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Monika Winklmeier*, Christian Wyss†

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Abstract

For an unbounded operator S on a Banach space the existence of invariant subspaces corresponding to its spectrum in the left and right half-plane is proved. The general assumption on S is the uniform boundedness of the resolvent along the imaginary axis. The projections associated with the invariant subspaces are bounded if S is strictly dichotomous, but may be unbounded in general. Explicit formulas for these projections in terms of resolvent integrals are derived and used to obtain perturbation theorems for dichotomy. All results apply, with certain simplifications, to bisectorial operators.

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1 Introduction

Let S be a densely defined linear operator S on a Banach space X such that the imaginary axis belongs to the resolvent set $\rho(S)$. A fundamental question is whether there exist closed invariant subspaces X_+ and X_- which correspond to the spectrum of S in the right and left half-plane \mathbb{C}_+ and \mathbb{C}_- , respectively. S is called *dichotomous* if these subspaces exist and yield a decomposition $X = X_+ \oplus X_-$. If in addition the restrictions $-S|_{X_+}$ and $S|_{X_-}$ generate exponentially decaying semigroups, then S is *exponentially dichotomous*. Dichotomy and exponential dichotomy have found a wide range of applications, e.g. to canonical factorisation and Wiener-Hopf integral operators [BGK86a, BGK86b], and to block operator matrices and Riccati equations

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[LT01, LRvdR02, RvdM04, TW14]. An extensive account may be found in the monograph [vdM08].

The investigation of dichotomous operators was started by H. Bart, I. Gohberg and M.A. Kaashoek in [BGK86b], where they established a sufficient condition for dichotomy: If a strip around the imaginary axis belongs to $\varrho(S)$, the resolvent $(S - \lambda)^{-1}$ is uniformly bounded on this strip, i.e.,

$$\sup_{|\operatorname{Re} \lambda| \leq h} \|(S - \lambda)^{-1}\| < \infty \quad (1)$$

for some $h > 0$, and the integral

$$Px = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda^2} (S - \lambda)^{-1} S^2 x \, d\lambda, \quad x \in \mathcal{D}(S^2), \quad (2)$$

extends to a bounded operator P on X , then S is dichotomous and P is the projection onto X_+ along X_- . On the other hand, there are simple examples where (1) holds, X_{\pm} exist, but $X_+ \oplus X_- \subset X$ is only dense and S is not dichotomous.

In our first main result, Theorem 4.1, we prove that (1) is in fact sufficient for the existence of invariant subspaces, even if S is not dichotomous: If (1) holds, then the subspaces

$$G_{\pm} = \{x \in X : (S - \lambda)^{-1}x \text{ has a bounded analytic extension to } \overline{\mathbb{C}_{\mp}}\}$$

are closed, S -invariant and satisfy $\sigma(S|_{G_{\pm}}) \subset \mathbb{C}_{\pm}$. Moreover, the integral (2) extends to a closed, possibly unbounded operator P , which is the projection onto G_+ along G_- . Finally, P is bounded if and only if S is dichotomous with respect to $X = G_+ \oplus G_-$. This decomposition has the additional property that the resolvents of $S|_{G_{\pm}}$ are uniformly bounded on \mathbb{C}_{\mp} , and we call S *strictly dichotomous* in this case.

One important class of operators satisfying (1) are *bisectorial* and *almost bisectorial* operators, for which $i\mathbb{R} \subset \varrho(S)$ and

$$\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|^{\beta}}, \quad \lambda \in i\mathbb{R} \setminus \{0\}, \quad (3)$$

with some $M > 0$ and $\beta = 1$ in the bisectorial, $0 < \beta < 1$ in the almost bisectorial case. In Theorem 5.6 we show that for bisectorial and almost bisectorial S equation (2) simplifies to

$$Px = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda} (S - \lambda)^{-1} S x \, d\lambda, \quad x \in \mathcal{D}(S), \quad (4)$$

and that the restrictions $S|_{G_{\pm}}$ are (almost) sectorial, i.e., they satisfy the resolvent estimate (3) on \mathbb{C}_{\mp} . The results for bisectorial S were to some extent obtained by W. Arendt and A. Zamboni in [AZ10]. In particular, they constructed closed projections by using a rearranged version of (4); our construction of P in

Theorem 4.1 is in fact a generalisation of their method to the weaker setting of (1). Note that bisectoriality does not imply dichotomy and hence unbounded P are still possible here as Example 5.8 shows.

In addition to the characterisation of dichotomy in Theorems 4.1 and 5.6, we also derive perturbation results. Theorem 7.1 states that if S is strictly dichotomous and T is another densely defined operator on X such that

- (i) $\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq h\} \subset \varrho(S) \cap \varrho(T)$,
- (ii) $\sup_{|\operatorname{Re} \lambda| \leq h} |\lambda|^{1+\varepsilon} \|(S - \lambda)^{-1} - (T - \lambda)^{-1}\| < \infty$ for some $\varepsilon > 0$, and
- (iii) $\mathcal{D}(S^2) \cap \mathcal{D}(T^2) \subset X$ is dense,

then T is strictly dichotomous too. A similar result was obtained in [BGK86b], but under significantly stronger conditions, namely $\varepsilon = 1$ in (ii) and $\mathcal{D}(T^2) \subset \mathcal{D}(S^2)$ instead of (iii). It is precisely the existence of the closed projections which allows us to use the more general condition (iii). If in addition S is (almost) bisectorial, then T is (almost) bisectorial too and condition (iii) may be relaxed to

- (iii') $\mathcal{D}(S) \cap \mathcal{D}(T) \subset X$ is dense,

see Theorem 7.3. In [TW14] this result was obtained for bisectorial S and $T = S + R$ where R is p -subordinate to S , i.e., $\mathcal{D}(S) \subset \mathcal{D}(R)$ and $\|Rx\| \leq c\|x\|^{1-p}\|Sx\|^p$, $x \in \mathcal{D}(S)$, with constants $c > 0$ and $0 \leq p < 1$. In this case (ii) holds with $\varepsilon = 1 - p$ and (iii') is trivially satisfied since $\mathcal{D}(S) = \mathcal{D}(T)$. On the other hand, (iii) and (iii') allow for a wider class of perturbations with $\mathcal{D}(S) \neq \mathcal{D}(T)$, e.g. if $T = S + R$ holds only in an extrapolation space, see Example 8.8.

Most of our results remain valid for non-densely defined S . In particular, this is true for the main Theorems 4.1 and 5.6, but not for the perturbation results. Unless explicitly stated otherwise, linear operators are not assumed to be densely defined.

There exist different approaches to dichotomy: For bisectorial S an equivalent condition for dichotomy in terms of complex powers of S is given in [DV89]. For generators of C_0 -semigroups, exponential dichotomy is equivalent to the hyperbolicity of the semigroup [KVL94], the latter being important e.g. in the study of nonlinear evolution equations. Finally, our approach is connected to (bounded as well as unbounded) functional calculus.

This article is organised as follows: In Section 2 we collect general facts about unbounded projections, in particular Lemma 2.3 on the existence of closed projections corresponding to invariant subspaces. Section 3 contains the definition and basic properties of dichotomous operators. Here we also show that dichotomy alone does not uniquely determine the decomposition $X = X_+ \oplus X_-$ while strict dichotomy does. In Section 4 we derive our main Theorem 4.1 in the general setting (1). The case of bisectorial and almost bisectorial operators is then considered in Section 5. For such operators we also provide some results on the location of their spectrum and derive yet another integral representation for

P , which was used in [LT01, TW14]. Section 6 is devoted to the subtle problem that the restrictions $S|_{G_{\pm}}$ are not necessarily densely defined even if S is. Following [BGK86b] we consider certain subspaces $M_{\pm} \subset G_{\pm}$ for which the parts of S in M_{\pm} are densely defined. We derive conditions for $M_{\pm} = G_{\pm}$, which is in turn equivalent to $S|_{G_{\pm}}$ being densely defined. The perturbation results are contained in Section 7, and in Section 8 we provide some additional examples to illustrate our theory. Finally, as an application, we show the dichotomy of a Hamiltonian operator matrix from control theory whose off-diagonal entries map into extrapolation spaces. In the previous results [LRvdR02, WJZ12, TW14] only settings without extrapolation spaces or with the additional assumption of a Riesz basis of eigenvectors could be handled.

We use the following notation: The open right and open left half-plane is denoted by \mathbb{C}_+ and \mathbb{C}_- , respectively. If X, Y are Banach spaces, then $T(X \rightarrow Y)$ denotes a linear operator from a (not necessarily dense) domain in X into Y . If $Z \subset X$, then we denote the restriction of T to Z by $T|_Z$. The set of all bounded linear operators from X to Y is denoted by $L(X, Y)$ and we set $L(X) = L(X, X)$. Let $U \subset \mathbb{C}$. We call a function $\varphi : U \rightarrow X$ *analytic* if it admits an analytic extension to some open neighbourhood of U . For the argument of a complex number we choose the range $-\pi < \arg z \leq \pi$ for $z \neq 0$ and set $\arg 0 = 0$.

2 Unbounded projections

Our main tool for investigating the dichotomy of an operator S are projections corresponding to invariant subspaces. In the general case, these projections will be unbounded and the direct sum of the corresponding subspaces is not the whole space X . Unbounded projections associated with bisectorial operators have been studied in [AZ10].

Definition 2.1. Let X be a Banach space. A (possibly unbounded) operator $P(X \rightarrow X)$ is called a *projection* if $\text{Im}(P) \subset \mathcal{D}(P)$ and $P^2 = P$.

In other words, P is a projection in the algebraic sense on the vector space $\mathcal{D}(P)$. If P is a projection, then

$$\mathcal{D}(P) = \text{Im}(P) \oplus \ker(P). \quad (5)$$

The *complementary projection* is given by $Q = I - P$, $\mathcal{D}(Q) = \mathcal{D}(P)$. In this case $\text{Im}(Q) = \ker(P)$, $\ker(Q) = \text{Im}(P)$.

Conversely, if $X_1, X_2 \subset X$ are linear subspaces such that $X_1 \cap X_2 = \{0\}$, then there are corresponding complementary projections P_1, P_2 with $\mathcal{D}(P_1) = \mathcal{D}(P_2) = X_1 \oplus X_2$, $\text{Im}(P_1) = X_1$ and $\text{Im}(P_2) = X_2$.

Remark 2.2. (i) A projection P is closed if and only if $\text{Im}(P)$ and $\ker(P)$ are closed subspaces.

(ii) A closed projection P is bounded if and only if $\text{Im}(P) \oplus \ker(P)$ is closed.

The next lemma gives a sufficient condition on a linear operator S that allows for the construction of a pair of closed complementary projections which commute with S and yield S -invariant subspaces.

Lemma 2.3. *Let $S(X \rightarrow X)$ be a closed operator with $0 \in \varrho(S)$. Suppose that there are bounded operators $A_1, A_2 \in L(X)$ satisfying*

$$A_1 + A_2 = S^{-2}, \quad (6)$$

$$A_1 A_2 = A_2 A_1 = 0, \quad (7)$$

$$A_j(S - \lambda)^{-1} = (S - \lambda)^{-1} A_j, \quad \lambda \in \varrho(S), j = 1, 2. \quad (8)$$

Then

- (i) $P_j := S^2 A_j$, $j = 1, 2$, are closed complementary projections, $\mathcal{D}(S^2) \subset \mathcal{D}(P_1) = \mathcal{D}(P_2)$ and $P_j x = A_j S^2 x$ for $x \in \mathcal{D}(S^2)$;
- (ii) the subspaces $X_j = \text{Im}(P_j)$ are closed, S - and $(S - \lambda)^{-1}$ -invariant for $\lambda \in \varrho(S)$ and satisfy $X_1 = \ker(A_2)$, $X_2 = \ker(A_1)$. Moreover

$$\sigma(S) = \sigma(S|X_1) \cup \sigma(S|X_2). \quad (9)$$

Proof. (i) Since S^2 is closed and A_j is bounded, P_j is closed. We have $x \in \mathcal{D}(P_j)$ if and only if $A_j x \in \mathcal{D}(S^2)$. Hence (6) implies that $\mathcal{D}(P_1) = \mathcal{D}(P_2)$ and that $P_1 + P_2 = I$ on $\mathcal{D}(P_1)$. From (8) it follows that

$$A_j S x = S A_j x \quad \text{for } x \in \mathcal{D}(S), j = 1, 2. \quad (10)$$

For $x \in \mathcal{D}(P_1)$, this implies $A_2 P_1 x = A_2 S^2 A_1 x = S^2 A_2 A_1 x = 0$ and hence $\text{Im}(P_1) \subset \mathcal{D}(P_2)$, $P_2 P_1 = 0$ and $P_1^2 = (I - P_2) P_1 = P_1$. Thus P_1 and P_2 are complementary projections. Additionally, (10) yields that if $x \in \mathcal{D}(S^2)$, then $x \in \mathcal{D}(P_1)$ and $A_j S^2 x = S^2 A_j x = P_j x$.

