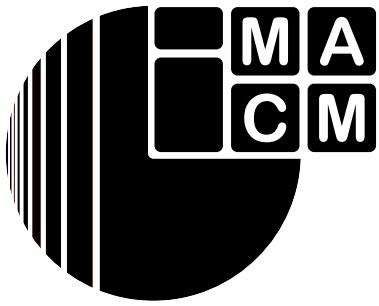

Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and Computational Mathematics (IMACM)

Preprint BUW-IMACM 14/14

Anton Arnold and Matthias Ehrhardt

A Transparent Boundary Condition for an Elastic Bottom in Underwater Acoustics

June 2014

http://www.math.uni-wuppertal.de

A Transparent Boundary Condition for an Elastic Bottom in Underwater Acoustics

Anton Arnold and Matthias Ehrhardt

Abstract This work is concerned with the derivation of a novel transparent boundary condition (TBC) for the coupling of the standard "parabolic" equation (SPE) in underwater acoustics (assuming cylindrical symmetry) with an elastic parabolic equation (EPE) for modelling the sea bottom extending hereby the existing TBCs for a fluid model of the seabed.

1 Introduction

"Parabolic" equation (PE) models appear in (underwater) acoustics as one-way approximations to the Helmholtz equation in cylindrical coordinates with azimuthal symmetry. These PE models have been widely used in the recent past for wave propagation problems in various application areas, e.g. seismology, optics and plasma physics but here we focus on their application to underwater acoustics, where PEs have been introduced by Tappert [18]. For more details we refer to [11].

In computational ocean acoustics one wants to determine the acoustic pressure $p(z,r)$ emerging from a time-harmonic point source situated in the water at $(z_s,0)$. The radial range variable is denoted by $r>0$ and the depth variable is $0<z<z_b$. The water surface is located at $z=0$, and the (horizontal) sea bottom at $z=z_b$. We point out that irregular bottom surfaces and sub-bottom layers can be included by simply extending the range of z. For an alternative strategy based on transformation techniques, including proofs of well-posedness in the case of upsloping and

Anton Arnold
Institut für Analysis und Scientific Computing, Technische Universität Wien, Wiedner Hauptstr. 8, 1040 Wien, Austria, e-mail: anton.arnold@tuwien.ac.at

Matthias Ehrhardt
Lehrstuhl für Angewandte Mathematik und Numerische Analysis, Fachbereich C Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, Gaußstr. 20, 42119 Wuppertal, Germany, e-mail: ehrhardt@math.uni-wuppertal.de

downsloping wedge-type domains in 2D and 3D we refer to [2, 7]. Further, the 3D treatment of a sloping sea bootom in a finite element context was presented in [17].

In the sequel we denote the local sound speed by $c(z,r)$, the density by $\rho(z,r)$, and the attenuation by $\alpha(z,r) \geq 0$. $n(z,r) = c_0/c(z,r)$ is the refractive index, with a reference sound speed c_0. The reference wave number is $k_0 = 2\pi f/c_0$, where f denotes the (usually low) frequency of the emitted sound.

1.1 The Parabolic Approximations

The acoustic pressure $p(z,r)$ satisfies the *Helmholtz equation*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \rho\frac{\partial}{\partial z}\left(\rho^{-1}\frac{\partial p}{\partial z}\right) + k_0^2 N^2 p = 0, \qquad r > 0, \tag{1}$$

with the complex refractive index (where α accounts for damping in the medium)

$$N(z,r) = n(z,r) + i\alpha(z,r)/k_0. \tag{2}$$

In the far field approximation ($k_0 r \gg 1$) the (complex valued) outgoing acoustic field

$$\psi(z,r) = \sqrt{k_0 r}\, p(z,r)\, e^{-ik_0 r} \tag{3}$$

satisfies the *one-way Helmholtz equation*:

$$\psi_r = ik_0\big(\sqrt{1-L} - 1\big)\psi, \qquad r > 0. \tag{4}$$

Here, $\sqrt{1-L}$ is a pseudo-differential operator, and L the *Schrödinger operator*

