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Christoph Hachtel, Andreas Bartel, Michael Günther

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Christoph Hachtel[1,*], Andreas Bartel, Michael Günther

Bergische Universität Wuppertal, Chair of Applied Mathematics/Numerical Analysis, Gaußstraße 20, 42119 Wuppertal, Germany

Abstract

In this paper we investigate coupled systems, where a MOR scheme is applied to one subsystem and a multirate method to the resulting system. We study the influence of the model order reduction on the stability of the coupled system and on the error and stability of a multirate time integration scheme. For balanced truncation model order reduction applied to the subsystem we derive a condition that guarantees the stability of the coupled system. A bound for the time domain error that is caused by the model order reduction is presented. For the time integration the multirate θ-method is used. We will show that this method is also stable for systems with order reduced subsystems.

Keywords: Multirate time stepping, model reduction, coupled systems, balanced truncation

1. Introduction

The numerical approximation of solutions of ordinary differential equations (ODEs) is usually computed at discrete time points. The distance between these time points

*Corresponding author

Email addresses: hachtel@math.uni-wuppertal.de (Christoph Hachtel), bartel@math.uni-wuppertal.de (Andreas Bartel), guenther@math.uni-wuppertal.de (Michael Günther)

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is determined by the dynamical properties of the ODE system. If there are only few components that provide fast dynamical changes, the whole system has to be integrated with an accordingly small stepsizes. This makes the numerical integration slow and inefficient. Multirate (MR) time integration schemes avoid this phenomena by integrating fast changing components with small step sizes and slow changing ones with larger step sizes. The crucial part here is the coupling between slow and fast subsystems. Multirate time integration schemes were introduced for BDF schemes in [1] with a simple coupling by inter-/extrapolation and were further developed and improved in e.g. [2, 3, 4, 5].

The application of a model order reduction (MOR) scheme to the slow components decrease the computational effort even further. Here, all slowly changing components are seen as one subsystem of a coupled system. This usually large dimensional, slowly changing subsystem is projected onto a small dimensional space.

Applying a MOR directly to the slow subsystem leaves the dimension of the coupling interface between the subsystems unchanged. This problem can be handled by an interface reduction, which is shown in [6]. There, a thermal-electric coupled system was integrated by a compound-step multirate method [7] and the slow subsystem, namely the temperature in a resistor, was reduced by a balanced truncation method, cf. [8]. The numerical multirate simulation of the thermal-electric coupled system shows a significant decrease of computational effort while the error to the non-reduced, single-rate reference solution was small. Nevertheless a detailed error and stability analysis was still missing.

In this work we will fill this gap by providing stability conditions for the system and the method and we will present an error estimation for the error between the multirate approximation using a reduced order, slow subsystem and the analytical solution of the original system. Here we will use the multirate θ-method by Hundsdorfer and Savcenco in [4] since this time integration scheme is more stable and suits better for stiff ODE systems.

The work is organised as follows: First we will introduce all necessary theory and notation and we will give an elegant and appropriate error splitting so that we can investigate the model order reduction error and the multirate error separately. The second part of the work is dedicated to the influence of the model order reduction. We will analyse the stability of the complete system with a reduced subsystem and we will give an error bound for the coupled system. In the third part we will show that the multirate θ-method can be also used for systems with a reduced order, slow subsystem since its influence on the stability of the multirate time integration scheme is quite small.

2. Problem Setting

We consider linear systems of ordinary differential equations

$$\dot{w}(t) = Aw(t), \qquad w(t_0) = w_0 \tag{1}$$

with $w(t) \in \mathbb{R}^n$, a matrix $A \in \mathbb{R}^{n\times n}$ and an initial condition $w_0 \in \mathbb{R}^n$. The system shall have a particular dynamic behaviour: A small number of components provides a high dynamical behaviour while the remaining components are changing much slower. By resorting the components we can achieve the following structure

$$\begin{pmatrix} \dot{w}_A(t) \\ \dot{w}_L(t) \end{pmatrix} = \begin{pmatrix} A_{AA} & A_{AL} \\ A_{LA} & A_{LL} \end{pmatrix} \begin{pmatrix} w_A(t) \\ w_L(t) \end{pmatrix}, \qquad w_0 = \begin{pmatrix} w_{A,0} \\ w_{L,0} \end{pmatrix} \tag{2}$$

