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PRICING SWING OPTIONS IN ELECTRICITY MARKETS WITH TWO STOCHASTIC FACTORS USING A PARTIAL DIFFERENTIAL EQUATIONS APPROACH*

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Abstract. In this paper we consider the valuation of swing options in electricity markets. This kind of contracts are modelled as path dependent options with multiple exercise rights. From the mathematical point of view the valuation of these products is posed as a sequence of free boundary problems where two exercise rights are separated by a time period.

In order to solve the pricing problem, we propose appropriate numerical methods based on a Lagrange-Galerkin discretization of the partial differential equation and an augmented Lagrangian active set method to cope with the early exercise feature. Moreover we derive appropriate artificial boundary conditions to treat numerically the unbounded domain. Finally some numerical results are presented in order to illustrate the proper behaviour of the numerical schemes.

Key words. Swing options, electricity price, option pricing, complementarity problem, Augmented Lagrangian Active Set formulation, Lagrange-Galerkin discretization, artificial boundary conditions

AMS subject classifications. 91G80, 35K65, 35K85, 65M25, 65M60, 90C33

Nowadays, due to the liberalization of the electricity markets, the electricity prices are determined by the principle of supply and demand. This absence of regularization affects the prices by increasing their volatility and introducing uncertainty. Due to this fact mainly companies, that buy directly electricity from an exchange, demand the existence of contracts which protect them against high prices but also give the possibility to get benefit from low prices. For an introduction of electricity market we address the reader to Chapter 2 of the recent book [1], for example.

In this paper we focus on one standard type of these contracts: swing options. Swing contracts are a kind of path dependent options that allow the holder to exercise a right multiple times over a period, with the constraint that each two consecutive exercise dates must be separated by a refracting period. That is, not all the rights can be exercised at a time. One example of this right could be to receive the payoff of a call option. Nevertheless, there are other possibilities such as the consideration of different payoff functions depending on the spot price, like calls and puts or calls with different strikes. Swing options are widely offered in the market and used by major energy companies, specially in electricity and fossil fuel markets. Sometimes the volume of the physical underlying commodity is also an additional state variable. A very interesting summary about the different features of swing option in practice and also its valuation can be found in Chapter 8 of [11], which is devoted to structured products based on fuels and commodities.

As indicated in [8], first discussions about swing options appeared in energy magazines [3] while a first rigorous treatment was addressed in [16]. The formulation of swing option prices in terms of multiple optimal stopping times represents a relevant step developed in [9], thus relating swing and American options. From this relation, the partial differential equation (PDE) approach to price swing options arises (for one exercise both options are equivalent). In this direction, we point out the references [10]

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and [27]. In [10] a one factor Ornstein-Uhlenbeck process for the logarithm of the commodity price is considered, a system of variational inequalities is posed and numerically solved. Also a one factor model is considered in [27]. Other alternative models to describe the evolution of the electricity prices are presented in [19] or in [2], where the author introduces a nonlinear Ornstein-Uhlenbeck process to model the spot prices which is jumpless but can incorporate spikes. More recently, in [14] a model with two stochastic factors is considered, although the pricing problem is solved by means of binomial trees. One of the innovative points of the present work is the consideration of numerical methods to solve the PDE formulation associated with a two factor model for electricity price. Hereby we consider the stochastic two factor model proposed in [14].

Financially, swing options can be equivalently handled as a portfolio of American type options with a waiting period (so called refracting period) between each two exercises. From the mathematical point of view, the swing option valuation problem can be posed as a sequence of free boundary problems, one for each right. Since in the obstacle function the value of the contract with one exercise right less is involved, additionally an initial boundary value problem (IBVP) restricted to a time interval of length equal to the refracting period needs to be solved.

In the literature, different numerical techniques have been employed to obtain the value of swing contracts. Binomial trees are considered in [16] but when the underlying is a one factor, seasonal, mean-reverting process. Also in [14] a binomial tree method is used when the spot price is modeled as the sum of a deterministic function to incorporate seasonality and two stochastic factors with the possibility of incorporating spikes. In other works, such as [20] or [24], the valuation of multiple stopping time problems is tackled with Monte Carlo simulation techniques. A first numerical solution of the PDE approach is provided in [10], in which the domain is truncated to a bounded domain and homogeneous Neumann boundary conditions are imposed. Additionally, finite elements or finite differences are combined with a PSOR algorithm to cope with the early exercise feature. Finite elements are also applied in [27] to solve the PDE problem when the spot price only depends on one stochastic factor whereas in [26] the solution of the PDE is discretized using finite difference schemes. Also in the case of electricity prices with one stochastic factor, swing options have been treated with Fourier based methods in [28]. Jump diffusion processes to describe the evolution of the underlying asset could be taken into account, thus leading to a partial integro-differential equation. In this setting a finite difference scheme combined with a dynamic programming technique has been used in [18] and an implicit-explicit finite difference scheme is proposed in [21]. As indicated in [8] practitioners usually value swing options by simulation techniques, however the rigorous error analysis associated with many simulation schemes is difficult.

In the present paper we propose the numerical solution of the two factor model by means of a Crank-Nicolson characteristic scheme for the time discretization combined with finite elements for the space discretization. To our knowledge the numerical solution of PDE models for swing options when two stochastic factors are considered in the electricity prices has not been addressed in the literature. The mathematical analysis of these discretization schemes has been already treated in [4, 5]. Furthermore, in order to deal with the inequalities associated with the early exercise feature of swing options we use the Augmented Lagrangian Active Set (ALAS) algorithm [17], which is far more efficient that the classical PSOR method or alternative duality or penalizations methods. Moreover, in order to obtain a numerical solution of the problem we need to replace the unbounded domain by a bounded one, hence appropriate boundary conditions are required. For this purpose, instead of using homogeneous Neumann boundary conditions (empirically motivated by the expression of the payoff function) as proposed in [10] for the one factor case, we derive more appropriate artificial boundary conditions (ABCs) based on the work of Halpern [13].

This paper is organized as follows. In Section 2 we describe the stochastic model for the spot electricity price under consideration and we state the mathematical problem that governs the valuation of swing contracts on this underlying. In Section 3 first we formulate the swing option pricing problem in a bounded domain after a localization procedure. Since we have to supply boundary conditions, we construct appropriate ABCs. Then, we introduce the discretization in time of the problem by using a Crank-Nicolson characteristic scheme and we state the variational formulation of the time discretized problem in order to apply finite elements. At the end of this section we describe the ALAS algoritm. Finally, in Section 4 we present some numerical results to illustrate our findings.

1. The mathematical model.

1.1. The electricity spot price model. In this work we assume (as in [14]) that under a risk neutral probability measure the electricity price, S_t, is a continuous time process

$$S_t = \exp(f(t) + \bar{X}_t + Y_t), \tag{1.1}$$

where f is a deterministic periodic function that represents the seasonality, $\bar{X}_t$ denotes the Ornstein-Uhlenbeck (OU) process with zero mean reversion level and mean reversion speed α

$$d\bar{X}_t \;=\; -\alpha \bar{X}_t dt \;+\; \sigma dW_t\,. \tag{1.2}$$

For the third component, Y_t, we consider the following stochastic differential equation (SDE)

$$dY_t \;=\; -\beta Y_t dt\,. \tag{1.3}$$

For convenience, we write the seasonal function f as a time-dependent mean reversion level of the process $\bar{X}_t$ and we introduce $X_t = \bar{X}_t + f(t)$. Next, we consider the following processes

$$M_t = \exp(X_t), \quad N_t = \exp(Y_t),$$

so that $S_t = M_t N_t$ and

$$\begin{aligned} dM_t &= \alpha\left(\mu(t) - \ln(M_t)\right) M_t dt \;+\; \sigma M_t dW_t\,, \\ dN_t &= -\beta \ln(N_t) N_t dt\,, \end{aligned} \tag{1.4}$$

with

$$\mu(t) = f(t) + \frac{1}{\alpha}\left(\frac{\sigma^2}{2} + f'(t)\right).$$

There exist other alternative models, either with one or two stochastic factors, to describe the spot electricity price evolution as the ones presented in [19].