(ii) Since S^2 is invertible, $X_1 = \ker(P_2) = \ker(A_2)$ and analogously $X_2 = \ker(A_1)$. Consequently these subspaces are closed and (8) and (10) imply that they are also S - and $(S - \lambda)^{-1}$ -invariant. To obtain (9), we show now the equivalent identity

$$\varrho(S) = \varrho(S|X_1) \cap \varrho(S|X_2).$$

From the invariance of X_j it is easily seen that for $\lambda \in \varrho(S)$ the operator $S|X_j - \lambda$ is bijective with inverse $(S - \lambda)^{-1}|X_j$; hence $\lambda \in \varrho(S|X_j)$. For the other inclusion let $\lambda \in \varrho(S|X_1) \cap \varrho(S|X_2)$. If $(S - \lambda)x = 0$, then $x \in \mathcal{D}(S^2) \subset \mathcal{D}(P_j)$ and therefore

$$0 = (S - \lambda)x = (S|X_1 - \lambda)P_1 x + (S|X_2 - \lambda)P_2 x.$$

This yields $P_1 x = P_2 x = 0$, i.e. $x = 0$. Hence $S - \lambda$ is injective. To show surjectivity, set

$$T = (S|X_1 - \lambda)^{-1} A_1 + (S|X_2 - \lambda)^{-1} A_2.$$

Then $(S - \lambda)T = A_1 + A_2 = S^{-2}$ from which we conclude that $\text{Im}(T) \subset \mathcal{D}(S^3)$ and $(S - \lambda)S^2 T = I$. \square

The following example illustrates a typical situation in which the projections from Lemma 2.3 are unbounded. It also shows that for fixed S there may be several possible choices for A_1, A_2 .

Example 2.4. Let S be the block diagonal operator on the sequence space $X = l^2$ given by

$$S = \text{diag}(S_1, S_2, \dots), \quad S_n = \begin{pmatrix} n & 2n^2 \\ 0 & -n \end{pmatrix}.$$

First observe that

$$(S_n - \lambda)^{-1} = \begin{pmatrix} (n - \lambda)^{-1} & 2n^2(n - \lambda)^{-1}(n + \lambda)^{-1} \\ 0 & -(n + \lambda)^{-1} \end{pmatrix}, \quad \lambda \neq \pm n.$$

Then $\sup_n \|(S_n - \lambda)^{-1}\| < \infty$ for $\lambda \notin \mathbb{Z} \setminus \{0\}$. Hence $\sigma(S) = \mathbb{Z} \setminus \{0\}$ and $(S - \lambda)^{-1} = \text{diag}((S_1 - \lambda)^{-1}, (S_2 - \lambda)^{-1}, \dots)$, $\lambda \in \varrho(S)$. The spectral projections P_n^+, P_n^- corresponding to the eigenvalues n and $-n$ of S_n are

$$P_n^+ = \begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix}, \quad P_n^- = \begin{pmatrix} 0 & -n \\ 0 & 1 \end{pmatrix}.$$

Moreover

$$S_n^{-1} = \begin{pmatrix} n^{-1} & 2 \\ 0 & -n^{-1} \end{pmatrix}, \quad S_n^{-2} = \begin{pmatrix} n^{-2} & 0 \\ 0 & n^{-2} \end{pmatrix}.$$

Let $A_n^\pm := S_n^{-2} P_n^\pm$. Then

$$A_n^+ = \begin{pmatrix} n^{-2} & n^{-1} \\ 0 & 0 \end{pmatrix}, \quad A_n^- = \begin{pmatrix} 0 & -n^{-1} \\ 0 & n^{-2} \end{pmatrix}$$

and we have $A_n^+ + A_n^- = S_n^{-2}$, $A_n^+ A_n^- = A_n^- A_n^+ = 0$ and $A_n^\pm (S_n - \lambda)^{-1} = (S_n - \lambda)^{-1} A_n^\pm$ for $\lambda \notin \mathbb{Z} \setminus \{0\}$.

Now we define an operator $A_1 \in L(X)$ by choosing for each n either A_n^+ or A_n^- ; for $A_2 \in L(X)$ we take the complementary choice. More explicitly let $\Lambda_1 \subset \mathbb{N}$ and $\Lambda_2 = \mathbb{N} \setminus \Lambda_1$. Set $\epsilon_n^j = +$ if $n \in \Lambda_j$ and $\epsilon_n^j = -$ if $n \notin \Lambda_j$. Then the operators

$$A_j = \text{diag}(A_1^{\epsilon_1^j}, A_2^{\epsilon_2^j}, A_3^{\epsilon_3^j}, \dots), \quad j = 1, 2, \quad (11)$$

satisfy all conditions in Lemma 2.3. The closed projections $P_j = S^2 A_j$ are unbounded and block diagonal with $P_j = \text{diag}(P_1^{\epsilon_1^j}, P_2^{\epsilon_2^j}, P_3^{\epsilon_3^j}, \dots)$. By Lemma 2.3, the subspaces $X_j = \text{Im}(P_j)$ are S -invariant. Clearly X_j is the closed linear hull of the eigenvectors of S for the eigenvalues $n \in \Lambda_j$ and the eigenvalues $-n$ for $n \notin \Lambda_j$. Note that $X_1 \oplus X_2 \neq X$. We will investigate this example further in Example 4.5.

Remark 2.5. Lemma 2.3 continues to hold if S^2 and S^{-2} are replaced throughout by S and S^{-1} . That is, if there exist $B_1, B_2 \in L(X)$ such that

$$B_1 + B_2 = S^{-1}, \quad B_1 B_2 = B_2 B_1 = 0, \\ B_j (S - \lambda)^{-1} = (S - \lambda)^{-1} B_j, \quad \lambda \in \varrho(S), \quad j = 1, 2,$$

then we obtain closed complementary projections $P_j = SB_j$ where, in particular, $\mathcal{D}(S) \subset \mathcal{D}(P_j)$, $X_j = \text{Im}(P_j)$ are S - and $(S - \lambda)^{-1}$ -invariant and (9) holds.

3 Dichotomous operators

Definition 3.1. Let X be a Banach space and let X_{\pm} be closed subspaces with $X = X_+ \oplus X_-$. An operator $S(X \rightarrow X)$ is called *dichotomous* with respect to the decomposition $X = X_+ \oplus X_-$ if

- (i) $i\mathbb{R} \subset \varrho(S)$,
- (ii) X_+ and X_- are S -invariant,
- (iii) $\sigma(S|X_+) \subset \mathbb{C}_+$, $\sigma(S|X_-) \subset \mathbb{C}_-$.

An operator S is called *strictly dichotomous* with respect to $X = X_+ \oplus X_-$ if, in addition,

- (iv) $\|(S|X_+ - \lambda)^{-1}\|$ is bounded on $\overline{\mathbb{C}_-}$, $\|(S|X_- - \lambda)^{-1}\|$ is bounded on $\overline{\mathbb{C}_+}$.

A dichotomous operator S is called *exponentially dichotomous* if $-S|X_+$ and $S|X_-$ generate exponentially decaying semigroups.

Clearly, exponential dichotomy implies strict dichotomy. Note that all dichotomous operators appearing in [LT01, LRvdR02, TW14] are in fact strictly dichotomous, see Remark 4.4.

Remark 3.2. For a given operator S , there may exist several decompositions of X with respect to which it is dichotomous (Example 3.3), but there exists at most one with respect to which it is strictly dichotomous (Lemma 3.7).

Example 3.3. Let X be a Banach space and let $S(X \rightarrow X)$ be any operator satisfying $\sigma(S) = \emptyset$. Then evidently S is dichotomous with respect to the two decompositions

$$X_+ = X, \quad X_- = \{0\} \tag{12}$$

and

$$X_+ = \{0\}, \quad X_- = X. \tag{13}$$

Examples for such operators are generators of nilpotent contraction semigroups, e.g. shift semigroups on bounded intervals. In this case, the resolvent $(S - \lambda)^{-1}$ is uniformly bounded on \mathbb{C}_+ and thus S is strictly dichotomous with respect to the decomposition (13) but not with respect to (12).

To obtain an example of a dichotomous operator with non-empty spectrum, we consider an operator given as the direct sum $S = S_0 \oplus S_1 \oplus S_2$ on $X = X_0 \oplus X_1 \oplus X_2$ where the S_j are linear operators on X_j such that

$$\sigma(S_0) = \emptyset, \quad \sigma(S_1) \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda \geq h\}, \quad \sigma(S_2) \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda \leq -h\}$$

for some $h > 0$. Then S is dichotomous with respect to the decompositions

$$X_+ = X_1 \oplus X_0, \quad X_- = X_2 \quad (14)$$

and

$$X_+ = X_1, \quad X_- = X_2 \oplus X_0. \quad (15)$$

Remark 3.4. Even in the case of bisectorial operators, dichotomy is not sufficient to determine the subspaces X_\pm uniquely, see Section 8.1.

Remark 3.5. Let S be a dichotomous operator with respect to $X = X_+ \oplus X_-$.

- (i) If x is a (generalised) eigenvector of S with eigenvalue $\lambda \in \mathbb{C}_\pm$, then $x \in X_\pm$.
- (ii) Suppose that S has a complete system of generalised eigenvectors. Then the spaces X_\pm are uniquely determined by S as the closures of the span of generalised eigenvectors of S whose eigenvalues belong to \mathbb{C}_\pm .

Lemma 3.6. (i) If S is dichotomous with respect to $X = X_+ \oplus X_-$, then $\mathcal{D}(S) = (\mathcal{D}(S) \cap X_+) \oplus (\mathcal{D}(S) \cap X_-)$, and S admits the block matrix representation

$$S = \begin{pmatrix} S|_{X_+} & 0 \\ 0 & S|_{X_-} \end{pmatrix}$$

with respect to $X = X_+ \oplus X_-$. In particular, $\sigma(S) = \sigma(S|_{X_+}) \cup \sigma(S|_{X_-})$ and the subspaces X_\pm are $(S - \lambda)^{-1}$ -invariant for $\lambda \in \varrho(S)$.

- (ii) If S is strictly dichotomous, then there exist $h > 0$ and $M > 0$ such that $\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq h\} \subset \varrho(S)$ and

$$M = \sup_{|\operatorname{Re} \lambda| \leq h} \|(S - \lambda)^{-1}\| < \infty. \quad (16)$$

Proof. (i) is proved as in [TW14, Lemma 2.4]. If S is strictly dichotomous, then a Neumann series argument implies that there exist $h > 0$ and $M > 0$ such that $\lambda \in \varrho(S|_{X_\pm})$ and $\|(S|_{X_\pm} - \lambda)^{-1}\| \leq M$ whenever $|\operatorname{Re} \lambda| \leq h$. (ii) then follows from the block matrix decomposition in (i). \square

We will now establish the uniqueness of the decomposition $X = X_+ \oplus X_-$ of a strictly dichotomous operator. To this end, let $S(X \rightarrow X)$ with $i\mathbb{R} \subset \varrho(S)$. We consider the two subspaces G_+ and G_- defined by

$$G_\pm = \{x \in X : (S - \lambda)^{-1}x \text{ has a bounded analytic extension to } \overline{\mathbb{C}_\mp}\}. \quad (17)$$

More explicitly, for $x \in X$ we consider the analytic function

$$\varphi_x : \varrho(S) \rightarrow X, \quad \varphi_x(\lambda) = (S - \lambda)^{-1}x.$$

Then $x \in G_+$ if and only if φ_x admits a bounded analytic extension to the closed left half-plane

$$\varphi_x^+ : \overline{\mathbb{C}_-} \cup \varrho(S) \rightarrow X.$$

Analogously, $x \in G_-$ if and only if φ_x admits a bounded analytic extension to the closed right half-plane

$$\varphi_x^- : \overline{\mathbb{C}_+} \cup \varrho(S) \rightarrow X.$$

Lemma 3.7. *Let $S(X \rightarrow X)$ with $i\mathbb{R} \subset \varrho(S)$. Then:*

- (i) $G_+ \cap G_- = \{0\}$.
- (ii) *If S is strictly dichotomous for the decomposition $X = X_+ \oplus X_-$, then $X_\pm = G_\pm$. In particular, X_\pm are uniquely determined.*

Proof. (i): Let $x \in G_+ \cap G_-$. Then $\varphi_x(\lambda) = (S - \lambda)^{-1}x$ can be extended to a bounded analytic function on \mathbb{C} which must be constant by Liouville's theorem. Hence $(S - \lambda_1)^{-1}x = (S - \lambda_2)^{-1}x$ for all $\lambda_1, \lambda_2 \in \varrho(S)$ which is only the case if $\varrho(S) = \emptyset$ or $x = 0$.

