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Abstract. This paper considers the numerical solution of the fully-nonlinear Black-Scholes problem, modelling the replication of contingent claims in illiquid markets. First, we present well-posedness discussion on the differential problem. Further, an unconditionally stable explicit finite difference scheme is proposed as consistency, positivity and convergence are studied. Numerical experiments validate our theoretical results.

1 Introduction

The derivation of the Black-Scholes (BS) option pricing model by Fisher Black and Myron Scholes is a starting point in modern computational finance. It is, however, based on several stylized unrealistic assumptions that oversimplify the market dynamics. Local and stochastic volatility BS models also neglect important characteristics such as transactions costs, feedback effects from the trading activity or market liquidity. These market features are implemented in the modified volatility BS problems given by a fully-nonlinear backward degenerate parabolic equation with the corresponding terminal condition [10,14,28]:

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\hat{\sigma}\left(S, \frac{\partial V}{\partial S}, \frac{\partial^2 V}{\partial S^2}\right)^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \quad S > 0,\, t \in [0, T) \\ V(S,T) = f(S), \quad S > 0, \end{aligned} \tag{1}$$

where S denotes the price of underlying asset, t stands for time, T - for maturity, $V(S,t)$ is the option price, $\hat{\sigma}$ is the (modified) volatility function, r - the risk free interest rate and $f(S)$ is the payoff function.

There are many results on the numerical solution of the linear BS problem and its generalizations [26,28]. However, few results can be found in the literature on the numerical solution of nonlinear BS equations of type (1). In Company et al. [6,7] the authors propose an explicit finite difference scheme that requires a restrictive stability condition on the time and spatial mesh sizes. Ankudinova and Ehrhardt [3] use a Crank-Nicolson method, combined with a high-order compact difference scheme, to construct a numerical scheme for the linearized BS equation using frozen values of the nonlinear volatility. Implicit numerical schemes for nonlinear option pricing problems with uncertain volatility have been analyzed in [18,22,29]. For more details on nonlinear BS problems and their numerical solution we refer the interested reader to the book [10].

This paper focuses on nonlinear models, pricing the replication of a European contingent claim in a market with imperfect liquidity as market liquidity is currently an issue of high concern in financial risk management. Most of the option pricing models assume that an option trader can not affect the underlying asset price in trading the underlying to replicate the option payoff which is reasonable only in a perfect liquid market. The market liquidity of assets affects their prices and expected returns as investors require higher return on assets with lower market liquidity as compensation for the higher cost of trading these assets. According to Frey and Polte [14] these models can be classified into two groups, illiquid market models with purely temporary price impact and with a permanent price impact on the underlying price dynamics for all market participants. In the later case hedging on the equilibrium price of the stock results in additional supply or demand. This effect is usually referred as *first-order effect* in the literature [15] if the hedging is based on the linear BS equation.

Here, we will deal with the model of Frey and Patie [13] including a liquidity parameter depending on the asset, that tries to cover *liquidity drops*, i.e. the market liquidity falls if the stock price decreases, cf. [15]. Secondly, we will consider the illiquid market model of Liu and Yong [23] that regularizes the partial differential equation close to maturity. More details on illiquid market models are found in the recent book [11].

1.1 The Frey and Patie setting

We now present the approach of Frey and Patie [13] in modelling the hedge cost when replicating the option payoff in illiquid markets. It is assumed that stock price dynamics follows the stochastic differential equation (SDE)

$$dS_t = \sigma S_{t-}\, dW_t + \rho\lambda(S_{t-})S_{t-}\, d\alpha_t^+,$$

where W_t is a standard Brownian motion, σ is the constant volatility, S_{t-} denotes the left limit $\lim_{S\to t, S<t} S_{t-}$ of the stock price S_{t-}, α_t denotes the number of shares in the portfolio at time t and $\lim_{\alpha_t^+\to t, S<t} \alpha_S$. The parameter $\rho \neq 0$ is a characteristic of the market independent of the payoff of the option; it is a measure for the feedback-effect of a large trader. The *liquidity profile of the market* is described by the continuous and positive function $\lambda(S)$ and chosen in a way to obtain the desired payoff. The values of ρ and $\lambda(S)$ must be calibrated from the observed option prices.

Hedging a terminal value claim with maturity T and payoff $f(S)$ results in a BS partial differential equation (PDE) for the hedge cost $V(S,t)$

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2\left(1-\rho\lambda(S)S\frac{\partial^2 V}{\partial S^2}\right)^2}\frac{\partial^2 V}{\partial S^2} = 0, \qquad (S,t)\in \Omega_T := \Omega\times(0,T], \\ &\qquad\qquad\qquad\qquad\qquad\qquad V(S,T) = f(S), \quad S\in\Omega := (0,+\infty), \end{aligned} \tag{2}$$

where the Gamma greek $\frac{\partial^2 V}{\partial S^2}(S,t)$ satisfies the assumption A4 of [13]

$$1-\rho\lambda(S)S\frac{\partial^2 V}{\partial S^2}(S,t) \geq \delta_0 > 0, \qquad (S,t)\in\Omega_T. \tag{3}$$

In view of the nonlinear PDE (2) this is a necessary condition for the well-posedness, cf. Section 1.2. In a financial interpretation A4 assumes that the variation of the large trader's trading strategy $\varphi(S_t,t)$ is small compared to the market, cf. [14]:

$$1-\rho\lambda(S_t)S_t\,\frac{\partial\varphi}{\partial S}(S_t,t)\geq 0 \qquad \text{a.s.} \tag{4}$$

Later one can argue that the delta hedging $\varphi = \frac{\partial V}{\partial S}$ is a reasonable trading strategy were the hedger is completely aware of the market feedback effect and changes his strategy accordingly, cf. [15]. This *full feedback* case leads to the fully nonlinear PDEs considered here.

The payoff function $f(S)$ is assumed to be a continuous piece-wise linear function and in the case of replicating a vanilla European option we have

$$V(S,T) = \begin{cases} \max(S-E,0), & \text{in the case of a call option} \\ \max(E-S,0), & \text{in the case of a put option} \end{cases} \tag{5}$$

where E denotes the strike price.

1.2 Well-posedness of the differential problem

The well-posedness analysis of fully-nonlinear degenerate parabolic problems is complicated in comparison with linear, semi- and quasi-linear problems. In the classical solution concept the uniqueness is often established by the comparison principle, following from some maximum principle where the nonlinear operator satisfies some kind of monotonicity property, in particular ellipticity [12]. However, due to the full nonlinearity, the existence a priori estimates has to be carried out in $C^{2,\alpha}(0<\alpha<1)$ Hölder spaces, cf., e.g. [23]. If the data and the boundary do not satisfy certain regularity and structure conditions classical solutions do not exist and these issues are further amplified by the fact that fully-nonlinear problems, in general, do not admit weak formulation in (weighted) Sobolev spaces.

The viscosity solution theory, introduced by P.-L. Lions several decades ago, provides a general framework for investigating the well-posedness of fully-nonlinear problems. It is of particular importance to financial applications as the viscosity solution is the *financially-relevant* solution [4,14,29]. We stress that *the ellipticity property of the nonlinear operator and the comparison principle are critical* for both the classical and generalized (viscosity) solution concept, see the Appendix. Moreover, by the comparison principle we have that *the solution of the initial-boundary value problems, corresponding to* (2) *and* (36) *is positive (short for non-negative) if* $f(S)$ *is positive* which is an important property of the differential solution (the hedge cost) in the financial setting.

We now recall the results of Agliardi et al. [2] on the local comparison principle (the authors also discuss its extension to the global comparison principle by constructing appropriate barrier functions) for nonlinear degenerate backward parabolic PDEs

$$u_t + x^2 G(x,t,p,q) = 0, \qquad (x,t) \in \Pi_T := (0,b) \times (0,T) \tag{6}$$

where $p \equiv \frac{\partial u}{\partial x}$, $q \equiv \frac{\partial^2 u}{\partial x^2}$ and $G \in C^2(\Pi_T \times \mathbb{R}^2)$. The parabolic part of the boundary is denoted by $\Gamma_T = I \cup II \cup III$, where $I = \{x = 0, 0 < t < T\}$, $II = \{0 < x < b, t = T\}$, $III = \{x = b, 0 < t < T\}$, $0 = x_{\min} \leq x_{\max} = b$.

Suppose that u is a classical solution of (6) in Π_T and

$$G_q \equiv \frac{\partial G}{\partial q}(x,t,\psi^{'}(x),\psi^{''}(x)) > 0 \qquad \text{for} \qquad 0 \leq x \leq b, \quad ' \equiv d/dx \tag{7}$$

where $u\,|_{II} = u(x,T) = \psi(x) \in C_p^2[0,b]$ and $\frac{\partial u}{\partial x} = \psi^{'}(x)$, $\frac{\partial^2 u}{\partial x^2} = \psi^{''}(x)$ for $t = T$. We have that under the condition (7) and also $G(x,t,0,0) = 0$ there exists a positive constant $\delta > 0$ such that $\max_{\overline{\Pi}_{T,T-\delta}} u = \max_{\Gamma_{T,T-\delta}} u$, where $\overline{\Pi}_{T,T-\delta} = \{\delta < t < T, a \leq x \leq b\}$, $\Gamma_{T,T-\delta}$ is the parabolic part of the boundary of $\Pi_{T,T-\delta}$. Similarly, $\min_{\overline{\Pi}_{T,T-\delta}} u = \min_{\Gamma_{T,T-\delta}} u$.

Remark 1. The result refers to the truncated problem (2) as for the numerical method we consider an artificial truncation of the domain so that $S \in [0,b]$. Consistent with their analysis of Barles et al. [5] we solve problems (2) with Dirichlet boundary conditions, compatible with the payoff:

$$V(0,t)f(0), \qquad V(b,t) = f(b) \tag{8}$$

Remark 2. Since the payoff function $V(S,T) = f(S) \in C_p^0(\Omega)$ one has to consider smoothing of the initial data to apply the comparison principle result. Frey [13] suggests that one may use, for example,

$$f(S) = \frac{1}{2}\left(S - E + \sqrt{(S-E)^2 + \alpha}\right)$$

for some small $\alpha > 0$.

