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Transparent boundary conditions for the hierarchies of parabolic equations where the solution of the n-th equation is used as an input term for the $n+1$-th equation are derived. The existence, uniqueness, and the well-posedness of the initial boundary value problem for the system of the coupled parabolic equations with the derived boundary conditions is established. Such coupled systems of the parabolic equations can be used to approximate the solution of the Helmholtz equation. The derived transparent boundary conditions may be therefore used for the simulation of the wave propagation in unbounded media.

1 Introduction

The parabolic equations theory is a powerful computational tool for the solution of various wave propagation problems (including the simulation of elastic, seismic and electromagnetic waves). The wide-angle parabolic equations (PEs) are traditionally derived by means of the operator square root approximation with a Padé series [1] (hereafter they are referred to as Padé wide-angle PE). Recently another approach to the wide-angle parabolic approximations was proposed [2]. The PE derivation [2] is based on the systematic use of the multiple-scale expansion method, and the resulting high-order parabolic approximations have the form of the system of parabolic equations (PEs), where the input term of the n-th PE is obtained from the solution of $n-1$-th PE [2]. It is important that for such parabolic approximations consistent interface and boundary conditions may be easily derived using the same multiple-scale asymptotic expansions [2]. In order to solve numerically wave propagation problems on unbounded domains using these new parabolic approximations one has to truncate the domain by introducing an artificial boundary (say $z=L$). The boundary conditions at such artificial boundary must be set up in such a way that the outgoing waves leave the computational domain without being reflected at $z=L$. Such boundary conditions are called *transparent boundary conditions* (TBCs). The theory of the TBS for the standard narrow-angle PE is well-developed (see [3] for details and references). The TBCs for the Padé wide-angle PE were also proposed in several papers including [4, 5, 6, 7, 8]. In this work we develop the TBCs for the wide-angle parabolic approximations derived in [2].

2 Parabolic approximations

The time-harmonic sound field $p(x,z)$ in an acoustical waveguide is described by the solution of the Helmholtz equation for the sound pressure [1]:

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right)+\frac{1}{\rho}\kappa^2 p=0\,, \qquad (1)$$

where z denotes the depth (the axis is directed downwards), and x is the horizontal spatial variable. The medium properties are represented by the wavenumber $\kappa=\kappa(x,z)=\omega/c$, where c is the sound speed, and the density $\rho=\rho(x,z)$. A point source at $x=0$, $z=z_s$ can be introduced into the equation (1) as an input term of the form $\delta(x)\delta(z-z_s)$. We consider high-order (wide-angle) parabolic approximations [2] $p_n(x,z)$ to the solution of (1)

$$p_n(x,z)=\mathrm{e}^{\mathrm{i}\kappa_0 x}\sum_{j=0}^{j=n}A_j(x,z)\,,$$

where κ_0 is the reference wavenumber, and $\kappa=\kappa+\nu(x,y)$. The envelope functions $A_j(x,z)$ are determined by

$$2\mathrm{i}\frac{1}{\rho}\kappa_0\frac{\partial A_j}{\partial x}+\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial A_j}{\partial z}\right)_z + \left[\mathrm{i}\left(\frac{1}{\rho}\kappa_0\right)_x+\frac{1}{\rho}\nu\right]A_j+\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial A_{j-1}}{\partial x}\right)=0\,, \tag{2}$$

where

$$A_{-1}(x,z)=0\,.$$

Note that equation (2) greatly simplifies in the case of a homogeneous medium, where it can be written as

$$2\mathrm{i}\kappa_0 A_{j,x}+A_{j,zz}+\nu A_j+A_{j-1,xx}=0\,. \tag{3}$$

It is clear that the medium beyond the artificial boundary must be homogeneous (otherwise the inhomogeneities will produce back-scattered waves which cannot be accounted using the TBCs). Hence in the sequel we restrict our attention to the construction of the TBCs for the simplified system of the coupled equations (3).