with fast changing variables $w_A(t) \in \mathbb{R}^{n_A}$, slow changing variables $w_L(t) \in \mathbb{R}^{n_L}$ and corresponding block matrices. We call the subsystem that describes w_A active or fast changing subsystem and the subsystem that describes w_L slow subsystems (in particular in electric circuit literature the latter is often referred to a latent subsystem). The coupling between the two subsystems is given by the off-diagonal blocks in (2). If the influence of the subsystems on each other is too strong this particular partitioning is not justified any more. Thus the assumption of a weakly coupled system, i.e.,

$$\|A_{AL}\| < \epsilon \qquad \text{and} \qquad \|A_{LA}\| < \epsilon \tag{3}$$

for any induced matrix norm and a fixed, small $\epsilon \in \mathbb{R}^+$ is reasonable. For stability reasons we assume that the system matrix $A = (a_{ij})_{i,j=1}^n$ is strict diagonal dominant and its diagonal entries are all negative

$$a_{ii} + \sum_{i\neq j} |a_{ij}| < 0 \qquad \text{for } i = 1, \dots, n. \tag{4}$$

Hence all eigenvalues of A have negative real part and the system is stable in sense of system theory.

The logarithmic matrix-norm for $M \in \mathbb{R}^{n\times n}$

$$\mu(M) := \lim_{h\to 0^+} \frac{\|I + hM\| - 1}{h} \tag{5}$$

has the following well known properties:

$$\|e^{Mt}\| \leq e^{\mu(M)t} \tag{6}$$

$$\mu(V^\top MV) \leq \mu(M) \tag{7}$$

for any matrix $V \in \mathbb{R}^{n\times r}$ with $V^\top V = I_r$. Next we will sketch the multirate θ-method and the necessary aspects of model order reduction.

2.1. Multirate Time Integration

Systems of ODEs like (2) can be efficiently integrated by multirate schemes, which use different step sizes for the subsystems: slow ones with a large macro-step τ and the active one with smaller micro-steps τ/m for $m \in \mathbb{N}$. The multirate θ-method was introduced in [4]. Here the coupling between the subsystem is achieved by refining the time grid for the active subsystem: Given the current time t_{n-1} and the macro step size τ, it first integrates the whole system (1) with a classical θ-method on $[t_{n-1}, t_{n-1} + \tau]$

$$\bar{w}_n = w_{n-1} + \theta\tau A w_{n-1} + (1-\theta)\tau A \bar{w}_n.$$

For the slow subsystem this approximation is accepted: $\bar{w}_{L,n} =: w_{L,n}$ (τ is adapted to the dynamics of the slow subsystem). The approximation of the active subsystem at t_n is achieved by computing m intermediate approximations with stepsize τ/m. Therefore the values of the slow subsystems at the intermediate time steps are necessary. They are approximated by linear or quadratic interpolation of $w_{L,n-1}$ and $w_{L,n}$. In [4], there is a detailed error and stability analysis for this multirate method.

2.2. Model Order Reduction (MOR)

Applying MOR to the slow subsystem will make the multirate scheme even more efficient. We sketch the general setting of MOR (we recommend [8] for any further details). Let be given a linear time invariant system (LTI)

$$\begin{aligned} \dot{x}(t) &= Mx(t) + Bu(t), \qquad x(t_0) = x_0 \\ y(t) &= Cx(t) \end{aligned} \tag{8}$$

with input $u(t) \in \mathbb{R}^p$, state-space vector $x(t) \in \mathbb{R}^n$, output $y(t) \in \mathbb{R}^m$, input matrix $B \in \mathbb{R}^{n\times p}$, system matrix $M \in \mathbb{R}^{n\times n}$, output matrix $C \in \mathbb{R}^{m\times n}$ and initial condition $x_0 \in \mathbb{R}^n$. Now the number of internal states is reduced while the input-output behaviour $u(t) \rightsquigarrow y(t)$ is approximated with sufficient accuracy. This is achieved by projecting the state-space vector and the matrices on a lower dimensional vector space. Therefore bi-orthogonal projection matrices $V, W \in \mathbb{R}^{n_L\times r}$ are computed. They satisfy $W^\top V = I$ and therefore $VW^\top$ is a projector. The reduced order model (ROM) reads:

$$\begin{aligned} \dot{x}_r(t) &= W^\top MVx_r(t) + W^\top Bu(t), \qquad x_r(t_0) = W^\top x_0 \\ \tilde{y}(t) &= CVx(t). \end{aligned}$$

