty.$$

Proof The implication (1) $\Rightarrow$ (2) follows as usual from the testing on fractional derivatives of reproducing kernels.

For (2) $\Rightarrow$ (1), write $K = \sup_{z\in\mathbb{D}}(1-|z|^2)^{\frac{1+\beta}{2}}\|F(I-\bar{z}T)^{-\beta-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{Y})}$ and let us first replace T by rT for some $0<r<1$. Write $\gamma = 2+\alpha$. By Theorem 2.4, $(rT)^*$ is the restriction of S_α^* to the invariant subspace $\mathcal{H}\subset A^2_\alpha(\mathbb{D},\mathcal{K})$, where $\mathcal{K}$ is another Hilbert space. Extend F trivially to $A^2_\alpha(\mathbb{D},\mathcal{K})$ by letting $F=0$ on $\mathcal{H}^\perp \subset A^2_\alpha(\mathbb{D},\mathcal{K})$. Then $F^*y\in\mathcal{H}$ for all $y\in\mathcal{Y}$. Then for each $z\in\mathbb{D}$ we obtain

$$\begin{aligned}
\|F(I-\bar{z}S_\alpha)^{-\beta-1}\|_{\mathcal{L}(A^2_\alpha(\mathbb{D},\mathcal{K}),\mathcal{Y})} &= \sup_{h\in A^2_\alpha(\mathcal{K}),\|h\|=1}\|F(I-\bar{z}S_\alpha)^{-\beta-1}h\|_{\mathcal{Y}} \\
&= \sup_{h\in A^2_\alpha(\mathcal{K}),\|h\|=1}\ \sup_{y\in\mathcal{Y},\|y\|=1}|\langle (I-\bar{z}S_\alpha)^{-\beta-1}h, F^*y\rangle| \\
&= \sup_{h\in A^2_\alpha,\|h\|=1}\ \sup_{y\in\mathcal{Y},\|y\|=1}|\langle h, (I-zS_\alpha^*)^{-\beta-1}F^*y\rangle| \\
&= \sup_{h\in A^2_\alpha,\|h\|=1}\ \sup_{y\in\mathcal{Y},\|y\|=1}|\langle h, (I-z(rT)^*)^{-\beta-1}F^*y\rangle| \\
&= \sup_{h\in \mathcal{H},\|h\|=1}\ \sup_{y\in\mathcal{Y},\|y\|=1}|\langle h, (I-z(rT)^*)^{-\beta-1}F^*y\rangle| \\
&= \|F(I-\bar{z}rT)^{-\beta-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{Y})} \\
&\leq K\frac{1}{(1-|rz|^2)^{\frac{1+\beta}{2}}} \\
&\leq K\frac{1}{(1-|z|^2)^{\frac{1+\beta}{2}}}.
\end{aligned} \tag{13}$$

Hence, by Proposition 3.2, F is an β-admissible observation operator for S_α.

Thus there exists a constant M such that for each $x\in\mathcal{H}$,

$$\begin{aligned}
\sum_{n=0}^\infty (1+n)^\beta\|F(rT)^nx\|^2_{\mathcal{Y}} &= \sum_{n=0}^\infty (1+n)^\beta \sup_{y\in\mathcal{Y},\|y\|=1}|\langle (rT)^nx, F^*y\rangle|^2_{\mathcal{Y}} \\
&= \sum_{n=0}^\infty (1+n)^\beta \sup_{y\in\mathcal{Y},\|y\|=1}|\langle x, ((rT)^n)^*F^*y\rangle|^2 \\
&= \sum_{n=0}^\infty (1+n)^\beta \sup_{y\in\mathcal{Y},\|y\|=1}|\langle x, (S_\alpha^n)^*F^*y\rangle|^2 \\
&= \sum_{n=0}^\infty (1+n)^\beta \sup_{y\in\mathcal{Y},\|y\|=1}|\langle S_\alpha^n x, F^*y\rangle|^2 \\
&= \sum_{n=0}^\infty (1+n)^\beta \|FS_\alpha^n x\|^2_{\mathcal{Y}} \leq M\|x\|^2
\end{aligned}$$

Here, the constant M depends only on K, α and β, but not on r. It therefore follows easily from the Monotone Convergence Theorem that

$$\sum_{n=0}^\infty (1+n)^\beta\|FT^nx\|^2_{\mathcal{Y}} \leq M\|x\|^2 \qquad (x\in\mathcal{H})$$

and F is a β-admissible observation operator for T. ■

By duality we obtain the following result.

