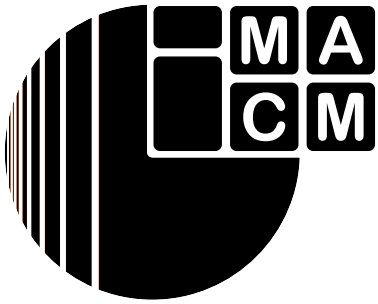

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Christoph Hachtel, Andreas Bartel, and Michael Günther

Abstract For systems of ordinary differential equations, where the components exhibit a largely differing dynamic behaviour, multirate methods exploit this structure to gain computational efficiency. A model order reduction applied to a single subsystem keeps the dimension in the coupling unchanged. In the case of stiff subsystems, where Jacobians are needed, the computational effort remains high. The here presented *interface reduction* approach is a promising way to turn the reduced dimension into an improved efficiency for multirate time domain simulation.

1 Introduction

The starting point is the following system of ordinary differential equations (ODEs)

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t) \tag{1}$$

with components of highly different dynamic behaviour. A multirate method exploits this special structure to compute the numerical approximation in a more efficient way. Thus, the system is split according to the dynamical behaviour:

$$\begin{aligned} \dot{\mathbf{y}}_A &= \mathbf{f}_A(\mathbf{y}_A, \mathbf{y}_L, t), \\ \dot{\mathbf{y}}_L &= \mathbf{f}_L(\mathbf{y}_A, \mathbf{y}_L, t), \end{aligned} \tag{2}$$

where $\mathbf{y}_A \in \mathbb{R}^{n_A}$ denote the fast changing, active components and $\mathbf{y}_L \in \mathbb{R}^{n_L}$ the slow changing, latent components (of $\mathbf{y}$). A multirate method as at applying an inherent time step to each subsystem, that is, the slow components are computed with a large macro step size H and the active components with a small micro step size h with $h \ll H$. A partitioning into more than two subsystems is possible. For simplicity of notation, we restrict ourselves to two subsystems. It is obvious that multirate methods can be even more efficient if the dimension of the slow subsystem is very large compared to the dimension of the active one.

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For a system with a fixed and given partitioning, a model order reduction (MOR) of the slow, high dimensional subsystem promises another gain of efficiency for the time domain simulation. Here, the challenging part is to combine the reduced dimension and the coupling interface with the other non-reduced subsystems.

The work is organised as follows. First we give an introduction to multirate compound step methods and we repeat briefly the well-known concepts of MOR. The combination of both techniques forms the heart of our work: multirate interface reduction. For this setting, a multiphysics application fits to illustrate the capabilities of this new approach. We are considering a regularised, academic test circuit with thermal active and dependent elements and give first numerical results.

2 Multirate Compound-Step Methods

Given the partitioned ODE (2), the crucial part of multirate schemes is the realisation of the coupling between the subsystems. In fact, this one of the distinguishing features of these techniques. In (2) and in all later equations, the coupling terms are printed in colour. Multirate integration schemes for implicit methods were first presented by Gear and Wells [1], where the coupling is simply achieved by inter- and extrapolating the unknown values. Though this approach seems to be a natural choice, several problems concerning the coupling terms appear. In the last years methods that achieve the multirate integration by using a dynamic refinement strategy became popular. These schemes integrate the whole system with a large step size H. By using error estimators, the step size is only refined for those components for which a given accuracy is not reached. Savcenco [2] uses embedded Runge-Kutta schemes for error estimation, Constantinescu and Sandu [3] are using Richardson extrapolation. These methods can handle systems for which a partitioning according to the dynamic behaviour is not known a priori.

Here we follow the idea of compound step methods, which were first developed using Runge-Kutta schemes by Kværnø and Rentrop [4] and then expanded to W-methods [5]. The main idea is to compute the macro-step $\mathbf{y}_L(t_0+H)$ and the first micro step $\mathbf{y}_A(t_0+h)$ coupled together in one compound step. The remaining micro steps $\mathbf{y}_A(t_0+ih)$, $i=2,\ldots,m$, can either be computed by interpolating the slow components or by using a dense output formula for the slow part. Compound step methods can be used for systems with a stronger coupling than the inter-/extrapolation methods of [1]. Mixed-multirate compound step methods [6] allow the usage of different integration schemes for compound and remaining micro steps. So the single methods can be chosen according to the properties of the subsystems.