There are several possibilities to compute V and W. We will follow the idea of balanced truncation. For this choice one obtains V and W by solving certain Lyapunov equations. There are methods that are more suitable for large scale systems by avoiding solving the Lyapunov equations directly, see e.g. [9]. Balanced truncation is stability preserving so that the reduced model of a stable system is again stable [10]. Another advantage of this method are the a-priori error bound

$$\|G_r - G\|_{\mathbb{H}_\infty} \leq= \gamma := 2(\sigma_{r+1} + \ldots + \sigma_{n_L}). \tag{9}$$

where G_r and G denote the transfer function of the reduced order and the original system, and $\sigma_{r+1}, \ldots, \sigma_{n_L}$ denote the truncated Hankel singular values of the system. The $\mathbb{H}_\infty$ of a transfer function G is defined as $\|G\|_{\mathbb{H}_\infty} = \sup_{\omega\in\mathbb{R}} \|G(i\omega)\|_2$. These error bounds are given in frequency domain. Since we aim at an error estimation directly in time domain, we follow another strategy, where we exploit a close link between balanced truncation and another MOR-method: proper orthogonal decomposition (POD). In POD a number of snapshots is calculated (these are vectors that describe the solution of the full order model at certain time points) and arranged in a snapshot matrix. Via a singular value decomposition the projection matrix is calculated. This method can be used for nonlinear systems. Applied to linear systems using a particular input-function the resulting projection matrix corresponds to a "one-sided" balanced truncation but can be easily completed to a classic balanced truncation projection matrix ([8], chapter 9.1.3). This allows us to use results for error estimation for POD-reduced systems like in [11]. Let x be the solution of a general, nonlinear system of ODEs $\dot{x}(t) = F(t, x)$ and V the projection matrix calculated with a POD (here it is $V = W$). Then for the squared 2-norm error holds

$$\int_0^T \|x(t) - VV^\top x(t)\|_2^2 dt = C_r \tag{10}$$

where C_r depends on the system and the rank of V, but can be calculated explicitly, see [11].

2.3. MOR for coupled systems and error splitting

Now we come back to the system of ODEs (2)

$$\dot{w}_A(t) = A_{AA}w_A(t) + A_{AL}w_L(t), \qquad w_A(t_0) = w_{A,0} \tag{11}$$
$$\dot{w}_L(t) = A_{LA}w_A(t) + A_{LL}w_L(t), \qquad w_L(t_0) = w_{L,0}. \tag{12}$$

Our goal is to apply a MOR method to the slow subsystem (12) and to investigate the error that is caused by the combination of the MOR and the multirate θ-method. Due to the coupling term $A_{AL}w_L$ the MOR error also influences the active subsystem. Therefore we cannot use the classic error and stability analysis of MOR, but we have to consider the aspects of MOR of coupled systems as described in [10]. To this end, the ODE system (11-12) has to be rewritten in the form of coupled LTI systems, where u_A and y_A denote the input/output of the active subsystems, and u_L, y_L denote the input/output of the slow subsystem, respectively.

$$
\begin{aligned}
\dot{w}_A &= A_{AA} w_A + I_{n_A} u_A & \dot{w}_L &= A_{LL} w_L + I_{n_L} u_L \\
y_A &= I_{n_A} w_A, & y_L &= I_{n_L} w_L \\
\text{IC: } w_A(t_0) &= w_{A,0} & w_L(t_0) &= w_{L,0}.
\end{aligned} \tag{13}
$$

The coupling is given by

$$
\begin{aligned}
u_A &= K_{AA} y_A + K_{AL} y_L = A_{AL} w_L \\
u_L &= K_{LA} y_A + K_{LL} y_L = A_{LA} w_A,
\end{aligned} \tag{14}
$$

using the notation of [10]. A model order reduction for the slow subsystem leads to

$$
\begin{aligned}
\dot{\tilde{w}}_A &= A_{AA} \tilde{w}_A + A_{AL} \tilde{y}_L & \dot{w}_{L,r} &= W^\top A_{LL} V w_{L,r} + W^\top A_{LA} \tilde{y}_A \\
\tilde{y}_A &= \tilde{w}_A, & \tilde{y}_L &= V w_{L,r} \\
\text{IC: } \tilde{w}_A(t_0) &= w_{A,0} & w_{L,r}(t_0) &= W^\top w_{L,0}.
\end{aligned} \tag{15}
$$