1.2. The PDE formulation. The price of any asset whose value is a function of time t and the stochastic factors M_t and N_t (the dynamics of which are described by equations (1.4)) is given by a stochastic process, $V_t = V(t, M_t, N_t)$, where V denotes a sufficiently smooth function. Then, by using a dynamic hedging methodology similar to the case of pension plans depending on salary (see [7], for example), the function V is the solution of a certain PDE problem. Thus, we can apply Itô's Lemma [15] to obtain the variation of V_t, dV_t, from time t to $t + dt$ for small dt. Hereafter, we suppress the dependence on t in order to simplify notation. More precisely, we have

$$dV = \left(\frac{\partial V}{\partial t} + \alpha\left(\mu(t) - \ln(M)\right) M \frac{\partial V}{\partial M} - \beta \ln(N) N \frac{\partial V}{\partial N} + \frac{1}{2}\sigma^2 M^2 \frac{\partial^2 V}{\partial M^2}\right) dt + \sigma M \frac{\partial V}{\partial M} dW. \tag{1.5}$$

Next, we build a risk free portfolio Π by buying one unit of the asset V_1 with maturity T_1 and selling Δ units of asset V_2 with maturity T_2. Thus, the resulting portfolio Π reads

$$\Pi = V_1 - \Delta V_2\,.$$

Note that the variation of the portfolio value between t and $t+dt$ is given by

$$d\Pi = dV_1 - \Delta dV_2 = (\dots)\ dt\ +\ \sigma M\left(\frac{\partial V_1}{\partial M} - \Delta\frac{\partial V_2}{\partial M}\right) dW, \tag{1.6}$$

where $(\dots)$ contains the drift term. Therefore, Π turns out to be risk-free for the following choice of Δ:

$$\Delta = \frac{\partial V_1/\partial M}{\partial V_2/\partial M}. \tag{1.7}$$

Moreover, for this choice of Δ, the variation of the risk-free portfolio is given by

$$\begin{aligned} d\Pi = \Bigg[&\frac{\partial V_1}{\partial t} - \beta\ln(N)N\frac{\partial V_1}{\partial N} + \frac{1}{2}\sigma^2M^2\frac{\partial^2 V_1}{\partial M^2} \\ &-\Delta\left(\frac{\partial V_2}{\partial t} - \beta\ln(N)N\frac{\partial V_2}{\partial N} + \frac{1}{2}\sigma^2M^2\frac{\partial^2 V_2}{\partial M^2}\right)\Bigg]\ dt. \end{aligned} \tag{1.8}$$

Next, by using the arbitrage free assumption, this variation is also given by $d\Pi = r\Pi dt$, where r is the deterministic risk free interest rate. Hence, we obtain the identity

$$\begin{aligned} &\left(\frac{\partial V_1}{\partial M}\right)^{-1}\left(rV_1 - \frac{\partial V_1}{\partial t} + \beta\ln(N)N\frac{\partial V_1}{\partial N} - \frac{1}{2}\sigma^2M^2\frac{\partial^2 V_1}{\partial M^2}\right) \\ =&\left(\frac{\partial V_2}{\partial M}\right)^{-1}\left(rV_2 - \frac{\partial V_2}{\partial t} + \beta\ln(N)N\frac{\partial V_2}{\partial N} - \frac{1}{2}\sigma^2M^2\frac{\partial^2 V_2}{\partial M^2}\right). \end{aligned} \tag{1.9}$$

Note that (1.9) holds for any pair of assets. Then, we can introduce the quantity

$$\bar{a}(t,M,N) = \left(\frac{\partial V}{\partial M}\right)^{-1}\left(rV - \frac{\partial V}{\partial t} + \beta\ln(N)N\frac{\partial V}{\partial N} - \frac{1}{2}\sigma^2M^2\frac{\partial^2 V}{\partial M^2}\right), \tag{1.10}$$

where it is convenient to write $\bar{a}(t,M,N) = \alpha\,(\mu(t) - \ln(M))\,M$.

By reordering the terms in (1.10) we obtain the following PDE in two spatial dimensions that governs the value of any asset depending on the two underlying stochastic factors M and N

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2M^2\frac{\partial^2 V}{\partial M^2} + \alpha\,(\mu(t) - \ln(M))\,M\frac{\partial V}{\partial M} - \beta\ln(N)N\frac{\partial V}{\partial N} - rV = 0. \tag{1.11}$$

For the particular case of electricity markets in which the payoff $\varphi(T,S)$ is a function depending on the electricity price, S, at maturity T, the equation (1.11) is supplied with the final condition

$$V(T,M,N) = \varphi(T,MN). \tag{1.12}$$

1.3. The swing option pricing problem. From the mathematical point of view, swing options in electricity markets can be modelled as financial products with multiple exercises of American type. Moreover, two exercise dates are separated by a constant refracting period $\delta > 0$. As it is mentioned in [9], the consideration of this refracting period avoids the exercise of all the rights at once, which would be optimal in the absence of this separation time. That is, without the refracting period δ, the swing option pricing problem could be reduced to the valuation of multiple American options.

Let us consider $p \in \mathbb{N}$ exercise rights. If we denote by $\mathcal{T}_{t,T}$ the set of all stopping times with values in $[0,T]$ and by $\mathcal{T}_{t,\infty}$ the set of all stopping times with values greater or equal than t, then we can define the set of admissible stopping time vectors in the following way (see [9] or [27], for example)

$$\mathcal{T}_t^{(p)} = \{\tau^{(p)} = (\tau_1,...,\tau_p) | \tau_i \in \mathcal{T}_{t,\infty} \text{ with } \tau_1 \le T \text{ a.s. and } \tau_{i+1} - \tau_i \ge \delta \text{ for } i = 1,...,p-1\}. \tag{1.13}$$

Note that at least one exercise right of the swing option with maturity T is exercised but it is not necessary to exercise all the rights. The investor could let an exercise right expire to get a benefit from better future prices. Thus, not all stopping times of a vector have their values in the interval $[0,T]$.

In [27] the risk free price of a swing option depending on one underlying factor is written as a multiple stopping time problem and it is proved in [9] that it can be translated to a sequence of single stopping time problems. Analogously, when the electricity price depends on two stochastic factors under a risk neutral probability measure Q, the price of a swing option with $p \in \mathbb{N}$ exercise rights is given by

$$V^{(p)}(t, M_t, N_t) = sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^Q \left[e^{-r(\tau - t)} \Phi^{(p)}(\tau, S_\tau) \right], \quad p \geq 1, \tag{1.14}$$

with $S_t = M_t N_t$ and

$$\Phi^{(p)}(t, S_t) = \begin{cases} \varphi(t, S_t) + \mathbb{E}^Q \left[e^{-r\delta} V^{(p-1)}(t+\delta, M_{t+\delta}, N_{t+\delta}) \right] & \text{if } t \leq T - \delta \\ \varphi(t, S_t) & \text{if } t > T - \delta. \end{cases}$$

