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Variational principles for self-adjoint operator functions arising from second-order systems

Birgit Jacob,[*]   Matthias Langer[†]   Carsten Trunk[‡]

Abstract

Variational principles are proved for the eigenvalues of a Cauchy problem given by a second-order linear differential operator equation of the form

$$\ddot{z}(t) + D\dot{z}(t) + A_0 z(t) = 0$$

in a Hilbert space H. Here A_0 is a uniformly positive operator and $A_0^{-1/2} D A_0^{-1/2}$ is a bounded non-negative operator in H. As usual, a scale of spaces H_s is related to A_0 and the inner product $\langle \cdot, \cdot \rangle$ in H extends to a mapping $\langle \cdot, \cdot \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} : H_{-\frac{1}{2}} \times H_{\frac{1}{2}} \to \mathbb{C}$. We associate with the second-order equation the form

$$\mathfrak{t}(\lambda)[x,y] = \langle \lambda^2 x + \lambda D x + A_0 x, y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}$$

with domain $H_{\frac{1}{2}}$ and a corresponding operator family $T(\lambda)$. Using form methods a generalized Rayleigh functional is defined and the eigenvalues above the essential spectrum of the generator associated with the Cauchy problem are characterized by a min-max and a max-min variational principle. The obtained results are illustrated with a damped beam equation.

1   Introduction

Variational principles are a very useful tool for the qualitative and numerical investigation of eigenvalues of self-adjoint operators and operator functions. For instance, the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ below the essential spectrum of a self-adjoint operator A with domain $\mathscr{D}(A)$ can be characterized using the Rayleigh functional

$$p(x) = \frac{\langle Ax, x \rangle}{\langle x, x \rangle}, \quad x \in \mathscr{D}(A), \, x \neq 0,$$

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via a min-max principle or a max-min principle:

$$\lambda_n = \min_{\substack{L \subset \mathscr{D}(A) \\ \dim L = n}} \max_{x \in L \setminus \{0\}} p(x) = \max_{\substack{L \subset H \\ \dim L = n-1}} \min_{\substack{x \in \mathscr{D}(A) \setminus \{0\} \\ x \perp L}} p(x).$$

Variational principles were first introduced by H. Weber, Lord Rayleigh, H. Poincaré, E. Fischer, G. Polya, and W. Ritz, H. Weyl, R. Courant (see, e.g. [4, 7, 20], and the references therein).

In this article we investigate variational principles for self-adjoint operator functions arising from second-order systems of the form

$$\ddot{z}(t) + D\dot{z}(t) + A_0 z(t) = 0 \tag{1.1}$$

on a Hilbert space H. It is assumed that the stiffness operator A_0 is a uniformly positive operator and that $A_0^{-1/2} D A_0^{-1/2}$ is a bounded non-negative operator in H, where in many applications the operator D encodes the damping. With these second-order systems we associate the operator function

$$T(\lambda) = \lambda^2 I + \lambda D + A_0$$

on H with domain $\mathscr{D}(T(\lambda)) = \left\{ x \in H_{\frac{1}{2}} \mid \lambda Dx + A_0 x \in H \right\}$, where $H_{\frac{1}{2}} := \mathscr{D}(A^{\frac{1}{2}})$, the corresponding sesquilinear form

$$\mathfrak{t}(\lambda)[x,y] = \left\langle \lambda^2 x + \lambda Dx + A_0 x, y \right\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \tag{1.2}$$

with domain $H_{\frac{1}{2}}$, and the operator matrix

$$\mathscr{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix}$$

on $\mathscr{H} = H_{\frac{1}{2}} \times H$ with domain $\mathscr{D}(\mathscr{A}) = \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \,\middle|\, A_0 z + Dw \in H \right\}$. The latter can be seen as companion matrix of the polynomial $T(\lambda)$; it is also the generator of a semigroup that corresponds to a first-order system if one rewrites (1.1).

We identify a disc $\Phi_{\gamma_0} \subset \mathbb{C}$ which is the largest disc around zero with an empty intersection with the essential spectrum of $\mathscr{A}$. For $\lambda \in \Phi_{\gamma_0}$ we show that the form $\mathfrak{t}(\lambda)$ is closed and sectorial and that the corresponding operator $T(\lambda)$ is m-sectorial. Moreover, on Φ_{γ_0} the spectrum (point spectrum) of $\mathscr{A}$ and the spectrum (point spectrum) of T coincide.

In [7] R. J. Duffin proved a variational principle for eigenvalues of a quadratic matrix polynomial, which was generalized in various directions to more general operator functions; see, e.g. the references in [9]. In [9] such a variational principle was proved for eigenvalues of operator functions whose values are possibly unbounded self-adjoint operators. Here we adapt this variational principle from [9] to our situation. Using the form $\mathfrak{t}(\lambda)$ we introduce a slightly more general definition of a generalized Rayleigh functional and we show that the variational principal generalizes to this situation. In

2

particular, for a fixed $x \in H_{\frac{1}{2}} \setminus \{0\}$, denote the two real solutions (if they exist) of the quadratic equation

$$\mathfrak{t}(\lambda)[x,x] = 0$$

by $p_-(x)$ and $p_+(x)$ such that $p_-(x) \le p_+(x)$ is satisfied and set $p_+(x) := \infty$, $p_-(x) := -\infty$ if there are no real solutions. Then the function p_+ plays the role of a generalized Rayleigh functional in our main theorem, which yields variational principles for the real eigenvalues of $\mathscr{A}$ or, what is equivalent, of T: in certain real intervals Δ above the essential spectrum of $\mathscr{A}$ in the disc Φ_{γ_0} with the property that Δ does not contain values of p_-, the spectrum of $\mathscr{A}$ is either empty or consists only of a finite or infinite sequence of isolated semi-simple eigenvalues of finite multiplicity of $\mathscr{A}$. Moreover, we show that these eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots$, counted according to their multiplicities, satisfy

$$\lambda_n = \max_{\substack{L \subset H_{1/2} \\ \dim L = n}} \min_{x \in L \setminus \{0\}} p_+(x) = \min_{\substack{L \subset H \\ \dim L = n-1}} \sup_{\substack{x \in H_{1/2} \setminus \{0\} \\ x \perp L}} p_+(x)$$

and, if $N < \infty$, we show for $n > N$ that

$$\sup_{\substack{L \subset \mathscr{D} \\ \dim L = n}} \min_{x \in L \setminus \{0\}} p_+(x) \le \inf \Delta \qquad \text{and} \qquad \inf_{\substack{L \subset H \\ \dim L = n-1}} \sup_{\substack{x \in \mathscr{D} \setminus \{0\} \\ x \perp L}} p_+(x) \le \inf \Delta.$$

A major application of this variational principle is a quite general interlacing principle which is the second main result of this article: if the stiffness operator A_0 decreases and the damping operator D increases, then the corresponding nth eigenvalue decreases compared with the nth eigenvalue of the unchanged system. We illustrate the obtained results with an example where we consider a beam equation with a damping such that A_0 corresponds to the fourth derivative on the interval $(0,1)$ (with some appropriate boundary conditions) and the damping D equals $-\frac{d}{dx} d \frac{d}{dx}$ with some smooth function d (and some boundary conditions).

We proceed as follows. In Section 2 we recall some basic notions of operators, operator functions and forms. The variational principle obtained in [9] is adapted to the setting of this paper in Section 3. Section 4 is devoted to general properties of the class of second-order systems studied in this paper. The main results of this paper are proved in Section 5. In particular, we study the form (1.2) and their relation to the operator matrix $\mathscr{A}$ and the operator function $T(\lambda)$. On a disc Φ_{γ_0} around zero, $\mathfrak{t}(\lambda)$ is a closed sectorial form and the spectrum (point spectrum) of $\mathscr{A}$ and the spectrum (point spectrum) of T coincide. Further, the variational principles for $\mathscr{A}$ are presented in Theorem 5.7. As an application of the variational principle we show interlacing properties of eigenvalues of two different second-order problems with coefficients which satisfy a specific order relation. Finally, in Section 6 we apply the obtained results to a damped beam equation.

2 Preliminaries

We recall in this section some basic notions for operators and operator functions. Let H be a Hilbert space. We denote the inner product on H by $\langle \cdot, \cdot \rangle$. A closed, densely defined

operator in H is called *Fredholm* if the dimension of its kernel and the (algebraic) co-dimension of its range are finite. The *essential spectrum* of a closed, densely defined operator S is defined by

$$\sigma_{\text{ess}}(S) := \big\{ \lambda \in \mathbb{C} \mid S - \lambda I \text{ is not Fredholm} \big\}.$$

Recall that a closed, densely defined operator T is called *sectorial* if its numerical range is contained in a sector $\{ z \in \mathbb{C} \mid \operatorname{Re} z \geq z_0, \, |\arg(z - z_0)| \leq \theta \}$ for some $z_0 \in \mathbb{R}$ and $\theta \in [0, \frac{\pi}{2})$. A sectorial operator T is called *m-sectorial* if $\lambda \in \rho(T)$ for some λ with $\operatorname{Re} \lambda < z_0$; see, e.g. [15, §V.3.10].

Let $\Delta \subset \mathbb{R}$ be an interval with

$$a = \inf \Delta \quad \text{and} \quad b = \sup \Delta, \qquad -\infty \leq a < b \leq \infty, \tag{2.1}$$

and let Ω be a domain in $\mathbb{C}$ such that $\Delta \subset \Omega$. On Ω we consider a family of closed, densely defined operators $T(\lambda)$, $\lambda \in \Omega$, in a Hilbert space H, where $T(\lambda)$ has domain $\mathscr{D}(T(\lambda))$.

In the following we shall assume that either $T(\lambda)$ or $-T(\lambda)$ is an m-sectorial operator for $\lambda \in \Omega$. Under this assumption the sesquilinear form $\langle T(\lambda) \cdot, \cdot \rangle$ is closable for $\lambda \in \Omega$, and we denote the closure by $\mathfrak{t}(\lambda)[\cdot, \cdot]$ with domain $\mathscr{D}(\mathfrak{t}(\lambda))$ and set $\mathfrak{t}(\lambda)[x] := \mathfrak{t}(\lambda)[x, x]$, which is the corresponding quadratic form. Recall (see, e.g. [15, §VII.4]) that $T := (T(\lambda))_{\lambda \in \Omega}$ is called a *holomorphic family of type* (B) if $T(\lambda)$ is m-sectorial for $\lambda \in \Omega$, the domain $\mathscr{D}(\mathfrak{t}(\lambda))$ of the quadratic form $\mathfrak{t}(\lambda)$ is independent of λ, which we denote by $\mathscr{D}$, and $\lambda \mapsto \mathfrak{t}(\lambda)[x]$ is holomorphic on Ω for every $x \in \mathscr{D}$.

The *spectrum* of the operator function T is defined as follows:

$$\sigma(T) := \big\{ \lambda \in \Omega \mid T(\lambda) \text{ is not bijective} \big\} = \big\{ \lambda \in \Omega \mid 0 \in \sigma(T(\lambda)) \big\}.$$

Similarly, the *essential spectrum* of the operator function T is defined as

$$\sigma_{\text{ess}}(T) := \big\{ \lambda \in \Omega \mid T(\lambda) \text{ is not Fredholm} \big\} = \big\{ \lambda \in \Omega \mid 0 \in \sigma_{\text{ess}}(T(\lambda)) \big\}.$$

A number $\lambda \in \Omega$ is called an *eigenvalue* of the operator function T if there exists an $x \in \mathscr{D}(T(\lambda))$, $x \neq 0$, such that $T(\lambda)x = 0$. The *point spectrum* is the set of all eigenvalues:

$$\sigma_{\text{p}}(T) := \big\{ \lambda \in \Omega \mid \exists x \in \mathscr{D}(T(\lambda)), \, x \neq 0, \, T(\lambda)x = 0 \big\}$$
$$= \big\{ \lambda \in \Omega \mid 0 \in \sigma_{\text{p}}(T(\lambda)) \big\},$$

where $\sigma_{\text{p}}(T(\lambda))$ denotes the point spectrum of the operator $T(\lambda)$ for fixed $\lambda \in \Omega$. The *geometric multiplicity* of an eigenvalue λ of the operator function T is defined as the dimension of $\ker T(\lambda)$.