For the multirate θ-method the interval $[t_0, t]$ is discretised in n time points. The error between the analytical solution $(w_A(t)^\top, w_L(t)^\top)^\top$ of the whole system (13) at time t and the approximation $(\tilde{w}_{A,n}^\top, w_{L,r,n}^\top)^\top$ by the multirate θ-method using a reduced order model for the slow subsystem (15) reads

$$
\begin{aligned}
E(t) = \begin{pmatrix} E_A(t) \\ E_L(t) \end{pmatrix} &= \begin{pmatrix} w_A(t) \\ w_L(t) \end{pmatrix} - \begin{pmatrix} \tilde{w}_{A,n} \\ V w_{L,r,n} \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} w_A(t) \\ w_L(t) \end{pmatrix} - \begin{pmatrix} \tilde{w}_A(t) \\ V w_{L,r}(t) \end{pmatrix}} + \underbrace{\begin{pmatrix} \tilde{w}_A(t) \\ V w_{L,r}(t) \end{pmatrix} - \begin{pmatrix} \tilde{w}_{A,n} \\ V w_{L,r,n} \end{pmatrix}} \\
&= \qquad\quad E_{MOR}(t) \qquad\qquad + \qquad\qquad E_{MR}(t).
\end{aligned} \tag{16}
$$

This error splitting allows us to apply a pure MOR-based error-analysis to E_{MOR} and to use aspects of the stability analysis of the θ-method of [4] for E_{MR}. First, we will investigate the stability of the coupled system. Then we will use the error splitting to derive an error bound for the MOR caused error. Finally we will study the influence of the order reduced subsystem on the stability of the multirate θ-method.

3. Model order reduction error for coupled systems

In this section we will analyse the influence of the model order reduction of the slow subsystem to the complete system. First we will investigate the asymptotic stability of the coupled system with order reduced subsystem. In the second part, we will derive an estimation for $E_{MOR}(t)$. To this end both systems (8) and (15) are solved over the interval $[0,T]$ with initial values $w_{i,0} = w_i(0)$ for $i \in \{A, L\}$. Then we will present an estimation for the integral of the error

$$\int_0^T \|E_{MOR}(t)\|_2^2 dt$$

in time domain without using frequency domain estimations.

3.1. Stability of the coupled system

A LTI system (8) is called asymptotic stable if all eigenvalues of the system matrix M have negative real part. Due to the assumption given in (4), both the complete system (1) and the two subsystems (11) and (12) are asymptotic stable. A reduced order model of an asymptotic stable LTI system, which is computed by balanced truncation, is again stable, cf. [10]. For the asymptotic stability of the complete system, an additional condition is provided in [10], such that also the complete system is again asymptotic stable. We will introduce all necessary notation and the condition for asymptotic stability and motivate why this condition can be fulfilled by the coupled, order reduced system (15). Let

$$G_i = C_i(I_{n_i} - A_{ii})^{-1}B_i \qquad i \in \{A, L\} \tag{17}$$

denote the transfer function of the subsystems and let X be the solution of the Lyapunov equation

$$\Psi X \Psi^\top - X = I \tag{18}$$

with matrices $\Psi = \Phi_2 \operatorname{diag}(\|G_A\|_{\mathbb{H}_\infty}, \|G_L\|_{\mathbb{H}_\infty})$, $\Phi_2 = (\|K_{ij}\|_2)_{i,j\in\{A,L\}}$ and a coupling matrix K like in (14). The complete system with order reduced slow subsystem is asymptotic stable if [10]

$$14\gamma\|\Phi_2\|_2\|X\|_2 < 1. \tag{19}$$

In our particular case the coupling matrix K reads

$$K = \begin{pmatrix} 0 & A_{AL} \\ A_{LA} & 0 \end{pmatrix}.$$

Using the assumption of a weakly coupled system given in (3), we get $\|\Phi_2\|_2 \leq \epsilon$. So this property of the system increases the stability of the complete system. It is also natural to assume that A_{LL} has fast decreasing Hankel singular values. Then γ becomes a small number which is controlled by the size of the reduced system.

Lemma 1. *Let be given a coupled linear system of ODEs like in* (13) *which fulfils the assumptions* (3) *and* (4)*. The reduced order model of the slow subsystem is achieved by balanced truncation and the resulting coupled system is given by* (15)*. The coupled system with a reduced order, slow subsystem is asymptotic stable if*

$$14\gamma\epsilon\|X\|_2 < 1 \tag{20}$$

with the above notation.

