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Abstract

We provide a detailed analysis of the classical and quantized theory of a multiplet of inhomogeneous Klein–Gordon fields, which couple to the spacetime metric and also to an external source term; thus the solutions form an affine space. Following the formulation of affine field theories in terms of presymplectic vector spaces as proposed in [Annales Henri Poincaré **15**, 171 (2014)], we determine the relative Cauchy evolution induced by metric as well as source term perturbations and compute the automorphism group of natural isomorphisms of the presymplectic vector space functor. Two pathological features of this formulation are revealed: the automorphism group contains elements that cannot be interpreted as global gauge transformations of the theory; moreover, the presymplectic formulation does not respect a natural requirement on composition of subsystems. We therefore propose a systematic strategy to improve the original description of affine field theories at the classical and quantized level, first passing to a Poisson algebra description in the classical case. The idea is to consider state spaces on the classical and quantum algebras suggested by the physics of the theory (in the classical case, we use the affine solution space). The state spaces are not separating for the algebras, indicating a redundancy in the description. Removing this redundancy by a quotient, a functorial theory is obtained that is free of the above mentioned pathologies. These techniques are applicable to general affine field theories and Abelian gauge theories. The resulting quantized theory is shown to be dynamically local.

Keywords: locally covariant quantum field theory, relative Cauchy evolution, quantum field theory on curved spacetimes, affine quantum field theory

MSC 2010: 81T20, 81T05

1 Introduction

Our understanding of quantum field theories on Lorentzian manifolds has made tremendous developments since the principle of general local covariance was introduced in [BFV03]. Its underlying physical idea, which roughly speaking says that any reasonable quantum field theory should be defined coherently on *all* spacetimes instead of focusing on formulations in individual spacetimes, is expressed mathematically in terms of category theory. The basic structure of interest is that of a covariant functor from a category of spacetimes (possibly with extra data such as fibre bundles) to a category of algebras, which is supposed to describe the association of observable algebras to spacetimes. The benefits from this new perspective on quantum field theory are substantial: On the one hand, many structural problems have been addressed and solved, for example the generalization of the famous spin-statistics theorem to curved spacetimes [Ver01] and the perturbative renormalization of quantum field theories, see e.g. [BDF09, HW01, HW02] for general developments and [FR13, BFR13] for perturbative gauge and gravity theories. On the other hand, the locally covariant framework also has had an impact on applications of quantum field theory to e.g. quantum energy inequalities [FP06, Few07] and cosmology [Ver12, PS13a, PS13b].

Another new and interesting aspect arising in the locally covariant framework is that internal symmetries of (quantum) field theories can be promoted to the functorial level. It has been proposed recently by one of us [Few13] that the automorphism group (of natural isomorphisms) of a locally covariant quantum field theory functor is a suitable generalization to curved spacetimes of the global gauge group in Minkowski space algebraic quantum field theory. Besides clarifying general properties of such automorphism groups, it has been shown in [Few13] that this concept indeed captures the usual orthogonal symmetries of the quantum theory of a multiplet of Klein–Gordon fields with equal masses.

In this paper we investigate how some of the above features are modified when the basic category is enriched from a category of spacetimes to include additional external sources. This particularly influences the *relative Cauchy evolution* [BFV03], which measures sensitivity of a theory to perturbations of the background structure, and plays a key role in the classification of the automorphism group [Few13] and also in defining the notion of a *dynamically local* theory [FV12a]. External sources provide additional degrees of freedom for the relative Cauchy evolution to exploit, leading to a richer framework.

Our investigation is conducted in the context of an example, namely the theory of a multiplet of *inhomogeneous* Klein–Gordon fields interacting with an external source, with underlying Lagrangian

$$\mathcal{L} = \sqrt{|\boldsymbol{g}|}\left(\frac{1}{2}\langle\nabla_a\phi,\nabla^a\phi\rangle - \frac{1}{2}m^2\langle\phi,\phi\rangle - \lambda\langle\boldsymbol{J},\phi\rangle\right)\,, \tag{1.1}$$

where $\boldsymbol{J}\in C^\infty(M,\mathbb{R}^p)$ is a classical and non-dynamical source term. The interest in this model comes from two directions. Physically, it represents an approximation to a model in which ϕ is coupled both to gravity and to other fields, but with the simplifying assumption that not only the metric but also the other fields have been 'frozen' as background structure represented by the external source; this would be an appropriate approximation in situations where the back-reaction of ϕ on both the metric and other fields can be neglected. Mathematically, interest arises because, in contrast to the homogeneous theory with $\boldsymbol{J}=0$, the equation of motion corresponding to this Lagrangian is not linear, but affine, and as a consequence the space of solutions is not a vector space, but rather an affine space. On the one hand, replacing linear structures by affine ones leads to the simplest 'non-linear' models of quantum field theories, which still can be treated exactly without the need for perturbative expansions [BDS12]. On the other hand, these affine structures are unavoidable in gauge theories [BDS13, BDHS13] as the space of connections on a principal bundle is intrinsically an affine space. Hence, inhomogeneous theories such as (1.1) can be regarded as toy-models for gauge theories, which reflect parts of their geometric structure. For these reasons, a general study of affine field theories was recently undertaken in [BDS12].

Formulating the (classical and quantum) inhomogeneous Klein–Gordon theory according to [BDS12], however, we have found that the resulting functors have two serious pathologies. First, their automorphism groups do not reflect the expected symmetries of this model: From the Lagrangian (1.1) one expects that the usual orthogonal symmetries are broken due to the presence of the (arbitrary) source terms $\boldsymbol{J}$, so that only a translation symmetry $\phi\mapsto\phi+\mu$, $\mu\in\mathbb{R}^p$, remains in the massless case $m=0$. By contrast, the functor constructed in [BDS12] always has a $\mathbb{Z}_2$ (sub)group of automorphisms, which has no corresponding interpretation at the level of the Lagrangian. Second, the functors of [BDS12] provide a description of a multiplet of p (mutually noninteracting) fields that is inequivalent to what it would provide for the composition of p copies of a single field. These defects convince us that there is a flaw in this earlier description of affine field theories and that the corresponding functor has to be modified.