One way to obtain forms is via a scale of spaces. For this let A_0 be a self-adjoint, positive definite linear operator on a Hilbert space H with domain $\mathscr{D}(A_0)$ such that $0 \in \rho(A_0)$. We define

$$H_{\frac{1}{2}} := \mathscr{D}(A_0^{1/2}) \text{ with the norm } \|x\|_{H_{\frac{1}{2}}} := \|A_0^{1/2} x\|_H \tag{2.2}$$

and

$$H_{-\frac{1}{2}} \text{ as the completion of } H \text{ with respect to the norm}$$

$$\|x\|_{H_{-\frac{1}{2}}} := \|A_0^{-1/2}x\|_H. \tag{2.3}$$

By continuity, A_0 and $A_0^{1/2}$ can be extended to isometric isomorphisms from $H_{\frac{1}{2}}$ onto $H_{-\frac{1}{2}}$ and from H onto $H_{-\frac{1}{2}}$, respectively. These extensions are also denoted by A_0 and $A_0^{1/2}$. The space $H_{-\frac{1}{2}}$ can be identified with the dual space of $H_{\frac{1}{2}}$ by identifying elements $x \in H_{-\frac{1}{2}}$ with functionals on $H_{\frac{1}{2}}$ as follows

$$\langle x,y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} := \langle A_0^{-1/2}x, A_0^{1/2}y \rangle, \qquad x \in H_{-\frac{1}{2}}, y \in H_{\frac{1}{2}}.$$

Note that, for $x \in H$, $y \in H_{\frac{1}{2}}$, we have

$$\langle x,y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle x,y \rangle_H. \tag{2.4}$$

Let A and B be bounded operators from $H_{\frac{1}{2}}$ to $H_{-\frac{1}{2}}$. Define sesquilinear forms $\mathfrak{a}$ and $\mathfrak{b}$ in H with domains $\mathscr{D}(\mathfrak{a}) = \mathscr{D}(\mathfrak{b}) = H_{\frac{1}{2}}$ by

$$\mathfrak{a}[x,y] := \langle Ax,y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}, \quad \mathfrak{b}[x,y] := \langle Bx,y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}, \quad x,y \in H_{\frac{1}{2}}.$$

Moreover, the corresponding quadratic forms are denoted by $\mathfrak{a}[x] := \mathfrak{a}[x,x]$ and $\mathfrak{b}[x] := \mathfrak{b}[x,x]$. Assume that the form $\mathfrak{a}$ is sectorial and closed in H. Here a form is called *sectorial* if its numerical range is contained in a sector $\{z \in \mathbb{C} \mid \operatorname{Re} z \geq z_0, \, |\arg(z - z_0)| \leq \theta\}$ for some $z_0 \in \mathbb{R}$ and $\theta \in [0, \frac{\pi}{2})$ (see, e.g. [15, V.§3.10]). Moreover, assume that the form $\mathfrak{b}$ is $\mathfrak{a}$-bounded with $\mathfrak{a}$-bound less than 1, i.e. there exist constants p,q with $p \geq 0$, $0 \leq q < 1$ such that

$$\big|\mathfrak{b}[x]\big| \leq p\|x\|^2 + q\big|\mathfrak{a}[x]\big| \qquad \text{for all } x \in H_{\frac{1}{2}}.$$

According to [15, Theorem VI.1.33] the sum

$$\mathfrak{t} := \mathfrak{a} + \mathfrak{b} \tag{2.5}$$

is a closed sectorial form with domain $\mathscr{D}(\mathfrak{t}) = H_{\frac{1}{2}}$. Note that, for all $x,y \in H_{\frac{1}{2}}$,

$$\mathfrak{t}[x,y] = \langle Ax,y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + \langle Bx,y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle Ax + Bx, y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}, \tag{2.6}$$

where the sum $Ax + Bx$ is calculated in $H_{-\frac{1}{2}}$. Hence, by the First Representation Theorem (see [15, Theorem VI.2.1]), there exists a representing operator T of the form $\mathfrak{t}$, i.e. an m-sectorial operator with $\mathscr{D}(T) \subset \mathscr{D}(\mathfrak{t})$ such that

$$\mathfrak{t}[x,y] = \langle Tx,y \rangle \qquad \text{for all } x \in \mathscr{D}(T), y \in \mathscr{D}(\mathfrak{t}). \tag{2.7}$$

Moreover, $\mathfrak{t}$ is the closure of the form $\langle T \cdot, \cdot \rangle$.

Lemma 2.1. *Assume that the form $\mathfrak{a}$ is sectorial and closed and the form $\mathfrak{b}$ is $\mathfrak{a}$-bounded with $\mathfrak{a}$-bound less than 1. Define the form $\mathfrak{t}$ as in (2.5) and the operator T as in (2.7). Then the operator T is given by*

$$\mathscr{D}(T) = \left\{ x \in H_{\frac{1}{2}} \mid Ax + Bx \in H \right\}, \tag{2.8}$$

$$Tx = Ax + Bx,$$

where $Ax + Bx$ is calculated as an expression in $H_{-\frac{1}{2}}$.

Proof. Denote the right-hand side of (2.8) by $\widetilde{\mathscr{D}}$. Let $x \in \mathscr{D}(T)$. Then $x \in \mathscr{D}(\mathfrak{t}) = H_{\frac{1}{2}}$, and we conclude from (2.6), (2.7) and (2.4) that, for all $y \in H_{\frac{1}{2}}$,

$$\langle Ax + Bx, y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \mathfrak{t}[x, y] = \langle Tx, y \rangle = \langle Tx, y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}.$$

Hence $Ax + Bx = Tx \in H$ and $x \in \widetilde{\mathscr{D}}$. That is, $\mathscr{D}(T) \subset \widetilde{\mathscr{D}}$.

Conversely, let $x \in \widetilde{\mathscr{D}}$. Then $x \in \mathscr{D}(\mathfrak{t})$ and $w := Ax + Bx \in H$. For $y \in H_{\frac{1}{2}}$ we have

$$\mathfrak{t}[x, y] = \langle Ax + Bx, y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle w, y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle w, y \rangle.$$

It follows from [15, Theorem VI.2.1 (iii)] that $x \in \mathscr{D}(T)$ and $Tx = w$. $\qquad\square$

Lemma 2.2. *Let R be a compact operator in H and ε an arbitrary positive number. Then there exists a constant $C \geq 0$ such that*

$$\|RA_0^{1/2} x\|^2 \leq \varepsilon \|A_0^{1/2} x\|^2 + C\|x\|^2 \qquad \text{for all } x \in H_{\frac{1}{2}}.$$

Proof. The operator $RA_0^{1/2} A_0^{-1/2} = R$ is a compact operator in H. Hence $RA_0^{1/2}$ is $A_0^{1/2}$-compact; see, e.g. [15, Section IV.1.3]. By [8, Corollary III.7.7], $RA_0^{1/2}$ has $A_0^{1/2}$-bound 0, which implies the assertion (see [15, §V.4.1]). $\qquad\square$

3 A general variational principle for self-adjoint operator functions

In this section we recall a general variational principle for eigenvalues of a self-adjoint operator function from [9] adapted to the present situation. Here we also show some additional statements. We mention that in [9] a more general class of operator functions is investigated.

For the rest of the paper $\mathscr{L}_I(S)$ denotes the spectral subspace of a self-adjoint operator S corresponding to the interval I. We suppose that one of the following two conditions is satisfied.

(I) Let Ω be a domain in $\mathbb{C}$ and $\Delta \subset \Omega \cap \mathbb{R}$ an interval with endpoints a, b as in (2.1). The family $(T(\lambda))_{\lambda \in \Omega}$ is a holomorphic family of type (B), $T(\lambda)$ is self-adjoint for $\lambda \in \Delta$ and there exists a $c \in \Delta$ such that $\dim \mathscr{L}_{(-\infty, 0)}(T(c)) < \infty$.

(II)   Let Ω be a domain in $\mathbb{C}$ and $\Delta \subset \Omega \cap \mathbb{R}$ an interval with endpoints a, b as in (2.1). The family $(-T(\lambda))_{\lambda \in \Omega}$ is a holomorphic family of type (B), $T(\lambda)$ is self-adjoint for $\lambda \in \Delta$ and there exists a $c \in \Delta$ such that $\dim \mathscr{L}_{(0,\infty)}(T(c)) < \infty$.

Note that under assumption (I) for $\lambda \in \Delta$ the operators $T(\lambda)$ are self-adjoint and sectorial, and, hence, bounded from below. Similarly, under assumption (II), the operators $T(\lambda)$ are bounded from above for $\lambda \in \Delta$. The condition $\dim \mathscr{L}_{(-\infty,0)}(T(c)) < \infty$ is equivalent to the fact that $\sigma(T(c)) \cap (-\infty, 0)$ consists of at most a finite number of eigenvalues of finite multiplicities. As in the previous section let $\mathscr{D}$ be the common domain of the quadratic forms $\mathfrak{t}(\lambda)$, which are the closures of the forms $\langle T(\lambda) \cdot, \cdot \rangle$, $\lambda \in \Omega$.

We shall also assume that one of the following two conditions $(\searrow)$, $(\nearrow)$ is satisfied.

$(\searrow)$ For every $x \in \mathscr{D} \setminus \{0\}$ the function $\lambda \mapsto \mathfrak{t}(\lambda)[x]$ is *decreasing at value zero* on Δ, i.e. if $\mathfrak{t}(\lambda_0)[x] = 0$ for some $\lambda_0 \in \Delta$, then

$$\mathfrak{t}(\lambda)[x] > 0 \qquad \text{for } \lambda \in (-\infty, \lambda_0) \cap \Delta,$$
$$\mathfrak{t}(\lambda)[x] < 0 \qquad \text{for } \lambda \in (\lambda_0, \infty) \cap \Delta.$$

$(\nearrow)$ For every $x \in \mathscr{D} \setminus \{0\}$ the function $\lambda \mapsto \mathfrak{t}(\lambda)[x]$ is *increasing at value zero* on Δ, i.e. if $\mathfrak{t}(\lambda_0)[x] = 0$ for some $\lambda_0 \in \Delta$, then

$$\mathfrak{t}(\lambda)[x] < 0 \qquad \text{for } \lambda \in (-\infty, \lambda_0) \cap \Delta,$$
$$\mathfrak{t}(\lambda)[x] > 0 \qquad \text{for } \lambda \in (\lambda_0, \infty) \cap \Delta.$$

If T satisfies $(\nearrow)$ or $(\searrow)$, then, for $x \in \mathscr{D} \setminus \{0\}$, the scalar function $\lambda \mapsto \mathfrak{t}(\lambda)[x]$ is either decreasing or increasing at a zero and, hence, it has at most one zero in Δ.

We now introduce the notion of a generalized Rayleigh functional p, which is a mapping from $\mathscr{D} \setminus \{0\}$ to $\mathbb{R} \cup \{\pm\infty\}$. If there is a zero λ_0 of the scalar function $\lambda \mapsto \mathfrak{t}(\lambda)[x]$ in Δ, then the corresponding value of a generalized Rayleigh functional $p(x)$ must equal this zero; $p(x) = \lambda_0$. Otherwise, there is some freedom in the definition. More precisely, we use the following definition.

Definition 3.1. Let Δ and Ω be as above. Moreover, let $T(\lambda)$, $\lambda \in \Omega$, be a family of closed operators in a Hilbert space H satisfying either (I) or (II) and which satisfies also $(\nearrow)$ or $(\searrow)$. In the case $(\searrow)$ a mapping $p : \mathscr{D} \setminus \{0\} \to \mathbb{R} \cup \{\pm\infty\}$ with the properties

$$p(x) \begin{cases} = \lambda_0 & \text{if } \mathfrak{t}(\lambda_0)[x] = 0, \\ < a & \text{if } a \in \Delta \text{ and } \mathfrak{t}(\lambda)[x] < 0 \text{ for all } \lambda \in \Delta, \\ \leq a & \text{if } a \notin \Delta \text{ and } \mathfrak{t}(\lambda)[x] < 0 \text{ for all } \lambda \in \Delta, \\ > b & \text{if } b \in \Delta \text{ and } \mathfrak{t}(\lambda)[x] > 0 \text{ for all } \lambda \in \Delta, \\ \geq b & \text{if } b \notin \Delta \text{ and } \mathfrak{t}(\lambda)[x] > 0 \text{ for all } \lambda \in \Delta. \end{cases}$$

is called a *generalized Rayleigh functional* for T on Δ. In the case $(\nearrow)$ a mapping $p : \mathscr{D} \setminus \{0\} \to \mathbb{R} \cup \{\pm\infty\}$ with the properties

$$
p(x) \begin{cases} = \lambda_0 & \text{if } \mathfrak{t}(\lambda_0)[x] = 0, \\ > b & \text{if } b \in \Delta \text{ and } \mathfrak{t}(\lambda)[x] < 0 \text{ for all } \lambda \in \Delta, \\ \geq b & \text{if } b \notin \Delta \text{ and } \mathfrak{t}(\lambda)[x] < 0 \text{ for all } \lambda \in \Delta, \\ < a & \text{if } a \in \Delta \text{ and } \mathfrak{t}(\lambda)[x] > 0 \text{ for all } \lambda \in \Delta, \\ \leq a & \text{if } a \notin \Delta \text{ and } \mathfrak{t}(\lambda)[x] > 0 \text{ for all } \lambda \in \Delta. \end{cases} \tag{3.1}
$$

is called a *generalized Rayleigh functional* for T on Δ.