Note that this condition can always be fulfilled by choosing r sufficiently large.

3.2. Error of order reduced coupled system

Next, we derive an estimate for the integral of the error $E_{MOR}(t)$ in (16). The error estimation is inspired by the work of Chaturantabut and Sorensen [11], who did a similar proof to derive an error bound for POD-DEIM method. They investigated a nonlinear, uncoupled system and we treat a linear system with multirate behaviour. To investigate the error, we decompose it once more

$$\begin{aligned}\|E_{MOR}(t)\|^2 = \left\| \begin{pmatrix} E_{A,MOR}(t) \\ E_{L,MOR}(t) \end{pmatrix} \right\|^2 &\leq \|E_{A,MOR}(t)\|^2 + \|E_{L,MOR}(t)\|^2 \\ &\leq \|E_{A,MOR}(t)\|^2 + \|\underbrace{w_L(t) - VW^\top w_L(t)}_{=:\rho(t)}\|^2 \\ &\quad + \|\underbrace{VW^\top w_L(t) - Vw_{L,r}(t)}_{=:\theta(t)}\|^2. \end{aligned} \tag{21}$$

It is clear that $E_{A,MOR}(t)$ and $\theta(t)$ depend on each other, the integral of $\|\rho(t)\|^2$ can be estimated according to (10). We denote this error bound by

$$\int_0^T \|\rho(t)\|_2^2 dt \leq C_r \tag{22}$$

with a constant $C_r \geq 0$ that depends on the dimension of the reduced order, slow subsystem. The crucial part for an estimation of the complete error will be an estimation of $\theta(t)$. We define $\hat{\theta}(t) = W^\top \theta(t)$. Note that it is $\|\hat{\theta}(t)\|_2 = \|\theta(t)\|_2$. Using the differential equation of w_L the differential equation for $\hat{\theta}(t)$ reads

$$\dot{\hat{\theta}}(t) = \hat{A}_{LL}\hat{\theta}(t) + W^\top A_{LA} E_{A,MOR}(t) \tag{23}$$

with $\hat{A}_{LL} = W^\top A_{LL} V$. The analytical solution is given by

$$\hat{\theta}(t) = \int_0^t e^{\hat{A}_{LL}(t-s)} W^\top A_{LA} E_{A,MOR}(s) ds \tag{24}$$

since $\hat{\theta}(0) = 0$. Writing $W^\top A_{LA} E_{A,MOR}(s)$ as a differential equation and computing the analytical solution gives

$$\begin{aligned} W^\top A_{LA} E_{A,MOR}(s) = & \underbrace{W^\top A_{LA} \int_0^t e^{A_{AA}(t-s)} A_{AL}\rho(s) ds}_{=:g(t)} \\ & + \underbrace{W^\top A_{LA} \int_0^t e^{A_{AA}(t-s)} A_{AL}\theta(s) ds}_{=:h(t)}, \end{aligned}$$

where we introduced functions g and h. Applying norms to (24) and inserting the latest results we get the following inequality

$$\|\hat{\theta}(t)\|_2 \leq \int_0^t \|e^{\hat{A}_{LL}(t-s)}\|_2 \|g(s)\|_2 ds + \int_0^t \|e^{\hat{A}_{LL}(t-s)}\|_2 \|h(s)\|_2 ds. \tag{25}$$

Our next step is to estimate the involved terms:

$$\|g(t)\|_2 \leq \|W^\top A_{LA}\|_2 \|A_{AL}\|_2 \int_0^t \|e^{A_{AA}(t-s)}\|_2 \|\rho(s)\|_2 ds.$$

Applying Cauchy-Schwarz inequality, using (6) and (10) and setting $\mu_A := \mu(A_{AA})$, we obtain

$$\begin{aligned} \|g(t)\|_2 &\leq C_r \|W^\top A_{LA}\|_2 \|A_{AL}\|_2 \left(\int_0^t (e^{\mu_A (t-s)})^2 ds \right)^{1/2} \\ &\leq C_r \|W^\top A_{LA}\|_2 \|A_{AL}\|_2 \left(q_{2\mu_A}(T)\right)^{1/2} =: \eta(r) = \eta \end{aligned}$$

with

$$q_\mu(t) = \int_0^t e^{\mu(t-s)} ds = \begin{cases} \frac{1}{\mu}(e^{\mu t} - 1) & \mu \neq 0 \\ t & \mu = 0 \end{cases}. \tag{26}$$