Let us now consider the Frey and Patie model (2). We have

$$G_q(S,q) = \frac{\sigma^2}{2\left(1 - \rho\lambda(S)Sq\right)^2}\frac{1 + \rho\lambda(S)Sq}{1 - \rho\lambda(S)Sq}$$

so the condition (7) for the local comparison principle takes the form

$$\frac{\sigma^2}{2\left(1-\rho\lambda(S)Sf''\right)^2}\frac{1+\rho\lambda(S)Sf^{''}}{1-\rho\lambda(S)Sf^{''}}>0 \qquad \text{for} \quad 0=S_{\min}\leq S\leq S_{\max}=b$$

This inequality will be fulfilled if, for example,

$$\left|f^{''}(S)\right|<\frac{1}{\rho\max|\lambda(S)S|}, \qquad S\in[0,b] \tag{9}$$

Then also (3) holds for the payoff as, indeed, it is a particular case of (9) for convex payoff (vanilla option) and nonnegative market liquidity function $\lambda(S)$. *The assumption* (3) *is now recognized as the ellipticity condition for the nonlinear operator of the Frey model* (2). We now observe the major issue of this model for the vanilla payoff (and also any other market instrument as their payoffs are, in general, linear combination of call and put payoffs): *the nonlinear operator of the Frey model* (2) *is not elliptic at the strike price near maturity. That is because the second spatial derivative, the Gamma greek, of the payoff is the Dirac δ-function (more precisely, δ-distribution), concentrated at the strike, and it violates* (3) *near maturity.*

The analyzed parabolicity issue of the Frey model is also consistent with the considerations, presented by Frey and Polte [14] where the authors rewrite (2) as a dynamic programming (HJB) equation for very small ρ. However, they fail to give clear and precise answer whether the model is credible or not.

Remark 3. We confirm these considerations by numerical simulations showing that the diffusive (*smoothing*) property of solution operator in time is not present at the strike for the Frey model. The smoothing property for solution $u(x,t)$ of the (model) heat equation with initial data $v(x)$ and solution operator in time $E(t)$ asserts that, cf. Larsson and Thomeè [21]:

$$\left\|D_t^j D^\alpha E(t)u\right\|_\infty \leq Ct^{-j-|\alpha|/2}\left\|v\right\|_\infty, \quad t>0 \tag{10}$$

where α stands for the multi-index for $d>1$. This property asserts that the solution of the heat equation is smooth for $t>0$ even if v is nonsmooth.

The aim of this paper is the numerical analysis of (2) by a stable explicit finite difference scheme. *Having in mind the presented considerations for the differential problem we investigate how the numerical solution inherits these well-posedness (ellipticity) issues. The application of unconditionally stable time stepping method is important for the analysis as we want to focus on the monotonicity of the scheme (the more restrictive property).* In the frame of parabolic problems the monotonicity of the scheme should be understood as discrete parabolicity [4,5], ie discrete maximum principle whence the comparison principle follows.

The generalization of the presented illiquid market modelling in multidimensional (multi-asset or stochastic volatility setting) is far from trivial where a very recent attempt in this direction is presented by Yazdanian and Pirvu [30].

We observe severe well-posedness issue with the one-dimensional Frey model (2) and for multidimensional problems we may outline that (beyond modelling) there are two major challenges - the analysis of the differential problem and the construction of a computationally efficient consistent and *monotone* scheme (in the general case of time- and space-dependent parameters and *cross derivative term* it is a quite difficult task) [12], often referred to as 'curse of dimensionality'.

Our paper is organized as follows. Section 2 analyzes the semi-discretization of the nonlinear problem (2) and the consequent linearization by the Picard iteration. In Section 3 we present and investigate the local Crank-Nicolson method (LCN) for the Frey and Patie model (2). A section to follow, Section 4, considers another modelling approach in illiquid markets, suggested by Liu and Yong [23], and its numerical solution by the proposed numerical method. Finally, numerical experiments are given in Section 5, discussing the properties of the numerical method and supplying numerical evidence to our theoretical analysis.

2 Spatial semi-discretization and linearization

This section is devoted to the analysis of the semi-discretization of the Frey and Patie PDE (2), equipped with the terminal condition (5) and boundary conditions (8). After performing time-reversal we introduce the spatial grid Ω_h with step $h = \triangle S$ by the nodes $S_i = ih$, $i = 0, \dots, M$ so that $Mh = b$, while we set $t_n = n\tau$, $n = 1, 2, \dots, N$, for the temporal step $\tau = T/N$.

Before we apply the spatial semi-discretization the following standard assumption on the regularity of the differential solution is made.

Assumption 1 *The solution of the problem* (2) *has continuous spatial derivatives up to fourth order.*

The corresponding *autonomous* ODEs system (by the method of vertical lines) for the semi-discrete solution $v(t) = [v_1(t), \dots, v_{M-1}(t)]^\top$ by the centered-in-space approximation

$$\frac{\partial^2 V}{\partial S^2}(S_i, t) = \frac{V(S_{i+1}, t) - 2V(S_i, t) + V(S_{i-1}, t)}{h^2} + O(h^2)$$

is obtained as

$$v'(t) = A(v(t))v(t) + g, \qquad t \in (0, T] \tag{11}$$

with

$$\begin{gathered} A(v) = \frac{1}{2h^2}\text{tridiag}\left(\beta_i(v), \alpha_i(v), \gamma_i(v)\right) \\ \alpha_i(v) = -2\hat{\sigma}_i^2(v)S_i^2, \;\; \beta_i(v) = \hat{\sigma}_i^2(v)S_i^2, \;\; \gamma_i(v) = \hat{\sigma}_i^2(v)S_i^2 \\ \hat{\sigma}_i^2(v) = \frac{\sigma^2}{\left(1 - \rho\lambda(S_i)S_i\Delta_i v\right)^2}, \end{gathered} \tag{12}$$

where $g \in \mathbb{R}^{M-1}$ is the vector, generated by the boundary conditions,

$$g = \frac{1}{2h^2}[\beta_1 f(0), 0, \dots, 0, \gamma_{M-1} f(b)]^\top$$

and Δ_i is the finite difference operator, corresponding to the approximation (2).

2.1 Properties of the semi-discrete nonlinear system

We now discuss the existence of a unique solution of the semi-discrete nonlinear ODE system (11). The consistency and stability properties of the scheme are proven. The qualitative behaviour of the semi-discrete solution is investigated by the comparison principle and convergence to the viscosity solution of the nonlinear PDE of Frey and Patie (2) is obtained.

Lemma 1. *The system* (11) *with initial data, corresponding to the payoff, has unique solution* $v(t) = [v_1(t), \dots, v_{M-1}(t)]^\top$ *in the convex domain*

$$\mathcal{D} := \{w \in \mathbb{R}^{M-1} : |\mathit{\Delta}_i w| < \frac{1}{\rho\,|\lambda(S_i)S_i|}, \;\; i = 1, \dots, M-1, t \in (0, T]\}$$

Proof. In the sequel we use the standard notations for the discrete maximum norm

$$\|w\|_\infty = \max_{1 \le i \le M-1} |w_i|\,, \qquad \|B\|_\infty = \max_{1 \le j \le M-1} \sum_{k=1}^{M-1} |b_{jk}|$$

for $w \in \mathbb{R}^{M-1}$ and $B \in \mathbb{R}^{(M-1)\times(M-1)}$. Moreover, we also introduce

$$\overline{\delta}_0 := \min_{\omega \in \mathcal{D}} \{1 - \rho\lambda(S_i)S_i\mathit{\Delta}_i w\},$$

so that we have $\overline{\delta}_0 > 0$. The Lipschitz continuity of the nonlinear operator $F(w) := A(w)v + g$

$$\|F(\tilde{w}) - F(w)\|_\infty \le \sup_{w \in \mathcal{D}} \|J_F(w)\|_\infty \|\tilde{w} - w\|_\infty$$

is now considered as $J_F(w)$ is the Jacobian matrix of F (note that $\|F(w)\|$ is bounded as $w \in \mathcal{D}$). One computes

$$\begin{aligned}
\frac{\partial F_i}{\partial w_i} &= \frac{1}{2h^2}\left(\alpha_i + \frac{\partial \alpha_i}{\partial w_i} w_i + \frac{\partial \beta_i}{\partial w_i} w_{i-1} + \frac{\partial \gamma_i}{\partial w_i} w_{i+1}\right) \\
\frac{\partial F_i}{\partial w_{i-1}} &= \frac{1}{2h^2}\left(\beta_i + \frac{\partial \alpha_i}{\partial w_{i-1}} w_i + \frac{\partial \beta_i}{\partial w_{i-1}} w_{i-1} + \frac{\partial \gamma_i}{\partial w_{i-1}} w_{i+1}\right) \\
\frac{\partial F_i}{\partial w_{i+1}} &= \frac{1}{2h^2}\left(\gamma_i + \frac{\partial \alpha_i}{\partial w_{i+1}} w_i + \frac{\partial \beta_i}{\partial w_{i+1}} w_{i-1} + \frac{\partial \gamma_i}{\partial w_{i+1}} w_{i+1}\right).
\end{aligned} \tag{13}$$

Further we have

$$\begin{aligned}
\frac{\partial \alpha_i}{\partial w_i} &= \frac{8}{h^2}\frac{\sigma^2\rho\lambda(S_i)S_i^3}{\overline{\delta}_0^3} \\
\frac{\partial \alpha_i}{\partial w_{i+1}} = \frac{\partial \alpha_i}{\partial w_{i-1}} = \frac{\partial \beta_i}{\partial w_i} = \frac{\partial \gamma_i}{\partial w_i} &= -\frac{4}{h^2}\frac{\sigma^2\rho\lambda(S_i)S_i^3}{\overline{\delta}_0^3} \\
\frac{\partial \beta_i}{\partial w_{i+1}} = \frac{\partial \beta_i}{\partial w_{i-1}} = \frac{\partial \gamma_i}{\partial w_{i+1}} = \frac{\partial \gamma_i}{\partial w_{i-1}} &= \frac{2}{h^2}\frac{\sigma^2\rho\lambda(S_i)S_i^3}{\overline{\delta}_0^3},
\end{aligned} \tag{14}$$

so that we derive

$$\frac{\partial F_i}{\partial w_i} = \frac{1}{2h^2}\left(\alpha_i + \frac{4\sigma^2\rho\lambda(S_i)S_i^3}{\overline{\delta}_0^3}\Delta_i w\right)$$
$$\frac{\partial F_i}{\partial w_{i-1}} = \frac{1}{2h^2}\left(\beta_i + \frac{2\sigma^2\rho\lambda(S_i)S_i^3}{\overline{\delta}_0^3}\Delta_i w\right)$$
$$\frac{\partial F_i}{\partial w_{i+1}} = \frac{1}{2h^2}\left(\gamma_i + \frac{2\sigma^2\rho\lambda(S_i)S_i^3}{\overline{\delta}_0^3}\Delta_i w\right)$$