(ii): For $x \in X_+$, strict dichotomy of S implies that $\varphi_x^+(\lambda) = (S|_{X_+} - \lambda)^{-1}x$ is a bounded analytic extension of $(S - \lambda)^{-1}x$ to $\overline{\mathbb{C}_-}$. Hence $x \in G_+$, i.e. $X_+ \subset G_+$; similarly $X_- \subset G_-$. From $X = X_+ \oplus X_-$ and $G_+ \cap G_- = \{0\}$ we thus obtain $X_\pm = G_\pm$. \square

Remark 3.8. (i) The subspaces G_\pm have been introduced in [BGK86b], but with the roles of G_+ and G_- exchanged and the additional conditions $\varphi_x^\pm(\lambda) \in \mathcal{D}(S)$ and $(S - \lambda)\varphi_x^\pm(\lambda) = x$. In the proof of Theorem 4.1 we will show that if in addition to $i\mathbb{R} \subset \varrho(S)$ an estimate (16) holds, then $\varphi_x^\pm(\lambda) = (S|_{G_\pm} - \lambda)^{-1}x$ and hence both conditions are automatically fulfilled.

- (ii) The estimate (16) also implies that the condition on the boundedness of the extensions φ_x^\pm in the definition of G_\pm can be weakened: If the analytic extensions of the resolvent to the left and right half-plane satisfy

$$\|\varphi_x^+(\lambda)\| \leq C|\lambda|^k, \quad \operatorname{Re} \lambda < -h,$$

and

$$\|\varphi_x^-(\lambda)\| \leq C|\lambda|^k, \quad \operatorname{Re} \lambda > h,$$

respectively, with some constants $k \in \mathbb{N}$ and $C > 0$, then they are bounded by the Phragmén-Lindelöf theorem below.

Theorem 3.9 (Phragmén-Lindelöf [Con78, Corollary VI.4.2]). *Let $a \geq 1/2$ and $\Sigma = \{z \in \mathbb{C} : |\arg z| < \frac{\pi}{2a}\}$. Consider an analytic function $f : \Sigma \rightarrow \mathbb{C}$ which is continuous on $\overline{\Sigma}$. If there exist constants $M, C > 0$ and $0 < b < a$ such that $|f(z)| \leq M$ for $z \in \partial\Sigma$ and $|f(z)| \leq Ce^{|z|^b}$ for $z \in \Sigma$, then also $|f(z)| \leq M$ for $z \in \Sigma$.*

4 Spectral splitting along the imaginary axis

In this section we prove our first spectral splitting result: If the resolvent of S is uniformly bounded along the imaginary axis, then the subspaces G_{\pm} defined in (17) are closed invariant subspaces of S corresponding to the spectrum in \mathbb{C}_{\pm} . Moreover we construct a pair of closed complementary projections P_{\pm} onto G_{\pm} .

We make the following assumption: there exists $h > 0$ such that

$$\{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq h\} \subset \varrho(S) \quad (18)$$

and

$$M := \sup_{|\operatorname{Re} \lambda| \leq h} \|(S - \lambda)^{-1}\| < \infty. \quad (19)$$

For this assumption to hold it is sufficient that $i\mathbb{R} \subset \varrho(S)$ and that the resolvent $(S - \lambda)^{-1}$ is uniformly bounded on $i\mathbb{R}$. If (18) and (19) hold for $h > 0$, then they also hold for some $h' > h$ with a possibly larger M . Both statements follow from a standard Neumann series argument.

Note that by Lemma 3.6(ii) every strictly dichotomous operator satisfies (18) and (19).

For every S which satisfies (18) and (19) we can define the operators

$$A_{\pm} = \frac{\pm 1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda^2} (S - \lambda)^{-1} d\lambda \quad (20)$$

as in [BGK86b]. Note that by (19) the integrals converge in the uniform operator topology, hence A_{\pm} are well-defined bounded linear operators. Due to Cauchy's theorem, the integrals on the right hand side are independent of h as long as (18) and (19) hold.

The next theorem extends the results from [BGK86b, Theorem 3.1]. In particular we obtain a spectral splitting also in the case of unbounded projections P_{\pm} .

Theorem 4.1. *Let $S(X \rightarrow X)$ be a linear operator on the Banach space X satisfying (18) and (19). Then:*

- (i) *The subspaces G_{\pm} defined in (17) are closed, S - and $(S - \lambda)^{-1}$ -invariant and satisfy*

$$\sigma(S) = \sigma(S|_{G_+}) \cup \sigma(S|_{G_-}), \quad (21)$$

$$\sigma(S|_{G_{\pm}}) = \sigma(S) \cap \mathbb{C}_{\pm}, \quad (22)$$

$$\|(S|_{G_{\pm}} - \lambda)^{-1}\| \leq M \quad \text{for } \lambda \in \overline{\mathbb{C}_{\mp}}, \quad (23)$$

where M is given by (19).

- (ii) *Let A_{\pm} as in (20). Then the operators $P_{\pm} = S^2 A_{\pm}$ are closed complementary projections satisfying $\mathcal{D}(P_+) = \mathcal{D}(P_-) = G_+ \oplus G_-$ and*

$$G_{\pm} = \operatorname{Im}(P_{\pm}) = \ker(A_{\mp}).$$

Hence P_{\pm} are the complementary projections corresponding to the direct sum $G_+ \oplus G_- \subset X$.

(iii) $\mathcal{D}(S^2) \subset \mathcal{D}(P_{\pm})$ and

$$P_{\pm}x = \frac{\pm 1}{2\pi i} \int_{\pm h-i\infty}^{\pm h+i\infty} \frac{1}{\lambda^2} (S-\lambda)^{-1} S^2 x d\lambda, \quad x \in \mathcal{D}(S^2). \quad (24)$$

Proof. We first show (ii). In [BGK86b] it has been shown that $A_+A_- = A_-A_+ = 0$ and $A_+ + A_- = S^{-2}$. For the convenience of the reader we sketch the proof. Using the resolvent identity and Fubini's theorem, we get

$$\begin{aligned} A_+A_- &= \int_{h-i\infty}^{h+i\infty} \int_{-h-i\infty}^{-h+i\infty} \frac{1}{\lambda^2 \mu^2} (S-\lambda)^{-1} (S-\mu)^{-1} d\mu d\lambda \\ &= \int_{h-i\infty}^{h+i\infty} \left(\int_{-h-i\infty}^{-h+i\infty} \frac{d\mu}{\mu^2(\lambda-\mu)} \right) \frac{1}{\lambda^2} (S-\lambda)^{-1} d\lambda \\ &\quad - \int_{-h-i\infty}^{-h+i\infty} \left(\int_{h-i\infty}^{h+i\infty} \frac{d\lambda}{\lambda^2(\lambda-\mu)} \right) \frac{1}{\mu^2} (S-\mu)^{-1} d\mu \end{aligned}$$

and hence $A_+A_- = 0$ since the integrals in parentheses vanish. Clearly $(S-\lambda)^{-1}$ commutes with A_{\pm} , so A_+ and A_- commute too and $A_-A_+ = 0$. By Cauchy's theorem we obtain

$$A_+ + A_- = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\lambda^2} (S-\lambda)^{-1} d\lambda = S^{-2}$$

where Γ is a small positively oriented circle around the origin. Therefore, by Lemma 2.3, $P_{\pm} = S^2 A_{\pm}$ are closed complementary projections with $\text{Im}(P_{\pm}) = \ker(A_{\mp})$.

We show next that $G_{\pm} = \ker(A_{\mp})$. Let $x \in G_+$. Then $(S-\lambda)^{-1}x$ has a bounded analytic extension φ_x^+ to \mathbb{C}_- . Consequently

$$A_-x = \frac{-1}{2\pi i} \int_{-h-i\infty}^{-h+i\infty} \frac{1}{\lambda^2} (S-\lambda)^{-1} x d\lambda = 0$$

by Cauchy's theorem. We thus have $G_+ \subset \ker(A_-)$. To show the converse inclusion, let $\text{Re } z < -h$ and consider the bounded operator

$$R_-(z) = \frac{1}{2\pi i} \int_{-h-i\infty}^{-h+i\infty} \frac{z^2}{\lambda^2(\lambda-z)} (S-\lambda)^{-1} d\lambda. \quad (25)$$

From $(S-z)(S-\lambda)^{-1} = I + (\lambda-z)(S-\lambda)^{-1}$ and the closedness of S we obtain $\text{Im}(R_-(z)) \subset \mathcal{D}(S)$ and

$$(S-z)R_-(z) = \frac{1}{2\pi i} \int_{-h-i\infty}^{-h+i\infty} \frac{z^2}{\lambda^2(\lambda-z)} d\lambda - z^2 A_- = I - z^2 A_-;$$

thus $(S - z)R_-(z) = I$ on $\ker(A_-)$. By Lemma 2.3, $\ker(A_-)$ is S - and $(S - \lambda)^{-1}$ - and hence also $R_-(z)$ -invariant. Since $(S - z)R_-(z) = R_-(z)(S - z)$ on $\mathcal{D}(S)$, we conclude that $\mathbb{C}_- \subset \varrho(S|_{\ker(A_-)})$ and for all $x \in \ker(A_-)$

$$(S|_{\ker(A_-)} - z)^{-1}x = \begin{cases} (S - z)^{-1}x, & |\operatorname{Re} z| \leq h, \\ R_-(z)x, & \operatorname{Re} z < -h. \end{cases} \quad (26)$$

The definition of $R_-(z)$ in conjunction with (19) implies that $\|R_-(z)\| \leq C|z|^2$ for $\operatorname{Re} z < -h$ with some constant C ; as remarked earlier, we may replace h by $h/2$ in (25). Together with (19), the Phragmén-Lindelöf Theorem 3.9 yields

$$\|(S|_{\ker(A_-)} - z)^{-1}\| \leq M, \quad z \in \overline{\mathbb{C}_-}. \quad (27)$$

As a consequence, if $x \in \ker(A_-)$, then $(S|_{\ker(A_-)} - z)^{-1}x$ is a bounded analytic extension of $(S - z)^{-1}x$ to $\overline{\mathbb{C}_-}$ and hence $x \in G_+$. We have thus shown that $G_+ = \ker(A_-)$ and the proof of $G_- = \ker(A_+)$ is analogous. In particular, we proved $\sigma(S|_{G_\pm}) \subset \mathbb{C}_\pm$, (23) and $\mathcal{D}(P_+) = \mathcal{D}(P_-) = \operatorname{Im}(P_+) \oplus \operatorname{Im}(P_-) = G_+ \oplus G_-$. All remaining statements in (i) and (iii) follow from Lemma 2.3. \square

Corollary 4.2. *Suppose $S(X \rightarrow X)$ satisfies (18) and (19). Then the following are equivalent:*

- (i) S is strictly dichotomous;
- (ii) $X = G_+ \oplus G_-$;
- (iii) $P_+ \in L(X)$.

In this case $X = G_+ \oplus G_-$ is the corresponding spectral decomposition.

Proof. In Lemma 3.7 we have already seen that strict dichotomy implies $X_\pm = G_\pm$ and hence $X = G_+ \oplus G_-$. Conversely, if $X = G_+ \oplus G_-$, then Theorem 4.1 implies that S is strictly dichotomous for the choice $X_\pm = G_\pm$. Finally (ii) \Leftrightarrow (iii) follows from $\mathcal{D}(P_+) = G_+ \oplus G_-$ and the closed graph theorem. \square

In Theorem 4.1 we did not assume that S is densely defined. If it is, then the projections P_\pm are densely defined too, and we obtain a nice criterion for strict dichotomy.

Corollary 4.3. *Let $S(X \rightarrow X)$ be densely defined and satisfy (18) and (19). Then $G_+ \oplus G_- \subset X$ is dense and the closed projections P_\pm are densely defined. Moreover, the following assertions are equivalent:*

- (i) S is strictly dichotomous;
- (ii) P_+ is bounded on some dense subspace of $\mathcal{D}(P_+)$;
- (iii) the operator P defined by

$$Px = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda^2} (S - \lambda)^{-1} S^2 x \, d\lambda, \quad x \in \mathcal{D}(S^2), \quad (28)$$

is bounded on some dense subspace of $\mathcal{D}(S^2)$.

In this case P_+ is the unique bounded extension of P to X .

Proof. The first assertions are immediate from $\mathcal{D}(S^2) \subset \mathcal{D}(P_\pm) = G_+ \oplus G_-$.

(i) \Rightarrow (ii): This follows from Corollary 4.2.

(ii) \Rightarrow (i): If P_+ is bounded on a dense subspace, then the closedness of P_+ implies that $\mathcal{D}(P_+) = X$. Thus $P_+ \in L(X)$ and S is strictly dichotomous.

(ii) \Leftrightarrow (iii) and the final assertion: This is clear since (24) implies that P is a restriction of P_+ . \square

Remark 4.4. Theorem 4.1 and Corollaries 4.2 and 4.3 generalise and extend the results from [BGK86b, Theorem 3.1].