Remark 3.2. One possible choice for p in the case $(\searrow)$ is the following (see [4, 9]). For $x \in \mathscr{D} \setminus \{0\}$ set

$$
p(x) = \begin{cases} \lambda_0 & \text{if } \mathfrak{t}(\lambda_0)[x] = 0, \\ -\infty & \text{if } \mathfrak{t}(\lambda)[x] < 0 \text{ for all } \lambda \in \Delta, \\ +\infty & \text{if } \mathfrak{t}(\lambda)[x] > 0 \text{ for all } \lambda \in \Delta, \end{cases}
$$

which was used as a definition of a generalized Rayleigh functional in [4, 9]. However, here we propose to use the Definition 3.1. This has the following advantage: if p is a generalized Rayleigh functional for T on Δ, then the same p remains a generalized Rayleigh functional in the sense of Definition 3.1 for T on a smaller interval Δ' with $\Delta' \subset \Delta$. Moreover, in many applications, including the one in Section 5, the operator function T is defined on a larger interval $\tilde{\Delta} \supset \Delta$ but satisfies, say, $(\searrow)$ only on Δ. If $\mathfrak{t}(\cdot)[x]$ has a zero λ_0 in $\tilde{\Delta}$ where $\lambda_0 < a$ and $\mathfrak{t}(\lambda)[x] < 0$ for all $\lambda \in \Delta$, one can set $p(x) := \lambda_0$.

For a generalized Rayleigh functional p as in Definition 3.1 we have for $\lambda \in \Delta$, $x \in \mathscr{D}(T(\lambda)) \setminus \{0\}$,

$$
T(\lambda)x = 0 \quad \Longrightarrow \quad p(x) = \lambda.
$$

If T satisfies $(\searrow)$, then for $x \in \mathscr{D} \setminus \{0\}$

$$
\begin{aligned}
\mathfrak{t}(\lambda)[x] > 0 &\quad \Longleftrightarrow \quad p(x) > \lambda, \\
\mathfrak{t}(\lambda)[x] < 0 &\quad \Longleftrightarrow \quad p(x) < \lambda;
\end{aligned}
$$

if T satisfies $(\nearrow)$, then for $x \in \mathscr{D} \setminus \{0\}$

$$
\begin{aligned}
\mathfrak{t}(\lambda)[x] > 0 &\quad \Longleftrightarrow \quad p(x) < \lambda, \\
\mathfrak{t}(\lambda)[x] < 0 &\quad \Longleftrightarrow \quad p(x) > \lambda.
\end{aligned}
$$

In [9, Theorem 2.1] a variational principle involving a generalized Rayleigh functional was derived. There the generalized Rayleigh functional was defined as in Remark 3.2 and not in the (slightly more general) way as in Definition 3.1. Therefore, the

variational principle in the following theorem is an adapted version of [9, Theorem 2.1] where a non-decreasing sequence of eigenvalues of an operator function is characterized. Moreover, in [9, Theorem 2.1] only the case (I), ($\searrow$) was considered (under slightly weaker assumptions on $\mathfrak{t}$).

Theorem 3.3. *Let Δ and Ω be as above. Moreover, let $T(\lambda)$, $\lambda \in \Omega$, be a family of closed operators in a Hilbert space H satisfying either* (I), ($\searrow$) *or* (II), ($\nearrow$), *let p be a generalized Rayleigh functional and assume that*

$$\Delta' := \begin{cases} \Delta & \text{if } \sigma_{\mathrm{ess}}(T) \cap \Delta = \emptyset, \\ \left\{ \lambda \in \Delta \mid \lambda < \inf\left(\sigma_{\mathrm{ess}}(T) \cap \Delta\right) \right\} & \text{if } \sigma_{\mathrm{ess}}(T) \cap \Delta \neq \emptyset, \end{cases}$$

is non-empty.

Then $\sigma(T) \cap \Delta'$ is either empty or consists only of a finite or infinite sequence of isolated eigenvalues of T with finite geometric multiplicities, which in the case of infinitely many eigenvalues in $\sigma(T) \cap \Delta'$ accumulates only at $\sup \Delta'$ (which equals $\inf(\sigma_{\mathrm{ess}}(T) \cap \Delta)$ if $\sigma_{\mathrm{ess}}(T) \cap \Delta \neq \emptyset$ and equals b otherwise).

If $\sigma(T) \cap \Delta'$ is empty, then set $N := 0$; otherwise, denote the eigenvalues in $\sigma(T) \cap \Delta'$ by $(\lambda_j)_{j=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, in non-decreasing order, counted according to their geometric multiplicities: $\lambda_1 \leq \lambda_2 \leq \cdots$. Choose $a' \in \Delta'$ so that in the case $N > 0$ it satisfies $a' \leq \lambda_1$. Then the quantity

$$\kappa := \begin{cases} \dim \mathscr{L}_{(-\infty,0)}\big(T(a')\big) & \text{if } (\mathrm{I}), (\searrow) \text{ are satisfied,} \\ \dim \mathscr{L}_{(0,\infty)}\big(T(a')\big) & \text{if } (\mathrm{II}), (\nearrow) \text{ are satisfied,} \end{cases}$$

is a finite number. Moreover, the nth eigenvalue λ_n, $n \in \mathbb{N}$, $n \leq N$, satisfies

$$\lambda_n = \min_{\substack{L \subset \mathscr{D} \\ \dim L = \kappa + n}} \sup_{x \in L \setminus \{0\}} p(x), \tag{3.2}$$

$$\lambda_n = \max_{\substack{L \subset H \\ \dim L = \kappa + n - 1}} \inf_{\substack{x \in \mathscr{D} \setminus \{0\} \\ x \perp L}} p(x). \tag{3.3}$$

For subspaces L with dimensions not considered in (3.2) and (3.3) the right-hand side of (3.2) and (3.3) gives values with the following properties: if $\kappa > 0$, then

$$\begin{aligned} \inf_{\substack{L \subset \mathscr{D} \\ \dim L = n}} \sup_{x \in L \setminus \{0\}} p(x) \leq a \\ \sup_{\substack{L \subset H \\ \dim L = n - 1}} \inf_{\substack{x \in \mathscr{D} \setminus \{0\} \\ x \perp L}} p(x) \leq a \end{aligned} \qquad \text{for } n = 1, \dots, \kappa; \tag{3.4}$$

if $N < \infty$, then

$$\begin{aligned} \inf_{\substack{L \subset \mathscr{D} \\ \dim L = n}} \sup_{x \in L \setminus \{0\}} p(x) \geq \sup \Delta' \\ \sup_{\substack{L \subset H \\ \dim L = n - 1}} \inf_{\substack{x \in \mathscr{D} \setminus \{0\} \\ x \perp L}} p(x) \geq \sup \Delta' \end{aligned} \qquad \text{for } n > \kappa + N \text{ with } n \leq \dim H. \tag{3.5}$$

Proof. Let us first consider the case when (I), $(\searrow)$ are satisfied. We apply [9, Theorem 2.1]. Since T is a holomorphic family of type (B), [9, Proposition 2.13] implies that conditions (i) and (ii) of [9, Theorem 2.1] are satisfied. It follows directly from (I) and $(\searrow)$ that (iii) and (iv) of [9, Theorem 2.1] are also satisfied. Now [9, Theorem 2.1] implies that $\sigma(T) \cap \Delta'$ is either empty or consists of a sequence of isolated eigenvalues that can accumulate at most at $\sup \Delta'$.

Set

$$
\Delta_1 := \begin{cases} \Delta' & \text{if } N = 0, \\ \{\mu \in \Delta' \mid \mu \leq \lambda_1\} & \text{otherwise.} \end{cases}
$$

In [9, Theorem 2.1] the number κ was defined as $\dim \mathscr{L}_{(-\infty,0)}\big(T(a'')\big)$ with a particular choice of $a'' \in \Delta_1$. However, the function

$$
\lambda \mapsto \dim \mathscr{L}_{(-\infty,0)}\big(T(\lambda)\big)
$$

is constant on Δ_1 by [9, Lemma 2.6]. Hence we choose an arbitrary $a' \in \Delta_1$ for the definition of κ, which by [9, Theorem 2.1 and Lemma 2.6] is a finite number:

$$
\kappa = \dim \mathscr{L}_{(-\infty,0)}\big(T(a')\big).
$$

Let us now prove (3.2). In [9] a special choice of a generalized Rayleigh functional was considered; see Remark 3.2. In order to distinguish it, we denote it by q, i.e. for $x \in \mathscr{D} \setminus \{0\}$ we set

$$
q(x) := \begin{cases} \lambda_0 & \text{if } \mathfrak{t}(\lambda_0)[x] = 0, \\ -\infty & \text{if } \mathfrak{t}(\lambda)[x] < 0 \text{ for all } \lambda \in \Delta, \\ +\infty & \text{if } \mathfrak{t}(\lambda)[x] > 0 \text{ for all } \lambda \in \Delta. \end{cases}
$$

We note that $p(x) = q(x)$ if $p(x) \in \Delta$ or $q(x) \in \Delta$. In [9, Theorem 2.1] it was proved that

$$
\lambda_n = \min_{\substack{L \subset \mathscr{D} \\ \dim L = \kappa + n}} \max_{x \in L \setminus \{0\}} q(x)
$$

for $n \in \mathbb{N}$, $n \leq N$. Let $n \in \mathbb{N}$ with $n \leq N$. There exists a subspace $L_0 \subset \mathscr{D}$ with $\dim L_0 = \kappa + n$ such that

$$
\max_{x \in L_0 \setminus \{0\}} q(x) = \lambda_n,
$$

which implies in particular that $q(x) \leq \lambda_n$ for all $x \in L_0 \setminus \{0\}$. If, for $x \in L_0 \setminus \{0\}$, we have $q(x) = -\infty$, then $p(x) \leq a$ by the definitions of p and q, and hence $p(x) \leq \lambda_n$. If, for $x \in L_0 \setminus \{0\}$, we have $q(x) \neq -\infty$, then $q(x) \in \Delta$ and hence $p(x) = q(x) \leq \lambda_n$. This implies that

$$
\sup_{x \in L_0 \setminus \{0\}} p(x) \leq \max_{x \in L_0 \setminus \{0\}} q(x) = \lambda_n. \tag{3.6}
$$

Let $L \subset \mathscr{D}$ be an arbitrary subspace with $\dim L = \kappa + n$. Then, by the definition of L_0,

$$
\max_{x \in L \setminus \{0\}} q(x) \geq \max_{x \in L_0 \setminus \{0\}} q(x) = \lambda_n.
$$

Hence there exists an $x_0 \in L \setminus \{0\}$ with $q(x_0) \geq \lambda_n$. If $q(x_0) = +\infty$, then $p(x_0) \geq b$ and, in particular, $p(x_0) \geq \lambda_n$. If $q(x_0) \neq +\infty$, then $q(x_0) \in \Delta$, which implies that $p(x_0) = q(x_0) \geq \lambda_n$. Hence

$$\sup_{x \in L \setminus \{0\}} p(x) \geq \lambda_n. \tag{3.7}$$

By (3.6) and (3.7) we obtain (3.2). Equation (3.3) is shown in a similar way.

Next we prove the first inequality in (3.4). Let $n \leq \kappa$ and let $\lambda \in \Delta_1$ be arbitrary. We have seen above that $\dim \mathscr{L}_{(-\infty,0)}(T(\lambda)) = \kappa$. Therefore we can choose an n-dimensional subspace of $\mathscr{L}_{(-\infty,0)}(T(\lambda))$, which we denote by L_0 and which is contained in $\mathscr{D}(T(\lambda)) \subset \mathscr{D}$. Since $\mathfrak{t}(\lambda)[x] < 0$ for all $x \in L_0 \setminus \{0\}$, we have

$$\inf_{\substack{L \subset \mathscr{D} \\ \dim L = n}} \sup_{x \in L \setminus \{0\}} p(x) \leq \sup_{x \in L_0 \setminus \{0\}} p(x) \leq \lambda.$$

This implies the first inequality in (3.4) since $\lambda \in \Delta_1$ was arbitrary. The second inequality in (3.4) is shown in a similar way.

We show the first inequality in (3.5). Let $n > \kappa + N$. If we have $\lambda_N = b = \sup \Delta'$, then (3.5) follows from (3.2). In all other cases, choose $\lambda \in \Delta'$ such that $\lambda > \lambda_N$ if $N > 0$. It follows from [9, Lemmas 2.6 and 2.7] that $\dim \mathscr{L}_{(-\infty,0)}(T(\lambda)) = \kappa + N$. Hence, for each subspace $L \subset \mathscr{D}$ with $\dim L = n$, there exists an $x_0 \in L \setminus \{0\}$ such that $\mathfrak{t}(\lambda)[x_0] \geq 0$. Therefore

$$\sup_{x \in L \setminus \{0\}} p(x) \geq p(x_0) \geq \lambda.$$

Since this is true for every such L, we have

$$\inf_{\substack{L \subset \mathscr{D} \\ \dim L = n}} \sup_{x \in L \setminus \{0\}} p(x) \geq \lambda,$$

which implies the validity of the first inequality in (3.5) as λ can be chosen arbitrarily close to $\sup \Delta'$; see [9, Lemma 2.6]. In a similar way one can show the second inequality in (3.5).

If instead of (I), ($\searrow$) the assumptions (II), ($\nearrow$) are satisfied, then the function $\widetilde{T}(\lambda) := -T(\lambda)$ satisfies the assumptions (I), ($\searrow$) and $\widetilde{p}(x) := p(x)$ is a generalized Rayleigh functional for $\widetilde{T}$ on Δ, see Definition 3.1. Hence we can apply the already proved statements to $\widetilde{T}$, which imply all assertions also in this situation as $\sigma_{\mathrm{p}}(\widetilde{T}) = \sigma_{\mathrm{p}}(T)$. □

Remark 3.4.

