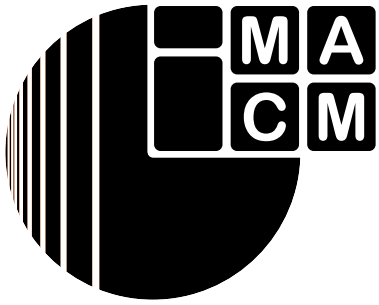

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Analysis of the Contraction Condition in the Co-Simulation of a Specific Electric Circuit

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Uncertainty Quantification in Co-Simulation for Coupled Electrical Circuits

Kai Gausling and Andreas Bartel

Abstract This paper combines co-simulation with uncertainty quantification in a numerical treatment. Our focus is mainly on the behavior of the stochastic quantities during the iterations in the co-simulation applied to an electric circuit with several uncertain parameters. For this purpose we classify the coupling structure of co-simulation model for a specific electric circuit by using standard theory. Next, we analyze the stability and rate of convergency for expectation and standard deviation of our circuit, while using the gPC expansion for the stochastic process.

1 Introduction

Co-simulation is an important method for coupled systems in time domain. Mainly, if dedicated simulation tools for the subsystems are available, then it is a relevant option. To enhance convergence, co-simulation works on certain time periods namely windows, where convergence can only be achieved by solving multiple times the subsystems. Co-simulation applied to coupled ordinary differential equations (ODEs) always convergences, see e.g. [4]. The situation is different for coupled differential-algebraic equations (DAEs). In such cases convergence can only be guaranteed if a contraction condition is fulfilled, see e.g. [1]. The theory of co-simulation shows that its convergence and stability is directly influenced by the sequence in which the subsystems are computed and also by the coupling interface, see e.g. [3].

Co-simulation operates on time windows $[T_n, T_n + H]$ and tries to compute the overall solution iteratively, by decoupled subsystems. Let (k) denote the current iteration, the old iterates are $(k-1)$. Such a co-simulation scheme can be encode by splitting functions $\mathbf{F}, \mathbf{G}$:

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$$\begin{aligned} \dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y},\mathbf{z}) \qquad \leftrightarrow \quad \dot{\tilde{\mathbf{y}}} = \mathbf{F}\left(\tilde{\mathbf{y}}^{(k)},\tilde{\mathbf{z}}^{(k)},\tilde{\mathbf{y}}^{(k-1)},\tilde{\mathbf{z}}^{(k-1)}\right) \\ 0 &= \mathbf{g}(\mathbf{y},\mathbf{z}) \qquad\quad\; 0 = \mathbf{G}\left(\tilde{\mathbf{y}}^{(k)},\tilde{\mathbf{z}}^{(k)},\tilde{\mathbf{y}}^{(k-1)},\tilde{\mathbf{z}}^{(k-1)}\right). \end{aligned} \tag{1}$$

Then the contraction condition reads:

$$\alpha := \|\mathbf{G}_{\tilde{z}^{(k)}}^{-1}\mathbf{G}_{\tilde{z}^{(k-1)}}\|_2 < 1, \tag{2}$$

where $\mathbf{G}_{\mathbf{z}^{(k)}}$, $\mathbf{G}_{\mathbf{z}^{(k-1)}}$ denote partials Jacobians of $\mathbf{G}$, see e.g. [1], [2]. It is still an open question how uncertainties in coupled systems change the contraction properties. In general, the contraction factor α may depends on components from the model. In such cases contraction depends on the balance between several parameters. Consequently, dealing with uncertain components in the co-simulation may change the contraction condition (2), that is, α will become stochastic.

Our paper is arranged as follows: In section 2 we consider a linear test circuit with uncertainties, where no algebraic constraints depends on old algebraic iterates (see section 3). Thus (2) holds for all further considerations. Section 4 provides an introduction to the gPC as one suitable technique. Section 5 gives insight into our simulation settings. Finally in section 6, we discuss our numerical simulation results, especially the rate of convergence when co-simulation shall be applied in stochastic approaches.