A similar argument leads to

$$\int_0^t \|e^{\hat{A}_{LL}(t-s)}\|_2 \|g(s)\|_2 ds \leq \eta \int_0^t \|e^{\hat{A}_{LL}(t-s)}\|_2 ds \leq \eta \int_0^T e^{\mu_L(T-s)} ds = \underbrace{\eta q_{\mu_L}(T)}_{=:\tilde{\eta}}$$

with $\mu_L = \mu(A_{LL})$. The function $h(t)$ can be estimated in terms of $\theta(t)$ as follows:

$$\begin{aligned}\|h(t)\|_2 &= \|W^\top A_{LA}\|_2 \|A_{AL}\|_2 \int_0^t \|e^{A_{AA}(t-s)}\|_2 \|\theta(s)\|_2 ds \\ &\leq \underbrace{\|W^\top A_{LA}\|_2 \|A_{AL}\|_2 \left(q_{2\mu_A}(T)\right)^{1/2}}_{=:\varphi} \left(\int_0^t \|\theta(s)\|_2^2 ds\right)^{1/2}.\end{aligned}$$

With this result, we get:

$$\begin{aligned}\int_0^t \|e^{\hat{A}_{LL}(t-s)}\|_2 \|h(s)\|_2 ds &\leq \varphi(q_{2\mu_L}(T))^{1/2} \left(\int_0^t \int_0^s \|\theta(\tau)\|_2^2 d\tau ds\right)^{1/2} \\ &\leq \varphi(q_{2\mu_L}(T))^{1/2} \left(\int_0^t \int_0^t \|\theta(\tau)\|_2^2 d\tau ds\right)^{1/2} \\ &\leq \varphi(q_{2\mu_L}(T))^{1/2} \left(t \int_0^t \|\theta(\tau)\|_2^2 d\tau\right)^{1/2}.\end{aligned}$$

Inserting all estimations into (25), we find

$$\|\hat{\theta}(t)\|_2 = \|\theta(t)\|_2 \leq \tilde{\eta} + \varphi(q_{2\mu_L}(T))^{1/2} \left(t \int_0^t \|\theta(\tau)\|_2^2 d\tau\right)^{1/2}.$$

Building the square of the previous inequality and using that $(a+b)^2 \leq 2a^2 + 2b^2$, we obtain

$$\|\theta(t)\|_2^2 \leq \underbrace{2\tilde{\eta}^2}_{=:\hat{\eta}} + \underbrace{2\varphi^2 q_{2\mu_L}(T) T}_{=:\hat{\varphi}} \int_0^t \|\theta(\tau)\|_2^2 d\tau = \hat{\eta} + \hat{\varphi} \int_0^t \|\theta(\tau)\|_2^2 d\tau.$$

Applying Gronwall's lemma, we get

$$\|\theta(t)\|_2^2 \leq \hat{\eta} e^{\hat{\varphi} t}. \tag{27}$$

The differential equation for $E_{A,MOR}$ is given by

$$\dot{E}_{A,MOR}(t) = A_{AA} E_{A,MOR}(t) + A_{AL} \left(\rho(t) + \theta(t)\right), \quad E_{A,MOR}(0) = 0.$$

With (27) we can estimate the analytical solution of this differential equation as follows:

$$\|E_{A,MOR}(t)\|_2^2 \leq \underbrace{2\|A_{AL}\|_2^2 q_{2\mu_A}(T)}_{=:\nu} \left(C_r + \int_0^t \|\theta(t)\|_2^2 dt \right). \tag{28}$$

Using (22), (27) and (28), we can estimate the final integral inequation

$$\begin{aligned}
\int_0^T \|E_{MOR}(t)\|_2^2 dt &\leq \int_0^T \|E_{A,MOR}(t)\|_2^2 dt + \int_0^T \|\rho(t)\|_2^2 dt + \int_0^T \|\theta(t)\|_2^2 dt \\
&\leq \nu C_r T + \nu T \int_0^T \|\theta(t)\|_2^2 dt + C_r + \int_0^T \|\theta(t)\|_2^2 dt \\
&= (1 + \nu T) \left(C_r + \int_0^T \|\theta(t)\|_2^2 dt \right) \qquad (29)
\end{aligned}$$

The integral can be further estimated by (25) and we get an estimation that only depends on the the system properties like the norms of of the partitioned matrices, the integration time T and the size of the reduced order slow subsystem r. The following proposition summerises the previous results.

