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Abstract Given a multiphysics problem with components of different dynamical behaviour reduction-multirate methods start with a model order reduction of the slow part system and apply than a multirate ODE-integration to the whole system. This approach lets us profit as much as possible from properties of the given system related to computational efficiency. In this paper we present the motivation and the idea behind this reduction-multirate approach.

1 Introduction

In general, the modeling of a multiphysical setting leads to a coupled system with largely differing dynamical behaviour. Possibly after a semi-discretization of the spatial variables, these models are often given by coupled systems of ODEs. Now, the existence of stiff parts suggest which type of time domain method should be applied. Furthermore the most active part, i.e., the part with the highest frequencies, determines the step size to be used.

Multirate ODE-solvers allow us to use different step sizes for each subsystems. The use of inherent step sizes for the subsystems with different dynamical behaviour gives us potential to enhance the numerical efficiency (the performance concerning computation time). The crucial part of a multirate solver is the coupling of the different scales, i.e., the computation of the coupling variables. We follow the idea of *compound-step* methods, which was first presented in [5]. Often the physics of the underlying systems justifies the usage of a certain coupling type. Although one saves computation time due to larger step sizes for latent components (macrostep), usually a large and stiff system remains to be solved in each macrostep.

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In the last years, model order reduction has been developed to a reliable technique to solve high dimensional systems of differential equations efficiently [8]. Until now there has been no work on combining model order reduction with multirate ODE-solvers. We present related ideas and concepts of the reduction-multirate methods and a suitable multiphysics example.

2 Mixed Multirate Methods

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Multirate integration schemes are interesting for systems of differential equations with parts of very different dynamic behaviour. As in most of the previous works about multirate methods we consider a system with very fast dynamic changes, the so-called active part, and a considerable slower part, the so-called latent behaviour, both parts depend on each other. In an ODE-framework that reads

$$\dot{\mathbf{y}}_A = \mathbf{f}_A\left(\mathbf{y}_A, \boxed{\mathbf{y}_L}\right) \quad \mathbf{y}_A(t_0) = \mathbf{y}_{A,0} \tag{1}$$

$$\dot{\mathbf{y}}_L = \mathbf{f}_L \left(\boxed{\mathbf{y}_A}, \mathbf{y}_L \right) \quad \mathbf{y}_L(t_0) = \mathbf{y}_{L,0}.$$
⁽²⁾

The coupling is illustrated by the boxes around the coupling terms. There are several approaches how to get such a partition. While [3] or [4] deal with a given monolithic system and partition it dynamically we are following the setting of a given partition like in [5] or [2]. This is justified since we are considering multiphysics problems and usually the underlying physical behaviour defines a certain dynamical behaviour. The idea of a mixed multirate intergration scheme is given in [2] and is based on the idea of multirate compound step Runge-Kutta methods first presented in [5]. In the latter the coupling is realized by integrating the latent and the active component coupled together but with different stepsizes: The latent component with a large macrostep H, the active one with a small microstep of size h = H/m. The remaining m-1 microsteps are computed with interpolating the latent component. Günther and Rentrop presented ROW-methods for multirate integration schemes in [6] and Bartel and Günther developed W-methods for compound-step multirate integrators in [7]. Here compound-step and remaining micro-steps are computed with the same integration scheme. In mixed multirate methods different schemes can be used for compound and micro-steps. That can be reasonable if the dimension of the active part is small compared to the whole system and a high accuracy is desired so a method of higher order can be applied to the remaining microsteps. This is exactly the case for the here presented topic so we follow [2] and apply a 2(3)-ROW-scheme for the compound step and a 3(4)-ROW-scheme for the remaining micro-steps. A set of coefficients can be also found in [2].

3 Model Order Reduction with Balanced Truncation

In multirate context we deal very often with a high-dimensional slow part of the ODE-system and only few active components. The question is whether we can gain efficiency not only by adapting stepsize but also to consider the dimension of the slow part: The idea is to apply a model order reduction before integrating the system. We assume a linear ODE-system for the slow part system so we can use the methods of linear model order reduction. We now present briefly the method of balanced truncation as it can be found in [8]. For a given linear time invariant (LTI) system

$$\dot{\mathbf{x}} = \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{u}(t) \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$$
(3)