2 Circuit Modeling and Uncertain Test Circuit

Usually, a mathematical model for electric circuits is obtained by modified nodal analysis (MNA), see e.g. [5]. This leads to the DAE:

$$\mathbf{E}(\mathbf{p})\,\dot{\mathbf{x}} + \mathbf{A}(\mathbf{p})\,\mathbf{x} = \mathbf{f}(t)\,,$$

where $\mathbf{E}$ describes the dynamic part, $\mathbf{A}$ the static part and $\mathbf{f}$ contains the time depended sources. We are searching for $\mathbf{x}$, which contains the node potentials and some branch currents. In addition the matrices $\mathbf{E}$ and $\mathbf{A}$ include some physical parameters $\mathbf{p} = (p_1,\ldots,p_q)^T$, which we assume to be uncertain. Furthermore we assume that all parameters are independent random variables in a corresponding probability space $(\Omega,\mathscr{F},\mathbf{P})$, with sample space Ω, $\sigma-$algebra $\mathscr{F}$ and probability measure $\mathbf{P}$. Here we use two distribution, namely Gaussian and uniform distribution.

Our test example is the 2-level RLC network given in Fig. 1, with uncertain components $\mathbf{p} = (R_1,\, R_2,\, C_1,\, C_2)^T$. We consider the two stochastic models for $\mathbf{p}$:

$$\begin{aligned} R_i &\sim \mathscr{N}(10\mathrm{k}\Omega, \sigma^2 R_i), \quad \text{or } R_i \sim \mathscr{U}(10\mathrm{k}\Omega - \delta R_i, 10\mathrm{k}\Omega + \delta R_i), \\ C_i &\sim \mathscr{N}(1\mathrm{pF}, \sigma^2 C_i), \qquad \text{or } C_i \sim \mathscr{U}(1\mathrm{pF} - \delta C_i, 1\mathrm{pF} + \delta C_i), \end{aligned} \tag{3}$$

(for $i = 1,2$). Furthermore we assume inductance $L = 1\mathrm{mH}$ and supply voltage $U_{in}(t) = 1\mathrm{V}\cdot\cos(\omega t)$ with an angular frequency $\omega = 2\pi\cdot 5\cdot 10^3$ Hz. Applying

MNA yields a DAE of index-1. To apply co-simulation, we use source coupling, see

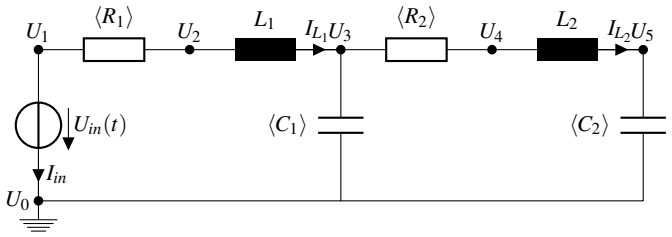

Fig. 1 Uncertain 2-level RLC circuit applied by supply voltage $U_{in}(t)$ (reference model). $\langle \cdot \rangle$ denotes the uncertain parameters.

e.g. [2], to decouple the system into two coupled networks at node U_3, see Fig. 2. Notice that both subsystems can be described by the same (index-1) DAE, with an additional current source for subsystem 1. The exchange of information between

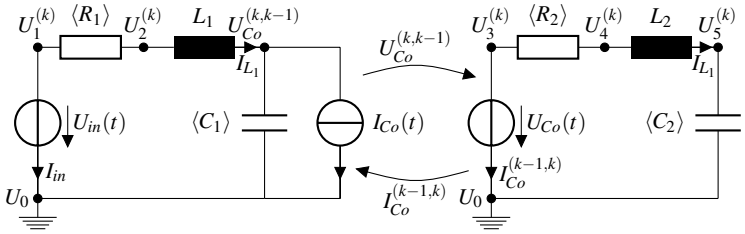

Fig. 2 Decoupled 2-level RLC network using source-coupling in a co-simulation of Gauss-Seidel type with uncertain components R_1, R_2, C_1, C_2. The coupling variables have two super-indices (e.g. $U_{Co}^{(k,k-1)}$). The first index gives the subsystem 1 first, the second index the subsystem 2 first case

both subsystem is organized by the additional variables U_{Co} and I_{Co}. Using Gauss-Seidel scheme, we have to define, which system is computed first.