Moreover, we start from:

$$V^{(0)}(t, M_t, N_t) = 0. \tag{1.15}$$

1.3.1. The free boundary problem. After making the time reversal change of variable, $\tau = T - t$, we introduce the function $u^{(p)}(\tau, M, N) = V^{(p)}(T-\tau, M, N)$ so that it solves the following complementarity problem:

$$\begin{aligned} \mathcal{L}[u^{(p)}] \leq 0 &\text{ in } (0,T) \times \mathbb{R}_+^2, \\ u^{(p)} \geq \Psi^{(p)} &\text{ in } (0,T) \times \mathbb{R}_+^2, \\ (\mathcal{L}[u^{(p)}])(u^{(p)} - \Psi^{(p)}) = 0 &\text{ in } (0,T) \times \mathbb{R}_+^2, \\ u^{(p)}(0,.) = \Psi^{(p)}(0,.) \qquad &\text{in } \mathbb{R}_+^2, \end{aligned} \tag{1.16}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}[F] = -\frac{\partial F}{\partial \tau} + \frac{1}{2}\sigma^2 M^2 \frac{\partial^2 F}{\partial M^2} + \alpha\left(\mu(T-\tau) - \ln(M)\right) M \frac{\partial F}{\partial M} - \beta \ln(N) N \frac{\partial F}{\partial N} - rF \tag{1.17}$$

and the p^{th} reward obstacle function $\Psi^{(p)}$ has the following form:

$$\Psi^{(p)}(\tau, S) = \begin{cases} \varphi(T-\tau, S) + w^{\tau,(p-1)}(\delta, M, N) & \text{for } \tau \in [\delta, T] \\ \varphi(T-\tau, S) & \text{for } \tau \in [0, \delta). \end{cases} \tag{1.18}$$

In (3.3), $w^{\tau,(p-1)}(\delta, M, N)$ denotes the value of the swing option with one exercise right less.

In order to obtain the value of $w^{\tau,(p-1)}(\delta, M, N)$ for $\tau \in [\delta, T]$, when $p = 1$ we note that

$$w^{\tau,(0)}(t, M, N) = 0 \quad \text{ for } (t, M, N) \in [0, \delta] \times \mathbb{R}_+^2,$$

whereas when $p > 1$ we need to solve the following PDE problem:

$$\begin{aligned} \mathcal{L}[w^{\tau,(p-1)}] = 0 &\text{ in } (0,\delta) \times \mathbb{R}_+^2, \\ w^{\tau,(p-1)}(0,\cdot) = u^{(p-1)}(\tau-\delta,\cdot) \qquad &\text{in } \mathbb{R}_+^2, \end{aligned} \tag{1.19}$$

where $\mathcal{L}$ is given by (1.17).

Note that due to the constant refracting period, the reward function (3.3) can be equivalently written as

$$\Psi^{(p)}(\tau,S)=\begin{cases}\varphi(T-\tau,S)+w^{\tau,(p-1)}(\delta,M,N) & \text{for } \tau\geq(p-1)\delta\\ \Psi^{(p-1)}(\tau,S) & \text{for } \tau<(p-1)\delta.\end{cases}\tag{1.20}$$

That is, in a period of length $(p-1)\delta$ we only can exercise $(p-1)$ rights due to the refracting period. That is why the value of the reward function with p exercise rights is equal to the value with $(p-1)$ rights at any time $\tau<(p-1)\delta$.

2. The numerical methods. In order to obtain a numerical approach of the value of a swing option with $p\in\mathbb{N}$ exercise rights, we need to solve a free boundary problem for each value of p. Additionally, for $p>1$, in order to obtain the value of the reward function, $\Psi^{(p)}(\tau,S)$, associated with each complementarity problem (1.16), the solution for certain times of an initial value problem is required. For the numerical solution of the PDEs (1.16) and (1.19), we propose a Crank-Nicolson characteristics time discretization scheme combined with a piecewise quadratic Lagrange finite element method. Thus, first a localization technique is used to cope with the initial formulation in an unbounded domain. For the additional inequality constraints associated with the complementarity problem (1.16), we propose a mixed formulation and an augmented Lagrangian active set (ALAS) technique.

2.1. Localization procedure and formulation in a bounded domain. In this section we replace the unbounded domain by a bounded one. In order to determine the required boundary conditions for the associated PDE problems we follow [23] which is based on the theory proposed by Fichera in [12]. More recently, this theory was also applied to degenerated parabolic PDEs which appear in finance in [6]. Let us introduce the notation:

$$x_0=\tau,\quad x_1=M\quad\text{and}\quad x_2=N,\tag{2.1}$$

and let us consider both x_1^∞ and x_2^∞ to be large enough suitably chosen real numbers. Let

$$\Omega=(0,x_0^\infty)\times(0,x_1^\infty)\times(0,x_2^\infty)$$

with $x_0^\infty=T$. Then, let us denote the Lipschitz boundary by $\Gamma=\partial\Omega$ such that $\Gamma=\bigcup_{i=0}^{2}(\Gamma_i^-\cup\Gamma_i^+)$, where

$$\Gamma_i^-=\{(x_0,x_1,x_2)\in\Gamma\mid x_i=0\},\quad \Gamma_i^+=\{(x_0,x_1,x_2)\in\Gamma\mid x_i=x_i^\infty\},\quad i=0,1,2.$$

Then, the operator defined in (1.17) can be written in the form:

$$\mathcal{L}[F]=\sum_{i,j=0}^{2}b_{ij}\frac{\partial^2F}{\partial x_ix_j}+\sum_{j=0}^{2}b_j\frac{\partial F}{\partial x_j}+b_0F,\tag{2.2}$$

where the involved data are given by

$$B=(b_{ij})=\begin{pmatrix}0&0&0\\0&\frac{1}{2}\sigma^2x_1^2&0\\0&0&0\end{pmatrix},\tag{2.3}$$

$$\vec{b}=(b_j)=\begin{pmatrix}-1\\g(x_0,x_1)\\h(x_2)\end{pmatrix},\quad b_0=-r,\tag{2.4}$$

where

$$g(x_0,x_1)=\begin{cases}0 & \text{if } x_1=0\\ \alpha\left(\mu(T-x_0)-\ln(x_1)\right)x_1 & \text{if } x_1\neq 0,\end{cases}\quad h(x_2)=\begin{cases}0 & \text{if } x_2=0\\ -\beta\ln(x_2)x_2 & \text{if } x_2\neq 0.\end{cases}$$

Thus, following [23], in terms of the normal vector to the boundary pointing inward Ω, $\vec{m}=(m_0,m_1,m_2)$, we introduce the following subsets of Γ:

$$\Sigma^0=\left\{x\in\Gamma/\sum_{i,j=0}^{2}b_{ij}m_im_j=0\right\},\quad \Sigma^1=\Gamma-\Sigma^0,$$

$$\Sigma^2=\left\{x\in\Sigma^0/\sum_{i=0}^{2}\left(b_i-\sum_{j=0}^{2}\frac{\partial b_{ij}}{\partial x_j}\right)m_i<0\right\}.$$

As indicated in [23] the boundary conditions at $\Sigma^1\bigcup\Sigma^2$ for the so-called first boundary value problem associated with (2.2) are required. Note that $\Sigma^1=\Gamma_1^+$ and $\Sigma^2=\Gamma_0^-$. Therefore, in addition to an initial condition (see section 1.3.1), we need to impose a boundary condition on Γ_1^+. For this purpose, in order to construct an ABC on this boundary we replace the operator (1.17) in the right exterior domain (i.e. for $x_1>x_1^\infty$) by the following one:

$$\bar{\mathcal{L}}[F]=-\frac{\partial F}{\partial\tau}+\frac{1}{2}\sigma^2(x_1^\infty)^2\frac{\partial^2F}{\partial x_1^2}+\alpha\left(\mu-\ln(x_1^\infty)\right)x_1^\infty\frac{\partial F}{\partial x_1}-rF,\tag{2.5}$$

where we assume that the coefficients are constant and there is no dependency on the variable x_2. Next, by applying the Laplace-method we can write the Laplace transformed right ABC as

$$\hat{F}_{x_1}(x_1^\infty,s)=\left(\frac{b}{2a}-\frac{1}{2a}\sqrt[+]{b^2+4(c+s)a}\right)\hat{F}(x_1^\infty,s).\tag{2.6}$$

where $a=\frac{1}{2}\sigma^2(x_1^\infty)^2$, $b=\alpha\left(\ln(x_1^\infty)-\mu\right)x_1^\infty$, $c=r$ and s is the dual variable of the Laplace transform. Here $\sqrt[+]{\cdots}$ denotes the branch of the square root with positive real part.