It is now obvious that $F(w)$ is Lipschitz continuous with Lipschitz constant $L = O\left(h^{-2}\right)$ for $w \in \mathcal{D}$ and therefore the assertion follows, cf. [17]. □

Remark 4. Let us remark that the condition

$$|\Delta_i v(0)| \leq \frac{1}{\rho\,|\lambda(S_i)S_i|}, \qquad i = 1, \dots, M-1$$

may be regarded as a semi-discrete analogue of (9). The condition for existence and uniqueness of the semi-discrete solution for convex payoff reduces to

$$1 - \rho\lambda(S_i)S_i\Delta_i v(t) > 0, \qquad i = 1, \dots, M-1, \quad t \in (0, T]$$

Further we need this important monotonicity property to motivate the convergence of the proposed discretizations.

Further we need the Gronwall lemma which is a standard tool in estimating the growth of functions that satisfy an integral inequality.

Lemma 2. *[16] (Gronwall) Let p and q be continuous real functions with $p \geq 0$. Let c be a non-negative constant. Assume that*

$$p(t) \leq q(t) + c\int_0^t p(s)\,ds \qquad \forall t \in [0, T]$$

Then we have the estimate

$$p(t) \leq e^{ct}q(t) \qquad \forall t \in [0, T]$$

Theorem 1. *The semi-discrete difference scheme* (11) *is consistent and stable.*

Proof. We define the spatial truncation error

$$\sigma_h(t) = V_h^{'}(t) - A(V_h(t))V_h(t) - g$$

where V_h is the projection of the PDE solution on the spatial grid. The consistency estimate is subject to similar considerations as given in the paper of

Company et al. [6] and we shall now briefly sketch its application to our problem. The semi-discrete difference scheme (11) is said to be *consistent of order q with* (2) if we have, cf. [19]

$$\|\sigma_h(t)\|_\infty = O(h^q) \qquad \text{uniformly for} \quad 0 \le t \le T$$

Starting with the following consideration

$$\sigma_h(t) = V_h^{'}(t) - AV_h(t) - g = V^{'}(t) - \frac{1}{2}\hat{\sigma}(S_i,t)S_i^2\frac{\partial^2 V}{\partial S^2}(S_i,t) - g$$
$$+\frac{1}{2}\hat{\sigma}(S_i,t)S_i^2\frac{\partial^2 V}{\partial S^2}(S_i,t) - AV_h(t) = \frac{1}{2}\hat{\sigma}(S_i,t)S_i^2\frac{\partial^2 V}{\partial S^2}(S_i,t) - AV_h(t)$$

we introduce the notation $\varDelta_i V_h(t) = x + \varDelta x$, where $x = \frac{\partial^2 V}{\partial S^2}(S_i,t)$ and $\varDelta x = O(h^2)$, according to (2).

If one considers the function $g_S(x) = \frac{x}{1-\rho\lambda(S)Sx}$ for a fixed value of the underlying asset variable S then $g_S(x)$ is a well-defined continuously differentiable function in any domain where $1-\rho\lambda(S)Sx \neq 0$ which corresponds to the well-posedness condition (37). Therefore, we obtain by the mean value theorem

$$\sigma_h(t) = \sigma^2 S_i^2\left(g_{S_i}(x+\varDelta x) - g_{S_i}(x)\right) = \sigma^2 S_i^2 g_{S_j}^{'}(x+\theta\varDelta x)\varDelta x, \ \ 0<\theta<1$$

and since $g_{S_j}^{'} = \frac{1+\rho\lambda(S)Sx}{(1-\rho\lambda(S)Sx)^3}$ is bounded uniformly in $t \in [0,T]$ we derive that

$$\|\sigma_h(t)\|_\infty = O(h^2)$$

so that the semi-discrete difference scheme is consistent of order 2 in space.

Next, we proceed with the stability estimate. The solution of (11) for $t \in [0,T]$ is also a solution of the integral equation

$$v(t) = v(0) + \int_0^t F(v(s))\,ds.$$

A small perturbation in the initial data results in a second solution

$$\tilde{v}(t) = \tilde{v}(0) + \int_0^t F(\tilde{v}(s))\,ds$$

and we arrive at

$$\|v(t)-\tilde{v}(t)\|_\infty \le \|v(0)-\tilde{v}(0)\|_\infty + \int_0^t \|F(v(s)) - F(\tilde{v}(s))\|_\infty\,ds$$
$$\le \|v(0)-\tilde{v}(0)\|_\infty + L\int_0^t \|v(s)-\tilde{v}(s)\|_\infty\,ds$$

By Lemma 2 we obtain

$$\|v(t)-\tilde{v}(t)\|_\infty \le e^{Lt}\,\|v(0)-\tilde{v}(0)\|_\infty \qquad \forall t\in[0,T] \tag{15}$$

This estimate (15) is, however, sub-optimal for stiff problems (large L). The following improvement is presented in Hairer et al. [17]

$$\|v(t) - \tilde{v}(t)\|_\infty \leq e^0 \|v(0) - \tilde{v}(0)\|_\infty \qquad \forall t \in [0, T]$$

since by (13) and (14) we have

$$\mu_\infty (J_F(v)) = \max_{1 \leq i \leq M-1} \left(\alpha_i + \beta_i + \gamma_i + \left(\frac{\partial \alpha_i}{\partial v_i} + \frac{\partial \beta_i}{\partial v_i} + \frac{\partial \gamma_i}{\partial v_i} \right) v_i \right.$$
$$\left. + \left(\frac{\partial \alpha_i}{\partial v_{i-1}} + \frac{\partial \beta_i}{\partial v_{i-1}} + \frac{\partial \gamma_i}{\partial v_{i-1}} \right) v_{i-1} + \left(\frac{\partial \alpha_i}{\partial v_{i+1}} + \frac{\partial \beta_i}{\partial v_{i+1}} + \frac{\partial \gamma_i}{\partial v_{i+1}} \right) v_{i+1} \right) = 0,$$

where $\mu_\infty(\cdot)$ is the logarithmic maximum norm. □

Definition 1. *[19] The system* (11) *is positive (short for "non-negativity preserving") if*

$$v(0) \geq 0 \qquad \textit{implies} \qquad v(t) \geq 0 \quad \forall t \geq 0$$

Theorem 2. *[19] Suppose that the nonlinear operator $F(v) = A(v)v + g$ is continuous and satisfies the Lipschitz condition with respect to v. Then the system* (11) *is positive if for any vector $v \in \mathbb{R}^{M-1}$ and $t \geq 0$*

$$v \geq 0, v_i = 0 \qquad \textit{implies} \qquad F_i(v) \geq 0, \quad i = 1, \dots, M-1 \tag{16}$$

Moreover, if also the following property is valid

$$\frac{\partial F_i(v)}{\partial v_j} \geq 0, \qquad i \neq j, \qquad i, j = 1, \dots, M-1 \tag{17}$$

we also have the comparison principle for the solution of the system (11)*, ie*

$$v(0) \leq \tilde{v}(0) \qquad \textit{implies} \qquad v(t) \leq \tilde{v}(t)$$

Proof. Since the off-diagonal elements β_i and γ_i in (12) are non-negative while the diagonal elements α_i are non-positive the requirement (16) is fulfilled.

We now consider the condition (17) as we obtain by (12), (13) and (14)

$$\frac{\partial F_i}{\partial v_{i-1}}(v) = \frac{1}{2h^2} \frac{\sigma^2 S_i^2}{(1 - \rho\lambda(S_i) S_i \varDelta_i v)^2} \left(1 + \frac{2\rho\lambda(S_i) S_i}{(1 - \rho\lambda(S_i) S_i \varDelta_i v)} \varDelta_i v \right)$$
$$= \frac{1}{2h^2} \frac{\sigma^2 S_i^2}{(1 - \rho\lambda(S_i) S_i \varDelta_i v)^2} \left(\frac{1 + \rho\lambda(S_i) S_i \varDelta_i v}{1 - \rho\lambda(S_i) S_i \varDelta_i v} \right)$$

Recalling $v \in \mathcal{D}$ so that we have $|\varDelta_i v| \leq \frac{1}{\rho|\lambda(S_i) S_i|}$ and therefore $\frac{\partial F_i}{\partial v_{i-1}}(v) > 0$. Analogously, we obtain $\frac{\partial F_i}{\partial v_{i+1}}(v) > 0$. □

Following Barles [4] and taking into account Theorems 1 and 2 we have the following corollary.

Corollary 1 *The semi-discrete solution of* (11) *converges to the viscosity solution of the nonlinear PDE* (2) *(if there is one).*

2.2 The Picard iteration

We now consider the solution of (11) for $t \in [t_n, t_n + 1]$. It is also the solution of the integral equation

$$v(t) = v(t_n) + \int_{t_n}^{t} (A(v(s))v(s) + g)\, ds$$

and will be approximated by the sequence a functions $v^0, v^1, v^2, \ldots$, where $v^0 = v(t_n)$ and

$$v^k(t) = v^0 + \int_{t_n}^{t} \left(A(v^{k-1}(s))v^k(s) + g\right) ds \tag{18}$$

which is called Picard iteration. By similar, yet simplified, considerations as in the nonlinear case we have that v^k exists and is bounded.