Linear (simply diagonal) implicit compound step methods like in [5] can be used for (at least moderately) stiff systems only by the computational cost of solving one system of linear equations per time step. The simplest version is based on the linear implicit Euler method, [3]: the multirate method reads for the compound step

$$\begin{pmatrix} h\dfrac{\partial \mathbf{f}_A}{\partial \mathbf{y}_A}-\mathbf{I}_A & \dfrac{h}{m}\dfrac{\partial \mathbf{f}_A}{\partial \mathbf{y}_L} \\ mH\dfrac{\partial \mathbf{f}_L}{\partial \mathbf{y}_A} & H\dfrac{\partial \mathbf{f}_L}{\partial \mathbf{y}_L}-\mathbf{I}_L \end{pmatrix} \begin{pmatrix} \mathbf{y}_A(t_0+h)-\mathbf{y}_A(t_0) \\ \mathbf{y}_L(t_0+H)-\mathbf{y}_L(t_0) \end{pmatrix} = \begin{pmatrix} h\mathbf{f}_A(\mathbf{y}_A(t_0),\mathbf{y}_L(t_0)) \\ H\mathbf{f}_L(\mathbf{y}_A(t_0),\mathbf{y}_L(t_0)) \end{pmatrix} \tag{3}$$

and for the remaining micro steps holds

$$\left(h\,\frac{\partial \mathbf{f}_A}{\partial \mathbf{y}_A}-\mathbf{I}_A\right)\mathbf{k}_{A,i} = -h\,\mathbf{f}_A(\mathbf{y}_A(t_0+ih),\tilde{\mathbf{y}}_L(t_0+ih)), \qquad i=1,\ldots,m-1 \tag{4}$$

with $\mathbf{k}_{A,i} = \mathbf{y}_A(t_0 + (i+1)h) - \mathbf{y}_A(t_0 + ih)$ and $\tilde{\mathbf{y}}_L$ the interpolated values of the slow components. The coupling between the slow and the active subsystem is realised by the off-diagonal elements in the coefficient matrix of the system of linear equations in (3). The multirate method (3–4) is of order one, but also in higher order compound step methods with underlying linear implicit integration schemes, e.g. [5,6], the coupling is partly realised by the off-diagonal blocks of the coefficient matrix. We can only expect high improvements in the computational effort by applying multirate schemes if the number of active components is much smaller than the number of slow components ($n_A \ll n_L$). So the question arises whether one can exploit this structure for a more efficient computation not only by using larger step sizes for the slow component but also reducing the dimension of the slow part by MOR.

3 Model Order Reduction (MOR)

Since we only expect small variations in the slow components, we assume here that the slow part is linear or at least linearised over a given macro step. Thus the partitioned system (2) can be rewritten with system matrix $\mathbf{A} \in \mathbb{R}^{n_L,n_L}$, input matrix $\mathbf{B} \in \mathbb{R}^{n_L,n_A}$ and output matrix $\mathbf{C} \in \mathbb{R}^{n_L,n_L}$:

$$\dot{\mathbf{y}}_A = \mathbf{f}_A(\mathbf{y}_A, \mathbf{y}_L, t) \tag{5}$$
$$\dot{\mathbf{y}}_L = \mathbf{A} \cdot \mathbf{y}_L + \mathbf{B} \cdot \mathbf{y}_A \tag{6}$$
$$\mathbf{y}_L = \mathbf{C} \cdot \mathbf{y}_L. \tag{7}$$

Now we apply MOR to the internal variable $\mathbf{y}_L$. To this end, the system matrices are projected on a low dimensional subspace by biorthogonal projection matrices $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n_L,r}$ with $r \ll n_L$. One ends up with a reduced slow subsystem

$$\dot{\mathbf{y}}_{L,r} = \mathbf{W}^T\mathbf{A}\mathbf{V} \cdot \mathbf{y}_{L,r} + \mathbf{W}^T\mathbf{B} \cdot \mathbf{y}_A \tag{8}$$
$$\mathbf{y}_{L,r} = \mathbf{C}\mathbf{V} \cdot \mathbf{y}_{L,r}. \tag{9}$$

The motivation of applying MOR is to obtain a small dimensional variable $\mathbf{y}_{L,r}$ while the output $\mathbf{y}_{L,r}$ shall be approximated sufficiently accurate. The way how the projection matrices $\mathbf{V}, \mathbf{W}$ are computed are defined by the MOR method, for further details see [7]. For the multirate-MOR setting, the usage of a certain MOR method is not mandatory so the user can choose his favorite method.