$$L = -k_0^{-2}\rho\,\partial_z(\rho^{-1}\partial_z) + V(z,r) \tag{5}$$

with the complex valued "potential"

$$V(z,r) = 1 - N^2(z,r) = 1 - \big[n(z,r) + i\alpha(z,r)/k_0\big]^2. \tag{6}$$

"Parabolic" approximations of (4) are formal approximations of the the pseudo–differential operator $\sqrt{1-L}$ by rational functions of L. This procedure yields a PDE that is easier to solve numerically than the pseudo-differential equation (4). For more details we refer to [18, 19]. The linear approximation of $\sqrt{1-\lambda}$ by $1 - \frac{\lambda}{2}$ gives the narrow angle or *standard "parabolic" equation* (SPE) of Tappert [18]

$$\psi_r = -\frac{ik_0}{2} L\psi, \qquad r > 0. \tag{7}$$

This Schrödinger equation (7) is a good description of waves with a propagation direction within about 15° of the horizontal. Rational approximations of the form

$$(1-\lambda)^{\frac{1}{2}} \approx f(\lambda) = \frac{p_0 - p_1\lambda}{1 - q_1\lambda} \tag{8}$$

with real p_0, p_1, q_1 yield the *wide angle "parabolic" equations* (WAPE)

$$\psi_r = ik_0\left(\frac{p_0 - p_1 L}{1 - q_1 L} - 1\right)\psi, \qquad r > 0. \tag{9}$$

offering an improved description of the wave propagation up to angles of about 40°.

Here we focus on a proper boundary condition (BC) at the sea bottom for the SPE (7) coupled to an elastic "parabolic" model for the sea bottom. At the water surface one usually employs a Dirichlet BC $\psi(z=0,r)=0$ and at the sea bottom one has to couple the wave propagation in the water to the wave propagation in the bottom.

1.2 The Coupling Condition

For the bottom region $z > z_b$ one usually use a fluid model (i.e. assuming that (7) or (9) with possibly different rational approximation (8) also hold for $z > z_b$) with constant parameters c_b, ρ_b, and α_b or with a linear squared refractive index [8, 12].

In [4] we analyzed this coupling of WAPEs with different parameters p_0, p_1, q_1 and it turned out that the coupled model is well-defined (and the resulting evolution equation is conservative in $L^2(\mathbb{R}^+;(\sigma\rho)^{-1}dz)$) if the *coupling condition*

$$p_1(z)/q_1(z) =: \mu = \text{const} \tag{10}$$

is satisfied. Hence, it is not advisable to couple the WAPE and the SPE (where $p_1 = 1/2; q_1 = 0$) numerically; in this case the evolution is not conservative in the dissipation-free case ($\alpha \equiv 0$) [4]. If the parameters p_0, p_1, q_1 are fixed in one medium, condition (10) still leaves two free parameters to choose a different rational approximation model of $(1-\lambda)^{\frac{1}{2}}$ in (8) for the second medium (cf. [9]). Hence, one can in fact obtain a better approximation in the second medium than with the originally intended "parabolic approximation".

1.3 Transparent Boundary Conditions

In practical simulations one is only interested in the acoustic field $\psi(z,r)$ in the water, i.e. for $0 < z < z_b$. While the physical problem is posed on the unbounded z-interval $(0,\infty)$, one wishes to restrict the computational domain in the z-direction by introducing an artificial boundary at or below the sea bottom. This artificial BC should of course change the model as little as possible, or ideally not at all.

In [14, 16] Papadakis derived *impedance BCs* or *transparent boundary conditions* (TBC) for the SPE and the WAPE, which completely solves the problem of restricting the z–domain without changing the physical model: complementing the WAPE (9) with a TBC at z_b allows to recover — on the finite computational domain $(0,z_b)$ — the exact half-space solution on $0<z<\infty$. As the SPE is a Schrödinger equation, similar strategies have been developed independently for quantum mechanical applications, cf. the review article [1].

Let us finally note, that Zhang and Tindle [22] proposed an alternative approach to the impedance BCs or TBCs of Papadakis. By minimizing the reflection coefficient at the water-bottom interface they derived in their *equivalent fluid approximation* an expression for a complex fluid density that can be used for modelling an elastic sea bottom in a classical fluid model. However, this approach yields only satisfactory results for low shear wave speeds [22].