Taking into account the approach of Halpern in [13] we use a first order Taylor approximation for small values of a of the square root term in (2.6) which leads to the following transformed boundary condition:

$$\hat{F}_{x_1}(x_1^\infty,s)\approx\left(\frac{b-|b|}{2a}-\frac{c+s}{|b|}\right)\hat{F}(x_1^\infty,s).\tag{2.7}$$

Finally, using an inverse Laplace transformation, for $b>0$ we obtain the following first order ABC:

$$\frac{\partial F}{\partial\tau}+b\frac{\partial F}{\partial x_1}+cF=0\quad\text{on}\quad\Gamma_1^+.\tag{2.8}$$

Taking into account the previous change of spatial variables we write the equation (1.19) in divergence form in the bounded spatial domain $\Omega=(0,x_1^\infty)\times(0,x_2^\infty)$. Thus, the IBVP takes the following form:

Find $w^{\tau,(p-1)}:[0,\delta]\times\Omega\to\mathbb{R}$ such that

$$\frac{\partial w^{\tau,(p-1)}}{\partial t}+\vec{v}\cdot\nabla w^{\tau,(p-1)}-Div(A\nabla w^{\tau,(p-1)})+lw^{\tau,(p-1)}=\widetilde{f}\text{in }(0,\delta)\times\Omega\,,\tag{2.9}$$

$$w^{\tau,(p-1)}(0,.)=u^{(p-1)}(\tau-\delta,.)\text{ in }\Omega,\tag{2.10}$$

$$\frac{\partial w^{\tau,(p-1)}}{\partial t}+\alpha\left(\ln(x_1^\infty)-\mu\right)x_1^\infty\frac{\partial w^{\tau,(p-1)}}{\partial x_1}+lw^{\tau,(p-1)}=0\text{on }(0,\delta)\times\Gamma_1^+\,.\tag{2.11}$$

Furthermore, for the complementarity problem associated with the swing option value, denoting by P the Lagrange multiplier, we can pose the mixed formulation:

Find $u^{(p)} : [0,T] \times \Omega \to \mathbb{R}$ such that

$$\frac{\partial u^{(p)}}{\partial \tau} + \vec{v} \cdot \nabla u^{(p)} - Div(A \nabla u^{(p)}) + l u^{(p)} + P = \widetilde{f} \text{in } (0,T) \times \Omega\,, \tag{2.12}$$

with the complementarity conditions

$$u^{(p)} \geq \Psi^{(p)}, \quad P \leq 0, \quad (u^{(p)} - \Psi^{(p)})P = 0 \quad \text{in } (0,T) \times \Omega \tag{2.13}$$

and the initial and boundary conditions

$$u^{(p)}(0,.) = \Psi^{(p)}(0,.) \text{ in } \Omega, \tag{2.14}$$

$$\frac{\partial u^{(p)}}{\partial \tau} + \alpha \left(\ln(x_1^\infty) - \mu\right) x_1^\infty \frac{\partial u^{(p)}}{\partial x_1} + l u^{(p)} = 0 \text{on } (0,T) \times \Gamma_1^+\,. \tag{2.15}$$

For both problems, the involved data is defined as follows:

$$A = \begin{pmatrix} \frac{1}{2}\sigma^2 x_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} \widetilde{g}(\tau, x_1) \\ \widetilde{h}(x_2) \end{pmatrix}, \quad l = r, \quad \widetilde{f} = 0,$$

$$\widetilde{g}(\tau, x_1) = \begin{cases} 0 & \text{if } x_1 = 0 \\ \left(\sigma^2 - \alpha\left(\mu(T-\tau) - \ln(x_1)\right)\right) x_1 & \text{if } x_1 \neq 0, \end{cases} \quad \widetilde{h}(x_2) = \begin{cases} 0 & \text{if } x_2 = 0 \\ \beta \ln(x_2) x_2 & \text{if } x_2 \neq 0. \end{cases} \tag{2.16}$$

2.2. Time discretization. First, we define the characteristics curve through $\mathbf{x} = (x_1, x_2)$ at time $\bar{\tau}$, $X(\mathbf{x}, \bar{\tau}; s)$, which satisfies:

$$\frac{\partial}{\partial s} X(\mathbf{x}, \bar{\tau}; s) = \vec{v}(X(\mathbf{x}, \bar{\tau}; s)), \quad X(\mathbf{x}, \bar{\tau}; \bar{\tau}) = \mathbf{x}. \tag{2.17}$$

In order to discretize in time the material derivative in the complementarity problem (2.12), let us consider a number of time steps $\bar{N}$, the time step $\Delta\tau = T/\bar{N}$ and the time mesh points $\tau^n = n\Delta\tau$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \bar{N}$. In order to obtain the initial condition for solving the problem (2.9) the time discretization has to be chosen such that $\delta/\Delta\tau \in \mathbb{N}$. So, we should choose $\bar{N}$ as a multiple of T/δ. In the discretization of the material derivative in the initial value problem (2.9) we consider a number of time steps equal to $\delta/\Delta\tau$.

The material derivative approximation by the characteristics method for both problems is given by:

$$\frac{DF}{D\tau} = \frac{F^{n+1} - F^n \circ X^n}{\Delta\tau},$$

where $F = u^{(p)}, w^{\tau,(p-1)}$ and $X^n(\mathbf{x}) := X(\mathbf{x}, \tau^{n+1}; \tau^n)$. For the case of $f = 0$ the components of $X^n(\mathbf{x})$ can be computed analytically:

$$X_1^n(\mathbf{x}) = \begin{cases} x_1 & \text{if } x_1 = 0 \\ \exp\left[(\exp(-\alpha\Delta\tau)(\sigma^2 + 2\alpha\ln(x_1)) - \sigma^2)/(2\alpha)\right] & \text{if } x_1 \neq 0, \end{cases}$$

$$X_2^n(\mathbf{x}) = \begin{cases} x_2 & \text{if } x_2 = 0 \\ \exp(\ln(x_2)\exp(-\beta\Delta\tau)) & \text{if } x_2 \neq 0. \end{cases}$$

However, for the general case where it is not possible to compute the characteristics curves analytically, some numerical ODE solvers can be used (see [4], for example).

Next, we consider a Crank-Nicolson scheme around $\big(X(\mathbf{x},\tau^{n+1};\tau),\tau\big)$ for $\tau=\tau^{n+\frac{1}{2}}$. So, the time discretized equation for $F=u^{(p)},w^{\tau,(p-1)}$ and $P=0$ can be written as follows:

Find F^{n+1} such that:

$$\frac{F^{n+1}(\mathbf{x})-F^n(X^n(\mathbf{x}))}{\Delta\tau}-\frac{1}{2}Div(A\nabla F^{n+1})(\mathbf{x})-\frac{1}{2}Div(A\nabla F^n)(X^n(\mathbf{x}))$$
$$+\frac{1}{2}(l\,F^{n+1})(\mathbf{x})+\frac{1}{2}(l\,F^n)(X^n(\mathbf{x}))=0. \tag{2.18}$$

Moreover, we also discretize the artificial boundary condition on Γ_1^+:

$$\frac{F^{n+1}(\mathbf{x})-F^n(\hat{X}^n(\mathbf{x}))}{\Delta\tau}+\frac{1}{2}(c\,F^{n+1})(\mathbf{x})+\frac{1}{2}(c\,F^n)(\hat{X}^n(\mathbf{x}))=0 \tag{2.19}$$

where $\hat{X}^n(\mathbf{x})=(-b\Delta\tau+x_1,x_2)^T$ in the case of $f=0$.