(i) Instead of assuming that T is a holomorphic family of type (B) it is sufficient to assume some weaker continuity properties. Also the domain of the quadratic form may depend on λ. For further details see [9], in particular, the assumptions (i) and (ii) there.

(ii) If the functional p is chosen such that it is continuous as a mapping from $\mathscr{D}$ into the extended real numbers $\mathbb{R} \cup \{\pm\infty\}$ and $p(cx) = p(x)$ for all $c \in \mathbb{C} \setminus \{0\}$ and

$x \in \mathscr{D}$, then the supremum in (3.2) is actually a maximum, i.e. the eigenvalue λ_n, $n \in \mathbb{N}$, $n \leq N$, satisfies

$$\lambda_n = \min_{\substack{L \subset \mathscr{D} \\ \dim L = \kappa + n}} \max_{x \in L \setminus \{0\}} p(x).$$

This follows from the fact that it is sufficient to take the supremum over the set $\{x \in L \mid \|x\| = 1\}$, which is compact. The same statement applies to (3.4) and (3.5).

A similar theorem holds if we replace in Theorem 3.3 the assumption (I), ($\searrow$) by (I), ($\nearrow$) and (II), ($\nearrow$) by (II), ($\searrow$), respectively, and change Δ' accordingly. This is done in the following theorem.

Theorem 3.5. *Let Δ and Ω be as above. Moreover, let $T(\lambda)$, $\lambda \in \Omega$, be a family of closed operators in a Hilbert space H satisfying either (I), ($\nearrow$) or (II), ($\searrow$), let p be a generalized Rayleigh functional and assume that*

$$\Delta' := \begin{cases} \Delta & \text{if } \sigma_{\mathrm{ess}}(T) \cap \Delta = \emptyset, \\ \{\lambda \in \Delta \mid \lambda > \sup(\sigma_{\mathrm{ess}}(T) \cap \Delta)\} & \text{if } \sigma_{\mathrm{ess}}(T) \cap \Delta \neq \emptyset, \end{cases}$$

is non-empty.

Then $\sigma(T) \cap \Delta'$ is either empty or consists only of a finite or infinite sequence of isolated eigenvalues of T with finite geometric multiplicities, which in the case of infinitely many eigenvalues in $\sigma(T) \cap \Delta'$ accumulates only at $\inf \Delta'$ (which equals $\sup(\sigma_{\mathrm{ess}}(T) \cap \Delta)$ if $\sigma_{\mathrm{ess}}(T) \cap \Delta \neq \emptyset$ and equals a otherwise).

If $\sigma(T) \cap \Delta'$ is empty, then set $N := 0$; otherwise, denote the eigenvalues in $\sigma(T) \cap \Delta'$ by $(\lambda_j)_{j=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, in non-increasing order, counted according to their geometric multiplicities: $\lambda_1 \geq \lambda_2 \geq \cdots$. Choose $b' \in \Delta'$ so that in the case $N > 0$ it satisfies $\lambda_1 \leq b'$. Then the quantity

$$\kappa := \begin{cases} \dim \mathscr{L}_{(-\infty, 0)}(T(b')) & \text{if (I), ($\nearrow$) are satisfied,} \\ \dim \mathscr{L}_{(0, \infty)}(T(b')) & \text{if (II), ($\searrow$) are satisfied,} \end{cases}$$

is a finite number. Moreover, the nth eigenvalue λ_n, $n \in \mathbb{N}$, $n \leq N$, satisfies

$$\lambda_n = \max_{\substack{L \subset \mathscr{D} \\ \dim L = \kappa + n}} \inf_{x \in L \setminus \{0\}} p(x), \tag{3.8}$$

$$\lambda_n = \min_{\substack{L \subset H \\ \dim L = \kappa + n - 1}} \sup_{\substack{x \in \mathscr{D} \setminus \{0\} \\ x \perp L}} p(x). \tag{3.9}$$

For subspaces L with dimensions not considered in (3.8) and (3.9) the right-hand side of (3.8) and (3.9) gives values with the following properties: if $\kappa > 0$, then

$$\sup_{\substack{L \subset \mathscr{D} \\ \dim L = n}} \inf_{x \in L \setminus \{0\}} p(x) \geq b$$

$$\inf_{\substack{L \subset H \\ \dim L = n - 1}} \sup_{\substack{x \in \mathscr{D} \setminus \{0\} \\ x \perp L}} p(x) \geq b \qquad \text{for } n = 1, \ldots, \kappa; \tag{3.10}$$

if $N < \infty$, then

$$\sup_{\substack{L \subset \mathscr{D} \\ \dim L = n}} \inf_{x \in L \setminus \{0\}} p(x) \leq \inf \Delta'$$

$$\inf_{\substack{L \subset H \\ \dim L = n-1}} \sup_{\substack{x \in \mathscr{D} \setminus \{0\} \\ x \perp L}} p(x) \leq \inf \Delta' \qquad \textit{for } n > \kappa + N \textit{ with } n \leq \dim H. \qquad (3.11)$$

Proof. The theorem follows from Theorem 3.3 applied to the function $\widehat{T}(\lambda) := T(-\lambda)$, $-\lambda \in \Omega$. With $\widehat{a} := -b$, $\widehat{b} := -a$ and $\widehat{\Delta} := \{-\lambda \mid \lambda \in \Delta\}$ all assumptions of Theorem 3.3 are satisfied, namely (I) and (II) remain the same and $(\searrow)$ turns into $(\nearrow)$ and vice versa. That is, $\widehat{T}$ satisfies either (I), $(\searrow)$ or (II), $(\nearrow)$. Then the mapping $\widehat{p}(x) := -p(x)$ is a generalized Rayleigh functional for $\widehat{T}$ on $\widehat{\Delta}$; see Definition 3.1. Since $\widehat{\lambda}_n = -\lambda_n$ for $\widehat{\lambda}_n \in \sigma_{\mathrm{p}}(\widehat{T})$, all assertions of Theorem 3.5 follow from Theorem 3.3. $\qquad \square$

Remark 3.6. If the functional p is chosen such that it is continuous and $p(cx) = p(x)$ for $c \in \mathbb{C} \setminus \{0\}$ and $x \in \mathscr{D}$ (see Remark 3.4), then the infimum in (3.8) is actually a minimum, i.e. the eigenvalue λ_n, $n \in \mathbb{N}$, $n \leq N$, satisfies

$$\lambda_n = \max_{\substack{L \subset \mathscr{D} \\ \dim L = \kappa + n}} \min_{x \in L \setminus \{0\}} p(x).$$

A similar statement applies to (3.10) and (3.11).

4 Framework

We study second-order systems of the following form

$$\ddot{z}(t) + D\dot{z}(t) + A_0 z(t) = 0. \qquad (4.1)$$

In common with [18, 19] we make the following assumptions throughout this paper.

(A1) The stiffness operator $A_0 : \mathscr{D}(A_0) \subset H \to H$ is a self-adjoint, positive definite linear operator on a Hilbert space H such that $0 \in \rho(A_0)$.

We define the Hilbert spaces $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$ as in (2.2) and (2.3), respectively. As in Section 2 we denote again by A_0 and $A_0^{1/2}$ the extensions of A_0 and $A_0^{1/2}$ to operators from $H_{\frac{1}{2}}$ to $H_{-\frac{1}{2}}$ and from H to $H_{-\frac{1}{2}}$, respectively. In addition to **(A1)** we impose further assumptions.

(A2) The damping operator $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is a bounded operator with

$$\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq 0, \qquad z \in H_{\frac{1}{2}}.$$

This condition is equivalent to the fact that the operator $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is such that $A_0^{-1/2} D A_0^{-1/2}$ is a bounded self-adjoint non-negative operator in H.

Note also that, for $x, y \in H_{\frac{1}{2}}$,

$$\langle Dx, y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \left\langle A_0^{-1/2} D A_0^{-1/2} A_0^{1/2} x, A_0^{1/2} y \right\rangle$$

$$= \left\langle A_0^{1/2} x, A_0^{-1/2} D A_0^{-1/2} A_0^{1/2} y \right\rangle = \overline{\langle Dy, x \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}. \qquad (4.2)$$

Equation (4.1) is equivalent to the following standard first-order equation

$$\dot{x}(t) = \mathscr{A} x(t)$$

in the space $\mathscr{H} := H_{\frac{1}{2}} \times H$ where $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ is given by

$$\mathscr{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix}, \qquad (4.3)$$

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \;\middle|\; A_0 z + Dw \in H \right\}. \qquad (4.4)$$

The operator $\mathscr{A}$ itself is not self-adjoint in the Hilbert space $\mathscr{H}$. It is easy to see (e.g. [19]) that $\mathscr{A}$ has a bounded inverse in $\mathscr{H}$ given by

$$\mathscr{A}^{-1} = \begin{bmatrix} -A_0^{-1} D & -A_0^{-1} \\ I & 0 \end{bmatrix}, \qquad (4.5)$$

where $A_0^{-1} D$ is considered as an operator acting in $H_{\frac{1}{2}}$. This together with the fact that $J \mathscr{A}$, where

$$J := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},$$

is a symmetric operator in the Hilbert space $\mathscr{H}$, implies the self-adjointness of $J \mathscr{A}$ in $\mathscr{H}$. Therefore,

$$\mathscr{A}^* = J \mathscr{A} J, \qquad \text{with } \mathscr{D}(\mathscr{A}^*) = J \mathscr{D}(\mathscr{A})$$

(see also [18, Proof of Lemma 4.5]) and

$$\operatorname{Re}\langle \mathscr{A} x, x \rangle \le 0 \quad \text{for } x \in \mathscr{D}(\mathscr{A}) \quad \text{and} \quad \operatorname{Re}\langle \mathscr{A}^* x, x \rangle \le 0 \quad \text{for } x \in \mathscr{D}(\mathscr{A}^*).$$

Hence $\mathscr{A}$ is the generator of a strongly continuous semigroup of contractions on the state space $\mathscr{H}$. This fact is well known; see, e.g. [2, 3, 6, 10, 16] or [18, Proposition 5.1].

For $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in H_{\frac{1}{2}} \times H$ we define an indefinite inner product on $\mathscr{H}$ by

$$\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right] := \left\langle J \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, x_2 \rangle_{H_{\frac{1}{2}}} - \langle y_1, y_2 \rangle.$$

Then $(\mathscr{H}, [\cdot, \cdot])$ is a Krein space and $\mathscr{A}$ is a self-adjoint operator with respect to $[\cdot, \cdot]$ (note that the latter is equivalent to the self-adjointness of $J\mathscr{A}$ in $\mathscr{H}$). Hence $\sigma(\mathscr{A})$ is symmetric with respect to $\mathbb{R}$, see, e.g. [5, Theorem VI.6.1]. For the basic theory of Krein spaces and operators acting therein we refer to [1] and [5]. In the following proposition we collect the above considerations.

Theorem 4.1. *If* **(A1)** *and* **(A2)** *are satisfied, then the operator $\mathscr{A}$ is self-adjoint in the Krein space $(\mathscr{H}, [\cdot, \cdot])$, its spectrum is contained in the closed left half plane and is symmetric with respect to the real line. The operator $\mathscr{A}$ has a bounded inverse and it is the generator of a strongly continuous semigroup of contractions on the state space $\mathscr{H}$.*

In the following we study the spectrum of $\mathscr{A}$. The last statement of Theorem 4.1 guarantees that the spectrum of $\mathscr{A}$ is contained in $\mathbb{C}_-$, where $\mathbb{C}_-$ denotes the closed left half plane $\{z \in \mathbb{C} \mid \operatorname{Re} z \leq 0\}$. Because of the first statement of Theorem 4.1 we even have $\sigma(\mathscr{A}) \subset \mathbb{C}_- \setminus \{0\}$. However, apart from this restriction and the symmetry with respect to the real line, the spectrum of $\mathscr{A}$ is quite arbitrary; see, e.g. [11, Examples 3.5 and 3.6] and we refer to Example 3.2 in [12].

For the rest of the paper we assume that, in addition to **(A1)** and **(A2)**, also the following condition is satisfied.

(A3) The operator A_0^{-1} is a compact operator in H.

Proposition 4.2. *Assume that* **(A1)**–**(A3)** *are satisfied. Consider $A_0^{-1}D$ as an operator acting in $H_{\frac{1}{2}}$ and $A_0^{-1/2}DA_0^{-1/2}$ as an operator acting in H, both being bounded operators. Then*

$$\sigma\big(A_0^{-1/2}DA_0^{-1/2}\big) = \sigma\big(A_0^{-1}D\big), \quad \sigma_{\text{ess}}\big(A_0^{-1/2}DA_0^{-1/2}\big) = \sigma_{\text{ess}}\big(A_0^{-1}D\big) \tag{4.6}$$

and

$$\sigma_{\text{ess}}(\mathscr{A}) = \left\{\lambda \in \mathbb{C} \setminus \{0\} \ \Big| \ \frac{1}{\lambda} \in \sigma_{\text{ess}}(-A_0^{-1}D)\right\} \tag{4.7}$$

$$= \left\{\lambda \in \mathbb{C} \setminus \{0\} \ \Big| \ \frac{1}{\lambda} \in \sigma_{\text{ess}}\big(-A_0^{-1/2}DA_0^{-1/2}\big)\right\} \tag{4.8}$$

$$\subset (-\infty, 0). \tag{4.9}$$

The spectrum in $\mathbb{C} \setminus \sigma_{\text{ess}}(\mathscr{A})$ is a discrete set consisting only of eigenvalues. Moreover, the set $\sigma(\mathscr{A}) \setminus \mathbb{R}$ has no finite accumulation point.