On each time level, $t \in [t_n, t_{n+1}]$, the following estimate is valid for $k = 1$

$$\left\|v(t) - v^1\right\|_\infty \le \left(\sum_{j=1}^{\infty} \frac{1}{j!} \left(L(t - t_n)\right)^j\right) (t - t_n) \max_{t_n \le s \le t} \left\|\frac{1}{2h^2} A(v^0)v^0\right\|_\infty$$

where $L > 0$ is the Lipschitz constant, associated with the nonlinear operator $A(v)$. Since $t - t_n \le \tau$ we have

$$\left\|v(t) - v^1\right\|_\infty \le \left(L\tau + O\left((L\tau)^2\right)\right) \tau L \left\|v^0\right\|_\infty = L^2\tau^2 \left\|v^0\right\|_\infty + O\left((L\tau)^3\right) \tag{19}$$

and second order of convergence in τ on each time level for a fixed $h > 0$.

It is now reasonable to allow the nonlinearities in (11) to lag one step behind and we obtain the following linear system

$$v'(t) = A_n v(t) + g, \quad t \in [t_n, t_{n+1}] \tag{20}$$

$$A_n = \frac{1}{2h^2} \operatorname{tridiag}\left(\beta_i^n, \alpha_i^n, \gamma_i^n\right), \tag{21}$$

where the solution of (20) is also the solution of (18). We use the notations

$$\alpha_i^n = -2\hat{\sigma}_{i,n}^2 S_i^2, \quad \beta_i^n = \hat{\sigma}_{i,n}^2 S_i^2, \quad \gamma_i^n = \hat{\sigma}_{i,n}^2 S_i^2$$

with

$$\hat{\sigma}_{i,n}^2 = \frac{\sigma^2}{\left(1 - \rho\lambda(S_i)S_i\Delta_i v(t_n)\right)^2}.$$

We define the solution of the linearized system at the final time level as the solution, obtained by successive resolution of (20) on each time level $[t_n, t_{n+1}]$, $i = 0, \ldots, N - 1$.

Lemma 3. *The solution of the linearized system* (20) *at the final time level, converges to the solution of the nonlinear system with rate of convergence 2 in* $\tau = t_{n+1} - t_n$, $n = 0, \ldots, N - 1$.

Proof. We define $\epsilon_{n+1} = v(t_{n+1}) - v^1(t_{n+1})$, $i = 0, \ldots, N-1$, and therefore, from (19), we have

$$\|\epsilon_1\|_\infty \leq L^2\tau^2\|v^0\|_\infty + O(L^3\tau^3) = L^2\tau^2\|f\|_\infty + O(L^3\tau^3)$$

For $\epsilon_2 = v(t_2) - v^1(t_2)$ we obtain

$$\begin{aligned}
\|\epsilon_2\|_\infty &\leq L^2\tau^2\|v^1(t_1)\|_\infty + O(L^3\tau^3) \\
&\leq L^2\tau^2\|-v(t_1) + v(t_1) + v^1(t_1)\|_\infty + O(L^3\tau^3) \\
&\leq L^2\tau^2\left(\|\epsilon_1\|_\infty + \|v(t_1)\|_{\infty, t\in[t_0,t_1]}\right) + O(L^3\tau^3) \\
&\leq L^2\tau^2\left(L^2\tau^2\|f\|_\infty + O(L^3\tau^3) + \|v(t_1)\|_\infty\right) + O(L^3\tau^3)
\end{aligned}$$

and therefore we have

$$\|\epsilon_2\|_\infty \leq L^2\tau^2\|v(t_1)\|_\infty + O(L^3\tau^3).$$

Successive application of these considerations yields

$$\|\epsilon_N\|_\infty \leq L^2\tau^2\|v(t_{N-1})\|_\infty + O(L^3\tau^3).$$

□

Further, we apply the following abuse of notations, see (21),

$$A_n = \operatorname{tridiag}\left(\beta_i^n, \alpha_i^n, \gamma_i^n\right)$$

and the solution of (20) reads, cf. Smith [25]

$$v(t) = -2h^2A_n^{-1}g + \exp\left(\frac{t-t_n}{2h^2}A_n\right)\left(v(t_n) + 2h^2A_n^{-1}g\right). \tag{22}$$

3 The LCN time stepping method for the Frey and Patie problem

The time stepping method is of particular importance to the numerical analysis of a differential problem as the stability of the scheme is a necessary condition for convergence of both linear and nonlinear problems.

Indeed, the standard fully-explicit scheme implies serious restrictions on the time step and in the nontrivial case of, e.g., singularly perturbed problems it is highly impractical as the grid Péclet number (the ratio of the convection and diffusion coefficients, scaled by h) becomes extremely large. Company et al. [6] propose fully-explicit finite difference schemes for the PDE (2). However the scheme is *stable only for the severe time step τ restriction*

$$\frac{\tau}{h^2} \leq \frac{1}{L(h)\sigma^2b^2}, \qquad L(h) = \frac{1}{(1-\rho m/h)^2} > 0 \tag{23}$$

where $m = \max\{S\lambda(S) : 0 \le S \le b\}$. Condition (23) is the necessary condition for L^∞-stability and monotonicity of the standard fully-explicit scheme.

Implicit time stepping prove to be more reliable for the problem (2), cf. Heider [18]. However, when using implicit schemes one has to invert large-sized matrices which is computationally expensive. The issues with the convergence of the Newton's iteration and whether the scheme is stable but not monotone or just unstable are not clearly answered as stability of the backward Euler scheme follows by the discrete maximum principle.

We consider now the application and the analysis of a distinct, yet also fully explicit, *unconditionally stable* approach to the time semi-discretization under the following assumption.

Assumption 2 *The solution of the problem* (2) *has continuous temporal derivatives up to second order.*

It is well-known that the Crank-Nicolson time-stepping method is based on the following approximation of the time propagator, cf. [25]:

$$\exp\left(\frac{\tau}{2h^2}A_n\right) \approx (I - \mu A_n)^{-1}(I + \mu A_n) \tag{24}$$

where $\mu = \frac{\tau}{4h^2}$. We now present the Lie-Trotter product formula:

Lemma 1 *[27] Let the matrix A can be denoted as $A = \sum_{i=1}^{M-1} A_i$. Then*

$$\exp\left(\frac{t}{h^2}A\right) = \lim_{\delta\to\infty}\left(\prod_{i=1}^{M-1}\exp\left(\frac{tA_i}{\delta h^2}\right)\right)^{\delta}, \quad \delta = 1, 2, \ldots$$

for any h, t.

The Lie-Trotter product formula is a corollary of the Baker-Campbell-Hausdorff formula (BCH) for $A = A_1 + A_2$ [19]

$$\begin{aligned} &\exp(\tau A_2)\exp(\tau A_1) = \exp(\tau\tilde{A}) \quad \text{with} \\ &\qquad \tilde{A} = A + \frac{1}{2}\tau[A_2, A_1] + \frac{1}{12}\tau^2\left([A_2,[A_2,A_1]] + [A_1,[A_1,A_2]]\right) + \ldots \end{aligned} \tag{25}$$

where $[A_2, A_1]$ denotes the commutator of A_2 and A_1. It follows from Lemma 1

$$\exp\left(\frac{\tau}{2h^2}A\right) \approx \prod_{i=1}^{M-1}\exp\left(\frac{\tau A_i}{2h^2}\right) \tag{26}$$

so (26) is a new approximation. In order to use this approximation we split the matrix A in (21) as follows:

$$A_1 = \begin{bmatrix} \alpha_1^n & \gamma_1^n & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ . & \ddots & \ddots & \ddots & . \\ 0 & \ldots & 0 & 0 & 0 \\ 0 & \ldots & 0 & 0 & 0 \end{bmatrix}, \qquad A_{M-1} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ . & \ddots & \ddots & . \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & \beta_{M-1}^n & \alpha_{M-1}^n \end{bmatrix}$$

$$A_i = \begin{bmatrix} 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ . & \ddots & \ddots & \ddots & \ddots & \ddots & . \\ 0 & \dots & \beta_i^n & \alpha_i^n & \gamma_i^n & \dots & 0 \\ . & \ddots & \ddots & \ddots & \ddots & \ddots & . \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \end{bmatrix}.$$

For any $i = 1, 2, \dots, M-1$, we obtain by (24)

$$\exp\left(\frac{\tau}{2h^2} A_i\right) \approx (I - \mu A_i)^{-1}(I + \mu A_i) \tag{27}$$

and further application of (26) and (27) results in

$$\exp\left(\frac{\tau}{2h^2} A\right) \approx \prod_{i=1}^{M-1} (I - \mu A_i)^{-1}(I + \mu A_i). \tag{28}$$

We now consider the matrix $I - \mu A_i$, $i = 2, \dots, M-2$ (similar considerations are valid for $i = 1$ and $i = M-1$). The approximation (28) is applicable to the problem (20) if the inverse matrix $(I - \mu A_{i-1})^{-1}$ exists.

Lemma 4. *The matrix $I - \mu A_{i-1}$ is a M-matrix.*

Proof. By (21) we have that

$$I - \mu A_i = \begin{bmatrix} 1 & \dots & \dots & \dots & \dots & \dots & 0 \\ . & \ddots & \ddots & \ddots & \ddots & \ddots & . \\ 0 & \dots & -\mu\beta_i^n & 1 - \mu\alpha_i^n & -\mu\gamma_i^n & \dots & 0 \\ . & \ddots & \ddots & \ddots & \ddots & \ddots & . \\ 0 & \dots & \dots & \dots & \dots & \dots & 1 \end{bmatrix}$$

so that $I - \mu A_i$ is a M-matrix if the following conditions are fulfilled

$$\begin{aligned} \textit{sign condition}: &\ 1 - \mu\alpha_i^n > 0,\ \beta_i^n \geq 0,\ \gamma_i^n \geq 0 \\ \textit{diagonal dominance}: &\ 1 - \mu\alpha_i^n - \mu\beta_i^n - \mu\gamma_i^n \geq 0 \end{aligned}$$

All of the above inequalities are trivial to check. □

Consequently, $I - \mu A_{i-1}$ is non-singular and we combine (20) and (28) to derive a new scheme

$$\acute{U}_{n+1} = -2h^2 A_n^{-1} g + \prod_{i=1}^{M-1} (I - \mu A_i)^{-1}(I + \mu A_i)\left(\acute{U}_n + 2h^2 A_n^{-1} g\right) \tag{29}$$

In order to improve the numerical accuracy of (29) we define $B_i = A_{M-i}$. By substituting B_i into (29) we deduce that

$$\grave{U}_{n+1} = -2h^2 A_n^{-1} g + \prod_{i=1}^{M-1} (I - \mu B_i)^{-1}(I + \mu B_i)\left(\grave{U}_n + 2h^2 A_n^{-1} g\right) \tag{30}$$

We take the mean value of (29) and (30) to obtain a more symmetric scheme

$$U_{n+1} = \frac{1}{2}\left(\prod_{i=1}^{M-1}(I-\mu A_i)^{-1}(I+\mu A_i) + \prod_{i=1}^{M-1}(I-\mu B_i)^{-1}(I+\mu B_i)\right) \cdot \left(U_n + 2h^2 A_n^{-1} g\right) - 2h^2 A_n^{-1} g \tag{31}$$

The presented method is referred to as the *local Crank-Nicolson (LCN) method* as proposed by Abduwali et al. [1,20].