$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x} \tag{4}$$

model order reduction computes rectangular biorthogonal projection matrices \mathbf{V}_r , \mathbf{W}_r so that the dimension r of reduced system matrices $\mathbf{W}_r^T \mathbf{A} \mathbf{V}_r$, $\mathbf{W}_r^T \mathbf{B}$, $\mathbf{C} \mathbf{V}_r$ is relevantly smaller than of the original system ($r \ll n$). While the output of the reduced system $\mathbf{y}_r(t)$ shall approximate the original output as good as possible. The idea of balanced truncation is now to keep all important states and truncate all states who need a lot energy to be reached and to be observed. Truncating states that are difficult to reach and to observe become equivalent if the system is balanced. One gets such a balanced system by solving Lyapunov-Equations and construct a suitable transformation matrix. Balanced truncation has many advantages over other MOR methods, e.g. i) the input and the output matrix are considered in the computation of projection matrices and ii) an efficient error estimate is available:

$$\|\mathbf{H} - \mathbf{H}_r\|_{\mathscr{H}_{\infty}} \le 2\sum_{i=r+1}^n \sigma_i \tag{5}$$

while **H** denotes the transfer function, σ the eigenvalues of the Gramian matrices of the balanced system and *r* is the dimension of the truncated system. Due to the fact that for this method the Lyapunov-Equations have to be solved, the method is less efficient for high dimensional problems.

4 A Multiphysics Application: An Electric-Thermal Problem

Benefits from multirate ODE-solvers can only be expected if applied to a system with differing dynamic behaviour. Here we consider an electric circuit in where the thermal behaviour of a resistor is included. This results in a coupled system of the network equations and the heat equation. While voltages change very fast, heating or cooling of devices is a much slower process. So this example suits for using multirate integration methods. Before applying the time integration, a semidiscretisation of space is performed for the heat equation. High accuracy demands as well as fine structures may lead to a large scale system. Therefore a model order reduction of the slow components will improve efficiency. The presented model is adapted from [1] with some modifications.

Circuit Modeling. The electric part is represented by the circuit diagram in Fig. 1. It is obvious that the corresponding nodal equations describe a relative stiff system of differential equations. So the mixed multirate ROW-method presented in section 2 is not a *natural choice*. To be able to apply this method to this circuit we use some *unphysical* parameters amongst others for the capacitances. Table 1 shows all relevant parameters. The ODE model reads

$$C_1 \dot{u}_3 = (u_2 - u_3)/R(T) - \dot{i}_{\rm di}(u_3 - u_4, T_{\rm di}) \tag{6}$$

$$C_2 \dot{u}_4 = i_{\rm di} (u_3 - u_4, T_{\rm di}) - u_4 / R_L \tag{7}$$

with the node voltages u_3 , u_4 and $u_2 = Av(t)$, the resistors's temperature *T* and the diode's temperature T_{di} . Between node two and three we consider a copper wire of length *l* and model it as a 1-D thermal dependent resistor. Let $a(x) = a_0 \cdot 1/(1 + (2/l)^2(l-x)x)$ denote the cross section of the wire while *x* represents the spatial coordinate; so at half of the length of the wire the cross section is half of the cross section at the ends. So we expect higher temperatures in the middle of the resistor. We assume a local resistance of the following type:

$$\rho(T) = r_0 (1 + \alpha (T - T_{\text{meas}})) \tag{8}$$

with thermal coefficient α and specific resistance r_0 at temperature T_{meas} . We get the total resistance R(T) by integrating the local resistance over the length of the wire *l* with respect to the cross section

$$R(T) = \int_0^l \frac{\rho(s, T(t, s))}{a(s)} ds = \int_0^l \tilde{\rho}(s, T(t, s)) ds.$$
(9)

The diode is also temperature dependent and has a strong nonlinear behaviour, for the characteristic curve and more details see [1].

Table 1 Parameters of the electric circuit

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decide	parameter	decide	capacity	-
amplification load resistance	$A = 300$ $R_L = 0.3 \text{k}\Omega$	capacity 1 capacity 2	$C_1 = 1F$ $C_2 = 100\mu F$	
pulsed voltage source $v(t) = \begin{cases} 0.5 \sin(\pi t / (2.5 \cdot 10^{-5} s)) \ [mV] \\ 0 \ [V] \end{cases}$		if $t < 2.5 \cdot 10^{-5} s$ otherwise		

Thermal Modeling and Coupling. The starting point of the thermal model is the 1-D heat equation for diffusive heat transport, which we use for the copper wire (resistor):

$$M'_{W}\dot{T} = \frac{\partial}{\partial x} \left(\Lambda(x) \frac{\partial T}{\partial x} \right) + \text{sources}$$
(10)

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with thermal mass of the wire M'_W and local 1-D conductivity $\Lambda(x) = \lambda(x) \cdot a(x)$. The sources term is comprised of two effects: (a) Local self heating due to the electric current. In fact, the dissipated power $P_W = u_R^2/R$ of the resistor results in heating the wire; (b) Cooling to the ambient temperature T_{env} , which given by Newton's cooling $C = -\gamma S'(T - T_{env})$ with surface S'. For further details see [1].