3 Abstract Coupling Analysis

To analyze coupled systems, a simple method is the fit standard theory of co-simulation, see e.g. [6]. To this end, we write the DAE model for the circuit given in Fig. 2 in semi-explicit form:

$$\dot{\mathbf{y}}_1 = \mathbf{f}_1(\mathbf{y}_1, \mathbf{z}_1, \mathbf{z}_2), \quad 0 = \mathbf{g}_1(\mathbf{y}_1, \mathbf{z}_1), \qquad \dot{\mathbf{y}}_2 = \mathbf{f}_2(\mathbf{y}_2, \mathbf{z}_2), \quad 0 = \mathbf{g}_2(\mathbf{y}_1, \mathbf{y}_2, \mathbf{z}_2). \quad (4)$$

The variables of the subsystems are allocated as follows:

$$\mathbf{y}_1 := \left[U_{Co},\, I_{L_1}\right]^T,\ \ \mathbf{z}_1 := \left[U_1,\, U_2,\, I_{in}\right]^T,\ \ \mathbf{y}_2 := \left[U_5,\, I_{L_2}\right]^T,\ \ \mathbf{z}_2 := \left[U_3,\, U_4,\, I_{Co}\right]^T.$$

Since $\partial \mathbf{g}_1/\partial \mathbf{z}_1$ and $\partial \mathbf{g}_2/\partial \mathbf{z}_2$ are regular. $\mathbf{y_1}$, $\mathbf{y_2}$ defines the differential and $\mathbf{z_1}$, $\mathbf{z_2}$ the algebraic variables. Depending on what subsystem is computed first, we obtain the following splitting schemes for **subsystem 1 first:**

$$\mathbf{F}(\cdot,\cdot,\cdot,\cdot) := \begin{bmatrix} \mathbf{f}_1(\mathbf{y}_1^{(k)}, \mathbf{z}_1^{(k)}, 0, \mathbf{z}_2^{(k-1)}) \\ \mathbf{f}_2(0, 0, \mathbf{y}_2^{(k)}, \mathbf{z}_2^{(k)}) \end{bmatrix},\ \mathbf{G}(\cdot,\cdot,\cdot,\cdot) := \begin{bmatrix} \mathbf{g}_1(\mathbf{y}_1^{(k)}, \mathbf{z}_1^{(k)}, 0, 0) \\ \mathbf{g}_2(\mathbf{y}_1^{(k)}, 0, \mathbf{y}_2^{(k)}, \mathbf{z}_2^{(k)}) \end{bmatrix}. \quad (5)$$

Notice that no algebraic constraints depends on old algebraic iterates. Thus the contraction factor α vanishes for the splitting scheme (5). Furthermore it becomes obvious, that introducing uncertainty in our co-simulation model does not manipulate the properties of contraction. Consequently convergence is guaranteed for the splitting schemes (5) involving uncertainties by using time step size $H < H_0$.

4 Generalized Polynomial Chaos (gPC)

To compute the stochastic quantities of our uncertain model, the gPC is applied. The gPC expansion involving a finite number of P summands reads:

$$f(\mathbf{p}) \approx f_{\mathrm{gPC}}(\mathbf{p}) := \sum_{j=0}^{P-1} f_j(t)\, \Phi_j(\mathbf{p}), \quad (6)$$

with unknown time dependent coefficient functions $f_j(t)$ and basis polynomials $\Phi_j(\mathbf{p})$, see e.g. [7]. The polynomial basis represents an orthogonal system, which depends of the random parameters. Due to the orthogonality of the basis, it is easy to show that the mean and variance of the response respectively read:

$$\mathbb{E}[f(\mathbf{p})] = f_0,\ \mathrm{Var}\left[f_{\mathrm{gPC}}(\mathbf{p})\right] = \sum_{j=1}^{P-1} f_j^2 \mathbb{E}\left[\Phi_j^2(\mathbf{p})\right]. \quad (7)$$

The costly part of the gPC expansion is to determine the unknown coefficient function. For this purpose we employ stochastic collocation. Finally, the total sensitivity coefficients, which denotes the interactions between several parameters, can be easily derive by regroup the coefficient functions and subsequent normalization.