The matrix $(I+\mu A_i)$ can be denoted by a simple form for $i = 2,3,\ldots,M-2$

$$(I+\mu A_i) = \begin{pmatrix} I_{i-2} & \cdot & \cdot \\ \cdot & \bar{R}_i & \cdot \\ \cdot & \cdot & I_{M-i-2} \end{pmatrix}, \qquad \bar{R}_i = \begin{pmatrix} 1 & 0 & 0 \\ \mu\beta_i^n & 1+\mu\alpha_i^n & \mu\gamma_i^n \\ 0 & 0 & 1 \end{pmatrix} \tag{32}$$

where I_i is the $i \times i$ identity matrix.

Similar to (32) we derive

$$(I-\mu A_i)^{-1} = \begin{pmatrix} I_{i-2} & \cdot & \cdot \\ \cdot & \hat{R}_i^{-1} & \cdot \\ \cdot & \cdot & I_{M-i-2} \end{pmatrix}, \ \hat{R}_i^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\mu\beta_i^n}{1-\mu\alpha_i^n} & \frac{1}{1-\mu\alpha_i^n} & \frac{\mu\gamma_i^n}{1-\mu\alpha_i^n} \\ 0 & 0 & 1 \end{pmatrix} \tag{33}$$

We obtain an explicit expression of U^{n+1} and, clearly, (31) is an explicit scheme.

Remark 5. The final scheme (31) shows that in the case of nonzero boundary conditions, i.e. $g \neq \mathbf{0}$, one has to invert the system matrix A_n in order to derive the numerical solution and the method is still not completely explicit. The efficient treatment of this issue is ongoing. However, the method is completely explicit for the case of zero boundary conditions, e.g. for the butterfly spread option [18]. Further, when considering the computational efficiency of the method one may recall, for example, the parallel computing of (29) and (30).

3.1 Positivity and stability

In this subsection we investigate the positivity property of the numerical solution of (31) and the stability of the fully discrete scheme. Stability analysis is a necessary part of the numerical analysis but positivity (some discrete maximum principle) also has to be considered when solving problems in finance since the prices and costs are always positive and this essential property should be preserved by the numerical method.

Theorem 3. *The numerical scheme* (31) *is unconditionally stable.*

Proof. Analogously to the considerations in [20] the application of the Gerschgorin theorem [25] to the matrix A_i implies that the non-zero matrix eigenvalues lie in the disc

$$\left|z + 2\sigma_{i,n}^2 S_i^2\right| \leq 2\sigma_{i,n}^2 S_i^2$$

and hence they are negative. Then, by the spectral mapping theorem,

$$|\eta_i| \leq \frac{|I + \mu\zeta_i|}{|I - \mu\zeta_i|} \leq 1$$

for any of the eigenvalues η_i of the the matrix $(I - \mu A_i)^{-1}(I + \mu A_i)$, corresponding to the eigenvalues ζ_i of A_i. Further, we have that $\prod_{i=1}^{M-1} |\eta_i| \leq 1$ and $\rho\left(\prod_{i=1}^{M-1}(I - \mu A_i)^{-1}(I + \mu A_i)\right) \leq 1$, where $\rho(A)$ denotes the spectral radius of the matrix A. Stability follows from this estimate, derived by the Gerschgorin theorem, as discussed in [25].

Theorem 4. *If $f(S)$ is positive and we assume that*

$$\frac{\tau}{2h^2} \leq \frac{1}{\hat{\sigma}_{i,n}^2 S_i^2} = \frac{\overline{\delta}_0^2}{\sigma^2 b^2}, \tag{34}$$

then the solution of (31) *is positive on each time level t_{n+1}, $n = 0, \dots, N-1$.*

Proof. We analyze the matrix $I + \mu A_i$, $i = 2, \dots, M-2$, (analogously for $i = 1$ and $i = M-1$)

$$I + \mu A_i = \begin{bmatrix} 1 \dots & \dots & \dots & \dots & \dots & 0 \\ . \ \ddots & \ddots & \ddots & \ddots & \ddots & . \\ 0 \dots & \mu\beta_i^n & 1 + \mu\alpha_i^n & \mu\gamma_i^n & \dots & 0 \\ . \ \ddots & \ddots & \ddots & \ddots & \ddots & . \\ 0 \dots & \dots & \dots & \dots & \dots & 1 \end{bmatrix}$$

All entries of the matrix $I + \mu A_i$ are positive by (34). Successful application of this consideration and Lemma 4 (one observes the positive entries of $(I - \mu A_i)^{-1}$ in (33)) yields a positive solution U_{n+1} of (31) on each time level, where U_0 is the restriction of $f(S)$ on the spatial grid, if $g = \mathbf{0}$.

If $g \neq \mathbf{0}$ we have that $A_n^{-1}g$ is negative since $-A_n$ is a M-matrix and g is positive. Therefore, since $U_n + 2h^2 A_n^{-1} g$ can be always considered positive for sufficiently small h, we have a positive solution of (31). □

Indeed, the condition (34) is the discrete analogue of the ellipticity Assumption A4 in [13] and also of the monotonicity condition, considered in Lemma 1 and 2. It enforces slightly relaxed bound on the temporal step τ than the one given by (23), obtained by Company et al. [6]. However, let us emphasize again that in our numerical analysis, *it is not a necessary condition for stability* and, as discussed in Section 5, it is solely a sufficient condition for positivity.

3.2 Consistency and convergence

In this section we discuss the consistency, monotonicity and convergence properties of the fully discrete scheme (31).

Lemma 5. *The local Crank-Nicolson method* (31) *has the second-order approximation in time.*

Proof. Starting from the following expansion formula

$$\exp\left(\frac{\tau}{2h^2}A_i\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\tau}{2h^2}A_i\right)^n,$$

the local Crank-Nicolson approximation (27) can be regarded as

$$(I - \frac{\tau}{2}\frac{1}{2h^2}A_i)^{-1}(I + \frac{\tau}{2}\frac{1}{2h^2}A_i) = I + \frac{\tau}{2h^2}A_i - \frac{\left(\frac{\tau}{2h^2}A_i\right)^2}{2} + \frac{\left(\frac{\tau}{2h^2}A_i\right)^3}{4} - \dots$$

and is a second-order approximation in time to $\exp\left(\frac{\tau}{2h^2}A_i\right)$. Considering (28), we derive

$$\prod_{i=1}^{M-1}(I - \mu A_i)^{-1}(I + \mu A_i) = \prod_{i=1}^{M-1}\left(\exp\left(\frac{\tau}{2h^2}A_i\right) - O\left(\left(\frac{\tau}{2h^2}\right)^2\right)\right)$$
$$= \prod_{i=1}^{M-1}\exp\left(\frac{\tau}{2h^2}A_i\right) - O\left(\left(\frac{\tau}{2h^2}\right)^2\right) = \exp\left(\frac{\tau}{2h^2}A\right) + O\left(\left(\frac{\tau}{2h^2}\right)^2\right)$$

since $\prod_{i=1}^{M-1}\exp\left(\frac{\tau}{2h^2}A_i\right)$ is a second-order approximation in time to $\exp\left(\frac{\tau}{2h^2}A\right)$. □

It follows that the LCN time stepping method is first-order consistent in time. This consideration also follows from the BCH formula (25).

Next, we investigate the error $e_n = U_{n+1} - v(t_{n+1})$, where $v(t_{n+1})$ is the solution of linearized system (20). Subtracting (22) from (31) we obtain

$$e_n = \left(\exp\left(\frac{\tau}{2h^2}A_n\right) + O\left(\left(\frac{\tau}{2h^2}\right)^2\right)\right)(U_n + 2h^2A_n^{-1}g) - \exp\left(\frac{\tau}{2h^2}A_n\right)$$
$$\cdot\left(v(t_n) + 2h^2A_n^{-1}g\right) = \exp\left(\frac{\tau}{2h^2}A\right)e_{n-1} + O\left(\left(\frac{\tau}{2h^2}\right)^2\right)2h^2A_n^{-1}g$$

and therefore

$$e_N = \exp\left(\frac{\tau}{2h^2}A_{N-1}\right)e_{N-1} + O\left(\left(\frac{\tau}{2h^2}\right)^2\right)2h^2A_{N-1}^{-1}g$$
$$= \exp\left(\frac{\tau}{2h^2}A_{N-1}\right)\left(\exp\left(\frac{\tau}{2h^2}A_{N-2}\right)e_{N-2} + O\left(\left(\frac{\tau}{2h^2}\right)^2\right)2h^2A_{N-2}^{-1}g\right)$$
$$+ O\left(\left(\frac{\tau}{2h^2}\right)^2\right)2h^2A_{N-1}^{-1}g = \prod_{i=0}^{N-1}\exp\left(\frac{\tau}{2h^2}A_i\right)e_0$$
$$+ O\left(\left(\frac{\tau}{2h^2}\right)^2\right)\left(2h^2A_{N-1}^{-1}g + \exp\left(\frac{\tau}{2h^2}A_{N-1}\right)2h^2A_{N-2}^{-1}g + \dots\right)$$

Finally, since the initial data $f(S)$ can be projected exactly on the grid, i.e., $e_0 = 0$, and by the condition for the temporal step (34) we obtain a first-order convergence in τ of the numerical solution to the solution of the linearized system.