Notice that the dimension of the output variable in (9), i.e., the dimension of the coupling interface slow to active, will be not reduced in this setting. In fact, with a non-reduced interface we cannot expect large improvements of the computational efficiency solving the system of linear equations in the compound step (3) by using a reduced slow subsystem. So we have to find a way to take the reduced dimension of the internal slow variable to the coupling interface to gain efficiency in the compound step.

4 Interface Reduction

Usually the (few) active components do not depend on the detailed information of every single slow component. So we may replace the coupling interface $\mathbf{y}_L$ in (5) by a low dimensional input $\mathbf{u}_A$. The same can be made for the slow part (7). Here we do not focus on the input, but we aim at a small dimensional output $\mathbf{z}_L$

replacing $\mathbf{y}_L$. Adopting the notation for coupled linear systems from [8], the modified multirate systems read

$$\dot{\mathbf{y}}_A = \mathbf{f}_A(\mathbf{y}_A, \mathbf{u}_A, t) \qquad \dot{\mathbf{y}}_L = \mathbf{A}\cdot\mathbf{y}_L + \mathbf{B}\cdot\mathbf{u}_L \tag{10}$$

$$\mathbf{u}_A = \mathbf{g}(\mathbf{z}_A, \mathbf{z}_L, \mathbf{u}, t) \qquad \mathbf{u}_L = \mathbf{K}_{LA}\cdot\mathbf{z}_A + \mathbf{K}_{LL}\cdot\mathbf{z}_L + \mathbf{H}\cdot\mathbf{u} \tag{11}$$

$$\mathbf{z}_A = \mathbf{h}(\mathbf{y}_A, t) \qquad \mathbf{z}_L = \mathbf{C}\cdot\mathbf{y}_L \tag{12}$$

with input $\mathbf{u}_X \in \mathbb{R}^{q_X}$, global input $\mathbf{u}$, output $\mathbf{z}_X \in \mathbb{R}^{p_X}$ and coupling matrices $\mathbf{K}_{LX}, X \in \{A, L\}$. In a monolithic system (1), the coupling functions $\mathbf{g}, \mathbf{h}$ and matrices $\mathbf{K}_{LA}, \mathbf{C}$ are not given by the system itself, but must be defined by the user exploiting some underlying properties, e.g. physical laws. These modifications in the multirate setting will not change the diagonal blocks in the compound step coefficient matrix (3), but for the off-diagonal blocks the mixed derivatives change into

$$\frac{\partial \mathbf{f}_A}{\partial \mathbf{y}_L} = \frac{\partial \mathbf{f}_A}{\partial \mathbf{u}_A}\cdot\frac{\partial \mathbf{g}}{\partial \mathbf{y}_L} = \frac{\partial \mathbf{f}_A}{\partial \mathbf{u}_A}\cdot\frac{\partial \mathbf{g}}{\partial \mathbf{z}_L}\cdot\frac{\partial \mathbf{z}_L}{\partial \mathbf{y}_L} = \frac{\partial \mathbf{f}_A}{\partial \mathbf{u}_A}\cdot\frac{\partial \mathbf{g}}{\partial \mathbf{z}_L}\cdot\mathbf{C} \tag{13}$$

$$\frac{\partial \mathbf{f}_L}{\partial \mathbf{y}_A} = \frac{\partial \mathbf{f}_L}{\partial \mathbf{u}_L}\cdot\frac{\partial \mathbf{u}_L}{\partial \mathbf{y}_A} = \frac{\partial \mathbf{f}_L}{\partial \mathbf{u}_L}\cdot\frac{\partial \mathbf{u}_L}{\partial \mathbf{z}_A}\cdot\frac{\partial \mathbf{z}_A}{\partial \mathbf{y}_A} = \mathbf{B}\cdot\mathbf{K}_{LA}\cdot\frac{\partial \mathbf{h}}{\partial \mathbf{y}_A}. \tag{14}$$

Inspecting the dimension of the matrix products on the right hand sides of (13) and (14) gives

$$(n_A \times q_A)\cdot(q_A \times p_L)\cdot(p_L \times n_L) \tag{15}$$

$$(n_L \times q_L)\cdot(q_L \times p_A)\cdot(p_A \times n_A). \tag{16}$$

In a multirate context the dimension n_A is supposed to be small. If the interface functions $\mathbf{g}, \mathbf{h}, \mathbf{K}_{LA}, \mathbf{C}$ are chosen such that the dimension of their codomains are small, then only one large dimension remains, namely the number of the slow components n_L. However, as we saw in Section 3, we can compute a reduced model of dimension r for the slow part and use matrices $\mathbf{B}_r$ and $\mathbf{C}_r$ in the mixed derivatives of (13–14).