In order to describe the essential spectrum further, set

$$\delta := \min \sigma\big(A_0^{-1/2}DA_0^{-1/2}\big), \qquad \gamma := \max \sigma\big(A_0^{-1/2}DA_0^{-1/2}\big). \tag{4.10}$$

If H is finite-dimensional and, hence, $\sigma_{\text{ess}}\big(A_0^{-1/2}DA_0^{-1/2}\big) = \emptyset$, set

$$\delta_0 := +\infty, \qquad \gamma_0 := 0; \tag{4.11}$$

otherwise, we set

$$\delta_0 := \min \sigma_{\mathrm{ess}}\big(A_0^{-1/2}DA_0^{-1/2}\big), \qquad \gamma_0 := \max \sigma_{\mathrm{ess}}\big(A_0^{-1/2}DA_0^{-1/2}\big). \qquad (4.12)$$

In the latter case we have $0 \leq \delta \leq \delta_0 \leq \gamma_0 \leq \gamma$.

In both cases the following statements are true:

- *if $\gamma_0 = 0$, then $\sigma_{\mathrm{ess}}(\mathscr{A}) = \emptyset$;*

- *if $\gamma_0 > 0$ and $\delta_0 = 0$, set $\delta_1 := \inf\big(\sigma_{\mathrm{ess}}\big(A_0^{-1/2}DA_0^{-1/2}\big) \setminus \{0\}\big)$. Then*

$$\inf \sigma_{\mathrm{ess}}(\mathscr{A}) = \begin{cases} -\infty & \text{if } \delta_1 = 0, \\ -\dfrac{1}{\delta_1} & \text{if } \delta_1 > 0, \end{cases}$$

$$\max \sigma_{\mathrm{ess}}(\mathscr{A}) = -\frac{1}{\gamma_0} \, ;$$

- *if $\delta_0 > 0$, then*

$$\min \sigma_{\mathrm{ess}}(\mathscr{A}) = -\frac{1}{\delta_0} \quad \text{and} \quad \max \sigma_{\mathrm{ess}}(\mathscr{A}) = -\frac{1}{\gamma_0} \, .$$

Proof. The equality in (4.7) was proved in [12, Theorem 4.1]. For $\lambda \in \mathbb{C}$ the relations

$$\ker\big(A_0^{-1/2}DA_0^{-1/2} - \lambda\big) = A_0^{1/2}\Big(\ker\big(A_0^{-1}D - \lambda\big)\Big)$$

$$\operatorname{ran}\big(A_0^{-1/2}DA_0^{-1/2} - \lambda\big) = A_0^{1/2}\Big(\operatorname{ran}\big(A_0^{-1}D - \lambda\big)\Big)$$

and the fact that $A_0^{1/2}$ is an isomorphism from $H_{\frac{1}{2}}$ onto H imply (4.6) and (4.8). The inclusion in (4.9) follows from assumption **(A2)**. The discreteness of the spectrum in $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathscr{A})$ follows from Fredholm theory and the fact that $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathscr{A})$ is a connected set and has non-empty intersection with $\rho(\mathscr{A})$, namely $0 \in \rho(\mathscr{A}) \cap (\mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathscr{A}))$ by (4.5). By [12, Corollary 5.2] no point from $\sigma_{\mathrm{ess}}(\mathscr{A})$ is an accumulation point of the non-real spectrum of $\mathscr{A}$, which shows that the non-real spectrum has no finite accumulation point. The remaining assertions are clear. $\qquad \square$

Note that, although A_0^{-1} is compact, the operator $\mathscr{A}^{-1}$ is in general not a compact operator in $\mathscr{H}$. In fact, $\mathscr{A}^{-1}$ is compact if and only if the operator D is compact as an operator acting from $H_{\frac{1}{2}}$ into $H_{-\frac{1}{2}}$; see [17, Lemma 3.2].

5   A quadratic operator polynomial

In the following we construct a quadratic operator polynomial $T(\lambda)$ that is connected with the operator $\mathscr{A}$. Throughout this section we assume that **(A1)**–**(A3)** from Section 4

are satisfied. We introduce two sesquilinear forms $\mathfrak{a}_0$ and $\mathfrak{d}$ with domains $\mathscr{D}(\mathfrak{a}_0) = \mathscr{D}(\mathfrak{d}) = H_{\frac{1}{2}}$, defined via

$$\mathfrak{a}_0[x,y] := \langle A_0 x, y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}},$$
$$\mathfrak{d}[x,y] := \langle Dx, y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}, \qquad x, y \in H_{\frac{1}{2}}.$$

By **(A1)**, we have $\mathfrak{a}_0[x,y] = \langle A_0^{1/2} x, A_0^{1/2} y \rangle$ and, hence, (see, e.g. [15, Example VI.1.13]) the form $\mathfrak{a}_0$ is a closed, symmetric, non-negative form in H. By **(A2)**, the form $\mathfrak{d}$ is symmetric (see (4.2)) and non-negative. Obviously, both forms are densely defined in H. Observe that, in general, the form $\mathfrak{d}$ is not a closed form in H. The corresponding quadratic forms are denoted by $\mathfrak{a}_0[x] := \mathfrak{a}_0[x,x]$ and $\mathfrak{d}[x] := \mathfrak{d}[x,x]$, $x \in H_{\frac{1}{2}}$. It follows from the definition of γ and δ that

$$\delta \mathfrak{a}_0[x] \leq \mathfrak{d}[x] \leq \gamma \mathfrak{a}_0[x], \qquad x \in H_{\frac{1}{2}}; \tag{5.1}$$

e.g. the second inequality is seen as follows:

$$\mathfrak{d}[x] = \langle Dx, x \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle A_0^{-1/2} D A_0^{-1/2} A_0^{1/2} x, A_0^{1/2} x \rangle$$

$$\leq \gamma \langle A_0^{1/2} x, A_0^{1/2} x \rangle = \gamma \mathfrak{a}_0[x].$$

Next we show that the form $\lambda \mathfrak{d}$ is relatively bounded with respect to the form $\mathfrak{a}_0$ with $\mathfrak{a}_0$-bound less than 1 if $|\lambda|$ is small enough. Set

$$\Phi_{\gamma_0} := \begin{cases} \left\{ z \in \mathbb{C} \,\middle|\, |z| < \dfrac{1}{\gamma_0} \right\} & \text{if } \gamma_0 \neq 0, \\ \mathbb{C} & \text{if } \gamma_0 = 0. \end{cases}$$

Lemma 5.1. *For $\lambda \in \Phi_{\gamma_0}$ the form $\lambda \mathfrak{d}$ is relatively bounded with respect to $\mathfrak{a}_0$ with $\mathfrak{a}_0$-bound less than* 1.

Proof. Obviously, for $\lambda = 0$ the assertion of Lemma 5.1 is true. Let $\lambda \in \Phi_{\gamma_0} \setminus \{0\}$ and choose $\gamma' \in \mathbb{R}$ such that $\gamma_0 < \gamma' < \frac{1}{|\lambda|}$. Denote by E the spectral function in H corresponding to the bounded selfadjoint operator $S := A_0^{-1/2} D A_0^{-1/2}$. Then, for $x \in H_{\frac{1}{2}}$, we have

$$|\mathfrak{d}[x]| = \langle Dx, x \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle A_0^{-1/2} Dx, A_0^{1/2} x \rangle = \langle S A_0^{1/2} x, A_0^{1/2} x \rangle$$

$$= \langle S E([0, \gamma']) A_0^{1/2} x, E([0, \gamma']) A_0^{1/2} x \rangle$$

$$+ \langle S E((\gamma', \infty)) A_0^{1/2} x, E((\gamma', \infty)) A_0^{1/2} x \rangle$$

$$\leq \gamma' \| E([0, \gamma']) A_0^{1/2} x \|^2 + \| S^{1/2} E((\gamma', \infty)) A_0^{1/2} x \|^2$$

$$\leq \gamma' \| A_0^{1/2} x \|^2 + \| S^{1/2} E((\gamma', \infty)) A_0^{1/2} x \|^2.$$

By the definition of γ_0 and the fact that $\gamma' > \gamma_0$ it follows that $E((\gamma',\infty))$ is a finite rank projection. Choose $\varepsilon > 0$ such that $|\lambda|(\gamma' + \varepsilon) < 1$, which is possible because $\gamma' < \frac{1}{|\lambda|}$. Then Lemma 2.2 applied to the finite rank operator $S^{1/2}E((\gamma',\infty))$ implies that there exists a $C \geq 0$ such that

$$\left|\lambda\mathfrak{d}[x]\right| \leq |\lambda|\gamma'\|A_0^{1/2}x\|^2 + |\lambda|\left(\varepsilon\|A_0^{1/2}x\|^2 + C\|x\|^2\right)$$

$$= |\lambda|(\gamma' + \varepsilon)\mathfrak{a}_0[x] + |\lambda|C\|x\|^2,$$

which shows that $\lambda\mathfrak{d}$ is $\mathfrak{a}_0$-bounded with $\mathfrak{a}_0$-bound less than 1. $\qquad\square$

For $\lambda \in \mathbb{C}$ we define the sesquilinear form $\mathfrak{t}(\lambda)$ with domain $\mathscr{D}(\mathfrak{t}(\lambda)) = H_{\frac{1}{2}}$ by

$$\begin{aligned}\mathfrak{t}(\lambda)[x,y] &:= \langle \lambda^2 x + \lambda Dx + A_0 x, y\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \\ &= \lambda^2\langle x,y\rangle + \lambda\langle Dx,y\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + \langle A_0 x,y\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}, \qquad x,y \in H_{\frac{1}{2}}, \quad (5.2)\end{aligned}$$

and the corresponding quadratic form by $\mathfrak{t}(\lambda)[x] := \mathfrak{t}(\lambda)[x,x]$ for $x \in H_{\frac{1}{2}}$.

Proposition 5.2. *For $\lambda \in \Phi_{\gamma_0}$ the form $\mathfrak{t}(\lambda)$ with domain $\mathscr{D}(\mathfrak{t}(\lambda)) = H_{\frac{1}{2}}$ is a closed sectorial form in H. The m-sectorial operator $T(\lambda)$ in H that is associated with $\mathfrak{t}(\lambda)$ is given by*

$$\mathscr{D}(T(\lambda)) = \left\{ x \in H_{\frac{1}{2}} \mid \lambda Dx + A_0 x \in H \right\}$$

and, for $x \in \mathscr{D}(T(\lambda))$,

$$T(\lambda)x = \lambda^2 x + \lambda Dx + A_0 x.$$

The family $T(\lambda)$, $\lambda \in \Phi_{\gamma_0}$, of m-sectorial operators is a holomorphic family of type (B), which satisfies $T(\overline{\lambda}) = T(\lambda)^*$ for $\lambda \in \Phi_{\gamma_0}$. For $\lambda \in \Phi_{\gamma_0} \cap \mathbb{R}$ the operators $T(\lambda)$ are self-adjoint and bounded from below.

Proof. Since $\mathfrak{a}_0$ is a closed symmetric non-negative form and, by Lemma 5.1, $\lambda\mathfrak{d}$ is bounded with respect to $\mathfrak{a}_0$ with $\mathfrak{a}_0$-bound less than 1, [15, Theorem VI.1.33] implies that $\mathfrak{t}(\lambda)$ is closed and sectorial for $\lambda \in \Phi_{\gamma_0}$. Hence by [15, Theorem VI.2.1] there exist m-sectorial operators $T(\lambda)$ that represent the forms $\mathfrak{t}(\lambda)$. The form of the domain and the action of $T(\lambda)$ follow from Lemma 2.1. The domain of $\mathfrak{t}(\lambda)$ is independent of λ, and the analyticity of $\lambda \mapsto \mathfrak{t}(\lambda)[x]$ is clear. Hence T is a holomorphic family of type (B). Since $\mathfrak{t}(\overline{\lambda})[x,y] = \overline{\mathfrak{t}(\lambda)[y,x]}$, we have $T(\overline{\lambda}) = T(\lambda)^*$; see [15, Theorem VI.2.5]. From this we obtain also the self-adjointness of $T(\lambda)$ for $\lambda \in \Phi_{\gamma_0} \cap \mathbb{R}$; moreover, $T(\lambda)$ is bounded from below in this case since it is m-sectorial. $\qquad\square$

Next we show that on Φ_{γ_0} the spectral problems for $\mathscr{A}$ and T are equivalent.