Let $\mathbf{A}=(A_0(x,z),A_1(x,z),\dots,A_n(x,z))$ be a solution to the reference initial boundary value problem (IBVP) for the system (3):

$$\begin{aligned}&2\mathrm{i}\kappa_0 A_{j,x}+A_{j,zz}+\nu A_j+A_{j-1,xx}=0\,,\\&A_0(0,z)=S(z)\,,\quad A_j(0,z)=0\,,\quad j=1,2,\dots\,,\\&A_j(x,0)=0\,,\\&\lim_{z\to\infty}|A_j(x,z)|=0\,.\end{aligned} \tag{4}$$

in the domain

$$\Omega=\{(x,z)|0\le z,0\le x\le x_{max}\}$$

with the initial condition $S(z)$ compactly supported on $[0,L]$ where L is sufficiently large. The main goal of this study is to develop the TBCs for (4) at $z=L$.

3 Transparent boundary conditions for the standard PEs

The TBCs for the standard paraxial equation

$$\mathrm{i}\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial y^2}=0\,. \tag{5}$$

(which is equivalent for the equation (3) for A_0) were proposed in [9], a fully discrete approach was developed in [5]. The TBCs for the equation (5) at $y=\pm a$ read as

$$\frac{\partial u}{\partial y}(x,\pm a)=\mp\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{\pi}}\frac{\partial}{\partial x}\int_0^x u(\xi,\pm a)\frac{d\xi}{\sqrt{x-\xi}}=0\,. \tag{6}$$

These conditions allow one to replace the Cauchy problem for (5) on the interval $y\in(-\infty,\infty)$ to a problem on the finite interval $y\in[-a,a]$. It is shown in [3, 5] that the Cauchy problem for (5) on $[-a,a]$ has a unique solution which coincides on this interval with the solution of the Cauchy problem for (5) on an unbounded interval $y\in(-\infty,\infty)$ (the initial conditions are assumed to be identical for both problems and the initial condition

$$u(0,y)=S(y)\in L^2[-a,a]$$

at $x=0$ is compactly supported on $[-a,a]$).

Our goal is to generalize the condition (6) to the case of the system (4).

4 TBCs for the higher-order parabolic approximations

4.1 Problem statement and the DtN operator

We seek to construct the artificial boundary conditions of the form

$$\mathcal{B}(A_j)=0\,, \tag{7}$$

for (4) at $z=L$ such that the solution $\mathbf{A}^t=(A^t_0(x,z),A^t_1(x,z),\dots,A^t_n(x,z))$ to the IBVP for the system (4) on the truncated domain $\Omega^t=\{(x,z)|0\le z\le L,0\le x\le x_{max}\}$ with the compactly supported on $[0,L]$ initial conditions and boundary conditions (7) at $z=L$ coincides with the solution of the reference IBVP.

The reference IBVP in the halfspace $z\ge 0$ is obviously equivalent to the two coupled systems of IBVPs:

$$\left\{\begin{aligned}&2\mathrm{i}\kappa_0 A^t_{j,x}+A^t_{j,zz}+\nu A^t_j+A^t_{j-1,xx}=0\,,\\&A^t_0(0,z)=S(z)\,,\quad A^t_j(0,z)=0,\ j=1,2,\dots\,,\\&A^t_j(x,0)=0,\quad j=0,1,\dots\,,\\&A^t_{j,z}(x,L)=A_{j,z}(x,L)\,,\end{aligned}\right. \tag{8}$$

where $(x,z)\in\Omega^t$ and

$$\left\{\begin{aligned}&2\mathrm{i}\kappa_0 A^r_{j,x}+A^r_{j,zz}+\nu_b A^r_j+A^r_{j-1,xx}=0\,,\\&A^r_j(0,z)=0,\quad j=0,1,\dots\,,\\&A^r_j(x,L)=A^t_j(x,L)\,,\\&\lim_{z\to\infty}|A^r_j(x,z)|=0\,,\end{aligned}\right. \tag{9}$$

where $(x, z) \in \Omega^r = [0, x_{max}] \times [L, \infty)$.