Thus,

$$F^{n+1}(\mathbf{x})=\frac{1-c\Delta\tau/2}{1+c\Delta\tau/2}F^n(\hat{X}^n(\mathbf{x}))\quad on\quad \Gamma_1^+. \tag{2.20}$$

In order to obtain the variational formulation of the semi-discretized problem, we multiply (2.19) by a suitable test function, integrate in Ω, use the classical Green formula and the following one [22]:

$$\int_\Omega Div(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x}=\int_\Gamma(\nabla X^n)^{-T}(\mathbf{x})\mathbf{n}(x)\cdot(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x}$$
$$-\int_\Omega(\nabla X^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\cdot\nabla\psi(\mathbf{x})d\mathbf{x}$$
$$-\int_\Omega Div((\nabla X^n)^{-T}(\mathbf{x}))(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} \tag{2.21}$$

Note that, when $f=0$, we have:

$$Div((\nabla X^n)^{-T}(\mathbf{x}))=\begin{pmatrix}\frac{1}{e_1}(\exp(\alpha\Delta\tau)-1)\\ \\ \frac{1}{e_2}(\exp(\beta\Delta\tau)-1)\end{pmatrix}, \tag{2.22}$$

where $e_1=\exp\left[(\exp(-\alpha\Delta\tau)(\sigma^2+2\alpha\ln(x_1))-\sigma^2)/(2\alpha)\right]$ and $e_2=\exp(\ln(x_2)\exp(-\beta\Delta\tau))$. In the general case $Div((\nabla X^n)^{-T}(\mathbf{x}))$ needs to be approximated. After the previous steps, we can write a variational formulation for the time discretized problem as follows:

Find $F^{n+1} \in H^1(\Omega)$ such that, for all $\psi \in H^1(\Omega)$ such that $\psi = 0$ on Γ_1^+:

$$\begin{aligned}\int_\Omega F^{n+1}(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_\Omega (\mathbf{A}\nabla F^{n+1})(\mathbf{x})\nabla\psi(\mathbf{x})d\mathbf{x} + \frac{\Delta\tau}{2}\int_\Omega lF^{n+1}(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} \\ = \int_\Omega F^n(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} - \frac{\Delta\tau}{2}\int_\Omega (\nabla X^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\nabla\psi(\mathbf{x})d\mathbf{x} \\ -\frac{\Delta\tau}{2}\int_\Omega lF^n(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} - \frac{\Delta\tau}{2}\int_\Omega Div((\nabla X^n)^{-T}(\mathbf{x}))(\mathbf{A}\nabla F^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x},\end{aligned} \tag{2.23}$$

where ∇X^n can be computed analytically in some cases. Other times it needs to be approximated (see [4], for example).

2.3. Finite elements discretization. For the spatial discretization we consider $\{\tau_h\}$, a quadrangular mesh of the domain Ω. Let $(T_1, \mathcal{Q}_2, \Sigma_{T_1})$ be a family of piecewise quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ denotes the space of polynomials defined in $T_1 \in \tau_h$ with degree less or equal than two in each spatial variable and Σ_{T_1} the subset of nodes of the element T_1. More precisely, let us define the finite elements space F_h by

$$F_h = \{\varphi_h \in \mathcal{C}^0(\bar\Omega) : \varphi_{h_{T_1}} \in \mathcal{Q}_2, \quad \forall T_1 \in \tau_h\}, \tag{2.24}$$

where $\mathcal{C}^0(\bar\Omega)$ is the space of piecewise continuous functions on $\bar\Omega$.

2.4. Augmented Lagrangian Active Set algorithm. Here the Augmented Lagrangian Active Set (ALAS) algorithm proposed in [17] is applied to the fully discretized in time and space mixed formulation (2.12)-(2.13). More precisely, after this full discretization procedure the discrete problem can be written in the form:

$$M_h u_h^{(p),n} + P_h^n = b_h^{n-1}, \tag{2.25}$$

with the discrete complementarity conditions

$$u_h^{(p),n} \geq \Psi_h^{(p),n}, \quad P_h^n \leq 0, \quad \left(u_h^{(p),n} - \Psi_h^{(p),n}\right) P_h^n = 0, \tag{2.26}$$

where P_h^n denotes the vector of the multiplier values and $\Psi_h^{(p),n}$ denotes the vector of the nodal values defined by the function $\Psi^{(p)}$.

The basic iteration of the ALAS algorithm consists of two steps. In the first step the domain is decomposed into active and inactive parts (depending on whether the constraints are active or not), and in the second step, a reduced linear system associated with the inactive part is solved. Thus, we use the algorithm for unilateral problems, which are based on the augmented Lagrangian formulation.

First, for any decomposition $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$, where $\mathcal{N} := \{1, 2, \ldots, N_{\text{dof}}\}$, let us denote by $[M_h]_{\mathcal{I}\mathcal{I}}$ the principal minor of the matrix M_h and by $[M_h]_{\mathcal{I}\mathcal{J}}$ the co-diagonal block indexed by $\mathcal{I}$ and $\mathcal{J}$. Thus, for each time τ_n, the ALAS algorithm computes not only $u_h^{(p),n}$ and P_h^n but also a decomposition $\mathcal{N} = \mathcal{J}^n \cup \mathcal{I}^n$ such that

$$\begin{aligned} M_h u_h^{(p),n} + P_h^n &= b_h^{n-1}, \\ [P_h^n]_j + \gamma \left[u_h^{(p),n} - \Psi^{(p)}\right]_j &\leq 0, \qquad \forall j \in \mathcal{J}^n, \\ [P_h^n]_i &= 0, \qquad \forall i \in \mathcal{I}^n, \end{aligned} \tag{2.27}$$

for a given positive parameter γ. In the above equations, $\mathcal{I}^n$ and $\mathcal{J}^n$ are the *inactive* and the *active* sets at time t_n respectively. More precisely, the iterative algorithm builds sequences $\left\{u_{h,m}^{(p),n}\right\}_m$, $\left\{P_{h,m}^n\right\}_m$, $\{\mathcal{I}_m^n\}_m$ and $\{\mathcal{J}_m^n\}_m$, converging to $u_h^{(p),n}$, P_h^n, $\mathcal{I}^n$ and $\mathcal{J}^n$, by means of the following procedure:

1. Initialize $u_{h,0}^{(p),n} = \Psi_h^{(p),n}$ and $P_{h,0}^n = \min\{b_h^n - M_h u_{h,0}^{(p),n}, 0\} \leq 0$. Choose $\gamma > 0$. Set $m = 0$.
2. Compute

$$\begin{aligned} Q_{h,m}^n &= \min\left\{0, P_{h,m}^n + \gamma\left(u_{h,m}^{(p),n} - \Psi_{h,m}^{(p),n}\right)\right\}, \\ \mathcal{J}_m^n &= \left\{j \in \mathcal{N}, \left[Q_{h,m}^n\right]_j < 0\right\}, \\ \mathcal{I}_m^n &= \{i \in \mathcal{N}, \left[Q_{h,m}^n\right]_i = 0\}. \end{aligned}$$

3. If $m \geq 1$ and $J_m^n = J_{m-1}^n$ then convergence is achieved. Stop.
4. Let $u^{(p)}$ and P be the solution of the linear system

$$\begin{aligned} &M_h u^{(p)} + P = b_h^{n-1}, \\ &P = 0 \text{ on } \mathcal{I}_m^n \quad \text{and} \quad u^{(p)} = \Psi_{h,m}^{(p),n} \text{ on } \mathcal{J}_m^n. \end{aligned} \tag{2.28}$$

   Set $u_{h,m+1}^{(p),n} = u^{(p)}$, $P_{h,m+1}^n = \min\{0, P\}$, $m = m + 1$ and go to Step 2.