Proposition 5.3. *Consider T as a function defined on Φ_{γ_0}. On Φ_{γ_0} the spectra and point spectra of $\mathscr{A}$ and T coincide, i.e.*

$$\sigma_p(\mathscr{A}) \cap \Phi_{\gamma_0} = \sigma(\mathscr{A}) \cap \Phi_{\gamma_0} = \sigma(T) = \sigma_p(T). \qquad (5.3)$$

For $\lambda_0 \in \sigma_p(\mathscr{A}) \cap \Phi_{\gamma_0}$ the geometric multiplicities coincide:

$$\dim \ker(\mathscr{A} - \lambda_0) = \dim \ker T(\lambda_0). \tag{5.4}$$

Moreover,

$$\sigma_{\mathrm{ess}}(T) = \emptyset.$$

If $\gamma_0 \neq 0$, then there are at most finitely many eigenvalues of $\mathscr{A}$ (and, hence, of T) in $\Phi_{\gamma_0} \setminus \mathbb{R}$.

Proof. First we show equality of the point spectra of $\mathscr{A}$ and T. For this, let $\lambda \in \Phi_{\gamma_0}$ and assume that $0 \in \sigma_p(T(\lambda))$. Then there exists $x \in \mathscr{D}(T(\lambda)) \setminus \{0\}$ with $\lambda^2 x + \lambda Dx + A_0 x = 0$. Therefore $\begin{pmatrix} x \\ \lambda x \end{pmatrix} \in \mathscr{D}(\mathscr{A})$ and

$$(\mathscr{A} - \lambda)\begin{pmatrix} x \\ \lambda x \end{pmatrix} = 0.$$

Conversely, if $\lambda \in \sigma_p(\mathscr{A})$ and if $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathscr{D}(\mathscr{A})$ is a corresponding eigenvector, one concludes that

$$y = \lambda x \quad \text{and} \quad A_0 x + Dy + \lambda y = 0. \tag{5.5}$$

Hence $x \in \mathscr{D}(T(\lambda))$ and $T(\lambda)x = 0$ with $x \neq 0$ because otherwise, $\begin{pmatrix} x \\ y \end{pmatrix} = 0$. Therefore the point spectra of $\mathscr{A}$ and T coincide in Φ_{γ_0}. Moreover, as the first component of an eigenvector $\begin{pmatrix} x \\ \lambda x \end{pmatrix} \in \mathscr{D}(\mathscr{A})$ of $\mathscr{A}$ satisfies $x \in \mathscr{D}(T(\lambda))$ and $T(\lambda)x = 0$ and vice versa, the statement on the geometric multiplicities follows.

Next assume that $\lambda \in \rho(\mathscr{A}) \cap \Phi_{\gamma_0}$. Then for $g \in H$ there exists $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathscr{D}(\mathscr{A})$ with

$$(\mathscr{A} - \lambda)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

From this one concludes that

$$y = \lambda x \quad \text{and} \quad A_0 x + Dy + \lambda y = g,$$

which shows that $x \in \mathscr{D}(T(\lambda))$ and $T(\lambda)x = g$. Hence $T(\lambda)$ is surjective and, by the already proved statement about the eigenvalues, $\lambda \in \rho(T)$. Proposition 4.2 implies that $\sigma_{\mathrm{ess}}(\mathscr{A}) \cap \Phi_{\gamma_0} = \emptyset$ which, together with $0 \in \rho(\mathscr{A})$ (see Theorem 4.1), gives the first equality in (5.3). Hence each point λ in Φ_{γ_0} is either an eigenvalue of $\mathscr{A}$ and, hence, of T, or belongs to the resolvent set of $\mathscr{A}$ and hence of T. This proves (5.3).

We show the statement about the essential spectrum of T. Let $\lambda \in \Phi_{\gamma_0}$. The statement is obvious for finite-dimensional H; hence let H be infinite-dimensional. By Lemma 5.1 there exist constants a, b such that $a \geq 0$, $0 \leq b < 1$ and

$$\left| \lambda \mathfrak{d}[x] \right| \leq a\|x\|^2 + b\mathfrak{a}_0[x], \quad x \in H_{\frac{1}{2}}.$$

Denote by L the spectral subspace for A_0 corresponding to the interval $\left[0, \frac{|\lambda|^2 + a}{1-b} + 1\right]$. Assume that $0 \in \sigma_{\mathrm{ess}}(T(\lambda))$. It follows from Proposition 4.2 and the definition of Φ_{γ_0} that $\lambda \notin \sigma_{\mathrm{ess}}(\mathscr{A})$ and $\overline{\lambda} \notin \sigma_{\mathrm{ess}}(\mathscr{A})$. Hence (5.4) implies that $\dim \ker T(\lambda) < \infty$ and

$$\dim \left(\operatorname{ran} T(\lambda)\right)^{\perp} = \dim \left(\ker T(\lambda)^*\right) = \dim \ker T(\overline{\lambda}) < \infty.$$

By [8, Theorem IX.1.3] there exists a singular sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \mathscr{D}(T(\lambda))$, $\|x_n\| = 1$, $x_n \rightharpoonup 0$ (i.e. x_n converges to 0 weakly) and $T(\lambda)x_n \to 0$ as $n \to \infty$. We decompose x_n as follows:

$$x_n = u_n + v_n, \qquad u_n \in L, \ v_n \perp L.$$

The projection onto L is weakly continuous and L is finite-dimensional by assumption **(A3)**; therefore the sequence $(u_n)_{n\in\mathbb{N}}$ converges strongly in H to 0, $A_0 u_n \to 0$ and $\|v_n\| \to 1$ as $n \to \infty$. We obtain

$$\left| \langle T(\lambda)x_n, x_n \rangle \right| = \left| \mathfrak{t}(\lambda)[x_n] \right| = \left| \lambda^2 + \lambda \mathfrak{d}[x_n] + \mathfrak{a}_0[x_n] \right|$$

$$\geq \mathfrak{a}_0[x_n] - \left(|\lambda^2| + |\lambda \mathfrak{d}[x_n]| \right) \geq (1-b)\mathfrak{a}_0[x_n] - \left(|\lambda^2| + a \right)$$

$$= (1-b)\left(\mathfrak{a}_0[u_n] + \mathfrak{a}_0[v_n] - \frac{|\lambda|^2 + a}{1-b} \right).$$

As $n \to \infty$, we have $\mathfrak{a}_0[u_n] \to 0$, and $\mathfrak{a}_0[v_n] \geq \left(\frac{|\lambda|^2 + a}{1-b} + 1 \right)\|v_n\|^2$ holds for every $n \in \mathbb{N}$. Hence

$$\liminf_{n\to\infty} |\langle T(\lambda)x_n, x_n \rangle| \geq (1-b) > 0,$$

which is a contradiction. Therefore $0 \notin \sigma_{\mathrm{ess}}(T(\lambda))$.

Finally, assume that $\gamma_0 > 0$. Suppose that there are infinitely many eigenvalues of $\mathscr{A}$ in $\Phi_{\gamma_0} \setminus \mathbb{R}$. Since $\Phi_{\gamma_0} \setminus \mathbb{R}$ is a bounded set, there exists a sequence of non-real eigenvalues of $\mathscr{A}$ which converges. However, this contradicts Proposition 4.2. Hence the last statement is proved. $\qquad\square$

In the following we prove variational principles for real eigenvalues of $\mathscr{A}$ or, what is equivalent (see Proposition 5.3), of T. To this end we introduce functionals p_+ and p_- which will serve as generalized Rayleigh functionals for T on appropriate intervals. For fixed $x \in H_{\frac{1}{2}} \setminus \{0\}$ consider the equation

$$\mathfrak{t}(\lambda)[x] = \lambda^2\|x\|^2 + \lambda\langle Dx, x\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \langle A_0 x, x\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = 0 \qquad (5.6)$$

as an equation in λ.

Definition 5.4. If (5.6) for $x \in H_{\frac{1}{2}} \setminus \{0\}$ does not have a real solution, then we set

$$p_+(x) := -\infty, \qquad p_-(x) := +\infty.$$

Otherwise, we denote the solutions of (5.6) by $p_\pm(x)$,

$$p_\pm(x) := \frac{-\langle Dx, x\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} \pm \sqrt{\langle Dx, x\rangle^2_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} - 4\|x\|^2\|A_0^{1/2}x\|^2}}{2\|x\|^2}$$

$$= \frac{-\mathfrak{d}[x] \pm \sqrt{\left(\mathfrak{d}[x]\right)^2 - 4\|x\|^2\mathfrak{a}_0[x]}}{2\|x\|^2}.$$

(5.7)

Note that the values of $p_+(x)$ and $p_-(x)$ belong to $(-\infty, 0) \cup \{\pm\infty\}$. Set

$$\mathscr{D}^* := \left\{ x \in H_{\frac{1}{2}} \setminus \{0\} \mid \exists \lambda \in \mathbb{R} \text{ such that } \mathfrak{t}(\lambda)[x] = 0 \right\}$$

$$= \left\{ x \in H_{\frac{1}{2}} \setminus \{0\} \mid p_\pm(x) \text{ are finite} \right\}$$

$$= \left\{ x \in H_{\frac{1}{2}} \setminus \{0\} \mid \langle Dx, x \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq 2\|x\| \, \|A_0^{1/2} x\| \right\} \tag{5.8}$$

and define

$$\alpha := \begin{cases} \max\left\{ \sup_{x \in \mathscr{D}^*} p_-(x), -\dfrac{1}{\gamma_0} \right\} & \text{if } \gamma_0 > 0, \\[2ex] \sup_{x \in \mathscr{D}^*} p_-(x) & \text{if } \gamma_0 = 0; \end{cases} \tag{5.9}$$

where we set $\sup_{x \in \mathscr{D}^*} p_-(x) = -\infty$ if $\mathscr{D}^* = \emptyset$.

We collect some of the properties of p_+, p_- and α in the following lemma. Note that $\gamma > 0$ if and only if $D \neq 0$.

Lemma 5.5. *Assume that $D \neq 0$. Then*

$$p_\pm(x) < -\frac{1}{\gamma} \qquad \text{for } x \in \mathscr{D}^*,$$

and hence

$$\alpha \leq -\frac{1}{\gamma}.$$

Proof. The assumption $D \neq 0$ implies that $\gamma > 0$. Let $x \in \mathscr{D}^*$. It follows from (5.1) that for $\lambda \in \left[-\frac{1}{\gamma}, \infty \right) \setminus \{0\}$,

$$\mathfrak{t}(\lambda)[x] = \lambda^2 \|x\|^2 + \lambda \mathfrak{d}[x] + \mathfrak{a}_0[x]$$

$$\geq \lambda^2 \|x\|^2 - \frac{\mathfrak{d}[x]}{\gamma} + \mathfrak{a}_0[x] \geq \lambda^2 \|x\|^2 > 0.$$

Since $\mathfrak{t}(0)[x] = \mathfrak{a}_0[x] > 0$, we therefore have $\mathfrak{t}(\lambda)[x] > 0$ for all $\lambda \in \left[-\frac{1}{\gamma}, \infty \right)$. This implies that $p_\pm(x) < -\frac{1}{\gamma}$. The statement on α follows from this and the inequality $-\frac{1}{\gamma_0} \leq -\frac{1}{\gamma}$. $\qquad\square$

In the next proposition we discuss when the set $\mathscr{D}^*$ is non-empty. Note that (i) in the following proposition contains a slight improvement of the fifth assertion in [13, Theorem 3.2].

Proposition 5.6. *For the set $\mathscr{D}^*$ we have the following implications.*

(i) *If*

$$A_0^{-1/2} D A_0^{-1/2} < 2 A_0^{-1/2}, \tag{5.10}$$

where the inequality is understood as a relation between two self-adjoint operators in the Hilbert space H (i.e. $\langle A_0^{-1/2}DA_0^{-1/2}x,x\rangle < 2\langle A_0^{-1/2}x,x\rangle$ for all $x \in H \setminus \{0\}$), then

$$\mathscr{D}^* = \emptyset \qquad \text{and we have} \quad \sigma_p(\mathscr{A}) \cap \mathbb{R} = \emptyset.$$

(ii) *If*

$$\left\| A_0^{-1/2}DA_0^{-1/2} \right\| > 2\left\| A_0^{-1/2} \right\|, \tag{5.11}$$

where the norms are the operator norm in the Hilbert space H, then

$$\mathscr{D}^* \neq \emptyset.$$

Proof. (i) Let $x \in H_{\frac{1}{2}} \setminus \{0\}$ be arbitrary and set $y := A_0^{1/2}x$. From the assumption (5.10) we obtain that

$$\left\langle A_0^{-1/2}DA_0^{-1/2}y,y \right\rangle < 2\langle A_0^{-1/2}y,y\rangle \leq 2\|y\|\, \|A_0^{-1/2}y\|,$$

which implies

$$\langle Dx,x\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} < 2\left\| A_0^{1/2}x \right\| \|x\|.$$

Together with (5.7) we can conclude that (5.6) has no real solution. Hence $\mathscr{D}^* = \emptyset$. To prove the last statement in (i), let λ be a real eigenvalue of $\mathscr{A}$ with corresponding eigenvector $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \in \mathscr{D}(\mathscr{A})$. Then

$$A_0 x + \lambda Dx + \lambda^2 x = 0,$$

by (5.5), which implies that $\mathfrak{t}(\lambda)[x] = 0$. The latter is not possible since $\mathscr{D}^* = \emptyset$.