To be able to apply the multirate ODE-integration scheme presented in chapter 2, we discretise space in the parabolic PDE (10) first (method of lines). We equip the wire with an equidistant grid I_h : $X_i = i \cdot h$, i = 0, ..., N with $X_N = N \cdot h = l$ and use a finite volume approach. For that we sub-divide the wire in cells of length *h* in the inner and h/2 at the boundaries. A schematic representation is given in Fig. 2. The heat conduction over one single cell can be simplified described by change of is inflow minus outflow. So we get the approximation

$$M'_{W,i}\dot{T}_{i} = \Lambda \frac{T_{i+1} - 2T_{i} + T_{i-1}}{h^{2}} + P'_{W,i} - \gamma S'_{W,i}(T_{i} - T_{env})$$
(11)

for the inner cells while *i* denotes the property of the *i*-th cell, i = 1, ..., N - 1. For the boundary cells we have

$$M'_{W,0}\dot{T}_0 = \Lambda (T_1 - T_0)/h + P'_{W,0} - \gamma S'_{W,0}(T_0 - T_{env})$$
(12)

$$M'_{W,N}\dot{T}_N = \Lambda (T_{N-1} - T_N)/h + P'_{W,N} - \gamma S'_{W,N} (T_N - T_{env}).$$
(13)

The diode is temperature dependent but without own thermal mass. So we just set the temperature at the end of the copper wire to be the temperature of the diode.

The coupling terms have been given indirectly in the above models.

(i) *Circuit to thermal:* Joule's law gives the dissipated power at the resistor. By adding an additional differential equation to the circuit equations,

$$\dot{e} = u_r \cdot \dot{i}_r = (u_2 - u_3)^2 / R(T) \tag{14}$$

total energy *e* is computed in each time step. And $P_W = e/\Delta t$ gives us the required power for some time step Δt .

(ii) *Thermal to circuit:* Since the resistance R(T) depends on the temperature profile *T* we need the temperature distribution in the resistor to compute it, for a





given distribution we use equation (9) to compute the total resistance. In addition the diode's current depends on the wire temperature of the last cell.

Numerical Results. To the coupled thermal-electric problem we apply the mixed multirate ODE-integration scheme from chapter 2. The acitve part is given by the circuit equations (6)-(7) and (14) while the latent part is given by the semidiscretised heat equation (11)-(13). The coupling is realised in the compound step of the multirate scheme and in the off-diagonal Jacobian matrices. Core of this work is the combination of the multirate integration with a model order reduction for the latent part given for example in chapter 3. In a early stage of research we restrict the setting to *linear* MOR. A thermal model is *a priori* non-linear, due to a relative short simulation time (0.1 seconds) and a even shorter timespan where voltage is applied in the circuit and several small assumptions like in equation (8). We can linearise the thermal behaviour of the resistor so that we get a linear system of the form

$$\dot{T} = \mathbf{A} \cdot T + \mathbf{B} \cdot [T_{env}, P_W] \tag{15}$$

$$[R(T), T_N] = \mathbf{C} \cdot T. \tag{16}$$

To get an impression of the electric behaviour of the system figure 3 shows the resulting voltage u_3 at node three. First we are investigating the influence of the linearisation of the thermal components. Figure 4 shows the temperature of the central cell in the resistor and of the diode in a linear and a nonlinear model. As we see the influence of the linearisation is negligible.

For this results the resistor was discretised in n = 10 cells. If we consider a more detailed discretisation the dimension of the system and the computation time would increase. For this case a model order reduction promises more efficiency in computation time. Due to that we will focus on model order reduction performed in a pre-processing step and the influences on the simulation results in the future research.

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5 Conclusions and Outlook

Combining a model order reduction with a multirate integration scheme can increase the efficiency of the broad algorithm. Since the multirate problem already provides a partitioning relating to the dynamics of the system we already know that there is no fast change in the latent component so that the error due to the model order reduction might be controllable. Especially for linear latent components the wide theory of linear model order reduction gives efficient and reliable error information.

The influence of the model order reduction on the multirate integrator ist not yet considered so it is desirable to have all-in-one multirate-MOR error bounds. After all in real-world multiphysics problems a latent linear component is not available so the question is whether a linearisation following linear MOR or a nonlinear MOR gives better results.

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