A second theme of this paper, then, is to propose a systematic way to improve the construction of the classical and quantum theory of the inhomogeneous multiplet of Klein–Gordon fields. The essential ingredient is the use of suitable state spaces (in the classical theory given by the solution space) and the characterization of the vanishing ideals induced by these state spaces in the abstract algebras considered in [BDS12]. These ideals reflect redundancies in that description, and we therefore quotient by the vanishing ideals to obtain improved functors. Our construction is free of the pathologies of [BDS12]: the functors have the expected automorphism groups and satisfy a natural composition property with respect to the size of the multiplet. This provides the foundation for a broader discussion of the properties of these models. It is worth emphasizing that our improved classical theory is not formulated in terms of presymplectic vector spaces, but in terms of Poisson algebras.[1] Similarly, the improved quantum theory is not given by CCR-algebras, but by quotients of such algebras. Finally, we would like to mention that the techniques developed in this work are important for and can be applied to generic affine field theories [BDS12] and, with slight modifications due to the presence of gauge invariance, also to Abelian gauge theories [BDS13, BDHS13].

The outline of this paper is as follows: In Section 2 we shall review briefly the techniques required for studying affine field theories [BDS12], focusing for simplicity on the explicit example given by the inhomogeneous multiplet of Klein–Gordon fields. The relative Cauchy evolution for this model is discussed in detail in Section 3; as a new feature compared to earlier studies, we study perturbations of both the metric $\boldsymbol{g}$ and the external source $\boldsymbol{J}$. The derivative of the relative Cauchy evolution along metric perturbations is calculated and it is shown how to identify it with the stress-energy tensor corresponding to the action given by (1.1). Furthermore, the derivative of the relative Cauchy evolution along external source perturbations is determined and identified with the $\boldsymbol{J}$-variation of the action given by (1.1). In Section 4 we compute the automorphism group of the functor describing the presymplectic vector spaces of the classical theory of a multiplet of inhomogeneous Klein–Gordon fields. We find that all endomorphisms of this functor (embeddings of the theory as a subtheory of itself) are in fact automorphisms (global gauge transformations), and that the automorphism group is isomorphic to $\mathbb{Z}_2$ in the massive case and to $\mathbb{Z}_2 \times \mathbb{R}^p$ for $m = 0$. The nontrivial $\mathbb{Z}_2$ automorphism does not describe a symmetry of the Lagrangian (1.1), suggesting that inhomogeneous field theories are not appropriately described by the presymplectic vector space functor developed in [BDS12]. This suggestion is strengthened in Section 5, where we study a composition property: Any pair $(\boldsymbol{M}, \boldsymbol{J})$ consisting of a spacetime $\boldsymbol{M}$ with source term $\boldsymbol{J} \in C^\infty(M, \mathbb{R}^p)$ may be split in a functorial way into two pairs $\big((\boldsymbol{M}, \boldsymbol{J}^q), (\boldsymbol{M}, \boldsymbol{J}^{p-q})\big)$, where the source $\boldsymbol{J}$ is split into the first q and last $p - q$ components. Treating the two pairs individually by the presymplectic vector space functor of [BDS12], we get a separate description of the first q and last $p - q$ components of the inhomogeneous Klein–Gordon field. We observe that the direct sum of these two presymplectic vector spaces is not isomorphic to the original presymplectic vector space, and as a consequence the theory obtained in the direct way is not naturally isomorphic to the one obtained after splitting. As the individual components of the inhomogeneous Klein–Gordon field have no mutual interactions, this behavior is pathological and strengthens our claim that the presymplectic vector space functor is not a satisfactory description of the inhomogeneous theory of a multiplet of Klein–Gordon fields.

In Section 6 we show how to resolve these issues by passing from the category of presymplectic vector spaces to that of Poisson algebras. The presymplectic vector space of [BDS12] has a canonical corresponding (abstract) Poisson algebra which can be represented naturally as an algebra of functionals on the affine space of solutions to the inhomogeneous Klein–Gordon equation. In this representation there arises a kernel, which has no corresponding analog at the level of the presymplectic vector spaces. We show that these kernels are natural Poisson ideals and hence we can modify our Poisson algebra functor by quotienting them out. The resulting improved Poisson algebra functor is shown to have the expected automorphism group (i.e. the trivial group for $m \neq 0$ and $\mathbb{R}^p$ for $m = 0$) and to satisfy the composition property. Hence, it is a better description of the classical theory of a multiplet of inhomogeneous Klein–Gordon fields. We extend these constructions to the quantum level in Section 7. The main idea is to characterize suitable state spaces for the CCR-algebras obtained by canonical quantization of our presymplectic vector spaces, which reflect the fact that the latter describe affine functionals on the solution space of the inhomogeneous theory. Quotienting by the intersection of the kernels of corresponding GNS representations, we obtain our improved (functorial) quantized theory, which has the correct automorphism group and satisfies the composition property. Furthermore, we prove that our improved theory satisfies the dynamical locality property introduced in [FV12a, FV12b]. The somewhat special case of the massless multiplet of inhomogeneous Klein–Gordon fields and its interpretation as a rather simple kind of gauge theory is discussed in Section 8. In Section 9 we add some concluding remarks, which

[1] In Appendix B, however, we explain a formulation using *pointed* presymplectic spaces.