This work is organized as follows: In §2 we review the TBC for the SPE coupled to an elastic bottom in the frequency domain and in §3 present in detail the analytic inverse Laplace transformation to obtain this TBC in the time domain. Finally, we draw a conclusion and summarize the basic inversion rules previously used.

2 The Transparent Boundary Condition for a Fluid Bottom

The basic idea of the derivation is to explicitly solve the equation in the sea bottom, which is the exterior of the computational domain $(0,z_b)$. The TBC for the SPE (or Schrödinger equation) was derived in [3, 14, 16] for various application fields:

$$\psi(z_b,r) = -(2\pi k_0)^{-\frac{1}{2}}\, e^{\frac{\pi}{4}i}\,\frac{\rho_b}{\rho_w}\int_0^r \psi_z(z_b,r-\tau)\, e^{i\omega_b\tau}\tau^{-\frac{1}{2}}\,d\tau, \tag{11}$$

with $\omega_b = k_0(N_b^2-1)/2$. This BC is nonlocal in the range variable r and involves a mildly singular convolution kernel. Equivalently, it can be written as

$$\psi_z(z_b,r) = -\left(\frac{2k_0}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\pi}{4}i}\, e^{i\omega_b r}\,\frac{\rho_w}{\rho_b}\,\frac{d}{dr}\int_0^r \psi(z_b,\tau)\, e^{-i\omega_b\tau}(r-\tau)^{-\frac{1}{2}}\,d\tau. \tag{12}$$

The r.h.s. of (12) can be expressed formally as a Riemann-Liouville fractional derivative of order $\frac{1}{2}$, cf. [3]:

$$\psi_z(z_b,r) = -\sqrt{2k_0}e^{-\frac{\pi}{4}i}\, e^{i\omega_b r}\,\frac{\rho_w}{\rho_b}\partial_r^{1/2}\big[\psi(z_b,r)\, e^{-i\omega_b r}\big]. \tag{13}$$

3 The Transparent Boundary Condition for an Elastic Bottom

The coupling of the SPE with an *elastic parabolic equation* (EPE) for the sea bottom was described in [5, 10, 21]. Papadakis et al. [15, 16] derived a TBC for this coupling in the frequency regime. It reads for the Laplace transformed wave field:

$$\hat{\psi}(z_b,s) = -\frac{\rho_b}{\rho_w}\frac{1}{k_0 N_s^4}\frac{1}{\sqrt[+]{M_p(s)}}\times \\ \times\left[\left(2M_s(s)+N_s^2\right)^2 - 4\sqrt[+]{M_p(s)}\sqrt[+]{M_s(s)}\left(M_s(s)+N_s^2\right)\right]\hat{\psi}_z(z_b,s), \quad (14)$$

with the notation

$$M_p(s) = 1-N_p^2 - i\frac{2}{k_0}s, \qquad M_s(s) = 1-N_s^2 - i\frac{2}{k_0}s. \quad (15)$$

Here, $N_p = n_p + i\alpha_p/k_0$ and $N_s = n_s + i\alpha_s/k_0$ denote the complex refractive indices for the compressional and shear waves in the bottom (cf. (2)).

4 The Transparent Boundary Condition in the Time Domain

In a tedious calculation the transformed TBC (14) can indeed be inverse Laplace transformed and it reads:

$$\psi(z_b,r) = \\ C\left[\int_0^r \psi_z(z_b,r-\tau)\,e^{i\omega_p\tau}g(\tau)\,d\tau - 2i\varphi\int_0^r \psi_{zr}(z_b,r-\tau)\,e^{i\omega_p\tau}\tau^{-\frac{1}{2}}\,d\tau\right], \quad (16)$$

with

$$C = -\frac{\rho_b}{\rho_w}\frac{2}{k_0^{5/2}N_s^4}\sqrt{\frac{2}{\pi}}\,e^{\frac{\pi}{4}i}, \quad \omega_p = \frac{k_0}{2}\left(N_p^2-1\right), \quad \varphi = -\frac{k_0}{2}\left(N_p^2-N_s^2\right),$$