The system (9) can be solved explicitly for a given input function $A^r_j(x, L) = A^t_j(x, L)$. Then we may compute the derivative of the obtained solution $A^r_{j,z}$ and use it as a Neumann condition for the problem (8). The mapping

$$\mathcal{DN} : A^t_j(x, L) \to A^t_{j,z}(x, L)$$

is called Dirichlet-to-Neumann operator, and the TBC may be written in the form

$$A^t_{j,z}(x, L) = \mathcal{DN}(A^t_j(x, L)) .$$

4.2 Laplace transformation

To obtain the explicit form of the $\mathcal{DN}$ operator we solve the IBVP (9). First we apply the Laplace transform to (9):

$$\begin{cases} \hat{A}^r_{j,zz} + (2\mathrm{i}\kappa_0\xi + \nu_b)\hat{A}^r_j = -\xi^2 \hat{A}^r_{j-1},\ z \in [L, \infty), \\ \hat{A}^r_j(\xi, L) = \hat{A}^t_j(\xi, L) , \\ \lim_{z\to\infty} |\hat{A}^r_j(\xi, z)| = 0 . \end{cases} \tag{10}$$

Introducing a new variable $t = z - L$ and setting $\hat{A}^r_j(\xi, z) = u_j(t)$, $2\mathrm{i}\kappa_0\xi + \nu_b = -w^2$, $\xi^2 = v$, $\hat{A}^t_j(\xi, L) = a_j$ in (10), we rewrite the BVP (10) as

$$\begin{cases} u''_j - w^2 u_j = -v u_{j-1} , \quad t \in [0, \infty) , \\ u_j(0) = a_j , \\ \lim_{t\to\infty} |u_j| = 0 . \end{cases} \tag{11}$$

Note that v, w are independent on t, and $w = \sqrt[+]{-2\mathrm{i}\kappa_0\xi - \nu_b}$ denotes the branch of the square root with positive real part.

The solution of (11) can be easily obtained using standard technique and reads

$$u_j(t) = \mathrm{e}^{-wt} \big(a_j + a_{j-1} v P_1(t, w) + a_{j-2} v^2 P_2(t, w) + \cdots + a_0 P_j(t, w)\big) , \tag{12}$$

where $P_k(t)$ for $k \geq 1$ are polynomials such that $u(t) = \mathrm{e}^{-wt} P_k(t)$ is a particular solution to the BVP with homogeneous boundary conditions, i.e.:

$$\begin{cases} u'' - w^2 u = -\mathrm{e}^{-wt} P_{k-1}(t) , \\ u(0) = 0 , \\ \lim_{t\to\infty} |u| = 0 . \end{cases}$$

Substituting $u(t)$ into the latter BVP, we obtain a BVP for $P_k(t)$

$$\begin{cases} P''_k - 2w P'_k = -P_{k-1} , \\ P_k(0) = 0 , \\ \lim_{t\to\infty} |P_k(t)\mathrm{e}^{-wt}| = 0 . \end{cases} \tag{13}$$

It is easy to check that the problem (13) has an explicit solution of the form

$$P_k(t) = \sum_{j=1}^{k} \alpha_{k,j} \frac{t^j}{w^{2k-j}} , \tag{14}$$

where the set of coefficients $\bar{\alpha}_{k+1} = \{\alpha_{k+1,j}\}$ may be computed from the set $\bar{\alpha}_k$ via the recursive formulae:

$$\begin{cases} \alpha_{k+1,k+1} = \frac{1}{2(k+1)} \alpha_{k,k} , \\ \alpha_{k+1,j} = \frac{j+1}{2} \alpha_{k+1,j+1} + \frac{1}{2j} \alpha_{k-1,j} ,\ j = 2, \ldots, k , \\ \alpha_{k+1,1} = \alpha_{k+1,2} . \end{cases} \tag{15}$$