5 Numerical Simulation

For all our investigations, a co-simulation is studied in one time window $[t_0, t_0 + H]$ with $t_0 = 0.4$ ms. To obtain an adequate quality of approximation on H, a gPC expansion with maximum polynomial order three is used, thus momenta up to order

three can be detected. We apply the stochastic collocation method which belongs to the family of non-intrusive methods. We use the Legendre-quadrature rule of order five based on tensor-product, which requires to solve the model 81 times. Notice that these are sample points of Ω.

Our algorithm works in the following manner: For each sample-point out of Ω, the reference model is solved in time domain up to t_0 to obtain initial values which are close to the solution. Now we start co-simulation with k iteration steps for each sample on $[t_0, t_0 + H]$ using the corresponding initial values. Furthermore, constant extrapolation of the initial value is used for the initial guess $\tilde{x}^{(0)}(t)$ on time window. Finally, we compute the stochastic momenta (depending on step k).

6 Numerical Results

Using MATLAB®we first investigate the error behavior related to the stochastic process in the output voltage U_5 using different levels of uncertainties for the splitting scheme (5). For this purpose, we consider a range of deviations for the resistances and capacitances, which are typical in electrical engineering. To this end we are focusing our attention on the error in the total sensitivity coefficients. The error of the solutions on the n-th time window after k iterations $\tilde{x}_c^{(k)}(t)$ is measured by comparing with a reference solution $x_m(t)$ computed by a monolithic simulation: $\Delta_n^{(k)}(t) = x_m(t) - \tilde{x}_c^{(k)}(t),\ \delta_n^{(k)} := \|\Delta_n^{(k)}\|_\infty$. Furthermore, we assume that the biggest discrepancy is located at the end of the time window. Please note that our co-simulation works only on the specified time window H, which means that there is no error transport between several time windows. For a quantitative evaluation we compute the mean error over all total sensitivity coefficients. As uncertainty we suppose uniform distributed parameters with a variation between $\delta R_i = 0.1\ (10\%) \ldots 0.5\ (50\%)$ for the resistances and $\delta C_i = 0.1\ (10\%) \ldots 0.5\ (50\%)$ for the capacitances around the nominal respective values.

Fig. 3 shows the mean error for $k = 1, 3, 5, 10$ iterations in the co-simulation. It becomes obvious, that for a high level of uncertainty the error increases. Furthermore, a continuous improvement in the mean error up to $k = 10$ can be observed in cases of high uncertainties for C_i and R_i. Accordingly, small uncertainties in our co-simulation model requires a smaller number of iterations, where the level of uncertainty in the capacitances mainly controls the rate of convergency.

Next we investigate the contraction and the rate of convergence for expectation, standard deviation and for the deterministic solution. For our observations, all node potentials $U_1, \ldots, U_5$ are involved. In the case of uniform distributed parameters, we suppose an variation about $\delta R_i = 0.1\ (10\%)$ for the resistances and $\delta C_i = 0.5\ (50\%)$ for the capacitances. Fig. 4 shows for splitting scheme (5). Thus all quantities have nearly the same rate of convergence for window sizes $10^{-8}\ s < H < 10^{-4}\ s$. It becomes obvious that a further reduction of the window size does not reduce the error in the expectation and standard deviation. This behavior differs to its deterministic solution, where a improvement up to the machine precision is achieved. The reasons