(ii) The number $\|A_0^{-1/2}DA_0^{-1/2}\|$ is an element of the closure of the numerical range of the self-adjoint operator $A_0^{-1/2}DA_0^{-1/2}$. Therefore, there exists a sequence (y_n) in H with $\|y_n\| = 1$ such that

$$\left\langle A_0^{-1/2}DA_0^{-1/2}y_n,y_n \right\rangle \to \|A_0^{-1/2}DA_0^{-1/2}\| \qquad \text{as } n \to \infty.$$

Assumption (5.11) implies that $\langle A_0^{-1/2}DA_0^{-1/2}y_{n_0},y_{n_0}\rangle > 2\|A_0^{-1/2}\|$ for some $n_0 \in \mathbb{N}$. Set $x := A_0^{-1/2}y_{n_0}$; then

$$\langle Dx,x\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \left\langle A_0^{-1/2}DA_0^{-1/2}y_{n_0},y_{n_0} \right\rangle > 2\|A_0^{-1/2}\|$$

$$\geq 2\|A_0^{-1/2}y_{n_0}\| = 2\|x\|\,\|A_0^{1/2}x\|.$$

Now we obtain from (5.8) that $x \in \mathscr{D}^*$; hence $\mathscr{D}^* \neq \emptyset$. $\qquad\square$

The following theorem is one of the main results of this paper. Recall that an eigenvalue is called semi-simple if the algebraic and geometric multiplicities coincide, i.e. if there are no Jordan chains.

Theorem 5.7. *Assume that* **(A1)–(A3)** *are satisfied. Let Δ be an interval with $\Delta \subset (\alpha, 0]$ and $\max \Delta = 0$. Then the set $\sigma(\mathscr{A}) \cap \Delta$ is either empty or consists only of a finite or infinite sequence of isolated semi-simple eigenvalues of finite multiplicity of $\mathscr{A}$. The case of infinitely many eigenvalues in $\sigma(\mathscr{A}) \cap \Delta$ occurs only if $\alpha = -\frac{1}{\gamma_0} = \inf \Delta$ and, in this case, the eigenvalues accumulate only at $-\frac{1}{\gamma_0}$.*

If $\sigma(\mathscr{A}) \cap \Delta$ is empty, then set $N := 0$; otherwise, denote the eigenvalues of $\mathscr{A}$ in Δ by $(\lambda_j)_{j=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, in non-increasing order, counted according to their multiplicities: $\lambda_1 \geq \lambda_2 \geq \cdots$. Then the nth eigenvalue λ_n, $n \in \mathbb{N}$, $n \leq N$, satisfies

$$\lambda_n = \max_{\substack{L \subset H_{1/2} \\ \dim L = n}} \min_{x \in L \setminus \{0\}} p_+(x) = \min_{\substack{L \subset H \\ \dim L = n-1}} \sup_{\substack{x \in H_{1/2} \setminus \{0\} \\ x \perp L}} p_+(x). \tag{5.12}$$

If $N < \infty$, then

$$\begin{aligned} \sup_{\substack{L \subset \mathscr{D} \\ \dim L = n}} \min_{x \in L \setminus \{0\}} p_+(x) &\leq \inf \Delta \\ \inf_{\substack{L \subset H \\ \dim L = n-1}} \sup_{\substack{x \in \mathscr{D} \setminus \{0\} \\ x \perp L}} p_+(x) &\leq \inf \Delta \end{aligned} \quad \text{for } n > N \text{ with } n \leq \dim H. \tag{5.13}$$

Proof. Except for the semi-simplicity, the first part of Theorem 5.7 follows from Proposition 4.2. In order to apply Theorem 3.5, we consider the operator function T defined on $\Omega := \Phi_{\gamma_0}$. Assumption (I) in Section 3 is satisfied because of Proposition 5.2 and because $T(0) = A_0$ is a positive definite operator in H. Next we show that $(\nearrow)$ is satisfied. For $x \in H_{\frac{1}{2}} \setminus \{0\}$, the function $\lambda \mapsto \mathfrak{t}(\lambda)[x]$ is increasing at value zero on Δ because it is convex and a zero in $(\alpha, 0]$ is the greater one of the two zeros of that function by the definition of α (note that a double-zero cannot lie in $(\alpha, 0]$). Hence $(\nearrow)$ is satisfied. Moreover, p_+ satisfies (3.1) in both cases $x \in \mathscr{D}^*$ and $x \notin \mathscr{D}^*$ by the definition of p_+. Therefore,

$$p(x) := p_+(x), \qquad x \in H_{\frac{1}{2}},$$

is a generalized Rayleigh functional for T on Δ, cf. Definition 3.1

By Proposition 5.3 the eigenvalues and their geometric multiplicities of T and $\mathscr{A}$ coincides in Δ and the interval Δ' in Theorem 3.5 equals now Δ. The quantity κ in Theorem 3.5 is determined as

$$\kappa = \dim \mathscr{L}_{(-\infty, 0)}(T(0)) = \dim \mathscr{L}_{(-\infty, 0)}(A_0) = 0.$$

Now the formulae in (5.12) and in (5.13) follow from (3.8), (3.9), Remark 3.6 and Proposition 5.3.

Let us finally show that the eigenvalues of $\mathscr{A}$ in $(\alpha, 0)$ are semi-simple. Assume that $\lambda \in (\alpha, 0)$ is an eigenvalue that has a Jordan chain, i.e. there exist vectors $\binom{x_0}{y_0}$, $\binom{x_1}{y_1} \in \mathscr{D}(\mathscr{A})$, both being non-zero, such that

$$(\mathscr{A} - \lambda) \binom{x_0}{y_0} = 0, \qquad (\mathscr{A} - \lambda) \binom{x_1}{y_1} = \binom{x_0}{y_0}.$$

It follows that $y_0 = \lambda x_0$ and $x_0 \neq 0$. Moreover, we have $x_0 \in \mathscr{D}(T(\lambda))$ and $T(\lambda)x_0 = 0$, cf. (5.5). From the second equation it follows that

$$y_1 = x_0 + \lambda x_1 \qquad \text{and} \qquad A_0 x_1 + D y_1 + \lambda y_1 = -\lambda x_0.$$

Substituting for y_1 we obtain

$$-\left(\lambda^2 + \lambda D + A_0\right) x_1 = (2\lambda + D)x_0$$

and hence, by (5.2) and the symmetry of $\mathfrak{t}(\lambda)$ for real λ,

$$\langle (2\lambda + D)x_0, x_0 \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = -\mathfrak{t}(\lambda)[x_1, x_0] = -\overline{\mathfrak{t}(\lambda)[x_0, x_1]}$$

$$= -\overline{\langle T(\lambda)x_0, x_1 \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}} = 0,$$

where we used that $x_0 \in \ker T(\lambda)$. The left-hand side of this equation is equal to $\mathfrak{t}'(\lambda)[x_0]$, which is positive because $\lambda \in (\alpha, 0)$ and there is no double-zero of $\lambda \mapsto \mathfrak{t}(\lambda)[x_0]$ in $(\alpha, 0]$. This is a contradiction and hence λ is semi-simple. $\qquad \square$

The next proposition provides a sufficient condition for the existence of eigenvalues in the interval $\left(-\frac{1}{\gamma_0}, 0\right)$.

Proposition 5.8. *Assume that* **(A1)–(A3)** *are satisfied and that* $\gamma_0 > 0$. *If*

$$\sigma\left(A_0^{-1/2} D A_0^{-1/2} - \frac{1}{\gamma_0} A_0^{-1}\right) \cap (\gamma_0, \infty) \neq \emptyset, \tag{5.14}$$

then

$$\sigma(\mathscr{A}) \cap \left(-\frac{1}{\gamma_0}, 0\right) \neq \emptyset. \tag{5.15}$$

Proof. Define the following operator function

$$R(\lambda) := A_0^{-1/2} D A_0^{-1/2} + \lambda A_0^{-1} + \frac{1}{\lambda} I, \qquad \lambda \in \mathbb{R} \setminus \{0\},$$

whose values are bounded operators in H. Assumption (5.14) implies that

$$\max \sigma\left(R\left(-\frac{1}{\gamma_0}\right)\right) = \max \sigma\left(A_0^{-1/2} D A_0^{-1/2} - \frac{1}{\gamma_0} A_0^{-1} - \gamma_0 I\right) > 0.$$

On the other hand, for $\lambda < 0$,

$$\max \sigma\left(R(\lambda)\right) \leq \gamma + \frac{1}{\lambda} \to -\infty \qquad \text{as } \lambda \to 0-.$$

Since $\max \sigma(R(\lambda))$ is continuous in λ (see, e.g. [15, Theorem V.4.10]), there exists a $\lambda_0 \in \left(-\frac{1}{\gamma_0}, 0\right)$ such that $\max \sigma(R(\lambda_0)) = 0$. The compactness of A_0^{-1} implies that

$$\max \sigma_{\mathrm{ess}}\left(R(\lambda_0)\right) = \gamma_0 + \frac{1}{\lambda_0} < 0.$$

Hence $0 \in \sigma_{\mathrm{p}}(R(\lambda_0))$, i.e. there exists a $y \in H \setminus \{0\}$ such that

$$A_0^{-1/2} D A_0^{-1/2} y + \lambda_0 A_0^{-1} y + \frac{1}{\lambda_0} y = 0.$$

Applying $A_0^{1/2}$ to both sides, multiplying by λ_0 and setting $x := A_0^{-1/2} y$ we obtain that

$$\lambda_0^2 x + \lambda_0 D x + A_0 x = 0.$$

This, together with (5.3), implies (5.15). $\qquad \square$

The converse of Proposition 5.8 is not true, i.e. (5.15) does not imply (5.14). This can be seen from the following example. Let $H = \ell^2$ and define the operators A_0 and D by

$$(A_0 x)_n = n x_n, \qquad (Dx)_n = \begin{cases} 2x_1, & n = 1, \\ \dfrac{n}{2} x_n, & n \geq 2, \end{cases}$$

where $x = (x_n)_{n=1}^\infty$. Then $\gamma_0 = \frac{1}{2}$,

$$\sigma\left(A_0^{-1/2} D A_0^{-1/2} - \frac{1}{\gamma_0} A_0^{-1}\right) = \left\{0, \frac{1}{2}\right\} \cup \left\{\frac{1}{2} - \frac{2}{n} \mid n \in \mathbb{N}, n \geq 2\right\},$$

which is disjoint from (γ_0, ∞). However, -1 is an eigenvalue of T with eigenvector $(1, 0, 0, \dots)$.

With the help of the form $\mathfrak{t}(\lambda)$ it is shown in the following proposition that a certain triangle belongs to the resolvent set of $\mathscr{A}$. We refer to [14, Theorem 3.2] for a full disc around zero with radius slightly less than $\frac{1}{\gamma}$ which is contained in $\rho(\mathscr{A})$.

Proposition 5.9. *Assume that $D \neq 0$. Then*

$$\left\{z \in \mathbb{C} \;\middle|\; -\frac{1}{\gamma} \leq \operatorname{Re} z < 0, \; \arg z \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)\right\} \setminus \left\{-\frac{1}{\gamma}\right\} \subset \rho(\mathscr{A}) \qquad (5.16)$$

where γ is defined in (4.10). If, in addition, $\gamma \neq \gamma_0$, then also $-\frac{1}{\gamma} \in \rho(\mathscr{A})$.

Proof. Since the set on the left-hand side of (5.16) is disjoint from the essential spectrum of $\mathscr{A}$ by Proposition 4.2, it is, according to Proposition 5.3, sufficient to show that the left-hand side contains no eigenvalues of T.

Let λ be in the set on the left-hand side of (5.16) or $\lambda = -\frac{1}{\gamma}$, and let $x \in \mathscr{D}(T(\lambda))$ with $\|x\| = 1$. Then $\operatorname{Re}(\lambda^2) > 0$ and hence, by (5.1),

$$\left|\langle T(\lambda) x, x \rangle\right| = \left|\mathfrak{t}(\lambda)[x]\right| \geq \operatorname{Re} \mathfrak{t}(\lambda)[x] = \mathfrak{a}_0[x] + \operatorname{Re}\left(\lambda \mathfrak{d}[x]\right) + \operatorname{Re} \lambda^2$$

$$> \mathfrak{a}_0[x] + (\operatorname{Re}\lambda)\mathfrak{d}[x] \geq (1 + \gamma \operatorname{Re}\lambda)\mathfrak{a}_0[x] \geq 0,$$

which shows that 0 is not an eigenvalue of $T(\lambda)$. $\qquad \square$

One can easily construct examples with eigenvalues λ of $\mathscr{A}$ satisfying $\arg\lambda \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ and $|\operatorname{Re}\lambda| < \frac{1}{\gamma}$. For example, let A_0 be a positive definite operator with compact resolvent and smallest eigenvalue $1/2$. For the choice $D = A_0$, we have $\gamma = 1$ and $\mathscr{A}$ has one eigenvalue λ satisfying $\operatorname{Re}\lambda = -1/4$ and $\arg\lambda \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$.