Rewriting the fully-discrete scheme (31) in the following form

$$U_{i,n+1} = H(U_{i-1,n}, U_{i,n}, U_{i+1,n}), \qquad i = 1, \dots, M-1 \tag{35}$$

we introduce the definition of monotone scheme, see Grossmann and Roos [16].

Definition 2. *The scheme* (35) *is* monotone *if H is nondecreasing in each argument.*

Theorem 5. *The scheme* (31) *is monotone and it also satisfies the discrete maximum and comparison principles. The numerical solution converges to the viscosity solution of the problem* (2) *(if there is one).*

Proof. By the condition (34) we deduce that all elements of the matrices (32) and (33) are positive. Therefore H is non-decreasing in each argument and the scheme is monotone. The discrete maximum and comparison principles [16] follow by the diagonal dominance of the matrices (32), (33).

Assembling all results together – convergence of the semi-discrete scheme, convergence of the linearized system, consistency, stability and monotonicity of the fully-discrete scheme – *convergence of the solution of fully-discrete scheme to the viscosity solution of* (2) *follows by [4].* □

Corollary 2 *A direct consequence of the monotonicity of the fully-discrete scheme is the monotonicity of the numerical solution U_{n+1} w.r.t. the spatial variable if $f(S)$ is monotone w.r.t. the spatial variable.*

4 The Liu and Yong model

In this section we discuss briefly another illiquid market model, proposed by Liu and Yong [23]. Let us consider the problem of hedging a terminal value claim with maturity T and payoff $f(S)$ for the stock price SDE for $t \geq 0$

$$dS(t) = \{\mu(t, S(t)) + \lambda(t, S(t))\eta(t)\}\, dt + \{\sigma(t, S(t)) + \lambda(t, S(t))\zeta(t)\}\, dW(t)$$

where $\mu(t, S(t))$ and $\sigma(t, S(t))$ are the expected return and the volatility, respectively, $\lambda(t, S(t))$ is the price impact function of the trader for some processes $\eta(t)$ and $\zeta(t)$. The nonlinear BS problem is given as

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2\left(1 - \lambda(S,t) S \frac{\partial^2 V}{\partial S^2}\right)^2} \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \qquad (S,t) \in \Omega_T \\ &\qquad\qquad\qquad\qquad V(S,T) = f(S), \qquad S \in \Omega := (0, +\infty) \end{aligned} \tag{36}$$

for the case of constant interest rate ($r \geq 0$) and reference volatility ($\sigma > 0$).

The existence and uniqueness of a classical (Hölder) solution of problem (36) and the comparison principle are studied by the well-posedness condition [23]

$$1-\lambda(S,t)S\frac{\partial^2 V}{\partial S^2}\geq \delta_0>0, \quad (S,t)\in \Omega\times[0,T] \tag{37}$$

if $f(e^x)$ is Lipschitz continuous and $e^{-\beta\sqrt{1+x^2}}f(e^x)$ is bounded for some $\beta\geq 0$.

The price impact function $\lambda(S,t)$ for the nonlinear problem (36),

$$\lambda(S,t)=\begin{cases}\frac{\gamma}{S}(1-e^{-\beta(T-t)}), & \underline{S}\leq S\leq \overline{S}\\ 0, \text{ otherwise}\end{cases} \tag{38}$$

reflects the assumption that as a trader buys, the stock price goes up and as the trader sells, the stock price goes down. The constant price impact coefficient $\gamma>0$ measures the price impact per traded share and $\underline{S}$ and $\overline{S}$ represent, respectively, the lower and upper limit of the stock price within which there is a price impact. The price impact function influences heavily both the differential problem and the numerical method. By inspecting (38) one observes that the PDE (36) *is obviously linear outside the interval* $(\underline{S},\overline{S})$.

The major difference between the presented two illiquid market problems is the following: *while the nonlinear operator of the Frey model* (2) *is not elliptic at the strike price near maturity the nonlinear operator of* (36) *is elliptic because of the choice* (38) *of the price impact function* $\lambda(S,t)$ *(as* $\lambda(S,T)=0$*).* This smoothing property is also present for (36) but there arises another question whether the smoothing is strong enough to damp the singularity of the Gamma in time, i.e. for which values of the parameters γ and β the price impact function $\lambda(S,t)$ (increasing w.r.t. time to maturity) can still control $S\frac{\partial^2 V}{\partial S^2}$ (decreasing) in time. We recall the results in [23] on this issue as detailed analysis of this issue is out of the scope of the current manuscript.

The numerical scheme for (36) is constructed analogously as for the Frey model. There are, however, few issues that has to be take into accounts. The degeneracy of the BS operator at $S=0$ influences the spatial discretization as the equation is convection-dominated in this neighbourhood. The basic and efficient approach to this issue is maximal use of central differencing [29], i.e. central approximation of the convection term far away from $S=0$ and upwind discretization in the few nodes where the convection dominance is present. The explicit time-dependence of the spatial discretization matrix also has to be resolved as the semi-discrete solution formula (22) *is not valid unless* $[A(t),A(s)]\neq 0$ (which is the general case), where $[A,B]:=AB-BA$ denotes the commutator of the matrices A and B [17]. We further refer to the Magnus series expansion for the treatment of this issue [24].

We investigate the considered problem (36) only experimentally as rigorous theoretical results are subject to detailed extension of the analysis already presented in the previous sections.

5 Numerical experiments

Numerical experiments are presented in this section in order to illustrate the stability and convergence properties of the method. We stress the fact that no smoothing techniques are applied to the terminal condition (5) and no grid refinement is used. The considered payoff corresponds to the vanilla call option.

We solve numerically the presented Frey and Patie model (FP) (2) and the Liu and Yong model (LY) (36) with the payoff (5). The parameters are:

1. *(FP)* $\lambda(S) = 1$. The strike price is $E = 100$, the volatility is $\sigma = 0.2$, the maturity date - $T = 0.25$ and the artificial boundary location is $b = 200$.
2. *(LY)* $\lambda(S,t)$ as in (38) with $\beta = 100$, $\gamma = 1$ and $\underline{S} = 20$, $\overline{S} = 80$. The strike price is $E = 50$, the volatility is $\sigma = 0.4$, the interest rate is $r = 0.06$, the maturity date - $T = 0.25$ and the artificial boundary location is $b = 200$.

In the tables below are presented the computed discrete maximum and $RMSE$ (*root mean square error*) norms of the error $E = U - V$, V is the restriction of the exact solution $V(S,t)$ on the grid, by the formulas

$$\|E\|_\infty = \max_i \|U_i^N - V_i^N\|, \ \|E\|_{RMSE} = \sqrt{\frac{1}{M_{br}} \sum_{i:S_i \in [0.8E;1.2E]} \left(U_i^N - V_i^N\right)^2},$$

where the area of interest to be tracked by the $RMSE$ norm is chosen to be $S_i \in [0.8E, 1.2E]$ (the area of practical interest) and M_{br} is the number of spatial nodes in this area.

The numerical rate of convergence (RC) is calculated using the double mesh principle

$$RC = \log_2(E^M/E^{2M}), \qquad E^M = \|V^M - U^M\|,$$

where $\|\cdot\|$ is the discrete norm, V^M and U^M are respectively the exact solution and the numerical solution, computed at the mesh with M sub-intervals.

The numerical results are focused on the particular ratio $:= \tau/(2h^2)$, considered in the monotonicity condition (34). For the simple case of $\rho = 0$ we have the linear Black-Scholes operator, where the interest rate and dividend rate are equal to 0. The condition (34) now reads as (with parameters as given in (FP))

$$\frac{\tau}{2h^2} \leq 0.000625 \tag{39}$$

Table 1 displays the convergence results of the presented LCN method for the discussed simple linear test equation w.r.t. to the reference solution, generated by the MATLAB function `blsprice(Price, Strike, Rate, Time, Volatility, Yield)`. We observe that even though (34) is violated there are no stability issues. The accuracy, of course, profits from smaller values of the ratio.

Figures 1 and 2 illustrate the numerical solution for $\rho = 0$ and $\rho = 0.06$, respectively, for $ratio = 0.1$ and $M = 1280$. We conclude that there are no stability issues both in the linear and nonlinear case. The conclusion completely corresponds to the *unconditional stability*, obtained in Section 3.1.

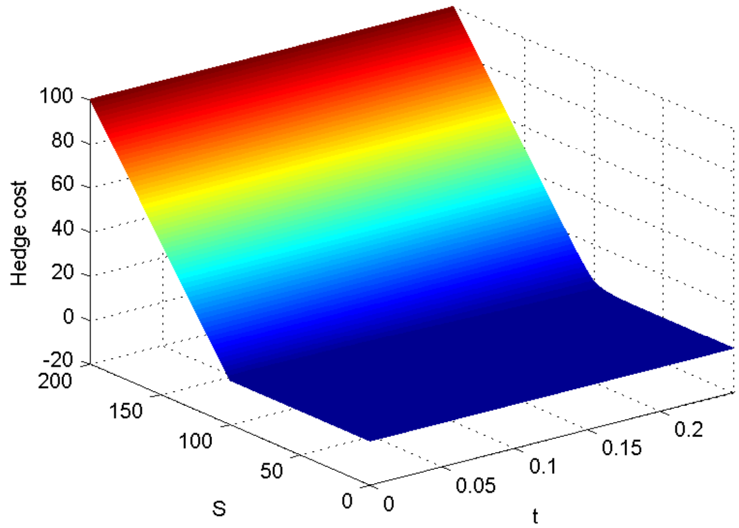

Fig. 1. $\rho = 0$

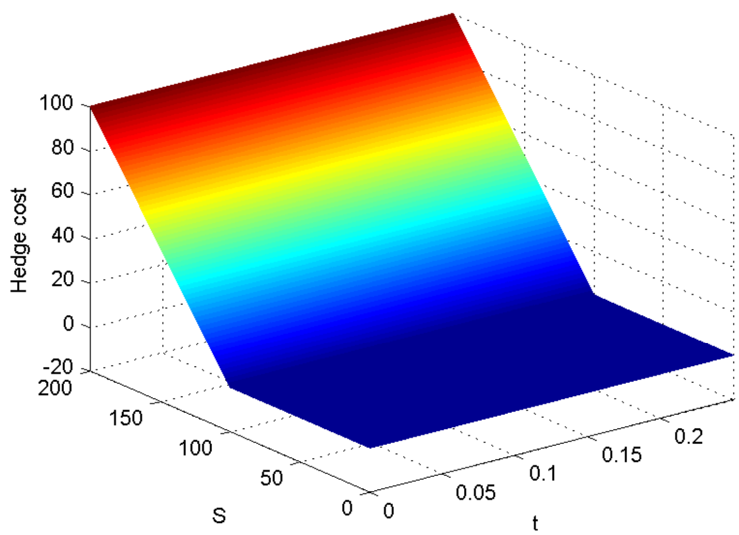

Fig. 2. $\rho = 0.06$

Table 1.