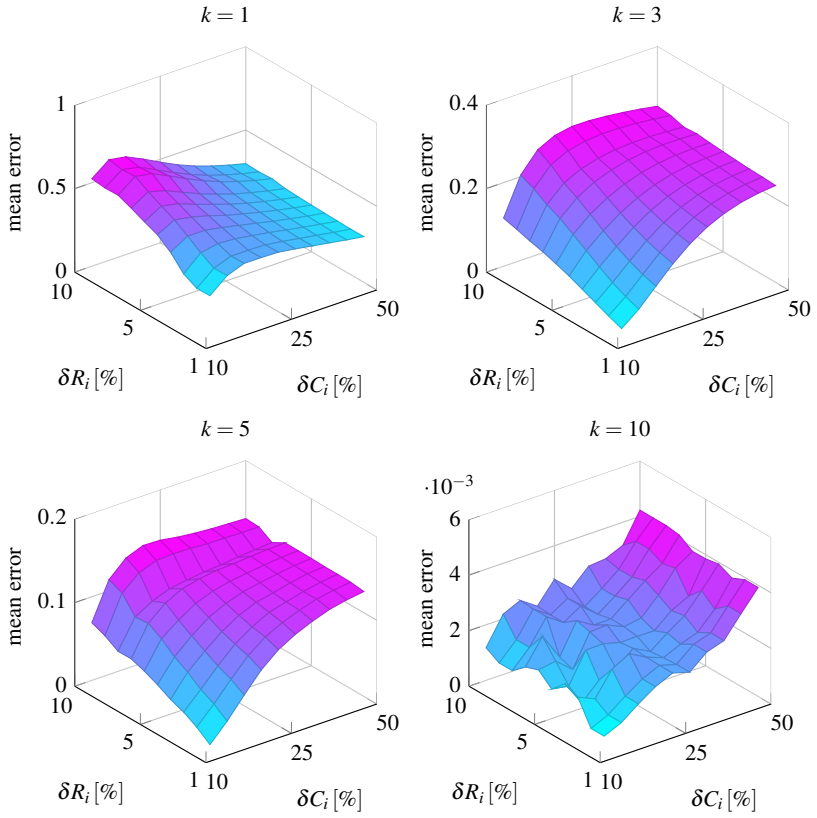

Fig. 3 Mean error over all total sensitivity coefficients obtained by comparing with a reference solution for $k = 1, 3, 5, 10$ iterations over $H = 0.1$ ms. Uniform distribution (legendre polynomials), $R_i \sim \mathscr{U}(10\text{k}\Omega - \delta R_i, 10\text{k}\Omega + \delta R_i)$, $C_i \sim \mathscr{U}(1\text{pF} - \delta C_i, 1\text{pF} + \delta C_i)$

for this are diverse: The usage of Gauss-Legendre quadrature formulas of order five produces a numerical quadrature error in each coefficient function $f_j(t)$ of (6). Furthermore, the accuracy of the stochastic process is limited by using a finite number of summands in the gPC expansion.

In order to investigate the performance of contraction, we decrease the window size by 10% down to $[0.4,\ 0.49]$ ms. For our tests the minimum error is bounded by the time integrator precision of 10^{-3} with which we solve the subsystems. Fig. 4 shows the performance of contraction for different quantities measured by the relative error. It becomes apparent that the performance for the expectation is much better than for the standard deviation. Here, expectation is already reproduced after five iteration steps, whereas the standard deviation requires about ten iterations to achieve the maximum precision of $\delta_n^{(k)} = 10^{-3}$. Due to the definition of the expectation, which is exactly represented by the first coefficient function f_0, see (7), it

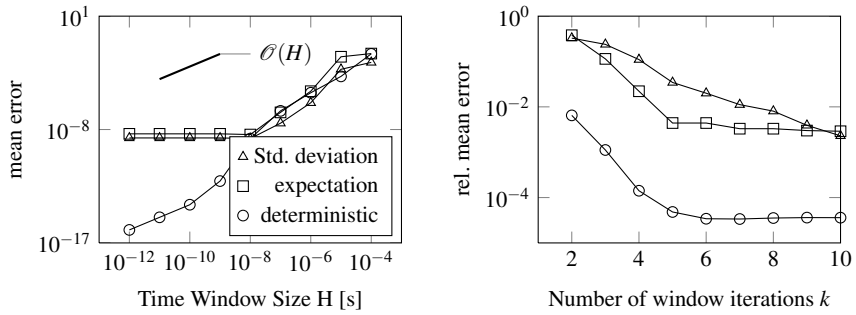

Fig. 4 (left) Convergence in expectation, standard deviation and in the deterministic solution concerning the node potentials U_1, U_2, U_3, U_4, U_5 after 0.4 ms for different time step sizes H with four iterations per time window. (right) Contraction measured by the relative error in dependence of the iterations k on the time window $[0.4,\ 0.49]$ ms.