should show that the techniques developed in this paper can be readily applied to generic affine field theories in the sense of [BDS12] and also to Abelian gauge theories [BDS13, BDHS13]. Furthermore, we compare our improved algebras with the algebras for inhomogeneous theories constructed (in a slightly ad-hoc way) by Hollands and Wald [HW05] and show that they are naturally isomorphic. Appendix A includes details on how to take the derivative of the relative Cauchy evolution and the stress-energy tensor. In Appendix B we give an alternative solution to the problems arising with the presymplectic vector space functor by introducing a category of *pointed* presymplectic spaces. Finally, Appendix C treats the quantization of our model by deformation methods. It turns out that our improved classical Poisson algebra is amenable to direct deformation quantization; alternatively, one may also apply an algebraic version of Fedosov's method – both lead to the improved quantum theory discussed in the text. We comment on the relationship between our approach and that of the recent paper [SDH12].

2 Preliminaries

2.1 Basics and notations

The model we study throughout this work is given by a multiplet of $p \in \mathbb{N}$ real scalar fields (with the same mass), which are minimally coupled to the Lorentzian metric and in addition coupled to an external source. We shall exclusively work in a category theoretical setting, which is an extension of the framework of locally covariant quantum field theory developed in [BFV03], see also [FV12a]. The basic category entering our construction is given by the following

Definition 2.1. The category LocSrc_p consists of the following objects and morphisms:

- The objects in LocSrc_p are pairs $(\boldsymbol{M}, \boldsymbol{J})$, where $\boldsymbol{M} = (M, \mathfrak{o}, \boldsymbol{g}, \mathfrak{t})$ is any oriented and time-oriented globally hyperbolic Lorentzian manifold (of signature $(+, -, \cdots, -)$ and with finitely many connected components) and $\boldsymbol{J} \in C^\infty(M, \mathbb{R}^p)$.
- The morphisms $f : (\boldsymbol{M}_1, \boldsymbol{J}_1) \to (\boldsymbol{M}_2, \boldsymbol{J}_2)$ in LocSrc_p are orientation and time-orientation preserving isometric embeddings $f : M_1 \to M_2$, such that $f[M_1] \subseteq M_2$ is causally compatible and open and such that $f^*(\boldsymbol{J}_2) = \boldsymbol{J}_1$, where f^* denotes the pull-back.

Any morphism whose image contains a Cauchy surface of the codomain will be called a *Cauchy morphism*; any functor from LocSrc_p to some other category is said to obey the *time-slice axiom* if it maps every Cauchy morphism to an isomorphism.

The configuration space of a multiplet of $p \in \mathbb{N}$ real scalar fields is given by the following contravariant functor $\mathfrak{C}^\infty_p : \mathsf{LocSrc}_p \to \mathsf{Vec}$ to the category of real vector spaces: To any object $(\boldsymbol{M}, \boldsymbol{J})$ in LocSrc_p we associate $\mathfrak{C}^\infty_p(\boldsymbol{M}, \boldsymbol{J}) := C^\infty(M, \mathbb{R}^p)$ and to any morphism $f : (\boldsymbol{M}_1, \boldsymbol{J}_1) \to (\boldsymbol{M}_2, \boldsymbol{J}_2)$ in LocSrc_p we associate the pull-back $\mathfrak{C}^\infty_p(f) := f^* : \mathfrak{C}^\infty_p(\boldsymbol{M}_2, \boldsymbol{J}_2) \to \mathfrak{C}^\infty_p(\boldsymbol{M}_1, \boldsymbol{J}_1)$.

We model the equations of motion for our theory, given by inhomogeneous Klein–Gordon equations, by a certain natural transformation. As a first step, remember that the homogeneous Klein–Gordon equation is described by the natural transformation $\mathrm{KG} = \{\mathrm{KG}_{\boldsymbol{M}}\} : \mathfrak{C}^\infty_p \Rightarrow \mathfrak{C}^\infty_p$ given by the Klein–Gordon operators

$$\mathrm{KG}_{\boldsymbol{M}} : \mathfrak{C}^\infty_p(\boldsymbol{M}, \boldsymbol{J}) \to \mathfrak{C}^\infty_p(\boldsymbol{M}, \boldsymbol{J}) \ , \quad \phi \mapsto \mathrm{KG}_{\boldsymbol{M}}(\phi) = \Box_{\boldsymbol{M}}(\phi) + m^2\phi \ . \tag{2.1}$$

Here $\Box_{\boldsymbol{M}}$ is the d'Alembert operator corresponding to $\boldsymbol{M}$ and $m \geq 0$ is a fixed mass. In order to couple the theory to the sources $\boldsymbol{J}$, which are part of the data of the category LocSrc_p, we have to relax the condition that morphisms in Vec are linear. Let us therefore introduce the category Aff of affine spaces over real vector spaces with affine maps as morphisms and the obvious forgetful functor $\mathfrak{Forget} : \mathsf{Vec} \to \mathsf{Aff}$. We compose $\mathfrak{C}^\infty_p$ with the functor $\mathfrak{Forget}$ and obtain as result a contravariant functor $\mathfrak{A}^\infty_p := \mathfrak{Forget} \circ \mathfrak{C}^\infty_p : \mathsf{LocSrc}_p \to \mathsf{Aff}$. The inhomogeneous Klein–Gordon operators are then described by the natural transformation $\mathrm{P} = \{\mathrm{P}_{(\boldsymbol{M},\boldsymbol{J})}\} : \mathfrak{A}^\infty_p \Rightarrow \mathfrak{A}^\infty_p$ given by

$$\mathrm{P}_{(\boldsymbol{M},\boldsymbol{J})} : \mathfrak{A}^\infty_p(\boldsymbol{M}, \boldsymbol{J}) \to \mathfrak{A}^\infty_p(\boldsymbol{M}, \boldsymbol{J}) \ , \quad \phi \mapsto \mathrm{P}_{(\boldsymbol{M},\boldsymbol{J})}(\phi) = \Box_{\boldsymbol{M}}(\phi) + m^2\phi + \boldsymbol{J} \ . \tag{2.2}$$

The solution spaces for these equations can be given a functorial form. Note that we do *not* assume that the solutions have spacelike compact support (there is no assumption on the support of $\boldsymbol{J}$).