- (i) Probably the main result of [BGK86b, Theorem 3.1] on spectral splitting for densely defined S is that if the operator P in (28) is bounded on $\mathcal{D}(S^2)$, then S is dichotomous. Corollary 4.3 shows that the boundedness on some dense subspace of $\mathcal{D}(S^2)$ is sufficient and that S is even strictly dichotomous in this case. Note that since [LT01, LRvdR02, TW14] all use [BGK86b, Theorem 3.1] to prove dichotomy, the corresponding operators in those papers are in fact strictly dichotomous.
- (ii) We have shown that also in the non-dichotomous case, i.e., when P_\pm are unbounded, G_\pm are closed invariant subspaces with $\sigma(S|_{G_\pm}) \subset \mathbb{C}_\pm$ and $\sigma(S|_{G_+}) \cup \sigma(S|_{G_-}) = \sigma(S)$. In [BGK86b] the closedness and S -invariance of G_\pm has only been obtained for exponentially dichotomous S .
- (iii) Using the Phragmén-Lindelöf theorem, we showed $G_\pm = \ker(A_\mp)$ whereas in [BGK86b] only the inclusion $G_\pm \subset \ker(A_\mp)$ was obtained.
- (iv) We showed that P_\pm as defined in Theorem 4.1 (ii) are closed projections even if they are unbounded. This is used e.g. in Corollary 4.3 where it is sufficient to check boundedness on any dense subspace. In Section 7 this allows us to prove perturbation results under weaker conditions on the domains of the involved operators.
- (v) In [BGK86b] it is always assumed that S is densely defined. Theorem 4.1 and Corollary 4.2 are valid also if S is not densely defined.

Example 4.5. We continue Example 2.4. A straightforward calculation shows that $\sup_{\lambda \in i\mathbb{R}} \|(S - \lambda)^{-1}\| \leq 3$ and hence (18) and (19) are satisfied. From Theorem 4.1, in particular (20), it follows that A_\pm and P_\pm are block diagonal and given by

$$\begin{aligned} A_+ &= \text{diag}(A_1^+, A_2^+, \dots), & A_- &= \text{diag}(A_1^-, A_2^-, \dots), \\ P_+ &= \text{diag}(P_1^+, P_2^+, \dots), & P_- &= \text{diag}(P_1^-, P_2^-, \dots), \end{aligned}$$

where A_n^\pm and P_n^\pm are as in Example 2.4. P_+ and P_- are unbounded and S is not dichotomous, compare Remark 3.5.

Remark 4.6. Since P_{\pm} are closed operators with $\mathcal{D}(S^2) \subset \mathcal{D}(P_{\pm})$, their restrictions $P_{\pm}|_{\mathcal{D}(S^2)}$ are bounded in the graph norm of S^2 . In the almost bisectorial case, we even have $\mathcal{D}(S) \subset \mathcal{D}(P_{\pm})$, and therefore $P_{\pm}|_{\mathcal{D}(S)}$ are bounded in the graph norm of S (see Theorem 5.6). In the general case it is not clear if $\mathcal{D}(S) \subset \mathcal{D}(P_{\pm})$ and the restriction $P_{\pm}|_{\mathcal{D}(S)}$ is bounded.

5 Bisectorial and almost bisectorial operators

In this section we investigate the spectral splitting of bisectorial and almost bisectorial operators. Their resolvent norm is not only bounded on the imaginary axis as assumed in Section 4, it even decays like $|\lambda|^{-\beta}$.

For $0 \leq \theta \leq \pi$ we define the sectors

$$\Sigma_{\theta} = \{z \in \mathbb{C} : |\arg z| \leq \theta\}. \quad (29)$$

Let us first recall the notion of sectorial and almost sectorial operators, see e.g. [Haa06, PS02].

Definition 5.1. A linear operator $S(X \rightarrow X)$ is called *sectorial* if there exist $0 < \theta < \pi$ and $M > 0$ such that $\sigma(S) \subset \Sigma_{\theta}$ and

$$\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|} \quad \text{for } \lambda \in \mathbb{C} \setminus \Sigma_{\theta}. \quad (30)$$

S is called *almost sectorial* if there exist $0 < \beta < 1$, $0 < \theta < \pi$ and $M > 0$ such that $\sigma(S) \subset \Sigma_{\theta}$ and

$$\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|^{\beta}} \quad \text{for } \lambda \in \mathbb{C} \setminus \Sigma_{\theta}. \quad (31)$$

Remark 5.2. There are subtle differences in the behaviour of sectorial and almost sectorial operators:

- (i) If S is sectorial with angle θ , then (30) also holds with a smaller angle $0 < \theta' < \theta$, though the constant M may be bigger for θ' as can be shown by a simple Neumann series argument.
- (ii) If S is almost sectorial, then $0 \in \varrho(S)$ (see e.g. [PS02, Remark 2.2]).
- (iii) If S is sectorial and $0 \in \varrho(S)$, then S is almost sectorial. This is true because $\|(S - \lambda)^{-1}\|$ is bounded in a neighbourhood of zero, and we easily obtain (31) from (30) (with a different constant M).

Note that (i) is not necessarily true for almost sectorial operators and that sectorial operators may have zero in their spectrum.

If we require the resolvent estimates only on the imaginary axis and allow spectrum on both sides of it, we obtain the definition of bisectorial and almost bisectorial operators.

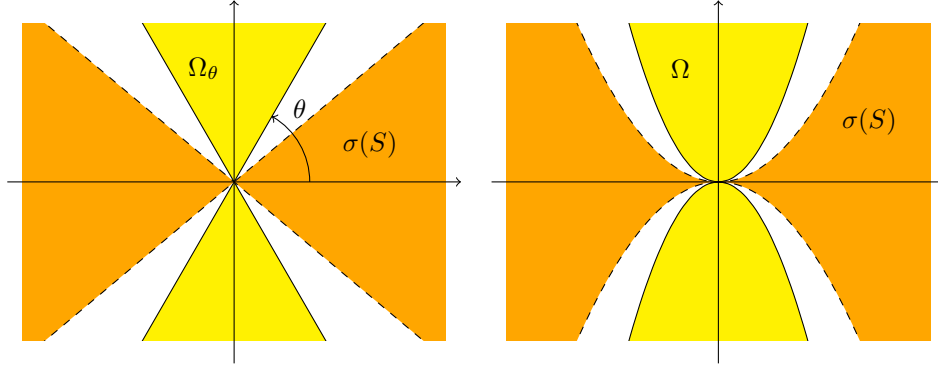


Figure 1: Location of the spectrum of a bisectorial and an almost bisectorial operator.

Definition 5.3. An operator $S(X \rightarrow X)$ is called *bisectorial* if $i\mathbb{R} \setminus \{0\} \subset \varrho(S)$ and

$$\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|}, \quad \lambda \in i\mathbb{R} \setminus \{0\}. \quad (32)$$

S is called *almost bisectorial* if $i\mathbb{R} \setminus \{0\} \subset \varrho(S)$ and there exists $0 < \beta < 1$ such that

$$\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|^\beta}, \quad \lambda \in i\mathbb{R} \setminus \{0\}. \quad (33)$$

Bisectorial operators have been studied e.g. in [AZ10, TW14].

The following results are analogous to the sectorial case. They imply that almost bisectorial operators and bisectorial operators with $0 \in \varrho(S)$ fulfil the assumptions of Theorem 4.1.

Remark 5.4. (i) If S is bisectorial, then there exists $0 < \theta < \pi/2$ such that the bisector

$$\Omega_\theta = \mathbb{C} \setminus (\Sigma_\theta \cup (-\Sigma_\theta)) = \{\lambda \in \mathbb{C} : \theta < |\arg \lambda| < \pi - \theta\}$$

belongs to $\varrho(S)$ and (32) holds on Ω_θ , see Figure 1.

- (ii) If S is almost bisectorial, then $0 \in \varrho(S)$.
- (iii) If S is bisectorial and $0 \in \varrho(S)$, then S is almost bisectorial.
- (iv) If S is almost bisectorial or bisectorial with $0 \in \varrho(S)$, then S satisfies (18) and (19) from Section 4.

Similar to Remark 5.4 (i), the resolvent estimate of an almost bisectorial operator actually holds inside a whole region around $i\mathbb{R}$:

$$\Omega = \{a + \mathrm{i}b \in \mathbb{C} \setminus \{0\} : |a| \leq \alpha |b|^\beta\}$$

belongs to the resolvent set, $\Omega \subset \varrho(S)$, and (33) holds for all $\lambda \in \Omega$ (typically with a larger constant M), see Figure 1.

Proof. Let $\lambda = a + \mathrm{i}b \in \Omega$. Then the identity

$$S - \lambda = (I - a(S - \mathrm{i}b)^{-1})(S - \mathrm{i}b)$$

and the estimate

$$\|a(S - \mathrm{i}b)^{-1}\| \leq \frac{M|a|}{|b|^\beta} \leq \alpha M < 1$$

imply $\lambda \in \varrho(S)$ and thus $\Omega \subset \varrho(S)$. Moreover

$$\|(S - \lambda)^{-1}\| \leq \frac{1}{1 - \|a(S - ib)^{-1}\|} \|(S - ib)^{-1}\| \leq \frac{M}{(1 - \alpha M)|b|^\beta}.$$

Now if also $|b| \geq 1$, then $|\lambda| \leq \alpha|b|^\beta + |b| \leq (1 + \alpha)|b|$ and hence

$$\|(S - \lambda)^{-1}\| \leq \frac{M(1 + \alpha)^\beta}{(1 - \alpha M)|\lambda|^\beta}, \quad \lambda = a + ib \in \Omega, \quad |b| \geq 1.$$

Since $0 \in \varrho(S)$ by Remark 5.4(ii) and $(S - \lambda)^{-1}$ is uniformly bounded on compact subsets of $\varrho(S)$, the proof is complete. \square

A similar result can be shown for an almost sectorial operator S : Let θ as in Definition 5.1. Then $\varrho(S)$ contains a parabola around every ray $\{re^{i\varphi} : r \geq 0\}$ with $\theta \leq |\varphi| \leq \pi$.

In the rest of this section we will investigate the spectral splitting properties of bisectorial and almost bisectorial operators. Compared with Theorem 4.1, we obtain simplified formulas for the projections $P_{\pm} = S^2 A_{\pm}$ and show that the restrictions $S|G_{\pm}$ to the spectral subspaces are sectorial and almost sectorial, respectively.

For an almost bisectorial operator S let

$$B_{\pm} = \frac{\pm 1}{2\pi i} \int_{\pm h-i\infty}^{\pm h+i\infty} \frac{1}{\lambda} (S-\lambda)^{-1} d\lambda \quad (34)$$

with $h > 0$ small enough. By (33) and Lemma 5.5 the integrals converge in the uniform operator topology. Hence B_{\pm} are well-defined bounded linear operators and, due to Cauchy's theorem, the integrals on the right hand side are independent of h for h small enough.

Theorem 5.6. *Let $S(X \rightarrow X)$ be almost bisectorial and let P_{\pm} as in Theorem 4.1. Then:*

(i) $P_{\pm} = SB_{\pm}$, $\mathcal{D}(S) \subset \mathcal{D}(P_{\pm})$ and

$$P_{\pm}x = \frac{\pm 1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda} (S - \lambda)^{-1} Sx d\lambda, \quad x \in \mathcal{D}(S). \quad (35)$$

(ii) $\pm S|G_{\pm}$ are almost sectorial with angle $\theta = \pi/2$ and unchanged constants M, β .

(iii) Let S be bisectorial with $0 \in \varrho(S)$ and let θ as in Remark 5.4 (i). Then $\pm S|G_{\pm}$ are sectorial with angle θ and unchanged constant M .

Proof. (i): Using the resolvent identity $(S - \lambda)^{-1} = S^{-1} + \lambda(S - \lambda)^{-1}S^{-1}$ we obtain from (20)

$$A_{\pm} = \frac{\pm 1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda^2} S^{-1} d\lambda + \frac{\pm 1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda} (S - \lambda)^{-1} S^{-1} d\lambda = B_{\pm} S^{-1}$$

because the first integral vanishes by Cauchy's theorem. Moreover $S^{-1}B_{\pm} = B_{\pm}S^{-1}$. Therefore $P_{\pm} = S^2A_{\pm} = S^2S^{-1}B_{\pm} = SB_{\pm}$. For $x \in \mathcal{D}(S)$ we get $B_{\pm}Sx = SB_{\pm}x$ and in particular $x \in \mathcal{D}(P_{\pm})$.