It is obvious that using this framework the number of function evaluation during one macro step increased due to the usage of input and output function. Hence we assume that the codomains of these functions are low dimensional this will not affect the computation time significantly.

Using this framework we expect higher efficiency in a time domain simulation. If we apply a MOR technique for which any error bounds are known also the error due to MOR can be handled. Nevertheless the replacement of $\mathbf{y}_X$ to $\mathbf{u}_X$ can influence the numerical properties of the integration method in particular the stability, which is not yet investigated.

5 Simulation

To apply the theoretical considerations of the above sections, we use as benchmark example the electric-thermal test circuit of [9] with the modifications given in [10]. It is a small electric circuit, in which some elements are modeled temperature dependent. The circuit diagram is given in Fig. 1. Due to electric current, the resistor $R(\mathbf{T})$ is heated and so the resistance of this device changes. The characteristic curve of the diode is also temperature dependent. The voltages are modeled by a nodal analysis using Kirchhoff's laws. For the temperature of the resistor (wire), the 1-D heat equation is semi-discretised using a finite volume approach, see Fig. 2. Finally we get a system of ordinary differential equations like in (1) in terms of the unknowns $\mathbf{y} = [u_3, u_4, e, \mathbf{T}]$, where u_3, u_4 denote the voltages at node 3 and 4, e is the dissipated energy in the thermal

Fig. 1 Circuit diagramm

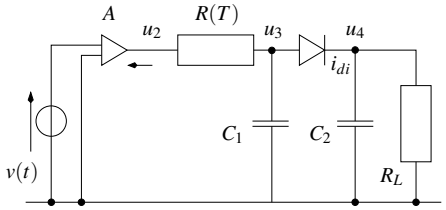
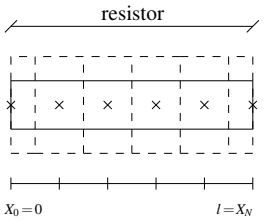


Fig. 2 Finite Volume Discretised Resistor



dependent resistor and $\mathbf{T}$ the vector of temperatures in the semi-discretised resistor. The multirate behaviour of this system is given by the physical properties: the voltages and the dissipated energy change very fast (with source of the network), and the temperature in the resistor changes much slower. Hence the partitioning according to the dynamical behaviour is quite natural:

$$[\dot{u}_3, \dot{u}_4, \dot{e}] = \mathbf{f}_A([u_3, u_4, e], \mathbf{T}, t) \tag{17a}$$

$$\dot{\mathbf{T}} = \mathbf{f}_L([u_3, u_4, e], \mathbf{T}, t). \tag{17b}$$

As mentioned in [10], the subsystem of the semi-discretised heat equation (17b) is not a-priori linear, but it can be easily linearised without loss of much accuracy. Hence a linear model order reduction of the thermal subsystem is possible.

For (17), the computational cost of the compound step (3) depends on the number of discretisation points of the spatial variable of the thermal subsystem. If a high accuracy is demanded this dimension can be large and the computational cost increases. So the question is how the coupling interface can be modified such that the dimension of the input of the active part and the output of the slow part is small.

The heating of the resistor, caused by the electric current, is computed by the dissipated power p. The electric subsystem is computing the total dissipated energy e in on macro step H. The ratio e/H defines the averaged power, which we use for coupling [9]. Hence we add an output function to the active subsystem: $\mathbf{h}([u_3, u_4, e], t) = e/H$. To compute e, we have either to calculate differences of e or we have to assign zero as the initial value for each macro step. If H is adjusted by a step size control, it has to be handled as an independent parameter.