Using the formulae (12) and (14) we immediately arrive at the solution $\hat{A}^r_j$ of the exterior problem (10) in the Laplace domain (ξ, z):

$$\hat{A}^r_j(\xi, z) = \mathrm{e}^{-w(\xi)(z-L)} \times \left(\sum_{k=0}^{j} \xi^{2k} P_k(z - L, w(\xi)) \hat{A}^t_{j-k}(\xi, L) \right) , \tag{16}$$

We differentiate the last equation with respect to z in order to obtain the $\mathcal{DN}$ operator:

$$\frac{\partial \hat{A}^r_j(\xi, z)}{\partial z} = \mathrm{e}^{-w(z-L)} \left(\sum_{k=0}^{j} \xi^{2k} (P'_k(z - L) - w P_k(z - L)) \hat{A}^t_{j-k}(\xi, L) \right) . \tag{17}$$

Let us recall that the coupling condition in the IBVP (8) reads $A^t_{j,z}(x, L) = A^r_{j,z}(x, L)$. Next we substitute the expression for $A^r_{j,z}(\xi, z)$ from (17) into this condition. Observing that

$$w(\xi)\mathrm{e}^{-w(\xi)(z-L)} \times \left. \left(\sum_{k=0}^{j} \xi^{2k} P_k(z - L, w(\xi)) \hat{A}^t_{j-k}(\xi, L) \right) \right|_{z=L} = w(\xi) \hat{A}^t_j(\xi, L) ,$$

and

$$\left. \sum_{k=1}^{j} \xi^{2k} P'_k(z - L, w) \hat{A}^t_{j-k}(\xi, L) \right|_{z=L} = \sum_{k=1}^{j} \xi^{2k} \frac{\alpha_{k,1}}{w^{2k-1}} \hat{A}^t_{j-k}(\xi, L) ,$$

we obtain the desired TBC in the Laplace domain:

$$\left.\frac{\partial \hat{A}^t_j(\xi,z)}{\partial z}\right|_{z=L} = -w(\xi)\hat{A}^t_j(\xi,L) + \sum_{k=1}^{j} \xi^{2k}\frac{\alpha_{k,1}}{w^{2k-1}}\hat{A}^t_{j-k}(\xi,L)\,. \tag{18}$$

In order to obtain the TBCs in the physical domain we compute the inverse Laplace transform $\mathcal{L}^{-1}$: $\hat{f}(\xi) \mapsto f(x)$ of the equation (18) using the well-known properties of $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}(\xi^{2k}\hat{f}(\xi)) = \frac{d^{2k}f}{dx^{2k}}\,,$$

$$\mathcal{L}^{-1}(\hat{g}(\xi-a)\hat{f}(\xi)) = \mathrm{e}^{ax}\int_0^x g(x-y)\mathrm{e}^{-ay}f(y)dy = \mathrm{e}^{ax}\mathcal{L}^{-1}(\hat{g}(\xi)\mathcal{L}(\mathrm{e}^{-ax}f(x)))\,,$$

$$\mathcal{L}^{-1}\left(\frac{\hat{f}(\xi)}{\xi^k}\right) = \int_0^y \ldots \int_0^{y_2} f(y_1)dy_1dy_2\ldots dy_k\,,$$

$$\mathcal{L}^{-1}\left(\sqrt[+]{\xi}\hat{f}(\xi)\right) = \frac{d}{dx}\int_0^x \frac{f(y)dy}{\sqrt{x-y}}\,,$$

and observing that

$$\frac{1}{w(\xi)^{2k-1}} = \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}(2k-1)}}{(2\kappa_0)^{k-\frac{1}{2}}}\frac{\sqrt[+]{\xi-\frac{\mathrm{i}\nu_b}{2\kappa_0}}}{\left(\xi-\frac{\mathrm{i}\nu_b}{2\kappa_0}\right)^k}\,.$$