Another application of Theorem 5.7 results in interlacing properties of eigenvalues of two different second-order problems with coefficients which satisfy a specific order relation. This is the content of the following theorem.

Theorem 5.10. *Let operators A_0, $\hat{A}_0$, D and $\hat{D}$ in the Hilbert space H be given so that A_0, D and $\hat{A}_0$, $\hat{D}$, respectively, satisfy assumptions* **(A1)–(A3)**. *Assume that $H_{\frac{1}{2}} = \hat{H}_{\frac{1}{2}}$, where $\hat{H}_{\frac{1}{2}}$ is defined as in* (2.2) *with A_0 replaced by $\hat{A}_0$ and assume that, for $x \in H_{\frac{1}{2}}$,*

$$
\begin{aligned}
\langle A_0 x, x\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} &\geq \langle \hat{A}_0 x, x\rangle_{\hat{H}_{-\frac{1}{2}} \times \hat{H}_{\frac{1}{2}}}, \\
\langle D x, x\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} &\leq \langle \hat{D} x, x\rangle_{\hat{H}_{-\frac{1}{2}} \times \hat{H}_{\frac{1}{2}}}.
\end{aligned}
\tag{5.17}
$$

Let $\hat{\mathscr{A}}$, $\hat{\delta}$, $\hat{\gamma}$, $\hat{\delta}_0$, $\hat{\gamma}_0$, $\hat{\mathfrak{t}}$, $\hat{p}_\pm$, and $\hat{\alpha}$ be defined as in (4.3)–(4.4), (4.10), (4.11), (4.12), (5.2), *and* (5.7)–(5.9), *respectively, where A_0 is replaced by $\hat{A}_0$ and D by $\hat{D}$. Then we have*

$$
\gamma \leq \hat{\gamma}, \qquad \gamma_0 \leq \hat{\gamma}_0, \qquad \delta \leq \hat{\delta}, \qquad \delta_0 \leq \hat{\delta}_0.
$$

Set

$$
\Delta := \left(\max\{\alpha, \hat{\alpha}\}, 0\right].
$$

Assume now that $\sigma(\mathscr{A}) \cap \Delta$ and $\sigma(\hat{\mathscr{A}}) \cap \Delta$ are non-empty. Let $(\lambda_n)_{n=1}^N$ and $(\hat{\lambda}_n)_{n=1}^{\hat{N}}$, $N, \hat{N} \in \mathbb{N} \cup \{\infty\}$, be the eigenvalues of $\mathscr{A}$ and $\hat{\mathscr{A}}$, respectively, in the interval Δ, both arranged in non-increasing order and counted according their multiplicities. Then

$$
\lambda_n \leq \hat{\lambda}_n \qquad \text{for } n \in \mathbb{N}, \, n \leq \inf\{N, \hat{N}\}.
$$

Proof. For finite-dimensional H the equations $\gamma_0 = \hat{\gamma}_0$ and $\delta_0 = \hat{\delta}_0$ follow by the definition of the corresponding quantities. For infinite-dimensional H the variational principle for self-adjoint operators characterizes the number γ_0 as follows:

$$
\begin{aligned}
\gamma_0 &= \max \sigma_{\mathrm{ess}}\left(A_0^{-1/2} D A_0^{-1/2}\right) \\
&= \inf_{n \in \mathbb{N}} \sup_{\substack{L \subset H \\ \dim L = n}} \inf_{x \in L \setminus \{0\}} \frac{\left\langle A_0^{-1/2} D A_0^{-1/2} x, x\right\rangle}{\|x\|^2} \\
&= \inf_{n \in \mathbb{N}} \sup_{\substack{L \subset H_{1/2} \\ \dim L = n}} \inf_{y \in L \setminus \{0\}} \frac{\langle D y, y\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\langle A_0 y, y\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}},
\end{aligned}
$$

where we made the substitution $y = A_0^{-1/2} x$. A similar formula is true for $\hat{\gamma}_0$. Now the relation $H_{\frac{1}{2}} = \hat{H}_{\frac{1}{2}}$ and the inequalities in (5.17) imply that $\gamma_0 \leq \hat{\gamma}_0$. The inequality

$\delta_0 \leq \hat{\delta}_0$ is proved analogously. The numbers γ, $\hat{\gamma}$, δ, $\hat{\delta}$ can also be characterized by similar formulae, e.g.

$$\gamma = \sup_{x \in H \setminus \{0\}} \frac{\langle A_0^{-1/2} D A_0^{-1/2} x, x \rangle}{\|x\|^2} = \sup_{y \in H_{1/2} \setminus \{0\}} \frac{\langle Dy, y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\langle A_0 y, y \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}},$$

which imply the inequalities $\gamma \leq \hat{\gamma}$ and $\delta \leq \hat{\delta}$.

The relations (5.17) imply that, for negative λ,

$$\mathfrak{t}(\lambda)[x] \geq \hat{\mathfrak{t}}(\lambda)[x], \qquad x \in H_{\frac{1}{2}}.$$

It follows that

$$p_-(x) \geq \hat{p}_-(x) \qquad \text{and} \qquad p_+(x) \leq \hat{p}_+(x), \qquad x \in H_{\frac{1}{2}} \setminus \{0\},$$

which, together with Theorem 5.7, imply the inequalities for the eigenvalues. $\qquad \square$

6   Example: beam with damping

We consider a beam of length 1 and study transverse vibrations only. Let $u(r,t)$ denote the deflection of the beam from its rigid body motion at time t and position r. We consider for the beam deflection a damping model which leads to the following description of the vibrations where $a_0 > 0$ is a real contant and $d \in C^1[0,1]$ satisfies $d(r) \geq d_0 \geq 0$ a.e:

$$\frac{\partial^2 u}{\partial t^2} + a_0 \frac{\partial^4 u}{\partial r^4} + \frac{\partial^2}{\partial t \partial r}\left[d \frac{\partial u}{\partial r} \right] = 0, \qquad r \in (0,1), t > 0. \tag{6.1}$$

Assuming that the beam is pinned, free to rotate and does not experience any torque at both ends, we have for all $t > 0$ the following boundary conditions

$$u\big|_{r=0} = u\big|_{r=1} = \frac{\partial^2 u}{\partial r^2}\bigg|_{r=0} = \frac{\partial^2 u}{\partial r^2}\bigg|_{r=1} = 0. \tag{6.2}$$

We consider the partial differential equation (6.1)–(6.2) as a second-order problem in the Hilbert space $H = L^2(0,1)$. In H we define the operator A_0 by

$$A_0 = a_0 \frac{\mathrm{d}^4}{\mathrm{d}r^4}, \quad \mathcal{D}(A_0) = \left\{ z \in H^4(0,1) \mid z(0) = z(1) = z''(0) = z''(1) = 0 \right\}.$$

It is easy to see that the operator A_0 satisfies assumptions **(A1)** and **(A3)** and that

$$H_{\frac{1}{2}} = \left\{ z \in H^2(0,1) \mid z(0) = z(1) = 0 \right\}$$

with inner product $\langle z, v \rangle_{H_{\frac{1}{2}}} = a_0 \langle z'', v'' \rangle$. The first eigenvalue μ_1 of A_0 is

$$\mu_1 = a_0 \pi^4,$$

and the operator $A_0^{1/2}$ is given by

$$A_0^{1/2} = -\sqrt{a_0}\,\frac{\mathrm{d}^2}{\mathrm{d}r^2}$$

with Dirichlet boundary conditions. Moreover, we have

$$\|z\|_{H_{\frac{1}{2}}}^2 \geq a_0\pi^4\|z\|^2 \quad \text{for } z \in H_{\frac{1}{2}}.$$

Let $x(t) = (u(\cdot,t),\dot{u}(\cdot,t))$. Then $\|x(t)\|_{H_{\frac{1}{2}}\times H}^2 = \|u''(\cdot,t)\|^2 + \|\dot{u}(\cdot,t)\|^2$ corresponds to the energy of the beam which justifies the choice of $L^2(0,1)$ as the Hilbert space for the analysis of the boundary value problem (6.1)–(6.2). We define the damping operator as

$$D := -\frac{\mathrm{d}}{\mathrm{d}r}\left[d\,\frac{\mathrm{d}}{\mathrm{d}r}\right].$$

Due to the fact that $d \in C^1[0,1]$, D is a linear bounded operator from $H_{\frac{1}{2}}$ to H. For $z \in H_{\frac{1}{2}}$ we have

$$\langle Dz,z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \langle dz',z'\rangle \geq d_0\pi^2\|z\|^2,$$

and thus the assumption **(A2)** holds as well. Since $DA_0^{-1/2}$ is a bounded operator in H and $A_0^{-1/2}$ is a compact operator in H, we see that $A_0^{-1/2}DA_0^{-1/2}$ is a compact operator in H. From this we obtain

$$\sigma_{\mathrm{ess}}\big(A_0^{-1/2}DA_0^{-1/2}\big) = \{0\}$$

and hence $\gamma_0 = \delta_0 = 0$. This, together with Proposition 4.2, yields

$$\sigma_{\mathrm{ess}}(\mathscr{A}) = \emptyset. \tag{6.3}$$

Finally, we apply the results of this paper to the damped beam equation.

Theorem 6.1. *Assume that*

$$d_0^2 \geq 4a_0 \qquad and \qquad d_0 + \sqrt{d_0^2 - 4a_0} > \|d\|_\infty. \tag{6.4}$$

Then $\mathscr{D}^* \neq \emptyset$ and

$$\alpha \leq -d_0\frac{\pi^2}{2}. \tag{6.5}$$

The set $\sigma(\mathscr{A}) \cap (\alpha,0)$ is non-empty and consists only of a finite sequence of isolated semi-simple eigenvalues of finite multiplicity of $\mathscr{A}$ counted according to their multiplicity: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ for some $N \in \mathbb{N}$. The nth eigenvalue λ_n, $1 \leq n \leq N$, satisfies (5.12) in Theorem 5.7, and the first eigenvalue λ_1 satisfies

$$\lambda_1 \geq \frac{1}{2}\left(-\|d\|_\infty\pi^2 + \pi^2\sqrt{d_0^2 - 4a_0}\right). \tag{6.6}$$

Proof. Denote by e_1 the eigenvector to the smallest eigenvalue of $A_0^{1/2}$ with $\|e_1\| = 1$, i.e. $e_1 = \sqrt{2}\sin(\pi\cdot)$ and $A_0^{1/2}e_1 = \sqrt{a_0}\pi^2 e_1$. Using (6.4) we obtain

$$\langle de_1', e_1' \rangle^2 - 4\|A_0^{1/2}e_1\|^2 \geq d_0^2\pi^4 - 4a_0\pi^4 \geq 0,$$

which by (5.8) implies that $\mathscr{D}^* \neq \emptyset$. Moreover,

$$p_+(e_1) = \frac{1}{2}\left(-\langle de_1', e_1' \rangle + \sqrt{\langle de_1', e_1' \rangle^2 - 4\|A_0^{1/2}e_1\|^2}\right)$$
$$\geq \frac{1}{2}\left(-\|d\|_\infty\pi^2 + \pi^2\sqrt{d_0^2 - 4a_0}\right). \tag{6.7}$$

For $x \in H_{\frac{1}{2}}$ we have $\langle x', x' \rangle \geq \pi^2\langle x, x \rangle$. Since $\gamma_0 = 0$, we obtain

$$\alpha = \sup_{x \in \mathscr{D}^*} p_-(x) \leq \sup_{x \in \mathscr{D}^*} -\frac{\langle Dx, x \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{2\|x\|^2} = \sup_{x \in \mathscr{D}^*} -\frac{d_0\langle x', x' \rangle}{2\|x\|^2} \leq -d_0\frac{\pi^2}{2}, \tag{6.8}$$

which proves (6.5). From (6.3) it follows that $\sigma(\mathscr{A}) \cap (\alpha, 0)$ is either empty or consists only of finitely many isolated eigenvalues of finite multiplicity. Since **(A1)–(A3)** are fulfilled, Theorem 5.7 implies that these eigenvalues are semi-simple and the nth eigenvalue λ_n, $1 \leq n \leq N$, satisfies (5.12). Relations (6.8), (6.4) and (6.7) imply that

$$\sup_{\substack{L \subset H_{1/2} \\ \dim L = 1}} \min_{x \in L\setminus\{0\}} p_+(x) = \sup_{\substack{x \in H_{1/2} \\ x \neq 0}} p_+(x) \geq p_+(e_1)$$
$$\geq \frac{1}{2}\left(-\|d\|_\infty\pi^2 + \pi^2\sqrt{d_0^2 - 4a_0}\right) > -d_0\frac{\pi^2}{2} \geq \alpha. \tag{6.9}$$

From this and (5.13) we obtain that $\sigma(\mathscr{A}) \cap (\alpha, 0) \neq \emptyset$. Moreover, λ_1 is equal to the left-hand side of (6.9), which yields (6.6). $\qquad\square$

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