Definition 2.2. The contravariant functor $\mathfrak{Sol}_p : \mathsf{LocSrc}_p \to \mathsf{Aff}$ is defined as follows: To any object $(\boldsymbol{M}, \boldsymbol{J})$ in LocSrc_p it associates the solution space

$$\mathfrak{Sol}_p(\boldsymbol{M}, \boldsymbol{J}) := \{\phi \in C^\infty(M, \mathbb{R}^p) : \mathrm{P}_{(\boldsymbol{M},\boldsymbol{J})}(\phi) = \mathrm{KG}_{\boldsymbol{M}}(\phi) + \boldsymbol{J} = 0\} \ , \tag{2.3}$$

which is an affine space over the vector space $\mathfrak{Sol}_p^{\mathrm{lin}}(\boldsymbol{M}) := \{\underline{\phi} \in C^\infty(M, \mathbb{R}^p) : \mathrm{KG}_{\boldsymbol{M}}(\underline{\phi}) = 0\}$. To any morphism $f : (\boldsymbol{M}_1, \boldsymbol{J}_1) \to (\boldsymbol{M}_2, \boldsymbol{J}_2)$ in LocSrc_p, $\mathfrak{Sol}_p$ associates the Aff-morphism given by the pull-back $\mathfrak{Sol}_p(f) := f^* : \mathfrak{Sol}_p(\boldsymbol{M}_2, \boldsymbol{J}_2) \to \mathfrak{Sol}_p(\boldsymbol{M}_1, \boldsymbol{J}_1)$.

2.2 The presymplectic vector space functor

We follow the prescription of [BDS12] in order to construct a covariant functor $\mathfrak{PS}_p : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ associating presymplectic vector spaces to objects in LocSrc_p, whose role is to label certain affine functionals on $\mathfrak{Sol}_p(\boldsymbol{M}, \boldsymbol{J})$, i.e. observables of the theory.[2] Here $\mathsf{PreSymp}$ denotes the category of real presymplectic vector spaces, with all morphisms being assumed to be injective. The aim is to have sufficiently many observables to separate the solutions, while also removing redundancy by identifying observables that vanish on all solutions. Accordingly, to any object $(\boldsymbol{M}, \boldsymbol{J})$ in LocSrc_p we associate the object $\mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J})$ in $\mathsf{PreSymp}$ given by the following construction: As a vector space,

$$\mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J}) := \big(C_0^\infty(M, \mathbb{R}^p) \oplus \mathbb{R}\big) / \mathrm{P}^*_{(\boldsymbol{M},\boldsymbol{J})}[C_0^\infty(M, \mathbb{R}^p)] \ , \tag{2.4}$$

where, for all $h \in C_0^\infty(M, \mathbb{R}^p)$,

$$\mathrm{P}^*_{(\boldsymbol{M},\boldsymbol{J})}(h) = \left(\mathrm{KG}_{\boldsymbol{M}}(h), \int_M \langle \boldsymbol{J}, h\rangle \ \mathrm{vol}_{\boldsymbol{M}}\right) \in C_0^\infty(M, \mathbb{R}^p) \oplus \mathbb{R} \ . \tag{2.5}$$

(One may also understand this construction as follows: $\mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J})$ is (isomorphic to) the vector space of compactly supported sections of the vector dual bundle of our configuration bundle $M \times \mathbb{R}^p \overset{\mathrm{pr}_1}{\to} M$ (in the category of affine bundles) modulo two quotients, which identify those elements corresponding to functionals which act trivial on all solutions. This viewpoint, which also leads naturally to the definitions of the presymplectic structure and morphisms given below, is spelled out in more detail in [BDS12].)

The presymplectic structure in $\mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J})$ is defined by, for all $[(\varphi, \alpha)], [(\psi, \beta)] \in \mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J})$,

$$\sigma_{(\boldsymbol{M},\boldsymbol{J})}\big([(\varphi, \alpha)], [(\psi, \beta)]\big) := \int_M \langle \varphi, \mathrm{E}_{\boldsymbol{M}}(\psi)\rangle \ \mathrm{vol}_{\boldsymbol{M}} \ , \tag{2.6}$$

where $\mathrm{E}_{\boldsymbol{M}} = \mathrm{E}^-_{\boldsymbol{M}} - \mathrm{E}^+_{\boldsymbol{M}}$ is the advanced-minus-retarded Green's operator for $\mathrm{KG}_{\boldsymbol{M}}$, and the Green's operators obey $\mathrm{supp}(\mathrm{E}^\pm_{\boldsymbol{M}}(\varphi)) \subseteq J^\pm_{\boldsymbol{M}}(\mathrm{supp}(\varphi))$.