Proposition. *Let be given a linear, stable, partitioned ODE system like in* (2) *which fulfils the assumptions* (3) *and* (4) *with the particular multirate behaviour of the subsystems and apply a balanced truncation model order reduction to the slow changing subsystem or rather the associated LTI system* (13)*. Then the coupled system with a reduced order, slow subsystem* (15) *of dimension r is again stable by choosing r sufficiently large and the error that is caused by model order reduction can be bounded in time domain by*

$$\int_0^T \|E_{MOR}(t)\|_2^2 dt \leq (1 + \nu T) \left(C_r + \frac{\hat{\eta}}{\hat{\varphi}} (e^{\hat{\varphi} T} - 1) \right).$$

The constants ν, C_r, $\hat{\eta}$, $\hat{\varphi}$ are explained in the calculation above.

Remark. *If the active subsystem does not depend on the slow one ($A_{AL} = 0$ and therefore $\nu = 0$) the error is a pure MOR error of the slow subsystem.*

At this point we want to remind the reader of assumption (3), so the additional error in (29) that is caused by the coupling will not dominate the MOR error.

4. Stability of multirate θ-method for order reduced subsystem and final result

Next we investigate the multirate error $E_{MR}(t)$ in (16). This is the error between the analytical solution and a multirate θ-method approximation of (15). For the multirate time integration, the interval $[0, T]$ is split into n macro-steps of length $\tau = T/n$, the macro-steps are refined once so the active subsystem is integrated with a micro-step size $\tau/2$. In terms of [4], the error E_n after n macro-steps can be expressed by the error at the previous macro-step E_{n-1}, an amplification matrix S and the local truncation error z_n:

$$E_n = SE_{n-1} + z_n. \tag{30}$$

The local truncation error is caused by the multirate θ-method. A short calculation shows that a model order reduction of the slow subsystem does not change the main properties of z_n and the order of the method remains 2 in case of $\theta = 0.5$ and 1 otherwise. We assume that the multirate θ-method is stable for the original system (2). That means that any product of S with itself is bounded by a moderate constant $M > 0$ such that

$$\|S^n\|_\infty < M \quad \forall\, n \geq 0. \tag{31}$$

Now we show that the stability of the multirate θ-method for the original, full order system can be used to derive stability of the multirate θ-method for a system with a reduced order, slow subsystem (15). To this end, we will use a perturbation condition which is given in [4]: Let (31) hold for (1) and let $\tilde{A}$ be the matrix of a perturbed system such that $\|A - \tilde{A}\|_\infty \leq L$ for a moderate constant L. Then the amplification matrix $\tilde{S}$ of the perturbed system can be bounded by

$$\|(\tilde{S})^n\|_\infty \leq Me^{CMT} \tag{32}$$

with a constant C depending on L and M. For the particular case of a balanced, slow subsystem, we will derive the system matrix $\tilde{A}$ and measure the perturbation L. Let the system matrices of the slow subsystem be of the following form

$$A_{LL} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad A_{LA} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \qquad A_{AL} = (C_1,\ C_2),$$

where $A_{11},\ B_1$ and C_1 describe the states of the system that will kept by the model order reduction. If the system is balanced, the projection matrices for balanced truncation are given by $V = W = (I,\ 0)^\top$, cf. [8], chapter 7.2. Then the system matrix of the coupled system with reduced order, slow subsystem reads

$$\begin{pmatrix} A_{AA} & C_1 \\ B_1 & A_{11} \end{pmatrix}.$$

We claim that

$$\|A_{12}\|_\infty \leq d_1 \|A_{22}\|_\infty \tag{33}$$

holds for a positive constant d_1. For the case of a symmetric matrix A_{LL} this statement is always true since $\|A_{12}\|_1 = \|A_{21}\|_\infty \leq \|A_{22}\|_\infty$. Similar to that is to claim the A_{LL} diagonal dominant by its columns. Before we can compute the size of the perturbation L, we have to identify the reduced order subsystem in the original vector space, i.e., we multiply the reduced matrices with the corresponding projection matrices:

$$\tilde{A} = \begin{pmatrix} A_{AA} & C_1 W^\top \\ V B_1 & V A_{11} W^\top \end{pmatrix} = \begin{pmatrix} A_{AA} & C_1 & 0 \\ B_1 & A_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{34}$$