and the kernel $g(\tau)$ given by

$$g(\tau) = -3\left(1-e^{i\varphi\tau}\right)\tau^{-\frac{5}{2}} + i\frac{k_0}{2}\left(3N_p^2-N_s^2-2N_s^2\,e^{i\varphi\tau}\right)\tau^{-\frac{3}{2}} \\ + \frac{k_0^2}{2}\left(N_p^4-N_p^2N_s^2+\tfrac{1}{2}N_s^4+N_p^2-N_s^2\right)\tau^{-\frac{1}{2}} = O\left(\tau^{-\frac{1}{2}}\right), \quad \text{for} \quad \tau\to\infty.$$

While this inverse transformation was carried out numerically in [15, 16], our novel analytical TBC in the time regime may simplify both the analysis and the numerical

solution of this coupled model. Let us remark that an asymptotic analysis of the elastic seabed was made by Makrakis [13].

4.1 Derivation of (16)

With the abbreviation $\hat{\psi}(s) := \hat{\psi}(z_b, s)$ and the notation

$$m_p(s) = \frac{k_0}{2} M_p(s) = -i\left[s - i\,\frac{k_0}{2}(N_p^2 - 1)\right], \tag{17}$$

$$m_s(s) = \frac{k_0}{2} M_s(s) = -i\left[s - i\,\frac{k_0}{2}(N_s^2 - 1)\right], \tag{18}$$

the transformed TBC (14) reads

$$\begin{aligned}
&\hat{\psi}(s)\\
&= -\frac{\rho_b}{\rho_w}\frac{1}{k_0 N_s^4}\left[\frac{k_0}{2}\frac{\left(\frac{4}{k_0}m_s(s) + N_s^2\right)^2}{\sqrt[+]{m_p(s)}} - 4\sqrt{\frac{2}{k_0}}\,\sqrt[+]{m_s(s)}\left(\frac{2}{k_0}m_s(s) + N_s^2\right)\right]\hat{\psi}_z(s)\\
&= -\frac{\rho_b}{\rho_w}\frac{8\sqrt{2}}{k_0^{5/2} N_s^4}\left[\frac{\left(m_s(s) + \frac{k_0}{4}N_s^2\right)^2}{\sqrt[+]{m_p(s)}} - 4\,\sqrt[+]{m_s(s)}\left(m_s(s) + \frac{k_0}{2}N_s^2\right)\right]\hat{\psi}_z(s),
\end{aligned} \tag{19}$$

where we denote the content of the square brackets by $f(s-\sigma)$ with

$$\sigma = i\frac{k_0}{2}\left(N_p^2 - 1\right). \tag{20}$$

We observe that we can write

$$m_p(s) = -i\,[s - \sigma], \qquad m_s(s) = -i\left[s - \sigma + i\,\frac{k_0}{2}(N_p^2 - N_s^2)\right]. \tag{21}$$

The next step is a shift in the argument of $\hat{\psi}_z(s)$ in (19) by σ:

$$\hat{\psi}(s+\sigma) = -\frac{\rho_b}{\rho_w}\frac{8\sqrt{2}}{k_0^{5/2} N_s^4}\, f(s)\,\hat{\psi}_z(z_b, s+\sigma), \tag{22}$$

Taking the branch with positive real part $\sqrt[+]{-i} = e^{-\frac{\pi}{4}i}$ we obtain for the kernel $f(s)$

$$\begin{aligned}
f(s) = &\frac{\left[-is + \frac{k_0}{2}(N_p^2 - N_s^2) + \frac{k_0}{4}N_s^2\right]^2}{e^{-\frac{\pi}{4}i}\,\sqrt[+]{s}}\\
&- e^{-\frac{\pi}{4}i}\,\sqrt[+]{s + i\frac{k_0}{2}(N_p^2 - N_s^2)}(-i)\,\sqrt[+]{s + i\frac{k_0}{2}(N_p^2 - N_s^2)}\left[s + i\frac{k_0}{2}N_p^2\right],
\end{aligned} \tag{23}$$