To any morphism $f : (\boldsymbol{M}_1, \boldsymbol{J}_1) \to (\boldsymbol{M}_2, \boldsymbol{J}_2)$ in LocSrc_p the functor $\mathfrak{PS}_p$ associates the morphism $\mathfrak{PS}_p(f) : \mathfrak{PS}_p(\boldsymbol{M}_1, \boldsymbol{J}_1) \to \mathfrak{PS}_p(\boldsymbol{M}_2, \boldsymbol{J}_2)$ in $\mathsf{PreSymp}$ that is canonically induced by the push-forward,

$$\mathfrak{PS}_p(f)\big([(\varphi, \alpha)]\big) := [(f_*(\varphi), \alpha)] \ , \tag{2.7}$$

for any $[(\varphi, \alpha)] \in \mathfrak{PS}_p(\boldsymbol{M}_1, \boldsymbol{J}_1)$, which is well-defined because

$$\left(f_*\big(\mathrm{KG}_{\boldsymbol{M}_1}(h)\big), \int_{M_1} \langle \boldsymbol{J}_1, h\rangle \ \mathrm{vol}_{\boldsymbol{M}_1}\right) = \mathrm{P}^*_{(\boldsymbol{M}_2,\boldsymbol{J}_2)}\big(f_*(h)\big) \ . \tag{2.8}$$

[2] In [BDS12] this functor was denoted by $\mathfrak{PhSp}$ and it was called the "phase space functor". We avoid this notation in our present work, since the presymplectic vector spaces obtained by $\mathfrak{PhSp}$ are just labeling spaces for functionals and not what one typically calls the phase space (i.e. the space of initial data or the space of solutions).

As mentioned, the role of the covariant functor $\mathfrak{PS}_p$ is to label affine functionals on the contravariant functor $\mathfrak{Sol}_p$. This manifests itself in a natural dual pairing: For each object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p the evaluation map

$$\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_{(\boldsymbol{M},\boldsymbol{J})} : \mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})\times\mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})\to\mathbb{R}\ ,\quad \big([(\varphi,\alpha)],\phi\big)\mapsto\bigg(\int_M\langle\varphi,\phi\rangle\ \mathrm{vol}_{\boldsymbol{M}}\bigg)+\alpha \tag{2.9}$$

is well-defined and linear in the left and affine in the right entry. Naturality means that the following diagram commutes for any morphism $f:(\boldsymbol{M}_1,\boldsymbol{J}_1)\to(\boldsymbol{M}_2,\boldsymbol{J}_2)$ in LocSrc_p

$$\begin{tikzcd}
\mathfrak{PS}_p(\boldsymbol{M}_1,\boldsymbol{J}_1)\times\mathfrak{Sol}_p(\boldsymbol{M}_2,\boldsymbol{J}_2) \arrow[r, "\mathrm{id}_{\mathfrak{PS}_p(\boldsymbol{M}_1,\boldsymbol{J}_1)}\times\mathfrak{Sol}_p(f)"] \arrow[d, "\mathfrak{PS}_p(f)\times\mathrm{id}_{\mathfrak{Sol}_p(\boldsymbol{M}_2,\boldsymbol{J}_2)}"'] & \mathfrak{PS}_p(\boldsymbol{M}_1,\boldsymbol{J}_1)\times\mathfrak{Sol}_p(\boldsymbol{M}_1,\boldsymbol{J}_1) \arrow[d, "\langle\!\langle\cdot,\cdot\rangle\!\rangle_{(\boldsymbol{M}_1,\boldsymbol{J}_1)}"] \\
\mathfrak{PS}_p(\boldsymbol{M}_2,\boldsymbol{J}_2)\times\mathfrak{Sol}_p(\boldsymbol{M}_2,\boldsymbol{J}_2) \arrow[r, "\langle\!\langle\cdot,\cdot\rangle\!\rangle_{(\boldsymbol{M}_2,\boldsymbol{J}_2)}"] & \mathbb{R}
\end{tikzcd} \tag{2.10}$$

Furthermore, the presymplectic structure in $\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$ coincides precisely with the Peierls bracket [Pei52] for the theory (1.1), on regarding elements of $\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$ as observables in this way.

We summarize the main properties of the covariant functor $\mathfrak{PS}_p$ defined by (2.4), (2.6) and (2.7), which follow immediately from the general treatment of affine field theories in [BDS12].

Proposition 2.3. *a) Let $(\boldsymbol{M},\boldsymbol{J})$ be any object in LocSrc_p. Then the null space $\mathfrak{N}_p(\boldsymbol{M},\boldsymbol{J})$ of the presymplectic structure in $\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$ is isomorphic to $\mathbb{R}$.*

b) The null space is functorial, i.e. $\mathfrak{N}_p:\mathsf{LocSrc}_p\to\mathsf{Vec}$ is a covariant functor.

c) The covariant functor $\mathfrak{PS}_p:\mathsf{LocSrc}_p\to\mathsf{PreSymp}$ satisfies the causality property and the time-slice axiom.

Proof. The proof of a) follows from [BDS12, Corollary 4.5.] and b) follows from [BDS12, Lemma 7.3.]. Item c) is a consequence of [BDS12, Theorem 5.5. and Theorem 5.6.]. □

We note that item c) of the proposition above means that $\mathfrak{PS}_p$ is a locally covariant classical field theory. Due to the nontrivial null space of the presymplectic structure (cf. item a)) this theory has distinct features which are not present in the homogeneous Klein–Gordon theory, where the null space is trivial.

2.3 Quantization

The theory $\mathfrak{PS}_p:\mathsf{LocSrc}_p\to\mathsf{PreSymp}$ may be quantized by composing it with the canonical commutation relation (CCR) functor (either in Weyl or polynomial form). Since these quantization functors preserve locality, causality and the time-slice axiom, we obtain a locally covariant quantum field theory in the sense of [BFV03, FV12a], with the small difference that our underlying geometric category is enhanced from Loc to LocSrc_p. For more details on the Weyl quantization functors for presymplectic vector spaces (and more general also presymplectic Abelian groups) we refer to the Appendix of [BDHS13]. The quantized theory of a multiplet of $p\in\mathbb{N}$ inhomogeneous Klein–Gordon fields is studied in detail in Section 7.