Thus we obtain the following TBCs

$$\frac{\partial A^t_j(x,z)}{\partial \mathbf{n}} = -\sqrt{\frac{2\kappa_0}{\pi}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\mathrm{e}^{\mathrm{i}\frac{\nu_b}{2\kappa_0}x}\frac{d}{dx}\int_0^x \frac{dy}{\sqrt{x-y}}\left(A^t_j(y,z)\mathrm{e}^{-\mathrm{i}\frac{\nu_b}{2\kappa_0}y}\right. \\ \left. - \sum_{k=1}^{j}\alpha_{k,1}(-2\mathrm{i}\kappa_0)^{-k}\int_0^y\int_0^{y_k}\ldots\int_0^{y_2}\mathrm{e}^{-\mathrm{i}\frac{\nu_b}{2\kappa_0}y_1}\frac{\partial^{2k}A^t_{j-k}(y_1,z)}{\partial y_1^{2k}}dy_1dy_2\ldots dy_k\right) \tag{19}$$

at $z=L$ and $z=0$ ($\mathbf{n}$ denotes the outward unit normal vector at $z=L$, $z=0$ respectively). We provide the values for the first few coefficients $\alpha_{k,1}$:

$$\alpha_{1,1}=\frac{1}{2}\,,\quad \alpha_{2,1}=\frac{1}{8}\,,\quad \alpha_{3,1}=\frac{1}{16}\,,\quad \alpha_{4,1}=\frac{5}{128}\,,$$

which are necessary to evaluate the sum in (19).

The TBCs (19) simplify significantly if we choose the reference wavenumber κ_0 in such a way that $\nu_b = 0$ (this condition is fulfilled if we set $\kappa_0(x) = \kappa_b$ in (2)). Under this assumption the multiple integral on the right-hand side of (19) vanishes, and the TBC becomes

$$\frac{\partial A^t_j(x,z)}{\partial \mathbf{n}} = -\sqrt{\frac{2\kappa_0}{\pi}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} \times \sum_{k=0}^{j}\alpha_{k,1}(-2\mathrm{i}\kappa_0)^{-k}\frac{d}{dx}\int_0^x \frac{\partial^k A^t_{j-k}(y,z)}{\partial y^k}\frac{dy}{\sqrt{x-y}}\,, \quad \text{at } z=0,L\,, \tag{20}$$

where $\alpha_{0,1} = -1$.

Note that the TBCs (19), (20) are a natural generalization of the TBC (6) for the paraxial PE (5) with the potential vanishing at the artificial boundary [9, 10].

All standard theorems about the existence, uniqueness, and the well-posedness of the IBVPs for the narrow-angle parabolic equations (see [3, 5, 11] for the details) with the transparent boundary conditions can be naturally generalized to the case of the IBVP (8) with the TBCs (20).

5 Numerical results

To test our TBCs (19) we consider the solution of the system (8) in the stripe $0 \le z \le L$ where $L = 200$ m, and the wavenumber corresponds to the frequency $f = 100$ Hz and the sound speed $c = 1500$ m/s. The TBC (20) is imposed at $z = L$. We use the optimal Gaussian initial condition [1] that simulates the field produced by a point source located at $z_s = 100$ m as the Cauchy data $S(z)$ for