where

$$\begin{aligned}\left[\cdots\right]^2 &= -\left[s+i\frac{k_0}{2}N_p^2 - i\frac{k_0}{4}N_s^2\right]^2 \\ &= -\left[s+i\frac{k_0}{2}(N_p^2-N_s^2)\right]\left[s+i\frac{k_0}{2}N_p^2\right] + \frac{k_0^2}{16}N_s^4,\end{aligned} \tag{24}$$

i.e. we have

$$\begin{aligned}f(s) &= e^{\frac{\pi}{4}i}\left\{\frac{1}{\sqrt[+]{s}}\left[\frac{k_0^2}{16}N_s^4 - \left[s+i\frac{k_0}{2}(N_p^2-N_s^2)\right]\left[s+i\frac{k_0}{2}N_p^2\right]\right]\right. \\ &\qquad \left. + \sqrt[+]{s+i\frac{k_0}{2}(N_p^2-N_s^2)}\left[s+i\frac{k_0}{2}N_p^2\right]\right\} \\ &= e^{\frac{\pi}{4}i}\left\{\left[\sqrt[+]{s+i\frac{k_0}{2}(N_p^2-N_s^2)} - \sqrt[+]{s} - i\frac{k_0}{2}(N_p^2-N_s^2)\frac{1}{\sqrt[+]{s}}\right]\cdot\right. \\ &\qquad \left.\cdot\left[s+i\frac{k_0}{2}N_p^2\right] + \frac{k_0^2}{16}N_s^4\frac{1}{\sqrt[+]{s}}\right\} \\ &= e^{\frac{\pi}{4}i}\left\{\left[\sqrt[+]{s-\gamma} - \sqrt[+]{s} + \gamma\frac{1}{\sqrt[+]{s}}\right]\left[s+i\frac{k_0}{2}N_p^2\right] + \frac{k_0^2}{16}N_s^4\frac{1}{\sqrt[+]{s}}\right\},\end{aligned} \tag{25}$$

with

$$\gamma = -i\frac{k_0}{2}(N_p^2-N_s^2). \tag{26}$$

Hence, inserting in (22) we obtain

$$\begin{aligned}\hat{\psi}(s+\sigma) = &\tilde{C}\left\{i\frac{k_0}{2}N_p^2\left[\sqrt[+]{s-\gamma} - \sqrt[+]{s} + \gamma\frac{1}{\sqrt[+]{s}}\right] + \frac{k_0^2}{16}N_s^4\frac{1}{\sqrt[+]{s}}\right\}\hat{\psi}_z(s+\sigma) \\ &+\tilde{C}\left[\sqrt[+]{s-\gamma} - \sqrt[+]{s} + \gamma\frac{1}{\sqrt[+]{s}}\right]\left\{s\,\hat{\psi}_z(s+\sigma)\right\},\end{aligned} \tag{27}$$

where

$$\tilde{C} = -\frac{\rho_b}{\rho_w}\frac{8\sqrt{2}}{k_0^{5/2}N_s^4}e^{\frac{\pi}{4}i}. \tag{28}$$

Next, an inverse Laplace transformation of (27) yields the convolution integral

$$\begin{aligned}\psi(r)\,e^{-\sigma r} = &\tilde{C}\int_0^r \psi_z(r-\tau)\,e^{-\sigma(r-\tau)}\,g_1(\tau)\,d\tau \\ &+\tilde{C}\int_0^r -\frac{\partial}{\partial\tau}\left[\psi_z(r-\tau)\,e^{-\sigma(r-\tau)}\right]g_2(\tau)\,d\tau,\end{aligned} \tag{29}$$