$M \times N$	ratio = 0.01				ratio = 0.001			
	E_∞^N	RC	$RMSE$	RC	E_∞^N	RC	$RMSE$	RC
160	4.716e-1	-	2.244e-1	-	1.269e-2	-	6.742e-3	-
320	1.287e-1	*1.874*	6.659e-1	*1.753*	3.185e-3	*1.995*	1.704e-3	*1.985*
640	3.195e-2	*2.010*	1.721e-2	*1.952*	7.970e-4	*1.999*	4.278e-4	*1.994*
1280	7.962e-3	*2.005*	4.331e-3	*1.991*	1.993e-4	*1.999*	1.072e-4	*1.997*

However, we also observe in Figure 2 that the solution does not show the smoothing property 10 that characterizes parabolic problems and that can be observed in Figure 1. In further support to this consideration we illustrate the corresponding numerical Gamma greek on Figures 3 and 4. The higher the value of the nonlinearity parameter ρ the stronger the nonlinearity in the PDE (2) and therefore the smaller δ_0 in the well-posedness condition (3) because of the non-smoothness of the payoff.

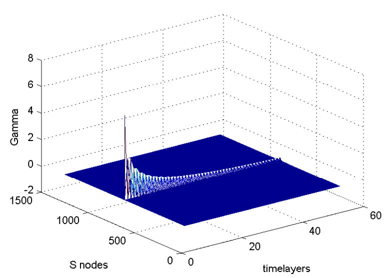

Fig. 3. $\rho = 0$

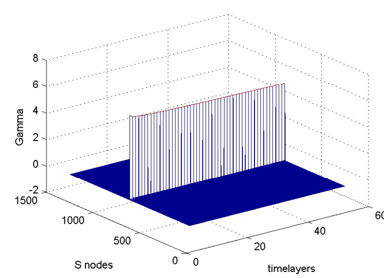

Fig. 4. $\rho = 0.06$

Let us now briefly consider the standard fully-explicit scheme, suggested in [6] and compare the numerical results. Figure 5 visualizes the numerical Gamma for $M = 640$ and $ratio = 0.001$ as we observe two major sources of oscillations, resulting from violating the stability condition (23). First, they appear around the strike, where the strength of these oscillations is controlled by the nonlinear parameter ρ and the size of the spatial step h (i.e. how accurately the numerical Gamma of the payoff approximates the analytical Gamma, the Dirac's δ-function). Away from the strike oscillations appear as the condition (23) is more restrictive for large values of S (away from the influence of the nonlinearity marginal values of Gamma are enough to destabilize the scheme).

Figure 6 shows the numerical Gamma for $M = 320$ and $ratio = 0.0001$. We observe stability (not present for $M = 640$ and the same value of $ratio$) but the figure implies that there is smoothing in time (deterioration of the strong norm

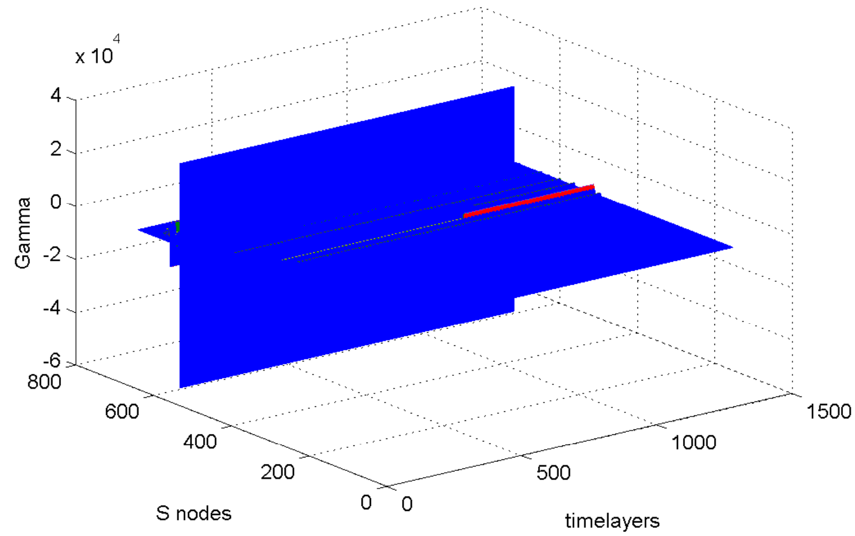

Fig. 5. $\rho = 0.06$

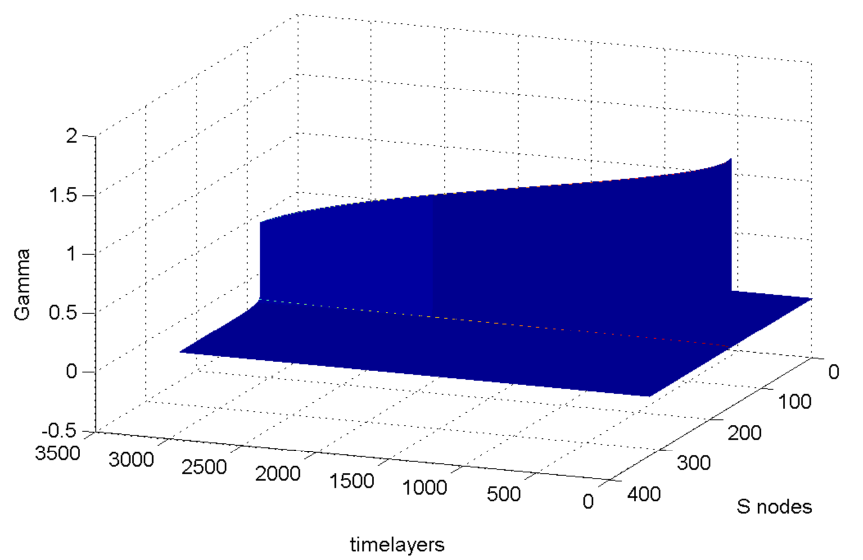

Fig. 6. $\rho = 0.06$

of the Gamma) - side effect of the inaccurate numerical approximation of the δ-function - and this property is not present for the differential problem. Indeed, the standard fully-explicit scheme is either unstable or, if decides to compute with the stability condition (23), it gives wrong qualitative information.

Taking these conclusions into account we compute the convergence of the numerical method, applied to (2) with parameters as given in (FP), for $\rho = 0.01$ in Table 2. *The error is calculated w.r.t. the numerical solution for* $M = 640$ and the size of τ is determined by the value of *ratio*.

Table 2.

	ratio = 0.001				ratio = 0.0001			
M	E_∞^N	RC	$RMSE$	RC	E_∞^N	RC	$RMSE$	RC
40	9.983e-1	-	6.625e-1	-	1.062e-1	-	5.853e-2	-
80	9.152e-1	*0.126*	6.631e-1	*0.089*	1.875e-2	*2.502*	1.045e-2	*2.486*
160	8.607e-1	*0.089*	5.984e-1	*0.058*	9.647e-3	*0.969*	7.142e-3	*0.548*
320	8.851e-1	*-.040*	6.203e-1	*-.052*	1.144e-3	*3.077*	8.964e-4	*2.994*

Clearly, we observe no convergence for $ratio = 0.001$ because the condition (39) is not even remotely satisfied. By the RMSE behaviour we observe that divergence is present in the strike region where the δ-function is concentrated. Actually, the condition (39) is even stricter in the nonlinear case (34) since $\overline{\delta}_0$ is smaller than 1 and it is getting smaller as $h \to 0$ (the more precisely the numerical Gamma approximates the $\partial^2 V/\partial S^2$ the smaller $\overline{\delta}_0$ is). Convergence is present for the case $ratio = 0.0001$ but *let us remind the reader and the convergence is computed w.r.t. the numerical solution for* $M = 640$*. Since the problem* (2) *is not parabolic at the strike and no comparison principle is present convergence to the viscosity solution cannot be established since we do not know if there is such a solution.*

We further investigate the influence of the liquidity parameter ρ and the space step h on the numerical solution. In Figures 7 and 8 we present the numerical solution for different values of ρ and h. The value of the hedge cost increases as ρ increases as also mentioned in [13].

It can observed, see Figures 9 and 10, that small values of h deteriorate the smoothing property of the numerical solution as the numerical Gamma copies the behaviour of the δ-function.

We now present the convergence results for the problem (LY), Table 3. The convergence is computed w.r.t. the numerical solution for $M = 1280$ and number of time levels, determined by the value of *ratio*.