And so we find

$$\|A - \tilde{A}\|_\infty = \left\| \begin{pmatrix} 0 & 0 & C_2 \\ 0 & 0 & A_{12} \\ B_2 & A_{21} & A_{22} \end{pmatrix} \right\|_\infty \tag{35}$$

using assumption (3) and (33) and the equivalence of the 2- and the ∞-norm yield

$$\leq 2\epsilon + d_1 \|A_{22}\|_\infty + \|A_{21}\|_\infty + d_2 \|A_{22}\|_2 \tag{36}$$

with positive constants d_1, d_2. Using assumption (4) we obtain

$$\|A - \tilde{A}\|_\infty \leq 2\epsilon + C_3 \sigma_{r+1} \tag{37}$$

with $d_3 = d_1 d_2 + 2d_2$ and σ_{r+1} the largest singular value of A_{22}. The previous results are summarised in the following lemma:

Lemma 2. *Let be given a system of ordinary differential equations like in* (2) *which fulfils the assumptions* (3), (4) *and* (33) *for that the multirate θ-method is stable and its slow subsystem is balanced. The reduced order, slow subsystem is achieved by balanced truncation. Then the multirate θ-method is also stable for the coupled system with reduced order, slow subsystem. The amplification matrix $\tilde{S}$ for the error is bounded by*

$$\|(\tilde{S})^n\|_\infty \leq M e^{CMT}$$

and C depends on the perturbation of the reduced order system matrix of the coupled system. The perturbation is bounded by

$$\|A - \tilde{A}\|_\infty \leq 2\epsilon + d_3\sigma_{r+1}$$

with the above notation.

The size of σ_{r+1} depends on the dimension of the reduced order, slow subsystem r. So r can be chosen in a way that σ_{r+1} is sufficiently small. Since the subsystems are only weakly coupled (cf. (3)) the influence of the coupling terms on the perturbation is weak. So the resulting perturbation is small and therefore stability of the θ-method can be also guaranteed for systems with reduced order, slow subsystem.

The results that are given in the previous lemma and proposition describe the influence of the model order reduction applied to the slow subsystem on the stability of the coupled system, on the stability of the multirate θ-method and on the split error in (16). The error that is caused by the MOR is here measured in an other norm than the error of the multirate θ-method. The following theorem summarises the previous results and adapts the norm of both split errors.

Theorem. *Let be given a partitioned system of ODEs* (2) *on the time interval* $[0, T]$ *that fulfils the assumptions* (3), (4) *and* (33). *Let the slow subsystem be balanced and let a balanced truncation MOR be applied to the slow subsystem. Let a multirate θ-method be applied to the partitioned ODE system with macro steps T/n and micro steps $T/2n$. Then the error between the analytical solution of the original, non-reduced system and the multirate θ-method approximation of the system with a reduced order, slow subsystem reads*

$$\int_0^T \|E(t)\|_2^2 dt \leq (1 - \nu T)\left(C_r + \frac{\hat{\eta}}{\hat{\varphi}}\left(e^{\hat{\varphi}T} - 1\right)\right) + d_4^2 M^2 T \left(\sum_{i=0}^{n-1} e^{CMt_i}\|z_i\|_2\right)^2$$

with the above notation for ν, C_r, $\hat{\eta}$, $\hat{\varphi}$, M, C and z_i, macro steps $t_i = i \cdot T/n$ and d_4 describes the equivalence between the 2- *and the* ∞*-norm.*

5. Conclusion

For a system of ODEs with a particular multirate behaviour a model order reduction for the slow changing subsystem combined with a multirate time integration increases the efficiency of the numerical time domain integration significantly as shown in [6]. In this work, we showed from theoretical point of view that both approaches can be combined in an stable way. We pointed out that the stability of the coupled system can be preserved by using balanced truncation MOR choosing the dimension r of the reduced order, slow subsystem in a proper way. We presented an error estimation in time domain that displays the influence of the MOR to the coupled system. Furthermore we showed that the stability of the multirate θ-method can be guaranteed also for coupled systems with a reduced order slow subsystem.

In this paper we investigated linear systems of ODEs. Whether the results of our work can be extended to the more general case of nonlinear ODEs or DAEs is an interesting field of future research.

To increase the efficiency of a combined multirate-MOR scheme a further investigation of the dimension reduction of the coupling interface is necessary. In future work we plan to study how the idea of interface reduction presented in [12] influence the efficiency and stability of multirate-MOR methods.

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