$$\begin{aligned} g_1(\tau) &= \mathscr{L}^{-1}\left\{ i\frac{k_0}{2}N_p^2\Big[\sqrt{s-\gamma}-\sqrt{s}+\gamma\frac{1}{\sqrt{s}}\Big] + \frac{k_0^2}{16}N_s^4\frac{1}{\sqrt{s}}\right\} \\ &= i\frac{k_0}{2}N_p^2\mathscr{L}^{-1}\left\{\sqrt{s-\gamma}-\sqrt{s}\right\} + \frac{k_0^2}{4}(N_p^2-\tfrac{1}{2}N_s^2)^2\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} \\ &= i\frac{k_0}{4\sqrt{\pi}}N_p^2(1-e^{\gamma\tau})\,\tau^{-\frac{3}{2}} + \frac{k_0^2}{4\sqrt{\pi}}(N_p^2-\tfrac{1}{2}N_s^2)^2\,\tau^{-\frac{1}{2}}, \end{aligned} \tag{30}$$

$$\begin{aligned} g_2(\tau) &= \mathscr{L}^{-1}\left\{\sqrt{s-\gamma}-\sqrt{s}+\gamma\frac{1}{\sqrt{s}}\right\} = \frac{1}{2\sqrt{\pi}}(1-e^{\gamma\tau})\,\tau^{-\frac{3}{2}} + \frac{\gamma}{\sqrt{\pi}}\,\tau^{-\frac{1}{2}} \\ &= \underbrace{\frac{1}{2\sqrt{\pi}}\left[(1-e^{\gamma\tau})\,\tau^{-\frac{3}{2}}+\gamma\,\tau^{-\frac{1}{2}}\right]}_{=g_3(\tau)} + \underbrace{\frac{\gamma}{2\sqrt{\pi}}\,\tau^{-\frac{1}{2}}}_{=g_4(\tau)}, \end{aligned} \tag{31}$$

$$\begin{aligned} I_3 &= \int_0^r -\frac{\partial}{\partial\tau}\left[\psi_z(r-\tau)\,e^{-\sigma(r-\tau)}\right] g_3(\tau)\,d\tau \\ &= \int_0^r \psi_z(r-\tau)\,e^{-\sigma(r-\tau)}g_3'(\tau)\,d\tau - \underbrace{\psi_z(r-\tau)\,e^{-\sigma(r-\tau)}g_3(\tau)\Big|_{\tau=0}^{\tau=r}}_{=0 \quad (\text{with } \psi_z(0)=0)}, \end{aligned} \tag{32}$$

$$\begin{aligned} \psi(r)\,e^{-\sigma r} =& \tilde{C}\int_0^r \psi_z(r-\tau)\,e^{-\sigma(r-\tau)}\big[g_1(\tau)+g_3'(\tau)\big]\,d\tau \\ &+\tilde{C}\int_0^r -\frac{\partial}{\partial r}\left[\psi_z(r-\tau)\,e^{-\sigma(r-\tau)}\right] g_4(\tau)\,d\tau \\ =& \tilde{C}\int_0^r \psi_z(r-\tau)\,e^{-\sigma(r-\tau)}\big[g_1(\tau)+g_3'(\tau)\big]\,d\tau \\ &+\tilde{C}\int_0^r \left[\psi_{zr}(r-\tau)\,e^{-\sigma(r-\tau)} - \sigma\psi_z(r-\tau)\,e^{-\sigma(r-\tau)}\right] g_4(\tau)\,d\tau, \end{aligned} \tag{33}$$

i.e.

$$\psi(r) = \tilde{C}\left[\int_0^r \psi_z(r-\tau)\,e^{\sigma\tau}\tilde{g}(\tau)\,d\tau + \int_0^r \psi_{zr}(r-\tau)\,e^{\sigma\tau}g_4(\tau)\,d\tau\right], \tag{34}$$

where

$$\tilde{g}(\tau) := \big[g_1(\tau)+g_3'(\tau)-\sigma g_4(\tau)\big]. \tag{35}$$

We calculate

$$\begin{aligned} g_3'(\tau) &= \frac{1}{2\sqrt{\pi}}\left[-\frac{3}{2}(1-e^{\gamma\tau})\,\tau^{-\frac{5}{2}} - \gamma e^{\gamma\tau}\,\tau^{-\frac{3}{2}} - \frac{\gamma}{2}\,\tau^{-\frac{3}{2}}\right] \\ &= \frac{1}{4\sqrt{\pi}}\left[-3(1-e^{\gamma\tau})\,\tau^{-\frac{5}{2}} - 2\gamma(\tfrac{1}{2}+e^{\gamma\tau})\,\tau^{-\frac{3}{2}}\right], \end{aligned} \tag{36}$$