(ii): Consider $\lambda^{\beta} = \exp(\beta \log \lambda)$ where \log is a branch of the logarithm on \mathbb{C}_{-} . Then the mapping $\lambda \mapsto \lambda^{\beta}$ is analytic on \mathbb{C}_{-} and continuous on $\overline{\mathbb{C}_{-}}$. The almost bisectoriality of S yields $\|\lambda^{\beta}(S|G_{+} - \lambda)^{-1}\| \leq M$ for $\lambda \in i\mathbb{R}$ since $(S|G_{+} - \lambda)^{-1} = (S - \lambda)^{-1}|G_{+}$. Here we used that $0 \in \varrho(S)$ by Remark 5.4 (ii). Since $\|(S|G_{+} - \lambda)^{-1}\|$ is bounded on \mathbb{C}_{-} by Theorem 4.1, the Phragmén-Lindelöf theorem implies $\|\lambda^{\beta}(S|G_{+} - \lambda)^{-1}\| \leq M$ for $\lambda \in \overline{\mathbb{C}_{-}}$. The proof for $S|G_{-}$ is analogous. With $\beta = 1$ we obtain (iii). \square

Statements (ii) and (iii) of the Theorem remain true when $S|G_{\pm}$ are replaced by the operators $S_{M\pm}$ from Section 6 as follows from Lemma 6.1 (iii).

Remark 5.7. The operators B_{\pm} satisfy the relations

$$B_{+} + B_{-} = S^{-1}, \quad B_{+}B_{-} = B_{-}B_{+} = 0.$$

These identities can be obtained either from the corresponding relations for A_{\pm} via $A_{\pm} = S^{-1}B_{\pm} = B_{\pm}S^{-1}$ or from (34) by direct computation. The latter approach, together with Remark 2.5, yields an alternative proof of Theorem 4.1 for almost bisectorial operators. In the bisectorial case, this was used in [AZ10]. There the sectoriality of the spectral parts $S|G_{\pm}$ was obtained ([AZ10, p. 215]), but not the S -invariance of G_{\pm} and the decomposition (21) of the spectrum.

To illustrate the situation of Theorem 5.6, we consider an almost bisectorial operator which is not dichotomous and whose projections P_{\pm} are (therefore) unbounded. This is a variant of Example 2.4 and 4.5.

Example 5.8. Let $0 < p < 1$ and consider the block diagonal operator S on $X = l^2$ given by

$$S = \text{diag}(S_1, S_2, \dots), \quad S_n = \begin{pmatrix} n & 2n^{1+p} \\ 0 & -n \end{pmatrix}.$$

$$\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|^{1-p}}, \quad \lambda \in i\mathbb{R} \setminus \{0\},$$

i.e., S is almost bisectorial. The spectral projections for S_n corresponding to the eigenvalues n and $-n$, respectively, are

$$P_n^+ = \begin{pmatrix} 1 & n^p \\ 0 & 0 \end{pmatrix}, \quad P_n^- = \begin{pmatrix} 0 & -n^p \\ 0 & 1 \end{pmatrix}.$$

Consequently, $P_{\pm} = \text{diag}(P_1^{\pm}, P_2^{\pm}, \dots)$ are unbounded and S is not dichotomous, compare Remark 3.5.

If in the above example we choose $p = 0$, then S becomes bisectorial *and* strictly dichotomous. In general however, bisectoriality (with $0 \in \varrho(S)$) does not imply dichotomy. A counterexample was given in [MY90] (see Example 8.2).

The identity (35) for the projections P_{\pm} from Theorem 5.6 can be rearranged to yield another integral representation. In the bisectorial case, this representation has been obtained and used extensively in [LT01, TW14].

Corollary 5.9. *Let $S(X \rightarrow X)$ be almost bisectorial. Then:*

(i) For every $x \in \mathcal{D}(S)$,

$$2P_+x - x = P_+x - P_-x = \frac{1}{\pi i} \int_{-i\infty}^{i\infty'} (S - \lambda)^{-1} x d\lambda; \quad (36)$$

in particular, the integral exists for all $x \in \mathcal{D}(S)$. Here the prime denotes the Cauchy principal value at infinity.

(ii) *If, in addition, S is densely defined and there exists a dense subspace $D \subset \mathcal{D}(S)$ such that*

$$\int_{i\infty}^{i\infty'} (S - \lambda)^{-1} x d\lambda, \quad x \in D,$$

defines a bounded operator, then P_+ is bounded and hence S is strictly dichotomous.

Proof. (i) From (35) we get for $x \in \mathcal{D}(S)$,

$$\begin{aligned} P_+ x &= \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda} (S - \lambda)^{-1} S x \, d\lambda = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \left(\frac{1}{\lambda} x + (S - \lambda)^{-1} x \right) d\lambda \\ &= \frac{1}{2} x + \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} (S - \lambda)^{-1} x \, d\lambda = \frac{1}{2} x + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} (S - \lambda)^{-1} x \, d\lambda. \end{aligned}$$

Note that in the last step we used Cauchy's integral theorem and (33). The assertion follows from $x = P_+x + P_-x$ for $x \in \mathcal{D}(P_+)$.

(ii) The assumption together with (i) implies that P_+ is bounded on the dense subspace D . Corollary 4.3 yields the claim. \square

Remark 5.10. In [LT01] the representation (36) was derived under the weaker condition $\lim_{t \rightarrow \pm\infty} \|(S - it)^{-1}\| = 0$ but with the additional assumption that the integral in (36) exists for every $x \in X$. By the uniform boundedness principle, the projections P_{\pm} are then bounded and S is strictly dichotomous.

Remark 5.11. An operator S is called *sectorially dichotomous* if it is dichotomous and $S|_{X_+}$ and $-S|_{X_-}$ are sectorial operators with angle $\theta \leq \pi/2$. Note that sectorial dichotomy implies bisectoriality [TW14, Lemma 2.12] as well as strict dichotomy. A question asked in [TW14] is the following: Is every bisectorial and dichotomous operator also sectorially dichotomous? If we assume strict dichotomy, the answer is yes since in this case $\pm S|_{G_{\pm}}$ are sectorial by Theorem 5.6 and strict dichotomy implies $G_{\pm} = X_{\pm}$.

The stronger assumption of strict dichotomy seems reasonable, for otherwise the spaces X_{\pm} are not unique. Moreover the main theorems in [TW14] actually yield strictly dichotomous operators, compare Remark 4.4.

6 The subspaces M_{\pm}

Recall that the subspaces G_{\pm} are S - and $(S - \lambda)^{-1}$ -invariant (Theorem 4.1). However, even if S is densely defined, the restrictions of S to G_{\pm} do not need to be densely defined, see Example 8.3. Let

$$M_{\pm} := \overline{\text{Im}(A_{\pm})} \quad (37)$$

and let $S_{M_{\pm}}$ be the part of S in M_{\pm} , i.e. S is the restriction of S to $\mathcal{D}(S_{M_{\pm}}) = \{x \in \mathcal{D}(S) \cap M_{\pm} : Sx \in M_{\pm}\}$. In [BGK86b] it is shown that $\mathcal{D}(S_{M_{\pm}})$ is dense in M_{\pm} if S is densely defined, cf. Lemma 6.2. In the following lemma we do not assume density of $\mathcal{D}(S)$.

Lemma 6.1. *Let $S(X \rightarrow X)$ be such that (18) and (19) hold. Then:*

- (i) $M_{\pm} \subset G_{\pm}$.
- (ii) M_{\pm} is $(S - \lambda)^{-1}$ -invariant for $\lambda \in \varrho(S)$ and $\mathcal{D}(S_{M_{\pm}}) = S^{-1}(M_{\pm})$.
- (iii) M_{\pm} is $(S|_{G_{\pm}} - \lambda)^{-1}$ -invariant for $\lambda \in \varrho(S|_{G_{\pm}})$,

$$\begin{aligned} \sigma(S_{M_{\pm}}) &= \sigma(S|_{G_{\pm}}), \\ (S_{M_{\pm}} - \lambda)^{-1}x &= (S|_{G_{\pm}} - \lambda)^{-1}x, \quad x \in M_{\pm}, \lambda \in \varrho(S_{M_{\pm}}). \end{aligned}$$

Proof. (i) follows from $\text{Im}(A_{\pm}) \subset \ker(A_{\mp}) = G_{\pm}$ and the closedness of G_{\pm} .

(ii) From $(S - \lambda)^{-1}A_{\pm} = A_{\pm}(S - \lambda)^{-1}$ it follows that $\text{Im}(A_{\pm})$ and hence M_{\pm} are $(S - \lambda)^{-1}$ -invariant. In particular $S^{-1}(M_{\pm}) \subset M_{\pm}$, which implies $\mathcal{D}(S_{M_{\pm}}) = S^{-1}(M_{\pm})$.

(iii) Let us prove the invariance of M_+ . Recall that $\varrho(S|_{G_+}) = \varrho(S) \cup \mathbb{C}_-$. For $\lambda \in \varrho(S)$ we have $(S|_{G_+} - \lambda)^{-1} = (S - \lambda)^{-1}|_{G_+}$, so the invariance of M_+ follows from (ii) in this case. For $\lambda \in \mathbb{C}_-$, the invariance follows from (26). The proof for M_- is analogous.

The invariance property of M_{\pm} immediately yields $\varrho(S|G_{\pm}) \subset \varrho(S_{M_{\pm}})$ and $(S_{M_{\pm}} - \lambda)^{-1}x = (S|G_{\pm} - \lambda)^{-1}x$ for $x \in M_{\pm}$ and $\lambda \in \varrho(S|G_{\pm})$. Now

$$\sigma(S) = \sigma(S|G_+) \dot{\cup} \sigma(S|G_-) \supset \sigma(S_{M_+}) \dot{\cup} \sigma(S_{M_-}).$$

So to prove $\sigma(S|G_{\pm}) \subset \sigma(S_{M_{\pm}})$ it suffices to show $\sigma(S) \subset \sigma(S_{M_+}) \cup \sigma(S_{M_-})$ or, equivalently, $\varrho(S_{M_+}) \cap \varrho(S_{M_-}) \subset \varrho(S)$. The proof is similar to the one for (9) in Lemma 2.3: If $\lambda \in \varrho(S_{M_+}) \cap \varrho(S_{M_-})$ and $(S - \lambda)x = 0$, then $x \in \mathcal{D}(S^2)$ and hence

$$x = y_+ + y_- \quad \text{with} \quad y_{\pm} = P_{\pm}x = A_{\pm}S^2x \in \text{Im}(A_{\pm}) \subset M_{\pm}.$$

In fact $y_{\pm} \in \mathcal{D}(S_{M_{\pm}})$ since $x \in \mathcal{D}(S^3)$ and $Sy_{\pm} = A_{\pm}S^3x$. Consequently

$$(S - \lambda)x = (S_{M_+} - \lambda)y_+ + (S_{M_-} - \lambda)y_- = 0$$

and thus $y_+ = y_- = x = 0$. The surjectivity of $S - \lambda$ is obtained by considering $T = (S_{M_+} - \lambda)^{-1}A_+ + (S_{M_-} - \lambda)^{-1}A_-$ and noting $(S - \lambda)S^2T = I$. \square

The following result has been obtained as part of [BGK86b, Theorem 3.1 and Lemma 3.3]. For the convenience of the reader we include the proof.

Lemma 6.2. *If $S(X \rightarrow X)$ is densely defined and satisfies (18) and (19), then $M_+ \oplus M_- \subset X$ is dense and the operators $S_{M_{\pm}}$ are densely defined.*

Proof. From $S^{-2} = A_+ + A_-$ we obtain $\mathcal{D}(S^2) \subset \text{Im}(A_+) \oplus \text{Im}(A_-) \subset M_+ \oplus M_-$ and thus the first assertion holds. For the second one note that $S^{-1}(\text{Im}(A_{\pm})) = A_{\pm}(\mathcal{D}(S))$ is dense in $\text{Im}(A_{\pm})$ and hence in M_{\pm} , and that $S^{-1}(\text{Im}(A_{\pm})) \subset S^{-1}(M_{\pm}) = \mathcal{D}(S_{M_{\pm}})$. \square

Despite the invariance of M_{\pm} under $(S - \lambda)^{-1}$, and the invariance of G_{\pm} under S and $(S - \lambda)^{-1}$, the subspaces M_{\pm} are in general not invariant under S itself; that is, the inclusion $S(\mathcal{D}(S) \cap M_{\pm}) \subset M_{\pm}$ does not need to hold. For densely defined operators S the S -invariance of M_{\pm} can be characterised as follows.

Theorem 6.3. *Let $S(X \rightarrow X)$ be densely defined satisfying (18) and (19). Then:*

(i) *The following equivalences hold:*

$$\begin{aligned} S|G_+ \text{ densely defined} &\iff M_+ = G_+ &\iff M_+ \text{ is } S\text{-invariant,} \\ S|G_- \text{ densely defined} &\iff M_- = G_- &\iff M_- \text{ is } S\text{-invariant.} \end{aligned}$$

In particular, $S_{M_{\pm}} = S|G_{\pm}$ if $M_{\pm} = G_{\pm}$.