It is important to note that, instead of solving the full linear system in (2.28), for $\mathcal{I} = \mathcal{I}_m^n$ and $\mathcal{J} = \mathcal{J}_m^n$ the following reduced system on the inactive set is solved:

$$\begin{aligned} [M_h]_{\mathcal{I}\mathcal{I}}\,[u^{(p)}]_{\mathcal{I}} &= \left[b^{n-1}\right]_{\mathcal{I}} - [M_h]_{\mathcal{I}\mathcal{J}}\left[\Psi^{(p)}\right]_{\mathcal{J}}, \\ [u^{(p)}]_{\mathcal{J}} &= \left[\Psi^{(p)}\right]_{\mathcal{J}}, \\ P &= b^{n-1} - M_h V. \end{aligned} \tag{2.29}$$

In [17], the authors proved the convergence of the algorithm in a finite number of steps for a Stieltjes matrix (i.e. a real symmetric positive definite matrix with negative off-diagonal entries, cf. [25]) and a suitable initialization (the same we consider in this paper). They also proved that $\mathcal{I}_m \subset \mathcal{I}_{m+1}$. Nevertheless, a Stieltjes matrix can be only obtained for linear elements but never for the here used quadratic elements because we have some positive off-diagonal entries arising from the stiffness matrix (actually we use a lumped mass matrix). However, we have obtained good results by using the ALAS algorithm with quadratic finite elements.

3. Numerical results. In this section we show some numerical results to illustrate the performance of the numerical methods, by comparing them with some examples in the literature. Note that this paper is the first one considering the numerical solution of the PDE associated to a two factor model for electricity prices. Thus, we mainly compare with an example in [14] considering two factors and a binomial method and also with an extension to two factors of the one factor stochastic model solved in [27] with finite elements.

3.1. Example 1. First we consider as in [14] the valuation of a swing option with up to $p = 20$ exercise rights where the rights correspond to the payoff of a call option. For this purpose, we need to specify a set of parameters, related to the market values of the data involved in the underlying factors, the initial conditions of the stochastic processes and the parameters of the payoff function. All of them are taken from [14] and are shown in Table 3.1. We have chosen these parameters in order to compare

Market parameters of the underlying factors	
Speed of mean reversion process M, α	7
Volatility, σ	1.4
Speed of mean reversion process N, β	200
Interest rate, r	0
Seasonality, f	0
Initial conditions	
Initial value of M, M_0	1
Initial value of N, N_0	1
Payoff function parameters	
Payoff, $\varphi(T,S)$	$(S-K)_+$
Strike, K	1

Table 3.1: Fixed parameters of the model for Example 1, cf. [14].

Computational domain	
x_1^∞	3K
x_2^∞	3K
ABC	
Coefficient b	20.13
Finite elements mesh data	
Number of elements	576
Number of nodes	2401
ALAS algorithm	
Parameter γ	10000

Table 3.2: Parameters of the numerical methods in Example 1.

the results we obtain with the ones in [14] where a binomial method has been used. Moreover, concerning the numerical methods we select the parameters collected in Table 3.2.

In Figure 3.1 we show the value per exercise right of the swing option when the maturity of the contract is one year and a right can be exercised at most once per day (i.e. the refracting period δ is one day). Moreover, we consider that the time step $\Delta\tau$ is also one day. For this example, in Figure 3.2 we represent the approximate location of the free boundary at origination (i.e. $t = 0$) when $p = 2$. In the white region it is optimal to exercise the option whereas the black region corresponds to the non early exercise region. In Figure 3.3 we show the exercise value (or obstacle) at origination for the same swing contract and data set. In Figure 3.4 the swing option value at origination is shown. Note that in the white region of Figure 3.2, the value of the swing option in Figure 3.4 coincides with the exercise value represented in Figure 3.3. Additionally, we present some results just changing $p = 2$ to $p = 6$ in the previous data. More precisely, in Figure 3.5 and 3.6 we observe that the exercise region is not too much affected for this change while the solution changes mainly due to the change in the new exercise value function.

Next, in Figure 3.7 we present the value per exercise right of the swing option when the maturity of the contract is two months, the refracting period δ is one day and the time step $\Delta\tau$ coincides with the refracting period. Finally, in Figure 3.8 we consider that the option has ten exercise opportunities per day (i.e. the refracting period is 0.1 days) and that the delivery period is six days. The time step $\Delta\tau$ is

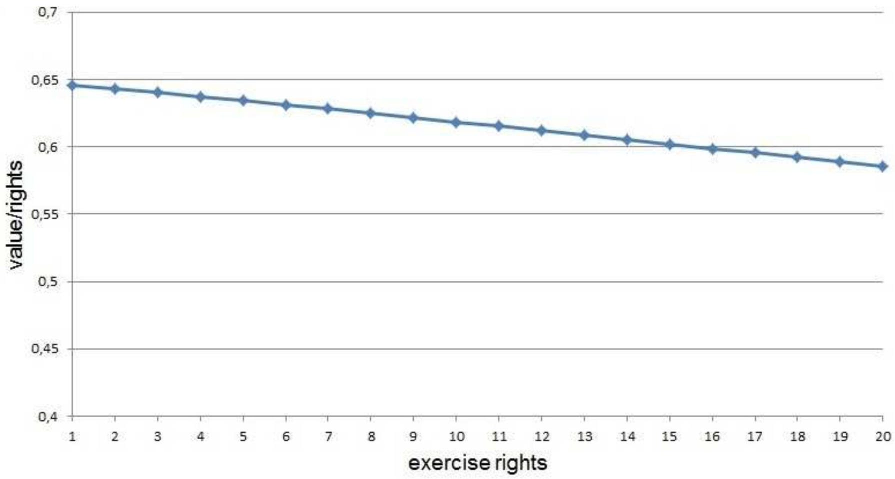

Fig. 3.1: Value per right of a swing option with 1 year to delivery in Example 1.

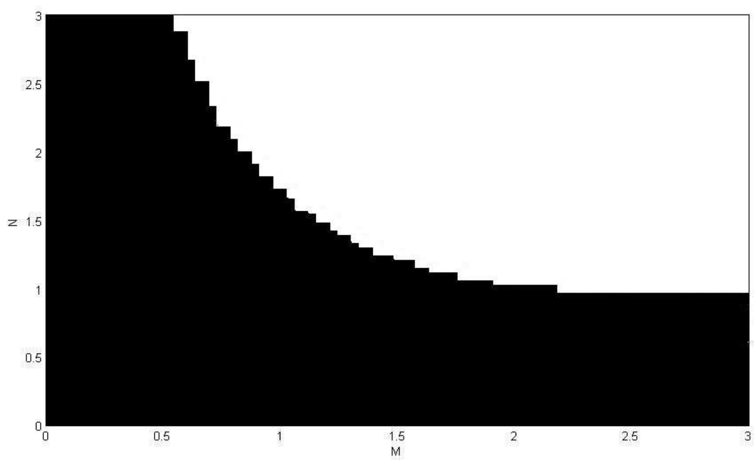

Fig. 3.2: Approximated free boundary in the grid at origination of a swing option with p=2 rights and 1 year to delivery in Example 1. Exercise region in white and non exercise region in black.

equal to the refracting period.

Figures 3.1, 3.7 and 3.8 are in full agreement with the analogous ones appearing in [14] obtained with a binomial methods. Furthermore, we can observe how the price per exercise right decreases with the number of exercise rights. It is what it is expected because p swing options with one exercise right (that would be equivalent to p American options) give more flexibility because you can exercise all the rights at once and consequently its price must be higher than the price of one swing option with p exercise rights. In Figure 3.8, the difference between two values per exercise right is smaller due to the value of the refracting period. As expected, when the value of the refracting period decreases the value of a swing option with p exercise rights tends to the value of p American options with 1 exercise right.