and

$$\sigma g_4(\tau) = \frac{\sigma\gamma}{2\sqrt{\pi}}\,\tau^{-\frac{1}{2}} = \frac{1}{4\sqrt{\pi}}\,\frac{k_0^2}{2}(N_p^2-1)(N_p^2-N_s^2)\,\tau^{-\frac{1}{2}}, \tag{37}$$

i.e. (35) gives finally

$$\begin{aligned}\tilde{g}(\tau) &= \frac{1}{4\sqrt{\pi}}\Big[ik_0N_p^2\big(1-e^{\gamma\tau}\big)\,\tau^{-\frac{3}{2}} + k_0^2\big(N_p^2-\tfrac{1}{2}N_s^2\big)^2\,\tau^{-\frac{1}{2}} - 3\big(1-e^{\gamma\tau}\big)\,\tau^{-\frac{5}{2}} \\ &\qquad + ik_0\big(N_p^2-N_s^2\big)\big(\tfrac{1}{2}+e^{\gamma\tau}\big)\,\tau^{-\frac{3}{2}} - \frac{k_0^2}{2}\big(N_p^2-1\big)\big(N_p^2-N_s^2\big)\,\tau^{-\frac{1}{2}}\Big] \\ &= \frac{1}{4\sqrt{\pi}}\Big[\frac{k_0^2}{2}\big(N_p^4-N_p^2N_s^2+\tfrac{1}{2}N_s^4+N_p^2-N_s^2\big)\,\tau^{-\frac{1}{2}} \\ &\qquad + i\frac{k_0}{2}\big(3N_p^2-N_s^2-2N_s^2\,e^{\gamma\tau}\big)\,\tau^{-\frac{3}{2}} - 3\big(1-e^{\gamma\tau}\big)\tau^{-\frac{5}{2}}\Big] \\ &= \mathrm{O}(\tau^{-\frac{1}{2}}), \quad \tau\to\infty.\end{aligned} \tag{38}$$

Finally, we define φ, ω, by setting $\gamma =: i\varphi$, $\sigma =: i\omega$ and

$$g(\tau) = 4\sqrt{\pi}\tilde{g}(\tau), \qquad C = \frac{\tilde{C}}{4\sqrt{\pi}}. \tag{39}$$

This completes the calculation of (16).

Conclusion and Outlook

First, we will make first numerical investigations for these new TBCs and investigate their superiority compared to using their formulation in transformed space. Next, instead of using an ad-hoc discretization of the analytic transparent BC we will construct discrete TBCs of the fully discretized half-space problem in the spirit of [4].

Appendix: Laplace–Transformations

$$\mathscr{L}^{-1}\big\{\sqrt{s-\gamma}-\sqrt{s}\big\} = \frac{1}{2\sqrt{\pi}}(1-e^{\gamma t})\,t^{-\frac{3}{2}}, \tag{40}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi}}\,t^{-\frac{1}{2}}, \tag{41}$$

$$\mathscr{L}^{-1}\{\hat{\psi}(s+\sigma)\} = \psi(t)\,e^{-\sigma t}, \tag{42}$$

$$\mathscr{L}^{-1}\{s\,\hat{\psi}(s+\sigma)\} = \frac{d}{dt}\big\{\psi(t)\,e^{-\sigma t}\big\} \quad \text{if } \psi(0)=0. \tag{43}$$