(ii) *If P_+ (or equivalently P_-) is bounded, then $M_{\pm} = G_{\pm}$.*

(iii) Suppose there exist $Q_n \in L(X)$, $n \in \mathbb{N}$, such that $\text{Im}(Q_n) \subset \mathcal{D}(S)$,

$$Q_n(S - \lambda)^{-1} = (S - \lambda)^{-1}Q_n \quad \text{for all } \lambda \in \varrho(S)$$

and

$$Q_n x \rightarrow x \quad \text{for all } x \in X.$$

Then $M_{\pm} = G_{\pm}$.

Proof. (i) From $G_+ = \text{Im}(P_+)$ and $P_+ = S^2 A_+$ we obtain

$$\mathcal{D}(S^2) \cap G_+ = S^{-2}(G_+) = \text{Im}(A_+) \subset M_+. \quad (38)$$

Suppose first that $S|_{G_+}$ is densely defined. Since $0 \in \varrho(S|_{G_+})$, $(S|_{G_+})^2$ is densely defined too, i.e. $\mathcal{D}(S^2) \cap G_+ \subset G_+$ is dense. Taking closures in (38), we thus obtain $G_+ \subset M_+$ and hence $G_+ = M_+$. Now let M_+ be S -invariant. From (38) we see $S^{-2}(G_+) \subset \mathcal{D}(S^2) \cap M_+$, hence, by the S -invariance of M_+ , we obtain $G_+ \subset M_+$, i.e. $G_+ = M_+$. The other implications are trivial and the case of M_- , G_- is analogous.

(ii) If P_{\pm} are bounded, then $X = G_+ \oplus G_-$. Since M_{\pm} are closed, it follows that $M_+ \oplus M_-$ is closed. By Lemma 6.2 it is dense in X , so we obtain $M_+ \oplus M_- = X$ and hence $M_{\pm} = G_{\pm}$.

(iii) The assumption that Q_n commutes with the resolvent implies $Q_n A_{\pm} = A_{\pm} Q_n$ and consequently $Q_n x \in \ker(A_-) = G_+$ for any $x \in G_+ = \ker(A_-)$. Since additionally $Q_n x \in \mathcal{D}(S)$ and $Q_n x \rightarrow x$, we obtain that $G_+ \cap \mathcal{D}(S)$ is dense in G_+ and thus $M_+ = G_+$ by (i). The proof of $M_- = G_-$ is analogous. \square

Remark 6.4. The conditions of Theorem 6.3(iii) hold for example if X is a Hilbert space which can be decomposed into an orthogonal direct sum of finite-dimensional subspaces X_k where each X_k is spanned by a set of eigenvectors of S . Then Q_n can be chosen as the orthogonal projection onto the subspace $X_1 \oplus \dots \oplus X_n$. Block diagonal operators as in Example 2.4 admit such orthogonal decompositions.

Although for a general densely defined operator S its restrictions $S|_{G_{\pm}}$ may fail to be densely defined, they always are if S is bisectorial with $0 \in \varrho(S)$.

Lemma 6.5. Let $S(X \rightarrow X)$ be densely defined and bisectorial with $0 \in \varrho(S)$. Then $S|_{G_{\pm}}$ are densely defined and $M_{\pm} = G_{\pm}$.

Proof. Since S is densely defined and $\|it(it - S)^{-1}\|$ is bounded for $t \in \mathbb{R}$, the identity

$$x = \lim_{t \rightarrow \infty} it(it - S)^{-1}x$$

holds for all $x \in X$, compare [Haa06, Proposition 2.1.1]. Moreover, $(it - S)^{-1}x \in G_{\pm} \cap \mathcal{D}(S)$ for $x \in G_{\pm}$, therefore $S|_{G_{\pm}}$ are densely defined and hence $M_{\pm} = G_{\pm}$ by Theorem 6.3. \square

Remark 6.6. If X is reflexive, then every sectorial (or bisectorial) operator is automatically densely defined, see e.g. [Haa06, Proposition 2.1.1]. From Theorem 5.6 we already know that if S is bisectorial, then the restrictions $\pm S|_{G_{\pm}}$ are sectorial; so they are densely defined if X is reflexive. The previous lemma ensures that this is true also in non-reflexive spaces.

Finally we show that the spaces M_{\pm} can be expressed in terms of the operators B_{\pm} from (34) if S is densely defined and almost bisectorial.

Lemma 6.7. *Let $S(X \rightarrow X)$ be densely defined and almost bisectorial. Let A_{\pm} and B_{\pm} be the operators defined in (20) and (34). Then*

$$M_{\pm} = \overline{\text{Im}(A_{\pm})} = \overline{\text{Im}(B_{\pm})}.$$

Proof. Since B_{\pm} is bounded and $\mathcal{D}(S)$ is dense,

$$\overline{\text{Im}(A_{\pm})} = \overline{\text{Im}(B_{\pm}S^{-1})} = \overline{\text{Im}(B_{\pm}|\mathcal{D}(S))} = \overline{\text{Im}(B_{\pm})}. \quad \square$$

7 Perturbation results

Our first perturbation result generalises [BGK86b, Theorem 5.1] where the stronger assumptions $\varepsilon = 1$, $\mathcal{D}(T^2) \subset \mathcal{D}(S^2)$ and exponential dichotomy of S were required.

Theorem 7.1. *Let $S(X \rightarrow X)$ be a densely defined and strictly dichotomous operator on the Banach space X . Suppose that $T(X \rightarrow X)$ is densely defined and that there exist $h > 0$, $\varepsilon > 0$ such that the following conditions hold:*

- (i) $\{\lambda \in \mathbb{C} : |\text{Re } \lambda| \leq h\} \subset \varrho(S) \cap \varrho(T)$;
- (ii) $\sup_{|\text{Re } \lambda| \leq h} |\lambda|^{1+\varepsilon} \|(S - \lambda)^{-1} - (T - \lambda)^{-1}\| < \infty$;
- (iii) $\mathcal{D}(S^2) \cap \mathcal{D}(T^2) \subset X$ dense.

Then T is strictly dichotomous too.

Proof. Since S is strictly dichotomous, $(S - \lambda)^{-1}$ is uniformly bounded for $|\text{Re } \lambda| \leq h'$ by (16) for some $0 < h' \leq h$. Condition (ii) implies that $(T - \lambda)^{-1}$ is also uniformly bounded for $|\text{Re } \lambda| \leq h'$. Let P_+^S , P_+^T be the projections corresponding to the spectrum in \mathbb{C}_+ of S and T , respectively. Using (24),

$$\frac{1}{\lambda^2}(S - \lambda)^{-1}S^2 = \frac{1}{\lambda^2}S + \frac{1}{\lambda}(S - \lambda)^{-1}S = \frac{1}{\lambda^2}S + \frac{1}{\lambda} + (S - \lambda)^{-1},$$

and the respective identity for T , we obtain for $x \in \mathcal{D}(S^2) \cap \mathcal{D}(T^2)$,

$$\begin{aligned} P_+^S x - P_+^T x &= \frac{1}{2\pi i} \int_{h'-i\infty}^{h'+i\infty} \left(\frac{1}{\lambda^2}(Sx - Tx) + ((S - \lambda)^{-1}x - (T - \lambda)^{-1}x) \right) d\lambda \\ &= \frac{1}{2\pi i} \int_{h'-i\infty}^{h'+i\infty} ((S - \lambda)^{-1}x - (T - \lambda)^{-1}x) d\lambda. \end{aligned}$$

By (ii) the last integral defines a bounded linear operator. Since P_+^S is bounded, P_+^T is bounded on the dense subspace $\mathcal{D}(S^2) \cap \mathcal{D}(T^2)$, and therefore T is strictly dichotomous by Corollary 4.3. \square

Remark 7.2. In [BGK86b, Theorem 5.1] it has been shown that, if S is exponentially dichotomous then so is T . This implication remains true in our more general setting, where we require (ii) for some $\varepsilon > 0$ instead of $\varepsilon = 1$, and (iii) instead of $\mathcal{D}(T^2) \subset \mathcal{D}(S^2)$. The proof of the exponential dichotomy of T is largely identical to the one in [BGK86b].

If the operator S is almost bisectorial, condition (iii) of Theorem 7.1 can be relaxed:

Theorem 7.3. *Let $S(X \rightarrow X)$ be densely defined, almost bisectorial and strictly dichotomous. Let $T(X \rightarrow X)$ be a densely defined operator and $\varepsilon > 0$ such that the following conditions hold:*

- (i) $i\mathbb{R} \subset \varrho(T)$;
- (ii) $\sup_{\lambda \in i\mathbb{R}} |\lambda|^{1+\varepsilon} \|(S - \lambda)^{-1} - (T - \lambda)^{-1}\| < \infty$;
- (iii) $\mathcal{D}(S) \cap \mathcal{D}(T) \subset X$ dense.

Then T is also strictly dichotomous and almost bisectorial with the same exponent β in the resolvent estimate (33). In particular, if S is bisectorial, then so is T .

Proof. Condition (ii) immediately implies that T satisfies an estimate (33) with the same β as for S . By Corollary 5.9 it suffices to show that

$$\int_{-i\infty}^{i\infty} (T - \lambda)^{-1} x d\lambda$$

defines a bounded linear operator on $\mathcal{D}(S) \cap \mathcal{D}(T)$. Since S is strictly dichotomous, the corresponding integral for S is bounded on $\mathcal{D}(S)$ by Corollary 5.9 (i). On the other hand, the difference of both integrals

$$\int_{-i\infty}^{i\infty} ((S - \lambda)^{-1} - (T - \lambda)^{-1}) d\lambda$$

converges in the uniform operator topology by (ii) and thus defines a bounded linear operator. \square

Remark 7.4. It is especially the relatively weak condition (iii) which makes Theorem 7.1 and 7.3 more generally applicable than comparable theorems from [BGK86b, LT01, TW14] where $\mathcal{D}(T^2) \subset \mathcal{D}(S^2)$ or $\mathcal{D}(T) = \mathcal{D}(S)$ was assumed. A situation where this generality is needed is the Hamiltonian operator matrix defined via extrapolation spaces in Example 8.8; in particular $\mathcal{D}(T) \neq \mathcal{D}(S)$ there.

We give an example showing that the unusual condition $\mathcal{D}(T^2) \subset \mathcal{D}(S^2)$ from [BGK86b] may fail even if $\mathcal{D}(S) = \mathcal{D}(T)$.

Example 7.5. Let S be an unbounded selfadjoint operator with strictly positive pure point spectrum, for instance the multiplication operator

$$S(l^2 \rightarrow l^2), \quad S(x_n)_{n \in \mathbb{N}} = (nx_n)_{n \in \mathbb{N}}$$

with domain $\mathcal{D}(S) = \{(x_n)_{n \in \mathbb{N}} \in l^2 : (nx_n)_{n \in \mathbb{N}} \in l^2\} \subsetneq l^2$. We take any $w \in l^2 \setminus \mathcal{D}(S)$ and define the bounded operator R on l^2 by $Rx = P_w x$ where P_w is the orthogonal projection onto w . Set $T := S + R$. Then $\mathcal{D}(T) = \mathcal{D}(S)$ and T is selfadjoint. On the other hand,

$$\begin{aligned} x \in \mathcal{D}(S^2) \cap \mathcal{D}(T^2) &\implies x \in \mathcal{D}(S^2) \wedge Tx = Sx + Rx \in \mathcal{D}(T) = \mathcal{D}(S) \\ &\implies x \in \mathcal{D}(S^2) \wedge Rx \in \mathcal{D}(S). \end{aligned}$$

Since $\text{Im}(R) \cap \mathcal{D}(S) = \{0\}$ by construction, it follows that $Rx = 0$, hence $x \in \text{span}\{w\}^\perp$. Consequently $\mathcal{D}(S^2) \cap \mathcal{D}(T^2) \subset \text{span}\{w\}^\perp$ and so $\mathcal{D}(S^2) \cap \mathcal{D}(T^2)$ cannot be dense in l^2 . Therefore

$$\mathcal{D}(S^2) \subsetneq \mathcal{D}(S^2) \cap \mathcal{D}(T^2) \quad \text{and} \quad \mathcal{D}(T^2) \subsetneq \mathcal{D}(S^2) \cap \mathcal{D}(T^2)$$

since S^2 and T^2 are densely defined, and we obtain $\mathcal{D}(T^2) \not\subset \mathcal{D}(S^2)$ as well as $\mathcal{D}(S^2) \not\subset \mathcal{D}(T^2)$. Note that S and T satisfy all conditions of Theorem 7.3, but not (iii) from Theorem 7.1.

One situation, where condition (ii) in Theorem 7.3 is fulfilled, is the case of so-called p -subordinate perturbations. The p -subordinate perturbations of bisectorial operators, in particular the change of their spectrum, have been studied in [TW14].