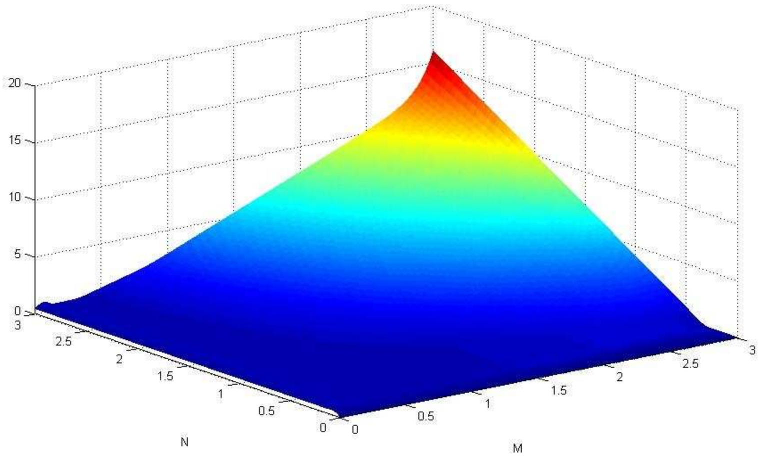

Fig. 3.3: Obstacle at origination of a swing option with p=2 rights and 1 year to delivery in Example 1.

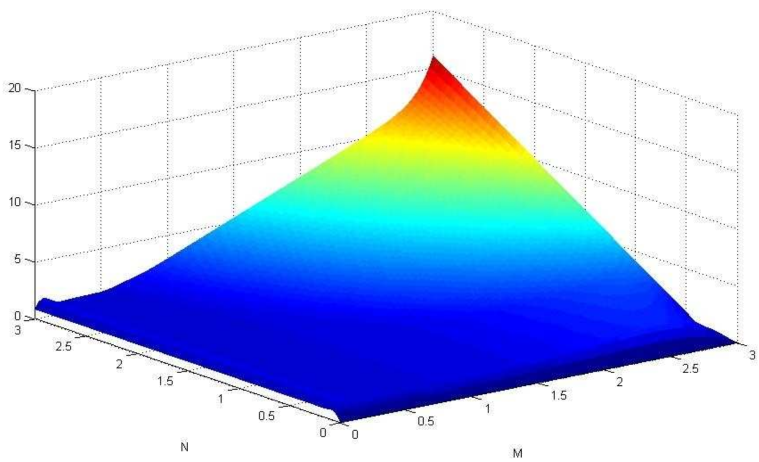

Fig. 3.4: Swing option value at origination with p=2 rights and 1 year to delivery in Example 1.

3.2. Example 2. In this section, unlike Example 1, we show some cases in which the seasonality function and the interest rate are different from zero. For this purpose, we consider a swing option with up to $p = 7$ rights, maturity 1 year and refracting period 0.1 years. Moreover, we consider the values for the parameters involved in the underlying factors which appear in Table 3.3. Most of them are taken from [27] for a one factor stochastic model for electricity prices, which in turn are taken from [19] and are obtained from daily electricity spot and future prices experimental observations. In order to pose a two factor model we consider different nonzero values for the parameter β. For the numerical solution we consider again the parameters in Table 3.2, except the coefficient b of the ABC, that in this case depends on time and it is always greater than zero. In this example, the time step is $\Delta\tau = 0.01$. On one hand, in Figures 3.9 and 3.10 we show the value of this option per exercise right when $\beta = 0.2$

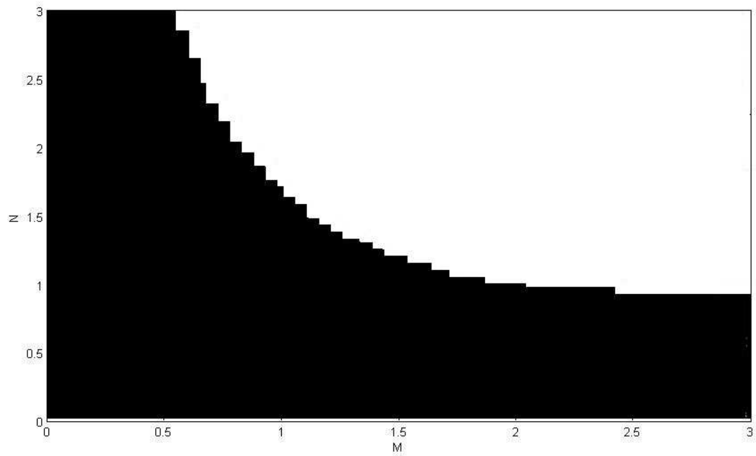

Fig. 3.5: Approximated free boundary in the grid at origination of a swing option with p=6 rights and 1 year to delivery in Example 1. Exercise region in white and non exercise region in black.

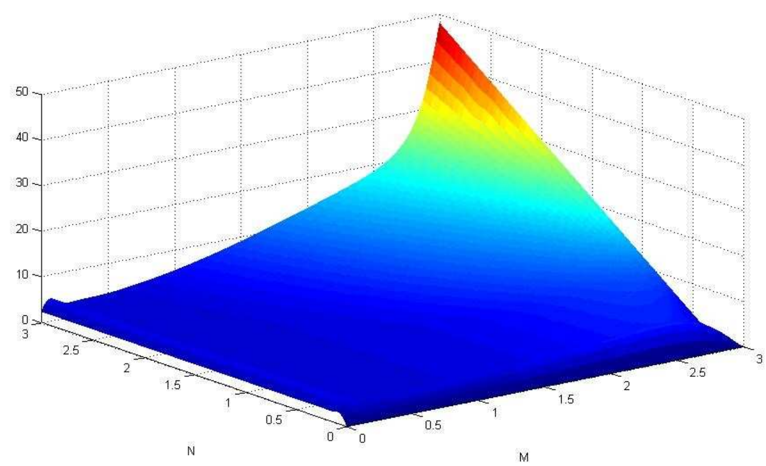

Fig. 3.6: Swing option value at origination with p=6 rights and 1 year to delivery in Example 1.

and $\beta = 2$, respectively, whereas on the other hand, in Figure 3.11 we represent its value for $\beta = 20$. Taking into account the three figures, we can observe that the value of the swing option increases when we increase the value of β.

4. Conclusions. In this paper we have considered the valuation of swing options in electricity markets by solving numerically a PDE based formulation. While the case of electricity prices driven by one stochastic factor has been considered in the literature with PDE methods, we have successfully addressed the case with two stochastic factors. Indeed, another novelty relies on the consideration of artificial boundary conditions (ABC) instead of the not so well justified homogeneous Neumann

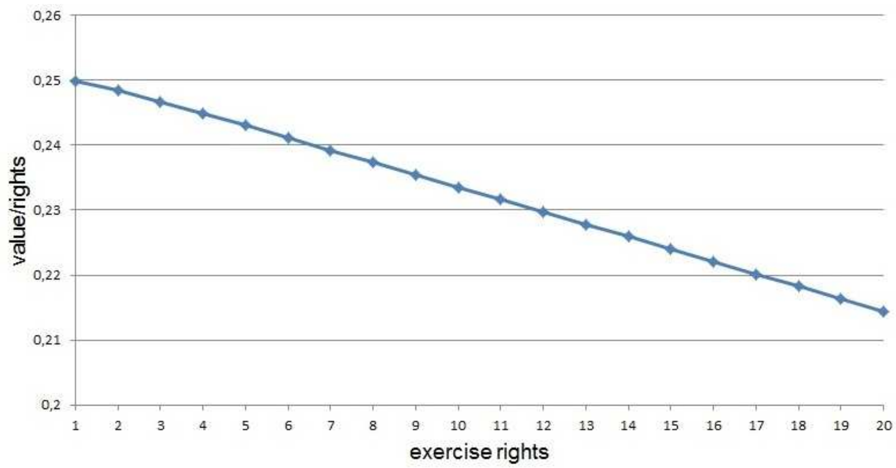

Fig. 3.7: Value per right of a swing option with 60 days to delivery in Example 1.