For the coupling interface slow to active, one has to consider the thermal dependent, physical parameters, which are necessary in the circuit model and which can be computed by a linear model. In our case, these are the total resistance $R(\mathbf{T})$ and the diode's temperature T_{di}. Additional input functions for the slow and the active part are not necessary with this choice of coupling interfaces. As global input variable $\mathbf{u}$ we have the source voltage $v(t)$ which is used in the active, electric subsystem only. These modifications in the interface of the coupled system (17) lead to

$$[\dot{u}_3, \dot{u}_4, \dot{e}] = \mathbf{f}_A([u_3, u_4, e], \mathbf{u}_A, t) \qquad \dot{\mathbf{T}} = \mathbf{A} \cdot \mathbf{T} + \mathbf{B} \cdot \mathbf{u}_L \tag{18a}$$

$$\mathbf{u}_A = [R(\mathbf{T}), T_{di}, v(t)]^T \qquad \mathbf{u}_L = p \tag{18b}$$

$$p = \mathbf{h}([u_3, u_4, e], t) = e/H \qquad [R(\mathbf{T}), T_{di}] = \mathbf{C} \cdot \mathbf{T}. \tag{18c}$$

For this system the off-diagonal blocks of the Jacobian matrix in the compound step (3) become much smaller. Inspecting the dimensions like in (15) gives for $\frac{\partial \mathbf{f}_A}{\partial \mathbf{y}_L}$ the matrix sizes $(3 \times 2) \cdot (2 \times n_L)$ and for $\frac{\partial \mathbf{f}_L}{\partial \mathbf{y}_A}$ the dimension $(n_L \times 1) \cdot (1 \times 3)$. Now, a model order reduction can decrease the number of thermal variables from n_L to a significant smaller number r. No large dimensional terms occur in this setting so we expect a

Fig. 3 Voltage at Node 3

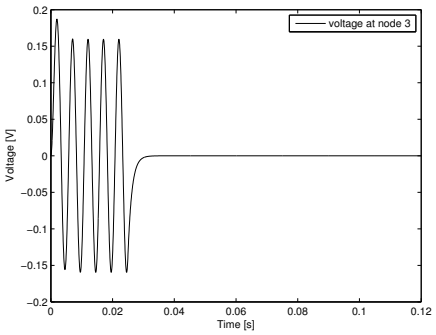
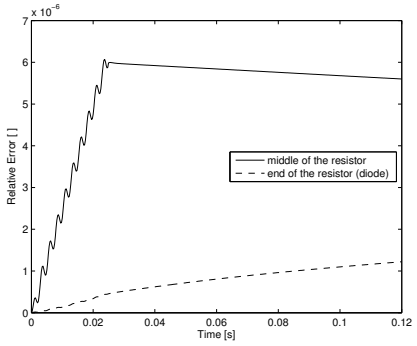


Fig. 4 Relative errors



large gain concerning the computational effort using compound step multirate methods for this multiphysics application.

For the simulation of the system we use the mixed multirate compound step method of [6] which consists of a third order for the compound and a fourth order linear implicit method for the remaining micro steps. We take the same parameters as in [10]. The resistor is discretised in $n_L = 50$ cells due to a demanded high accuracy so a model order reduction for the thermal part system becomes reasonable. For the model order reduction we applied a balanced truncation technique (see [7]) and reduce the thermal subsystem to $r = 5$. As ratio between micro and macro steps we have $m = 5$. As reference solution we used a full order thermal model ($n_L = 50$) and a single rate linear implicit method which corresponds to the third order linear implicit method of the compound step in the multirate solver. For the reference solution we choose the step size equal to the micro step size of the multirate scheme. We implemented the system and the integration method in Matlab 2013a. As expected the computation time decreases significantly: For a simulation time interval $[0., 12s]$, the non-reduced reference solution needs $31.62s$ while the multirate, interface-reduced system only needs $2.25s$. Here, we are interested in two physical sizes: One is the temperature of the diode and the other is the highest temperature in the resistor which is found at its middle. Figure 4 shows the relative error of the multirate solution to the reference solution of these two physical sizes. Figure 3 shows the voltage curve at node three. Compared to the gain of efficiency this error seems to be quite small but it looks quite surprising: It is similar to the temperature profile of the corresponding cells during the time simulation, the error seems to be caused by a constant factor. Here is further work necessary to be find the source of this interface reduction error, possible reasons are the model order reduction, the multirate method or the chosen interfaces.

6 Conclusion

Using a reduced order model for time domain simulation with multirate compound step methods, the computational time was not improved as expected. In this paper we pointed out why a MOR in a unmodified multirate framework does not lead to computational improvements. Furthermore, by introducing interface reduction, we presented a way how the reduced dimension in a model order reduced subsystem can be exploited also for multirate compound step methods. We put up interface reduction approach to an academic multiphyics test system. The observed error is not yet understand, its sources will be analysed in the future.

Hence interface reduction modifies the multirate ODE framework stability of the multirate compound step method (cf. [11]) cannot be guaranteed any more so further work about this open point is necessary.

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