Definition 7.6. Let $S(X \rightarrow X)$, $R(X \rightarrow X)$ be linear operators. R is called p -subordinate to S with $0 \leq p \leq 1$ if $\mathcal{D}(S) \subset \mathcal{D}(R)$ and there exists $c > 0$ such that

$$\|Rx\| \leq c\|x\|^{1-p}\|Sx\|^p, \quad x \in \mathcal{D}(S).$$

Corollary 7.7. Let $S(X \rightarrow X)$ be densely defined, almost bisectorial with exponent $\beta > 1/2$ and strictly dichotomous. Let R be p -subordinate to S with $p < 2\beta - 1$ and let $T = S + R$. If $i\mathbb{R} \subset \varrho(T)$, then T is strictly dichotomous and almost bisectorial with the same exponent β . Moreover, if S is bisectorial, then so is T .

Proof. First note that $\mathcal{D}(T) = \mathcal{D}(S)$ since $\mathcal{D}(S) \subset \mathcal{D}(R)$. Hence condition (iii) in Theorem 7.3 is satisfied and it remains to show that (ii) holds too. Consider $\lambda \in i\mathbb{R}$ and the identity

$$T - \lambda = (I + R(S - \lambda)^{-1})(S - \lambda).$$

with M as in (33), $\tilde{c} > 0$ appropriate, and $|\lambda|$ large. Note that $p < 2\beta - 1 \leq \beta$. Hence, for $\lambda \in i\mathbb{R}$, $|\lambda|$ large, this implies $\lambda \in \varrho(T)$ and

Consequently,

where $2\beta - p > 1$.

8 Examples

In Example 3.3 we saw that the decomposition of a dichotomous operator is not necessarily unique. The following example shows that this is even possible for bisectorial operators on Hilbert spaces.

For instance, if S is the generator of a nilpotent contraction semigroup on X , then $-S$ is accretive with $\sigma(S) = \emptyset$.

Every accretive operator S has a square root $S^{1/2}$ which is sectorial with any angle $\theta > \pi/4$, see [Haa06, Proposition 3.1.2]. In particular, $S^{1/2}$ is bisectorial.

Here $S^{1/2}$ is strictly dichotomous only with respect to the first choice $X_+ = X$, $X_- = \{0\}$.

Then A_0 is non-densely defined, $\sigma(A_0) = \emptyset$, and A_0 is accretive. As in Section 8.1 it follows that A_0 has a square root, $A = A_0^{1/2}$, where A is sectorial with any angle $\theta > \pi/4$ and $\sigma(A) = \emptyset$. Moreover, A is non-densely defined too (otherwise $A_0 = A^2$ had to be densely defined). In fact,

$$\overline{\mathcal{D}(A)} = \overline{\mathcal{D}(A_0)} = \{f \in X : f(0) = 0\}.$$

Let $w \in X$, $w(t) = 1$ constant. Hence $w \in X \setminus \overline{\mathcal{D}(A)}$ and $X = \overline{\mathcal{D}(A)} \oplus \text{span}\{w\}$. Fix $0 < s < 1/2$ and consider the rank one operator $B : \mathcal{D}(B) \subset l^2 \rightarrow X$,

$$B\alpha = \sum_{k=1}^{\infty} k^s \alpha_k \cdot w, \quad \mathcal{D}(B) = \left\{ \alpha = (\alpha_k)_{k=1}^{\infty} \in l^2 : \sum_{k=1}^{\infty} k^s |\alpha_k| < \infty \right\}.$$

Let $q \geq 1$ and let $C : \mathcal{D}(C) \subset l^2 \rightarrow l^2$,

$$C\alpha = (-k^q \alpha_k)_{k=1}^{\infty}, \quad \mathcal{D}(C) = \left\{ \alpha = (\alpha_k)_{k=1}^{\infty} \in l^2 : (-k^q \alpha_k)_{k=1}^{\infty} \in l^2 \right\}.$$

Lemma 8.4. *The operators B and C are densely defined, C is selfadjoint, $\sigma(C) = \{-k^q : k \in \mathbb{N}\}$, $\mathcal{D}(C) \subset \mathcal{D}(B)$, and B is $1/q$ -subordinate to C .*

Proof. The first assertions are clear. The last assertion follows because for all $\alpha \in \mathcal{D}(C)$

$$\begin{aligned} \|B\alpha\| &\leq \sum_{k=1}^{\infty} k^s |\alpha_k| \leq \left(\sum_{k=1}^{\infty} \frac{1}{k^{2-2s}} \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} k^2 |\alpha_k|^2 \right)^{\frac{1}{2}}, \\ \sum_{k=1}^{\infty} k^2 |\alpha_k|^2 &\leq \left(\sum_{k=1}^{\infty} |\alpha_k|^2 \right)^{1-\frac{1}{q}} \left(\sum_{k=1}^{\infty} k^{2q} |\alpha_k|^2 \right)^{\frac{1}{q}}. \end{aligned} \quad \square$$

On $X \times l^2$ consider the operator

$$S \begin{pmatrix} f \\ \alpha \end{pmatrix} = \begin{pmatrix} A(f - B\alpha) \\ C\alpha \end{pmatrix}, \quad \mathcal{D}(S) = \left\{ \begin{pmatrix} f \\ \alpha \end{pmatrix} \in X \times \mathcal{D}(C) : f - B\alpha \in \mathcal{D}(A) \right\}.$$

Proposition 8.5. *The operator S has the following properties:*

- (i) S is densely defined.
- (ii) $\sigma(S) = \sigma(C)$.
- (iii) S is almost bisectorial.
- (iv) $G_+ = X \times \{0\}$, $S|_{G_+}$ is sectorial and $\mathcal{D}(S|_{G_+}) = \mathcal{D}(A) \times \{0\}$. In particular, $S|_{G_+}$ is not densely defined.

Proof. For the proof of (i) note that $\overline{\mathcal{D}(A)} \times \{0\} \subset \overline{\mathcal{D}(S)}$ and $(w, k^{-s} e_k) \in \mathcal{D}(S)$ where (e_k) is the standard orthonormal basis in l^2 . This shows $(w, 0) \in \mathcal{D}(S)$, hence $X \times \{0\} \subset \overline{\mathcal{D}(S)}$. Finally, for every $x \in \mathcal{D}(C)$, $(Bx, x) \in \mathcal{D}(S)$ and thus $(0, x) \in \mathcal{D}(S)$, so we showed that S is densely defined.

(ii) can be shown by direct computation. Moreover, for $\lambda \in \varrho(S)$,

$$(S - \lambda)^{-1} \begin{pmatrix} f \\ \alpha \end{pmatrix} = \begin{pmatrix} (A - \lambda)^{-1}f + A(A - \lambda)^{-1}B(C - \lambda)^{-1}\alpha \\ (C - \lambda)^{-1}\alpha \end{pmatrix}. \quad (39)$$

On $i\mathbb{R}$ the norms $\|\lambda(A - \lambda)^{-1}\|$ and $\|A(A - \lambda)^{-1}\|$ as well as the analogous expressions for C are uniformly bounded. The subordination property of B yields the estimate $\|B(C - \lambda)^{-1}\| \leq c|\lambda|^{-\beta}$, $\lambda \in i\mathbb{R} \setminus \{0\}$, with $\beta = 1 - \frac{1}{q}$ and $c > 0$. Therefore

$$\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|^\beta}, \quad \lambda \in i\mathbb{R} \setminus \{0\}, \quad \beta = 1 - \frac{1}{q},$$

with some $M > 0$, which proves (iii).

For the proof of (iv) let us first calculate G_+ . Since for every $f \in X$,

$$(S - \lambda)^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} (A - \lambda)^{-1}f \\ 0 \end{pmatrix}$$

is a bounded analytic function on $\overline{\mathbb{C}_-}$, we have $(f, 0) \in G_+$. On the other hand, if $(f, \alpha) \in G_+$, then (39) implies that $(C - \lambda)^{-1}\alpha$ has a bounded analytic extension to \mathbb{C}_- , which is possible only if $\alpha = 0$. Thus

$$G_+ = X \times \{0\},$$

and therefore $\mathcal{D}(S|_{G_+}) = \mathcal{D}(A) \times \{0\}$. It follows that $S|_{G_+}$ is not densely defined and $S|_{G_+} \cong A$, so $S|_{G_+}$ is sectorial. \square

8.4 A non-densely defined non-sectorial but almost sectorial operator

As a simple example of a non-densely defined operator on a Hilbert space that satisfies an estimate (31) with $\beta < 1$ but is not sectorial, we consider now an ordinary differential operator on the Sobolev space $H^1([0, 1])$. In a Banach space setting, operators of this form have been considered in [SS86]. A non-densely defined operator which is even sectorial and defined on a non-reflexive Banach space is the operator A from Example 8.3.

Example 8.6. For $m \geq 1$ consider the operator A_0 on $L^2([0, 1])$ given by

$$\begin{aligned} A_0 f &= (-1)^m f^{(2m)}, \\ \mathcal{D}(A_0) &= \{f \in H^{2m}([0, 1]) : f^{(j)}(0) = f^{(j)}(1) = 0, j = 0, \dots, m-1\}. \end{aligned}$$

The operator A_0 is positive selfadjoint. Let A be the part of A_0 on the Sobolev space $H^1([0, 1])$, i.e.,

$$\begin{aligned} A f &= A_0 f = (-1)^m f^{(2m)}, \\ \mathcal{D}(A) &= \{f \in H^{2m+1}([0, 1]) : f^{(j)}(0) = f^{(j)}(1) = 0, j = 0, \dots, m-1\}. \end{aligned}$$

We easily see that A is closed and $\sigma_p(A_0) = \sigma_p(A)$. Moreover, if $\lambda \in \varrho(A_0)$ then $A - \lambda$ is bijective with inverse $(A - \lambda)^{-1} = (A_0 - \lambda)^{-1}|H^1([0, 1])$ and hence $\sigma_p(A) = \sigma(A) = \sigma(A_0)$.

Since A_0 is selfadjoint, it is sectorial with arbitrary small angle $\theta > 0$. Let $g \in L^2([0, 1])$, $|\arg \lambda| \geq \theta > 0$ and set $f = (A_0 - \lambda)^{-1}g$. Then the identity $(-1)^m f^{(2m)} = \lambda f + g$ yields

$$\|f^{(2m)}\|_{L^2} \leq (1 + |\lambda| \|(A_0 - \lambda)^{-1}\|) \|g\|_{L^2} \leq (1 + M) \|g\|_{L^2}$$

where $M > 0$ is the constant from the sectoriality estimate (30). Since $|f|_{2m} = \|f\|_{L^2} + \|f^{(2m)}\|_{L^2}$ is an equivalent norm on $H^{2m}([0, 1])$ and $\|(A_0 - \lambda)^{-1}\|$ is uniformly bounded for $|\arg \lambda| \geq \theta$, this implies

$$\|(A_0 - \lambda)^{-1}g\|_{H^{2m}} \leq c_1 \|g\|_{L^2}, \quad |\arg \lambda| \geq \theta,$$

with $c_1 > 0$ depending on $\theta > 0$. Using the interpolation inequality $\|f\|_{H^k} \leq c_2 \|f\|_{L^2}^{1-k/n} \|f\|_{H^n}^{k/n}$, we obtain for $g \in H^1([0, 1])$

$$\begin{aligned} \|(A_0 - \lambda)^{-1}g\|_{H^1} &\leq c_2 \|(A_0 - \lambda)^{-1}g\|_{L^2}^{1-\frac{1}{2m}} \|(A_0 - \lambda)^{-1}g\|_{H^{2m}}^{\frac{1}{2m}} \\ &\leq \frac{c_3}{|\lambda|^{1-\frac{1}{2m}}} \|g\|_{L^2} \leq \frac{c_3}{|\lambda|^{1-\frac{1}{2m}}} \|g\|_{H^1} \end{aligned}$$

with $c_3 > 0$ depending on $\theta > 0$. Consequently A is almost sectorial,

$$\|(A - \lambda)^{-1}\| \leq \frac{c_3}{|\lambda|^\beta}, \quad |\arg \lambda| \geq \theta, \quad \beta = 1 - \frac{1}{2m}. \quad (40)$$

Moreover A even has a compact resolvent: indeed our calculations imply that $(A_0 - \lambda)^{-1}$ is a bounded operator from $L^2([0, 1])$ to $H^1([0, 1])$ and the embedding $H^1([0, 1]) \hookrightarrow L^2([0, 1])$ is compact. Finally note that A is not densely defined since the closure of $\mathcal{D}(A)$ in $H^1([0, 1])$ is $H_0^1([0, 1])$. In particular, the estimate (40) cannot be improved to $\beta = 1$, i.e., A is not a sectorial operator because sectorial operators in reflexive spaces are densely defined. On the other hand, (40) can be improved to $\beta = 1 - \frac{1}{4m}$ as indicated in [SS86].