3 Relative Cauchy evolution of the functor $\mathfrak{PS}_p$

Relative Cauchy evolution encodes the sensitivity of a theory to variations in the background structures; in this it closely resembles the action. Apart from its intrinsic interest, understanding the relative Cauchy evolution is also an integral step in characterizing the automorphism groups of our functors in Section 4. We base our analysis on the refined construction given in [FV12a], which we now review and adapt to our present setting.

Given any object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p, we can consider its perturbation by elements $(\boldsymbol{h},\boldsymbol{j})\in\Gamma^\infty_0(T^*M\vee T^*M)\times C^\infty_0(M,\mathbb{R}^p)$, where $\Gamma^\infty_0(T^*M\vee T^*M)$ denotes the vector space of compactly supported sections of the symmetric tensor product of the cotangent bundle (i.e. symmetric tensor fields). Explicitly, given

$(\boldsymbol{h},\boldsymbol{j}) \in \Gamma_0^\infty(T^*M \vee T^*M) \times C_0^\infty(M,\mathbb{R}^p)$ such that $\boldsymbol{g}+\boldsymbol{h}$ is time-orientable, we define $(\boldsymbol{M}[\boldsymbol{h}],\boldsymbol{J}[\boldsymbol{j}]) := \big((M,\mathfrak{o},\boldsymbol{g}+\boldsymbol{h},\mathfrak{t}_{\boldsymbol{h}}),\boldsymbol{J}+\boldsymbol{j}\big)$, where $\mathfrak{t}_{\boldsymbol{h}}$ is the unique time-orientation for $\boldsymbol{g}+\boldsymbol{h}$, such that $\mathfrak{t}_{\boldsymbol{h}} = \mathfrak{t}$ outside the support of $\boldsymbol{h}$. If $(\boldsymbol{M}[\boldsymbol{h}],\boldsymbol{J}[\boldsymbol{j}])$ is an object in LocSrc_p, i.e. if $\boldsymbol{M}[\boldsymbol{h}]$ is globally hyperbolic, we say that $(\boldsymbol{h},\boldsymbol{j})$ is a globally hyperbolic perturbation and write $(\boldsymbol{h},\boldsymbol{j}) \in H(\boldsymbol{M},\boldsymbol{J})$. Evidently $H(\boldsymbol{M},\boldsymbol{J})$ contains an open neighborhood of $\{0\}\times C_0^\infty(M,\mathbb{R}^p)$ in the usual test-section topology.

For any object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p and any $(\boldsymbol{h},\boldsymbol{j}) \in H(\boldsymbol{M},\boldsymbol{J})$ we define the sets

$$M^\pm := M \setminus J^\mp_{\boldsymbol{M}}\big(\mathrm{supp}(\boldsymbol{h}) \cup \mathrm{supp}(\boldsymbol{j})\big), \tag{3.1}$$

which are causally compatible, open and globally hyperbolic subsets of $\boldsymbol{M}$ and $\boldsymbol{M}[\boldsymbol{h}]$. We define $\boldsymbol{M}^\pm[\boldsymbol{h},\boldsymbol{j}] := \boldsymbol{M}|_{M^\pm} = \boldsymbol{M}[\boldsymbol{h}]|_{M^\pm}$ and $\boldsymbol{J}^\pm[\boldsymbol{h},\boldsymbol{j}] := \boldsymbol{J}|_{M^\pm} = (\boldsymbol{J}+\boldsymbol{j})|_{M^\pm}$. Notice that $(\boldsymbol{M}^\pm[\boldsymbol{h},\boldsymbol{j}],\boldsymbol{J}^\pm[\boldsymbol{h},\boldsymbol{j}])$ are objects in LocSrc_p and further that the canonical inclusions of underlying manifolds yield Cauchy morphisms

$$i^\pm_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}] : \big(\boldsymbol{M}^\pm[\boldsymbol{h},\boldsymbol{j}],\boldsymbol{J}^\pm[\boldsymbol{h},\boldsymbol{j}]\big) \to \big(\boldsymbol{M},\boldsymbol{J}\big)\,, \tag{3.2a}$$

$$j^\pm_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}] : \big(\boldsymbol{M}^\pm[\boldsymbol{h},\boldsymbol{j}],\boldsymbol{J}^\pm[\boldsymbol{h},\boldsymbol{j}]\big) \to \big(\boldsymbol{M}[\boldsymbol{h}],\boldsymbol{J}[\boldsymbol{j}]\big)\,. \tag{3.2b}$$

Since, by Proposition 2.3, $\mathfrak{PS}_p : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ satisfies the time-slice axiom, we can construct isomorphisms $\tau^\pm_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}] : \mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J}) \to \mathfrak{PS}_p(\boldsymbol{M}[\boldsymbol{h}],\boldsymbol{J}[\boldsymbol{j}])$ in $\mathsf{PreSymp}$ by

$$\tau^\pm_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}] := \mathfrak{PS}_p(j^\pm_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}]) \circ \big(\mathfrak{PS}_p(i^\pm_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}])\big)^{-1}\,. \tag{3.3}$$

The relative Cauchy evolution of $\mathfrak{PS}_p$ induced by $(\boldsymbol{h},\boldsymbol{j}) \in H(\boldsymbol{M},\boldsymbol{J})$ is defined as the automorphism

$$\mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}] := \big(\tau^-_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}]\big)^{-1} \circ \tau^+_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}] \in \mathrm{Aut}(\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J}))\,, \tag{3.4}$$

and may be computed as follows. Owing to the time-slice axiom, any element in $\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$ may be written in the form[3] $[(\varphi,\alpha)]_{(\boldsymbol{M},\boldsymbol{J})}$ with φ supported in M^+, whereupon

$$\tau^+_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}]([(\varphi,\alpha)]_{(\boldsymbol{M},\boldsymbol{J})}) = [(\varphi,\alpha)]_{(\boldsymbol{M}[\boldsymbol{h}],\boldsymbol{J}[\boldsymbol{j}])}\,. \tag{3.5}$$