Stable convergence for the given values of *ratio* is observed although the monotonicity condition is unlikely to be satisfied. In comparison with the Frey and Patie problem we are now computing convergence on a finer mesh without implying serious restriction on the time step τ. It is evident that small values

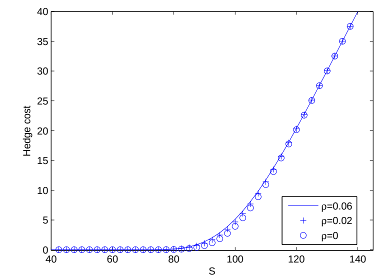

Fig. 7. $M = 80$, ratio=0.001

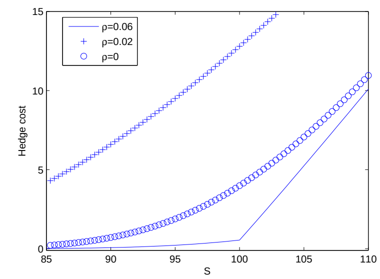

Fig. 8. $M = 640$, ratio=0.001

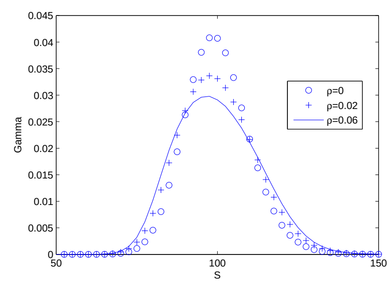

Fig. 9. $M = 80$, ratio=0.001

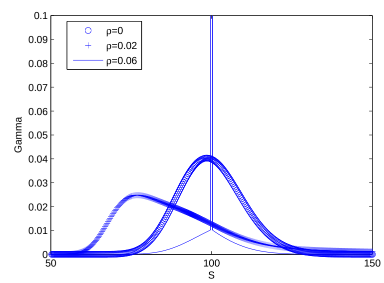

Fig. 10. $M = 640$, ratio=0.001

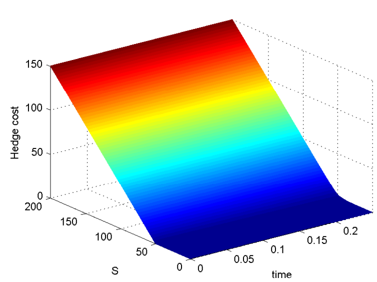

Fig. 11. $M = 1280$, ratio=0.1

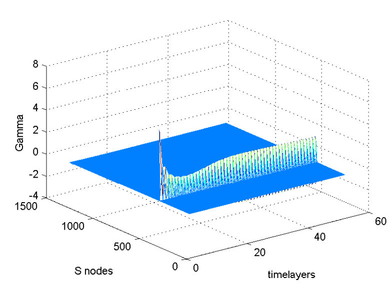

Fig. 12. $M = 1280$, ratio=0.1

Table 3.

	ratio = 0.001				ratio = 0.0001			
M	E_∞^N	RC	$RMSE$	RC	E_∞^N	RC	$RMSE$	RC
40	9.988e-2	-	6.685e-2	-	5.662e-1	-	5.334e-2	-
80	4.477e-2	*1.143*	2.890e-2	*1.121*	2.785e-2	*1.023*	2.607e-2	*1.033*
160	1.717e-2	*1.383*	1.288e-2	*1.167*	1.273e-2	*1.130*	1.220e-2	*1.096*
320	6.409e-3	*1.422*	5.387e-3	*1.257*	5.372e-3	*1.244*	5.231e-3	*1.221*
640	1.979e-3	*1.695*	1.728e-3	*1.640*	1.774e-3	*1.599*	1.556e-3	*1.749*

of h do not have any deteriorating impact on the numerical solution even for relatively large $ratio$. This can be related to the smoothing effect that the choice of price impact factor function $\lambda(S, t)$ (38) has both on the differential and discrete problems. Our conclusion is further supported by Figures 11 and 12 that one may compare with Figures 2 and 4. It is interesting to observe that the smoothing effect is present only for small time to maturity.

The impact of the parameter γ on the numerical solution and the Gamma greek is illustrated by Figures 13 and 14. The parameter's role is analogous to ρ in the Frey and Patie model – the higher the value the more illiquid the market is (and the higher the hedge cost is) and the stronger the nonlinearity in the differential problem is. We now observe deterioration of absolute value of the Gamma even for small values of the space step h.

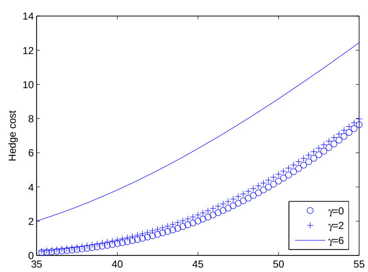

Fig. 13. $M = 640$, ratio=0.001

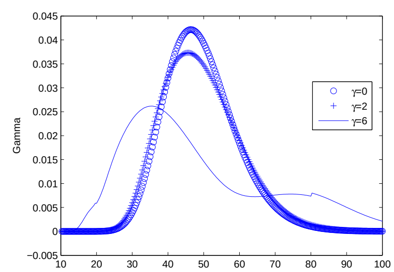

Fig. 14. $M = 640$, ratio=0.001

6 Conclusion

In this paper we present the numerical analysis of the fully-nonlinear problem, modelling the replication of contingent claims in illiquid markets. The discrete scheme is shown to be consistent and unconditionally stable. However, due to the non-parabolic nature of Frey and Patie model at the strike the monotonicity condition in practice can not be satisfied for small space discretization step. For the Liu and Yong model we observe stable convergence results and parabolic behaviour of the numerical solution even without strongly considering the monotonicity restriction on the time step.

Further we will combine the presented numerical method with the transparent boundary condition (TBC) method [9].

7 Acknowledgement

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8 Appendix: Introduction to Viscosity Solution Theory

We briefly introduce the reader to the viscosity solution theory following [8,12,31].

Let us consider the following Cauchy problem (here $\mathcal{D}u$ denotes the derivative of $u(x,t)$ in the spatial variables where $\mathcal{D}^2 u$ denotes the Hessian matrix)

$$u_t + \mathcal{F}(x,t,\mathcal{D}u,\mathcal{D}^2 u) = 0 \qquad \text{in} \quad Q = \mathbb{R}^n \times (0,T] \tag{40}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^n \tag{41}$$

We further assume that $\mathcal{F}$ is degenerate elliptic and proper according to the following definition.

Definition 3. *Eq.* (40) *is* degenerate parabolic *if F is* degenerate elliptic

$$\mathcal{F}(x,t,z,q,X+Y) \leq \mathcal{F}(x,t,z,q,X) \quad \forall Y \geq 0, \quad X,Y \in \mathbb{S}^n$$

and $\mathcal{F}(x,t,z,q,X) \in C(\overline{J})$, where $\mathbb{S}^n$ denotes the space of $n \times n$ symmetric matrices with the usual ordering and $J = Q \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$, $X,Y \in \mathbb{S}^n$. If there are positive constants θ and Θ such that

$$\theta \operatorname{tr} Y \leq \mathcal{F}(x,t,z,q,X) - \mathcal{F}(x,t,z,q,X+Y) \leq \Theta \operatorname{tr} Y \quad \forall Y \geq 0$$

where $\operatorname{tr} Y$ *denotes the trace of the matrix Y we then say that $\mathcal{F}$ is* uniformly elliptic *and eq.* (40) *is* uniformly parabolic. *Also, $\mathcal{F}$ is* proper *if $\mathcal{F}$ satisfies:*

$$\exists c_0 > 0 \quad : \quad c_0 r + F(x,t,r,q,X) \leq c_0 s + F(x,t,s,q,X) \quad \textit{for} \quad r \leq s$$

and $\forall (x,t,r,q,X), (x,t,s,q,X) \in J$.

Let us consider $u \in C^{2,1}(Q)$ and

$$u_t(x,t) + \mathcal{F}(x,t,u(x,t),\mathcal{D}u(x,t),\mathcal{D}^2 u(x,t)) \leq 0 \quad \text{for all} \quad (x,t) \in Q$$

i.e. u is classical subsolution of $u_t + \mathcal{F} = 0$ and $\mathcal{F}$ is degenerate parabolic. Suppose that $\phi \in C^{2,1}(Q)$ and (x,t) is a local maximum of $u - \phi$ in Q so that $u_t(x,t) = \phi_t(x,t), \mathcal{D}u(x,t) = \mathcal{D}\phi(x,t)$ and $\mathcal{D}^2 u(x,t) \leq \mathcal{D}^2\phi(x,t)$ and *by the ellipticity property* we have

$$\begin{aligned}\phi_t(x,t) &+ \mathcal{F}(x,t,u(x,t),\mathcal{D}\phi(x,t),\mathcal{D}^2\phi(x,t)) \\ &\leq u_t(x,t) + \mathcal{F}(x,t,u(x,t),\mathcal{D}\phi(x,t),\mathcal{D}^2\phi(x,t)) \leq 0\end{aligned}$$

The inequality $\phi_t(x,t) + \mathcal{F}(x,t,u(x,t),\mathcal{D}\phi(x,t),\mathcal{D}^2\phi(x,t)) \leq 0$ does not depend on the derivatives of u so we may consider an arbitrary function u to be (some kind of generalized) *subsolution* of $u_t + \mathcal{F} \leq 0$ if

$$\phi_t(x,t) + \mathcal{F}(x,t,u(x,t),\mathcal{D}\phi(x,t),\mathcal{D}^2\phi(x,t)) \leq 0$$

whenever $\phi \in C^{2,1}(Q)$ and (x,t) is a local maximum of $u - \phi$. The viscosity *supersolution* is defined analogously by replacing 'maximum' with 'minimum'.

A viscosity solution of problem (40), (41) is both a viscosity subsolution and a viscosity supersolution and a classical solution is also a viscosity solution.

We say that a *comparison principle* (extension of the maximum principle) between the viscosity subsolution $u(x,t)$ and the supersolution $v(x,t)$ holds if $u(x,0) \leq v(x,0)$ implies $u(x,t) \leq v(x,t)$ in Q. Further assumptions on the non-linear operator $\mathcal{F}$ are, however, needed, e.g. $\mathcal{F}$ is locally uniformly continuous in q, $\mathcal{F}$ is Lipschitz continuous in x and satisfies the Kruzhkov's structure condition as well as the standard assumption for the growth at infinity of (41), cf. Zhan [31].

The existence and uniqueness results for viscosity solution of (40) are based on the comparison principle and carried out by two major techniques - approximate method via the stability property by the Arzela-Ascoli compactness theorem or the Perron method by the construction of suitable barrier functions. Regularity results are also provided by a maximum principle and construction of barriers of solutions, cf. the discussions in [12,31].