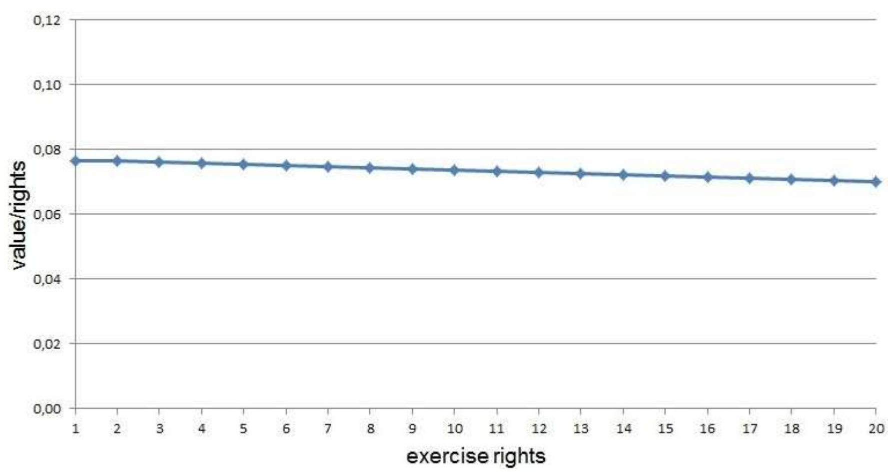

Fig. 3.8: Value per right of a swing option with 6 days to delivery in Example 1.

boundary conditions already used in the one factor case.

The swing option mainly consists of a path dependent option with multiple exercise rights. The right consists of receiving the payoff of a call option. The valuation problem has been posed as a sequence of free boundary problems, one for each right. Additionally, an initial value problem has to be solved due to the fact that the value of a swing option with one exercise less is involved in the definition of the obstacle function.

In order to obtain a numerical solution of the problem, we have proposed appropriate numerical methods based on Lagrange-Galerkin formulations combined with the ALAS algorithm to deal with the early exercise feature. Previously, as we have to confine the unbounded domain, appropriate artificial boundary conditions are constructed. Finally, we show some numerical results in order to illustrate the behaviour of the proposed methods.

Market parameters of the underlying factors	
Speed of mean reversion process M, α	0.016
Volatility, σ	0.086
Speed of mean reversion process N, β	0.2, 2, 20
Interest rate, r	0.05
Seasonality, f	$4.867 + 0.306\cos\left((t + 0.836)\frac{2\pi}{365}\right)$
Initial conditions	
Initial value of M, M_0	10
Initial value of N, N_0	1
Payoff function parameters	
Payoff, $\varphi(T, S)$	$(S - K)_+$
Strike, K	10

Table 3.3: Fixed parameters of the model with seasonality in Example 2.

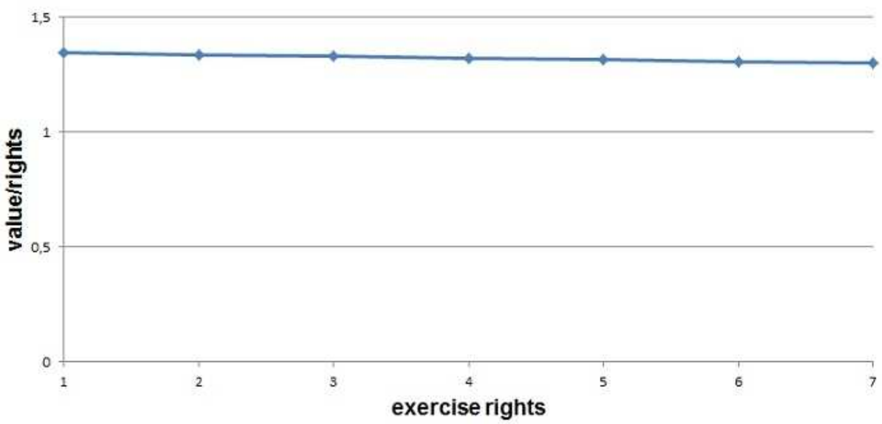

Fig. 3.9: Value per right of a swing option with 1 year to delivery when $\beta = 0.2$ in Example 2.

It is important to note some advantages of numerical methods for PDE valuation of swing options with respect to alternative Monte Carlo or lattice methods. Thus, the first ones provide the surface of swing prices at origination for the whole set of electricity spot prices, while the alternative approaches obtain one swing option price for each spot price. Also the methods we propose exhibit a clear advantage in the computation of the exercise boundary and exercise region simultaneously with the computation of swing option prices for a set of electricity spot prices. The use of Monte Carlo or lattice methods for this purpose would require a lot of additional computations.

As a future work, we plan to incorporate possible spikes in the electricity prices. For this purpose, jump diffusion processes are required to describe the evolution of the underlying factors, thus leading to partial integro-differential equation problems instead of PDE ones.

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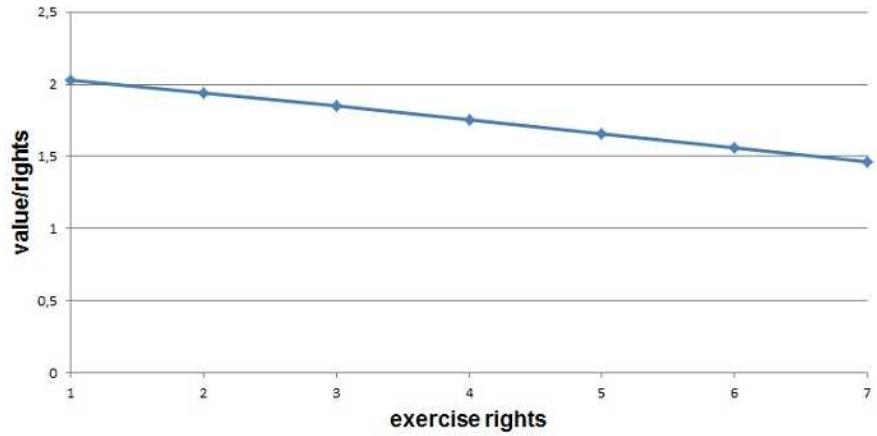

Fig. 3.10: Value per right of a swing option with 1 year to delivery when $\beta = 2$ in Example 2.

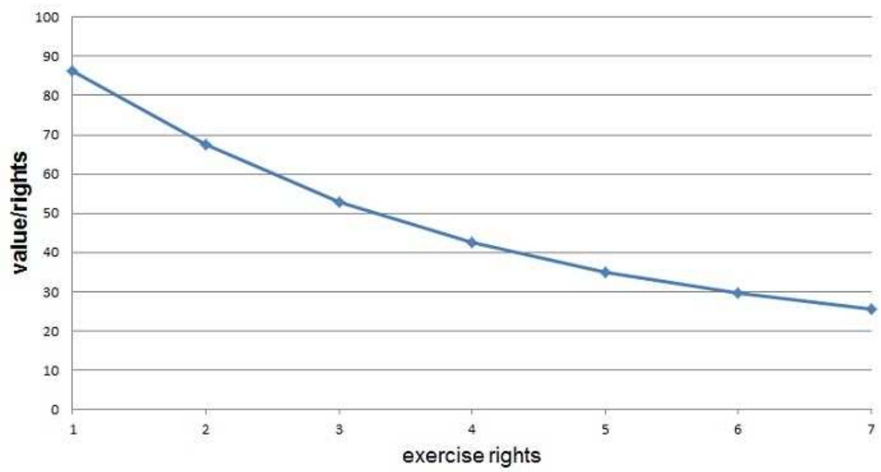

Fig. 3.11: Value per right of a swing option with 1 year to delivery when $\beta = 20$ in Example 2.

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