In turn, given a representative $(\varphi',\alpha') \in [(\varphi,\alpha)]_{(\boldsymbol{M}[\boldsymbol{h}],\boldsymbol{J}[\boldsymbol{j}])}$ so that φ' has support in M^-, the relative Cauchy evolution of $[(\varphi,\alpha)]_{(\boldsymbol{M},\boldsymbol{J})}$ is

$$\mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}]\big([(\varphi,\alpha)]_{(\boldsymbol{M},\boldsymbol{J})}\big) = \big(\tau^-_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}]\big)^{-1}\big([(\varphi,\alpha)]_{(\boldsymbol{M}[\boldsymbol{h}],\boldsymbol{J}[\boldsymbol{j}])}\big) = [(\varphi',\alpha')]_{(\boldsymbol{M},\boldsymbol{J})}\,. \tag{3.6}$$

Thus it remains to find a suitable representative (φ',α'). By a standard argument, see e.g. [FV12b, Lemma 3.1.], we can find a smooth function $\chi \in C^\infty(M)$, such that $\varphi' = \varphi - \mathrm{KG}_{\boldsymbol{M}[\boldsymbol{h}]}\big(\chi\,\mathrm{E}^-_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\big)$ has support in M^- and such that $\chi\,\mathrm{E}^-_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)$ has compact support. Explicitly, we take any two Cauchy surfaces $\Sigma^\pm$ in $\boldsymbol{M}^-[\boldsymbol{h},\boldsymbol{j}]$ such that $\Sigma^+\cap\Sigma^- = \emptyset$, Σ^+ is in the future of Σ^- and $\mathrm{supp}(\varphi)\cup\mathrm{supp}(\boldsymbol{h})\cup\mathrm{supp}(\boldsymbol{j})$ is in the future of Σ^+. Any χ such that $\chi|_{J^+_{\boldsymbol{M}[\boldsymbol{h}]}(\Sigma^+)} \equiv 1$ and $\chi|_{J^-_{\boldsymbol{M}[\boldsymbol{h}]}(\Sigma^-)} \equiv 0$ then leads by the formula above to a φ' with the desired properties. Using (2.5) and now dropping the labels on equivalence classes (which from now on are all taken with respect to $(\boldsymbol{M},\boldsymbol{J})$) we obtain for the relative Cauchy evolution

$$\mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}]\big([(\varphi,\alpha)]\big) = \Big[\Big(\varphi - \mathrm{KG}_{\boldsymbol{M}[\boldsymbol{h}]}\big(\chi\,\mathrm{E}^-_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\big), \alpha - \int_M \Big\langle \boldsymbol{J}+\boldsymbol{j}, \chi\,\mathrm{E}^-_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\Big\rangle\,\mathrm{vol}_{\boldsymbol{M}[\boldsymbol{h}]}\Big)\Big]\,. \tag{3.7}$$

As $\chi\,\mathrm{E}^-_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)$ is compactly supported, we may use the equivalence relation with respect to $(\boldsymbol{M},\boldsymbol{J})$ to obtain

$$\begin{aligned}&\Big(\mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}] - \mathrm{id}_{\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})}\Big)\big([(\varphi,\alpha)]\big)\\ &\quad= \Big[\Big((\mathrm{KG}_{\boldsymbol{M}} - \mathrm{KG}_{\boldsymbol{M}[\boldsymbol{h}]})\big(\chi\,\mathrm{E}^-_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\big), \int_M \Big\langle (1-\rho_{\boldsymbol{h}})\boldsymbol{J} - \rho_{\boldsymbol{h}}\,\boldsymbol{j}, \chi\,\mathrm{E}^-_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\Big\rangle\,\mathrm{vol}_{\boldsymbol{M}}\Big)\Big]\,,\end{aligned} \tag{3.8}$$

[3] For clarity, in this discussion we shall indicate the LocSrc_p object with respect to which the equivalence relation is understood.

in which we have also written $\rho_{\boldsymbol{h}} \in C_0^\infty(M)$ for the unique function such that $\mathrm{vol}_{\boldsymbol{M}[\boldsymbol{h}]} = \rho_{\boldsymbol{h}}\, \mathrm{vol}_{\boldsymbol{M}}$, explicitly $\rho_{\boldsymbol{h}} = \sqrt{|\boldsymbol{g}+\boldsymbol{h}|}/\sqrt{|\boldsymbol{g}|}$. Noting that $\chi = 1$ and $\mathrm{E}^+_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi) = 0$ on $\mathrm{supp}(\boldsymbol{j}) \cup \mathrm{supp}(\boldsymbol{h})$, we may replace both occurrences of $\chi\, \mathrm{E}^-_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)$ by $\mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)$, obtaining

$$\begin{aligned}&\left(\mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}] - \mathrm{id}_{\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})}\right)([(\varphi,\alpha)])\\ &= \left[\Big((\mathrm{KG}_{\boldsymbol{M}} - \mathrm{KG}_{\boldsymbol{M}[\boldsymbol{h}]})(\mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)), \int_M \Big(\langle -\boldsymbol{j}, \mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\rangle + \langle (1-\rho_{\boldsymbol{h}})(\boldsymbol{J}+\boldsymbol{j}), \mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\rangle\Big)\, \mathrm{vol}_{\boldsymbol{M}}\Big)\right],\end{aligned} \tag{3.9}$$