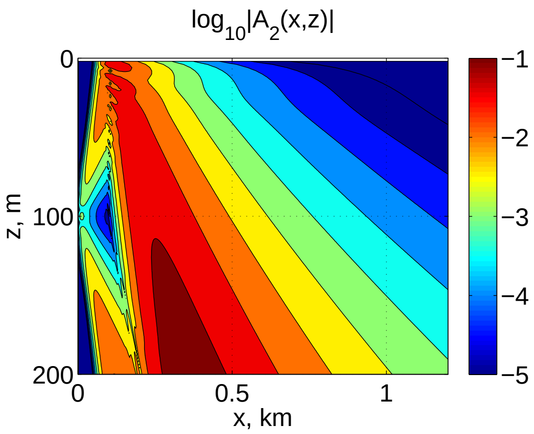

Figure 1: Analytical solution $A_2(x,z)$ computed by the Fourier method

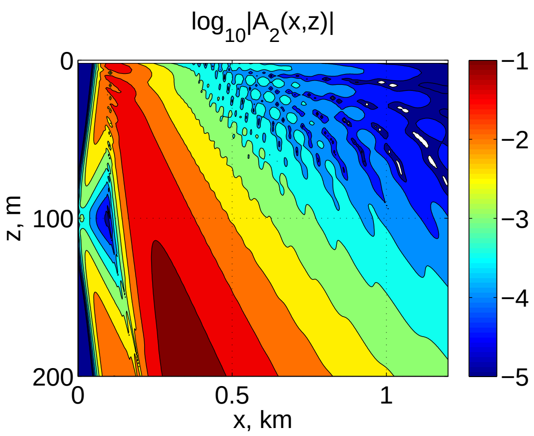

Figure 2: Numerical solution $A_2(x,z)$ computed by the finite-difference discretization of the IBVP (8) with the TBC (20)

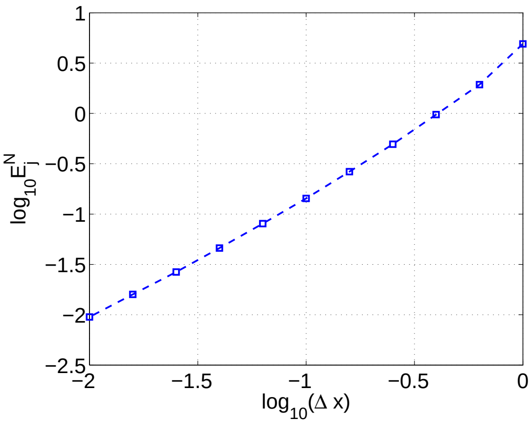

Figure 3: Errors for the solution $A_2(x,z)$ (computed at $x = 1200$ m) as a function of the mesh size Δx.

the first equation (i.e. the equation for A_0) in the system (4). In the case of a homogeneous medium and the Gaussian Cauchy data the system (4) has a simple analytical solution (which can be easily obtained using the Fourier method to write the free space solution and then putting the image source at $z = -z_s$).

The numerical solution is computed using the Crank-Nicholson-type scheme for the PE equations in (8) and the generalization of the Baskakov-Popov discretization [9] for the TBC (20). It was shown recently [10] that this discretization allows to maintain the unconditional stability of the Crank-Nicholson scheme. Note that this is not necessarily true for the arbitrary numerical implementation of the non-local TBCs of the type (6). In cases of some specific discretizations the stability region of the resulting numerical scheme may be somewhat bizarre as it is shown in [12, 13].

As an example we compare the analytical and the numerical solutions of the third equation of the system (4), i.e. the equation for $A_2(x,z)$. They are shown in Fig. 1 and 2 respectively. Note that the amplitude of the reflected wave is much smaller than that of the incident one. In Fig. 3 we plotted the errors of the numerical scheme (which are attributed to the reflection) for different values of the mesh size Δx (the ratio $(\Delta z)^2/\Delta x = 2$ was kept constant). It is clear that the errors decrease steadily with the mesh size, and this confirms that the TBCs (20) are indeed valid for the system (4).

6 Conclusion

In this work the TBCs for the system of the coupled PEs (4) were proposed. This system represents a wide-angle parabolic approximation for the Helmholtz equation (1), and as such it is useful for the solution of various wave propagation problems. The derived TBCs allow to use these parabolic approximation in situations where the medium has no physical boundaries (such problems are ubiquitous e.g. in the underwater acoustics).

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