is resolved with a higher quadrature order than the standard deviation. Hence, in contrast to the standard deviation no approximation error is caused by using a finite number of coefficient functions in (6). This can explain Fig. 4 right.

An example is given in Fig. 5 and Fig. 6, where the expectation and standard deviation is presented only for the node potentials U_3 and U_5 over the time window $[0.4,\ 0.49]$ ms. As uncertainty we choose $R_i \sim \mathcal{U}(10\text{k}\Omega - 10\%, 10\text{k}\Omega + 10\%)$, $C_i \sim \mathcal{U}(1\text{pF} - 50\%, 1\text{pF} + 50\%)$. Here, expectation is well approximated already after $k = 3$ iteration steps, whereas the standard deviation requires more than five iterations to achieve an error of approximately $\delta_n^{(k)} = 10^{-3}$. In addition there are oscillations in the standard deviation of U_3 over H, which are not be further analysed in this paper. However, tests have shown that the oscillation can be minimize by reducing the window size. All our investigation holds also by using Gaussian distribution settings given in (3).

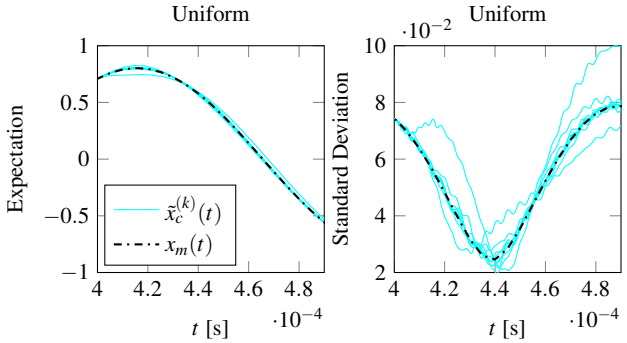

Fig. 5 Expectation and standard deviation for U_3 using different numbers of iteration steps for subsystem 1 first where uniform distributed components are introduced.

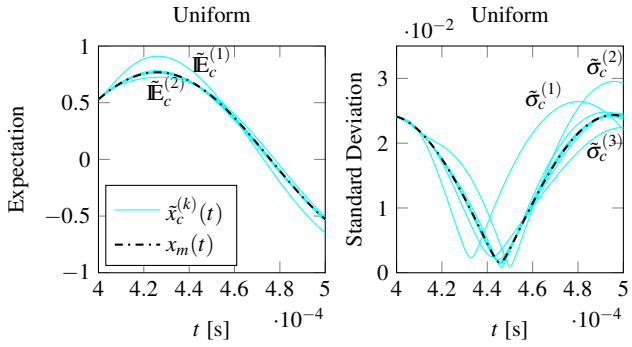

Fig. 6 Expectation and standard deviation for U_5 using different numbers of iteration steps for subsystem 1 first where uniform distributed components are introduced.

7 Conclusions

We have shown for our test case, that the number of iterations which are needed to achieve a predefined accuracy is mainly controlled by the level of uncertainty. Co-simulation models with higher uncertainties naturally require a larger number of iterations. Furthermore, uncertain time-dependent components have a greater impact than uncertain static components regarding the rate of convergency. Concerning our test example, the speed of contraction for expectation and standard deviation differs from each other. Thus, different stochastic quantities requires a different number of iterations to archive a suitable accuracy in co-simulation.

It is a future aim to combine co-simulation and UQ for electrical circuits, where the contraction factor α does not vanish.

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