8.5 A densely defined operator S with non-densely defined almost sectorial $S|_{G_\pm}$

The following is a variant of Example 8.3 in a Hilbert space setting. Here $S|_{G_+}$ is non-densely defined and almost sectorial.

Example 8.7. Let A be the operator from Example 8.6 acting on $X = H^1([0, 1])$. Then $\overline{\mathcal{D}(A)} = H_0^1([0, 1])$. Let $w_1(t) = t$, $w_2(t) = 1 - t$, so that

$$X = \overline{\mathcal{D}(A)} \oplus \text{span}\{w_1, w_2\}.$$

For $0 < s < 1/2$ define $B : \mathcal{D}(B) \subset l^2 \rightarrow X$ by

$$B\alpha = \sum_{k=1}^{\infty} (2k)^s \alpha_{2k} \cdot w_1 + \sum_{k=1}^{\infty} (2k-1)^s \alpha_{2k-1} \cdot w_2,$$

$$\mathcal{D}(B) = \left\{ \alpha = (\alpha_k)_{k=1}^{\infty} \in l^2 : \sum_{k=1}^{\infty} k^s |\alpha_k| < \infty \right\}.$$

As in Example 8.3 consider also the selfadjoint operator $C(\alpha_k) = (-k^q \alpha_k)$ on l^2 and then $S : \mathcal{D}(S) \subset X \times l^2 \rightarrow X \times l^2$,

$$S \begin{pmatrix} f \\ \alpha \end{pmatrix} = \begin{pmatrix} A(f - B\alpha) \\ C\alpha \end{pmatrix}, \quad \mathcal{D}(S) = \left\{ \begin{pmatrix} f \\ \alpha \end{pmatrix} \in X \times \mathcal{D}(C) : f - B\alpha \in \mathcal{D}(A) \right\}.$$

We obtain again that B is $1/q$ -subordinate to C , that S is densely defined and that $(S - \lambda)^{-1}$ is given by (39) where now $\sigma(S) = \sigma(A) \cup \sigma(C)$. Together with the estimate $\|(A - \lambda)^{-1}\| \leq c|\lambda|^{-\beta}$ on $i\mathbb{R}$, $\beta = 1 - \frac{1}{2m}$, we then arrive at

$$\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|^{\gamma}}, \quad \lambda \in i\mathbb{R} \setminus \{0\}, \quad \gamma = 1 - \frac{1}{2m} - \frac{1}{q},$$

with some $M > 0$. Theorems 4.1 and 5.6 thus yield the S -invariant subspaces G_{\pm} . As in Example 8.3, we derive $G_+ = X \times \{0\}$ and $S|_{G_+} \cong A$, which is not densely defined. Here $S|_{G_+}$ is not sectorial but almost sectorial with $\beta = 1 - \frac{1}{2m}$.

8.6 Hamiltonian operator matrices

We can apply our theory to the Hamiltonian operator matrix appearing in systems theory in the case of so-called unbounded control and observation operators. We obtain criteria ensuring that the Hamiltonian is bisectorial and strictly dichotomous. Our setting generalises the ones in [TW14] and [WJZ12] since we do not require a basis of (generalised) eigenvectors of the Hamiltonian and at the same time allow the control and observation operators to map out of the state space.

Example 8.8. Let A be a sectorial operator on a Hilbert space H with angle less than $\pi/2$ and $0 \in \varrho(A)$. Let $H_1 = \mathcal{D}(A)$ be equipped with the graph norm and H_{-1} the completion of H with respect to the norm $\|A^{-1} \cdot\|$. Let H_1^d, H_{-1}^d be the corresponding spaces for A^* . Moreover, we consider intermediate spaces in the sense of Lions and Magenes [LM72, Chapter 1]¹,

$$H_s = [H_1, H]_{1-s}, \quad H_{-s} = [H, H_{-1}]_s, \quad s \in [0, 1],$$

Again, H_s^d and H_{-s}^d are defined analogously. In the special case when A is self-adjoint, we obtain the fractional domain spaces $H_s = H_s^d = \mathcal{D}(A^s) = \mathcal{D}((A^*)^s)$,

¹This is equivalent to taking complex interpolation spaces, see [LM72, Chapter 1, §14]. Note that all involved spaces are Hilbert spaces.

$s \in [-1, 1]$, compare [WJZ12, §3]. In general however, $H_s \neq H_s^d$. The intermediate spaces yield bounded extensions

$$A : H_{1-s} \longrightarrow H_{-s}, \quad A^* : H_{1-s}^d \longrightarrow H_{-s}^d, \quad s \in [0, 1].$$

Using the scalar product $(\cdot|\cdot)$ of H , we can identify the dual of H_{-s} with H_s^d . This means that the scalar product of H extends to a sesquilinear form $(x|y)_{-s,s*}$, $x \in H_{-s}$, $y \in H_s^d$. Similarly, the dual of H_s is identified with H_{-s}^d . This is also referred to as taking duality with respect to the pivot space H , see e.g. [TW09, §§2.9, 2.10].

Let U, Y be Hilbert spaces and consider for some fixed $0 < s < 1/2$ bounded operators $B : U \longrightarrow H_{-s}$, $C : H_s \longrightarrow Y$, the control and observation operators, respectively. With the above duality identifications, their adjoints are bounded operators $B^* : H_s^d \longrightarrow U$, $C^* : Y \longrightarrow H_{-s}^d$. The Hamiltonian is now given by the operator matrix

$$T = \begin{pmatrix} A & BB^* \\ C^*C & -A^* \end{pmatrix}.$$

Let $V_s = H_s \times H_s^d$. The Hamiltonian induces the bounded linear mapping $T : V_{1-s} \longrightarrow V_{-s}$, which we consider, for the moment, as an unbounded operator on V_{-s} with domain V_{1-s} . We use the decomposition

$$T = S + R, \quad S = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}, \quad R = \begin{pmatrix} 0 & BB^* \\ C^*C & 0 \end{pmatrix},$$

where $S : V_{1-s} \longrightarrow V_{-s}$ and $R : V_s \longrightarrow V_{-s}$ are bounded. Let $\lambda \in i\mathbb{R} \setminus \{0\}$. In the following estimates, c denotes a positive constant which is independent of λ , but may change from estimate to estimate. By the sectoriality of A , we have

$$\|(A - \lambda)^{-1}\| \leq c|\lambda|^{-1}, \quad \|A(A - \lambda)^{-1}\| \leq 1 + |\lambda| \|(A - \lambda)^{-1}\| \leq c,$$

and the second inequality implies

$$\|(A - \lambda)^{-1}\|_{H \rightarrow H_1} \leq c, \quad \|(A - \lambda)^{-1}\|_{H_{-1} \rightarrow H} \leq c.$$

Interpolation yields

$$\|(A - \lambda)^{-1}\|_{H \rightarrow H_s} \leq c|\lambda|^{-(1-s)}, \quad \|(A - \lambda)^{-1}\|_{H_{-s} \rightarrow H} \leq c|\lambda|^{-(1-s)},$$

and then

$$\|(A - \lambda)^{-1}\|_{H_{-s} \rightarrow H_s} \leq c|\lambda|^{-(1-2s)}.$$

Since an analogous estimate holds for $(A^* - \lambda)^{-1}$, we obtain

$$\|(S - \lambda)^{-1}\|_{V_{-s} \rightarrow V_s} \leq c|\lambda|^{-(1-2s)}. \quad (41)$$

Consider the identity

$$T - \lambda = (I + R(S - \lambda)^{-1})(S - \lambda).$$

From (41) we obtain

$$\|R(S - \lambda)^{-1}\|_{V_{-s} \rightarrow V_{-s}} \leq \|R\|_{V_s \rightarrow V_{-s}} \|(S - \lambda)^{-1}\|_{V_{-s} \rightarrow V_s} \leq c|\lambda|^{-(1-2s)}$$

and since $s < 1/2$ we get that, for $\lambda \in i\mathbb{R}$ and $|\lambda|$ large, $I + R(S - \lambda)^{-1}$ is an isomorphism on V_{-s} ; consequently $\lambda \in \varrho(T)$ with

$$\|(T - \lambda)^{-1}\|_{V_{-s} \rightarrow V_{-s}} \leq c\|(S - \lambda)^{-1}\|_{V_{-s} \rightarrow V_{-s}} \leq c|\lambda|^{-1}.$$

Moreover

$$(T - \lambda)^{-1} - (S - \lambda)^{-1} = -(T - \lambda)^{-1}R(S - \lambda)^{-1}$$

and hence

$$\|(T - \lambda)^{-1} - (S - \lambda)^{-1}\|_{V_{-s} \rightarrow V_{-s}} \leq c|\lambda|^{-(2-2s)}.$$

We consider now the part of T in V_s , which we denote again by T . Then, similar to the above, the identity

$$T - \lambda = (S - \lambda)(I + (S - \lambda)^{-1}R)$$

and the estimate

$$\|(S - \lambda)^{-1}R\|_{V_s \rightarrow V_s} \leq c|\lambda|^{-(1-2s)}$$

yield

$$\begin{aligned} \|(T - \lambda)^{-1}\|_{V_s \rightarrow V_s} &\leq c|\lambda|^{-1}, \\ \|(T - \lambda)^{-1} - (S - \lambda)^{-1}\|_{V_s \rightarrow V_s} &\leq c|\lambda|^{-(2-2s)}, \end{aligned}$$

for $\lambda \in i\mathbb{R}$, $|\lambda|$ large.

For the rest of this example, we consider T as an operator on V , i.e., we take the part of T in V . Applying interpolation to the above results, we get that, if $\lambda \in i\mathbb{R}$ and $|\lambda|$ large enough, then $\lambda \in \varrho(T)$ and

$$\|(T - \lambda)^{-1}\| \leq c|\lambda|^{-1}, \quad (42)$$

$$\|(T - \lambda)^{-1} - (S - \lambda)^{-1}\| \leq c|\lambda|^{-(2-2s)}. \quad (43)$$

Next we show that T is bisectorial. In view of (42) it suffices to show that $i\mathbb{R} \subset \varrho(T)$. This will be done using the same technique as in [TW14, Lemma 4.5]. Suppose $it \in i\mathbb{R}$ is contained in the approximate point spectrum $\sigma_{\text{app}}(T)$, i.e., there exists a sequence $(x_n, y_n) \in V_{1-s}$ such that $\|x_n\|^2 + \|y_n\|^2 = 1$ and $(T - it)(x_n, y_n) \rightarrow 0$ in V as $t \rightarrow \infty$. This implies

$$\begin{aligned} ((A - it)x_n | y_n)_{-s, s*} + (BB^*y_n | y_n)_{-s, s*} &\rightarrow 0, \\ (C^*Cx_n | x_n)_{-s*, s} - ((A^* + it)y_n | x_n)_{-s*, s} &\rightarrow 0. \end{aligned}$$

Summing both equations and taking the real part gives us

$$\|B^*y_n\|_U^2 + \|Cx_n\|_Y^2 = (BB^*y_n | y_n)_{-s, s*} + (C^*Cx_n | x_n)_{-s*, s} \rightarrow 0. \quad (44)$$

From $(T - it)(x_n, y_n) \rightarrow 0$ it follows that $x_n + (A - it)^{-1}BB^*y_n \rightarrow 0$ as a limit in H . Since $(A - it)^{-1}B$ is bounded as an operator $U \rightarrow H$ and $B^*y_n \rightarrow 0$ by (44), this yields $x_n \rightarrow 0$. Analogously, using $Cx_n \rightarrow 0$, we get $y_n \rightarrow 0$, which is a contradiction to $\|x_n\|^2 + \|y_n\|^2 = 1$. Therefore $\sigma_{\text{app}}(T) \cap i\mathbb{R} = \emptyset$. Since $\partial\sigma(T) \subset \sigma_{\text{app}}(T)$ and $i\mathbb{R} \cap \varrho(T) \neq \emptyset$ we conclude that in fact $i\mathbb{R} \subset \varrho(T)$.

We can now invoke Theorem 7.3 to show the strict dichotomy of T . Indeed the assumptions on A imply that S is bisectorial and strictly dichotomous. Moreover, by $i\mathbb{R} \subset \varrho(T)$ and (43) the conditions (i) and (ii) of Theorem 7.3 are satisfied too. Hence if $\mathcal{D}(S) \cap \mathcal{D}(T) = V_1 \cap \mathcal{D}(T) \subset V$ is dense, then T is strictly dichotomous. Note that a typical setting in control theory is $H = L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$, A is an elliptic differential operator and the control and observation operators act on traces of functions on the boundary of Ω . In this case $\mathcal{D}(S) \neq \mathcal{D}(T)$ but $C_0^\infty(\Omega)^2 \subset \mathcal{D}(S) \cap \mathcal{D}(T)$, i.e., $\mathcal{D}(S) \cap \mathcal{D}(T)$ is in fact dense in V .

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