Theorem 3.4. *Let $\beta \in (-1,0)$. Let $\mathcal{H}$, $\mathcal{U}$ be Hilbert spaces and let $T \in \mathcal{L}(\mathcal{H})$ be a γ-hypercontraction for some $\gamma > 1$. Let $E \in \mathcal{L}(\mathcal{U},\mathcal{H})$. Then the following are equivalent:*

1. *E is a β-admissible control operator for T.*
2. $$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\frac{1+\beta}{2}}\|(I-\bar{z}T)^{-\beta-1}E\|_{\mathcal{L}(\mathcal{H},\mathcal{Y})} < \infty.$$

Remark 3.5. *Theorem 3.3 in particular shows Wynn's result [24] for β-admissibility of normal discrete contractive semigroups, also for infinite-dimensional output space.*

4 Continuous-time β-admissibility

We consider a continuous-time control system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0,\, t \geq 0, \\ y(t) &= Cx(t),\, t \geq 0. \end{aligned}$$

Here A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. Writing $\mathcal{H}_1 = D(A)$ and $\mathcal{H}_{-1} = D(A^*)^*$, we suppose that $B \in \mathcal{L}(\mathcal{U},\mathcal{H}_{-1})$ and $C \in \mathcal{L}(\mathcal{H}_1,\mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces as well.

Definition 4.1. *Let $\beta > -1$.*

1. *B is called a β-*admissible control operator for $(T(t))_{t\geq 0}$*, if there exists a constant $M > 0$ such that*
$$\left\| \int_0^\infty T(t)Bu(t)\,dt \right\| \leq M\|u\|_{L^2_\beta(0,\infty;\mathcal{U})}$$
for every $u \in L^2_\beta(0,\infty;\mathcal{U})$.
2. *C is called a β-*admissible observation operator for $(T(t))_{t\geq 0}$*, if there exists a constant $M > 0$ such that*
$$\int_0^\infty t^\beta \|CT(t)x\|^2\,dt \leq M\|x\|^2_{\mathcal{H}}$$
for every $x \in \mathcal{H}_1$.

Remark 4.2. Similarly as for discrete-time systems it can be shown for $\beta \in (-1,1)$ that B is a β-admissible control operator for $(T(t))_{t\geq 0}$ if and only if B^* is a $(-\beta)$-admissible observation operator for $(T^*(t))_{t\geq 0}$. Note that in [8] a different definition of weighted admissibiity for control operators was given, for which the duality does not hold in this form. We refer to the comments following [8, Rem. 1.2] for more information.

The following result is proven in [26, Propositions 2.1 and 2.2] for $\beta \in (0,1)$. The trivial extension to the case $\beta > 0$ is given for completeness. For $\alpha > -1$ we write $A^2_\alpha(\mathbb{C}_+)$ for the Bergman space on the right half-plane corresponding to the measure $x^\alpha\,dx\,dy$.

Proposition 4.3. *Let $\beta > 0$. Suppose that A generates a contraction semigroup on $\mathcal{H}$ and that $C \in \mathcal{L}(D(A), \mathcal{Y})$. Define the cogenerator $T \in \mathcal{L}(\mathcal{H})$ by $T := (I + A)(I - A)^{-1}$ and $F := C(I - A)^{-(1+\beta)} \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$. Then the following statements hold.*

1. *C is a β-admissible observation operator for $(T(t))_{t \geq 0}$ if and only if F is a β-admissible observation operator for T.*

2. *The resolvent condition* (3) *for (F, T) holds if and only if*

$$\sup_{\lambda \in \mathbb{C}_+} (Re\, \lambda)^{\frac{1+\beta}{2}} \| C(\lambda - A)^{-(1+\beta)} \| < \infty.$$