References

1. X. Antoine, A. Arnold, C. Besse, M. Ehrhardt and A. Schädle, *A Review of Transparent and Artificial Boundary Conditions Techniques for Linear and Nonlinear Schrödinger Equations*, Commun. Comput. Phys. **4** (2008), 729–796. (open-access article)
2. D.C. Antonopoulou, V.A. Dougalis and G.E. Zouraris, *Galerkin methods for parabolic and Schrödinger equations with dynamical boundary conditions and applications to underwater acoustics*, SIAM J. Numer. Anal. **47** (2009), 2752–2781.
3. A. Arnold, *Numerically Absorbing Boundary Conditions for Quantum Evolution Equations*, VLSI Design **6** (1998), 313–319.
4. A. Arnold and M. Ehrhardt, *Discrete transparent boundary conditions for wide angle parabolic equations in underwater acoustics*, J. Comp. Phys. **145** (1998), 611–638.
5. M.D. Collins, *A higher-order parabolic equation for wave propagation in an ocean overlying an elastic bottom*, J. Acoust. Soc. Am. **86** (1989), 1459–1464.
6. T.W. Dawson, D.J. Thomson and G.H. Brooke, *Nonlocal boundary conditions for acoustic PE predictions involving inhomogeneous layers*, in 3rd European Conference on Underwater Acoustics, FORTH–IACM, Heraklion, 1996, pp. 183–188.
7. V.A. Dougalis, F. Sturm and G.E. Zouraris, *On an initial-boundary value problem for a wide-angle parabolic equation in a waveguide with a variable bottom*, Math. Meth. Appl. Sci. **32** (2009), 1519–1540.
8. M. Ehrhardt and R.E. Mickens, *Solutions to the Discrete Airy Equation: Application to Parabolic Equation Calculations*, J. Comput. Appl. Math. **172** (2004), 183–206.
9. R.R. Greene, *The rational approximation to the acoustic wave equation with bottom interaction*, J. Acoust. Soc. Am. **76** (1984), 1764–1773.
10. R.R. Greene, *A high-angle one-way wave equation for seismic wave propagation along rough and sloping interfaces*, J. Acoust. Soc. Am. **77** (1985), 1991–1998.
11. F.B. Jensen, W.A. Kuperman, M.B. Porter and H. Schmidt, *Computational Ocean Acoustics*, AIP Press, New York, 1994.
12. M.F. Levy, *Transparent Boundary Conditions for Parabolic Equation Solutions of Radiowave Propagation Problems*, IEEE Trans. Antennas Propag. **45** (1997), 66–72.
13. G.N. Makrakis, *Asymptotic study of the elastic seabed effects in ocean acoustics*, Applicable Analysis **66** (1997), 357–375.
14. J.S. Papadakis, *Impedance formulation of the bottom boundary condition for the parabolic equation model in underwater acoustics*, NORDA Parabolic Equation Workshop, NORDA Tech. Note 143, 1982.
15. J.S. Papadakis, M.I. Taroudakis, P.J. Papadakis and B. Mayfield, *A new method for a realistic treatment of the sea bottom in the parabolic approximation*, J. Acoust. Soc. Am. **92** (1992), 2030–2038.
16. J.S. Papadakis, *Impedance Bottom Boundary Conditions for the Parabolic–Type Approximations in Underwater Acoustics*, in Advances in Computer Methods for Partial Differential Equations VII, eds. R. Vichnevetsky, D. Knight, and G. Richter, IMACS, New Brunswick, NJ, 1992, pp. 585–590.
17. F. Sturm and N.A. Kampanis, *Accurate treatment of a general sloping interface in a finite-element 3D narrow-angle PE model*, J. Comput. Acoust. **15** (2007), 285–318.
18. F.D. Tappert, *The parabolic approximation method*, in *Wave Propagation and Underwater Acoustics*, Lecture Notes in Physics 70, eds. J.B. Keller and J.S. Papadakis, Springer, New York, 1977, pp. 224–287.
19. D.J. Thomson, *Wide–angle parabolic equation solutions to two range–dependent benchmark problems*, J. Acoust. Soc. Am. **87** (1990), 1514–1520.
20. D.J. Thomson and M.E. Mayfield, *An exact Radiation Condition for Use with the A Posteriori PE Method*, J. Comp. Acous. **2** (1994), 113–132.
21. B.T. Wetton and G.H. Brooke, *One-way wave equations for seismoacoustic propagation in elastic waveguides*, J. Acoust. Soc. Am. **87** (1990), 624–632.
22. Z.Y. Zhang and C.T. Tindle, *Improved equivalent fluid approximations for a low shear speed ocean bottom*, J. Acoust. Soc. Am. **98** (1995), 3391–3396.