Proof 1. F is β-admissible for T if and only if $\Lambda : A^2_{\beta-1}(\mathbb{D}) \to \mathcal{L}(\mathcal{H}, \mathcal{Y})$ defined initially on reproducing kernels by $\Lambda f = F f(T)$ extends to a bounded linear operator. On the other hand, C is β-admissible for A if and only if $\tilde{\Lambda} : A^2_{\beta-1}(\mathbb{C}_+) \to \mathcal{L}(\mathcal{H}, \mathcal{Y})$ defined initially on reproducing kernels by $\tilde{\Lambda}(g) = Cg(-A)$ extends to a bounded linear operator. That the two conditions are equivalent follows from the fact that for any $\beta > 0$ there is an isomorphism $J_\beta : A^2_{\beta-1}(\mathbb{D}) \to A^2_{\beta-1}(\mathbb{C}_+)$ for which $\Lambda = \tilde{\Lambda} \circ J_\beta$ holds on each reproducing kernel.

2. Follows directly from the identities

$$D(I - \bar{z}T)^{-(1+\beta)} = \frac{CR\left(\frac{1-\bar{z}}{1+\bar{z}}, A\right)^{1+\beta}}{(1 + \bar{z})^{1+\beta}}, \qquad z \in \mathbb{D}$$

and

$$\mathrm{Re}\left(\frac{1-z}{1+z}\right) |1 + z|^2 = (1 - |z|^2), \qquad z \in \mathbb{D}.$$

■

Our main theorems concerning continuous-time systems are as follows.

Theorem 4.4. *Let $\beta > 0$. Let $(T(t))_{t \geq 0}$ be a contraction semigroup on $\mathcal{H}$ such that the adjoint of the cogenerator T^* is γ-hypercontractive for some $\gamma > 1$. Then the following are equivalent:*

1. *C is β-admissible observation operator for $(T(t))_{t \geq 0}$.*

2. $$\sup_{\lambda \in \mathbb{C}_+} (Re\, \lambda)^{\frac{1+\beta}{2}} \| C(\lambda - A)^{-(1+\beta)} \| < \infty.$$

Proof The statement of the theorem follows from Proposition 4.3 together with Theorem 3.3. ■

Remark 4.5. *T^* is γ-hypercontractive if every operator $T^*(t)$, $t \geq 0$, is γ-hypercontractive. If A generates a contraction semigroup of normal operators, then the adjoint of the cogenerator of $(T(t))_{t \geq 0}$ is γ-hypercontractive for each $\gamma \geq 1$, see Section 2.*

By duality we obtain the following result.

Theorem 4.6. *Let $\beta \in (-1,0)$. Let $(T(t))_{t\geq 0}$ be a contraction semigroup on $\mathcal{H}$ such that the cogenerator T is γ-hypercontractive for some $\gamma > 1$. Then the following are equivalent:*

1. *B is β-admissible control operator for $(T(t))_{t\geq 0}$.*
2. $$\sup_{\lambda\in\mathbb{C}_+} (Re\,\lambda)^{\frac{1+\beta}{2}} \|(\lambda - A)^{-(1+\beta)}B\| < \infty.$$

Theorems 4.4 and 4.6 give positive results for $\beta > 0$ and adjoints of γ-hypercontractions in the case of observation operators, and for $\beta < 0$ and γ-hypercontractions in the case of control operators. The remaining possibilities for $\beta \in (-1,0) \cup (0,1)$ can be shown not to hold by means of various counterexamples. For $\beta \in (-1,0)$ the counterexample for normal semigroups given in [25] shows that there is no positive result for observation operators in either the γ-hypercontractive or adjoint γ-hypercontractive case. For $\beta \in (0,1)$, there is a counterexample in [25] based on the unilateral shift, which is 2-hypercontractive, see Figure 3. By Remark 4.2, these provide appropriate counterexamples for control operators as well.

	T γ-hypercontr. for some $\gamma > 1$	T^* γ-hypercontr. for some $\gamma > 1$
$\beta \in (-1,0)$	Counterexample [25]	Counterexample [25]
$\beta \in (0,1)$	Counterexample [25]	Conjecture holds by Theorem 4.4

Figure 3: Weighted Weiss conjecture for observation operators

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