after a further slight rearrangement. Note that (3.9) applies only for representatives where φ is supported in M^+. In this form, it is easy to see what the functional derivative of the relative Cauchy evolution with respect to $\boldsymbol{h}$ and $\boldsymbol{j}$ *ought* to be, simply by expanding to first order in these quantities. This procedure gives

$$\frac{d}{ds}\mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[s\boldsymbol{h}, s\boldsymbol{j}]([(\varphi,\alpha)])\Big|_{s=0} =: -\left(\mathcal{T}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h}] + \mathcal{J}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{j}]\right)([(\varphi,\alpha)]), \tag{3.10}$$

where

$$\mathcal{T}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h}]([(\varphi,\alpha)]) = \left[\left(\mathrm{KG}'_{\boldsymbol{M}[\boldsymbol{h}]}(\mathrm{E}_{\boldsymbol{M}}(\varphi)), \int_M \frac{1}{2} g^{ab} h_{ab}\, \langle \boldsymbol{J}, \mathrm{E}_{\boldsymbol{M}}(\varphi)\rangle\, \mathrm{vol}_{\boldsymbol{M}}\right)\right], \tag{3.11a}$$

$$\mathcal{J}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{j}]([(\varphi,\alpha)]) = \left[\left(0, \int_M \langle \boldsymbol{j}, \mathrm{E}_{\boldsymbol{M}}(\varphi)\rangle\, \mathrm{vol}_{\boldsymbol{M}}\right)\right], \tag{3.11b}$$

and $\mathrm{KG}'_{\boldsymbol{M}[\boldsymbol{h}]} = \frac{d}{ds}\mathrm{KG}_{\boldsymbol{M}[s\boldsymbol{h}]}\big|_{s=0}$.[4] Formulae (3.11a) and (3.11b) hold for arbitrary representatives (φ,α). Note that elements in $\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$ which are of the form $[(0,\alpha)]$, $\alpha \in \mathbb{R}$, are left unchanged under the relative Cauchy evolution. In Appendix A, we shall show how (3.10) holds rigorously in the weak-$*$ topology induced by the pairing (2.9). Moreover, we obtain the formula

$$\langle\langle \mathcal{T}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h}]([(\varphi,\alpha)]), \phi\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} = \frac{1}{2}\frac{d}{ds}\int_M h_{ab}\, T^{ab}_{(\boldsymbol{M},\boldsymbol{J})}[\phi + s\,\mathrm{E}_{\boldsymbol{M}}(\varphi)]\, \mathrm{vol}_{\boldsymbol{M}}\Big|_{s=0}, \tag{3.12}$$

where the stress-energy tensor is[5]

$$T^{ab}_{(\boldsymbol{M},\boldsymbol{J})}[\phi] := -\frac{2}{\sqrt{|\boldsymbol{g}|}}\frac{\delta S}{\delta g_{ab}(x)} = \left\langle \nabla^a\phi, \nabla^b\phi\right\rangle - \frac{1}{2}g^{ab}\langle \nabla_c\phi, \nabla^c\phi\rangle + \frac{1}{2}m^2 g^{ab}\langle\phi,\phi\rangle + g^{ab}\langle \boldsymbol{J},\phi\rangle, \tag{3.13}$$

and S is the classical action obtained from the Lagrangian (1.1) (with $\lambda = 1$). Similarly, it is clear from (3.11b) that

$$\langle\langle \mathcal{J}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{j}]([(\varphi,\alpha)]), \phi\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} = \frac{d}{ds}\int_M \langle \boldsymbol{j}, \phi + s\,\mathrm{E}_{\boldsymbol{M}}(\varphi)\rangle\, \mathrm{vol}_{\boldsymbol{M}}\Big|_{s=0}. \tag{3.14}$$

These formulae establish a close link between the relative Cauchy evolution and the action; indeed,

$$\frac{d}{ds}\left\langle\left\langle \mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[s\boldsymbol{h}, s\boldsymbol{j}]([(\varphi,\alpha)]), \phi\right\rangle\right\rangle_{(\boldsymbol{M},\boldsymbol{J})}\Big|_{s=0} = \frac{\delta^2 S}{\delta\phi\delta\boldsymbol{g}}(\mathrm{E}_{\boldsymbol{M}}(\varphi)\otimes\boldsymbol{h}) + \frac{\delta^2 S}{\delta\phi\delta\boldsymbol{J}}(\mathrm{E}_{\boldsymbol{M}}(\varphi)\otimes\boldsymbol{j}), \tag{3.15}$$

where the functional derivatives are evaluated at $\phi \in \mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})$, and on the background $(\boldsymbol{M},\boldsymbol{J})$, and we differentiate with respect to the covariant metric tensor.

[4]Note that the derivative of the relative Cauchy evolution involves *minus* the derivative of the Klein–Gordon operator. The BFV paper [BFV03] contains an error [or unconventional terminology] on p.61, where an advanced solution is given support in the causal future of the source, leading to the opposite overall sign in the analogous expression for the derivative on p.62 and hence in their Theorem 4.3.

[5]The minus sign before the functional derivative appears because we differentiate with respect to the covariant form of the metric.

At this point the following remark is in order: The stress-energy tensor (3.13) is *not* covariantly conserved for generic $(\boldsymbol{M}, \boldsymbol{J})$ and generic solutions ϕ of the inhomogeneous Klein–Gordon equation, since

$$\nabla_a T^{ab}_{(\boldsymbol{M},\boldsymbol{J})}[\phi] = \left\langle \nabla^b \boldsymbol{J}, \phi \right\rangle \ . \tag{3.16}$$

Modifying $T^{ab}_{(\boldsymbol{M},\boldsymbol{J})}$ by adding a constant functional, which would not change the derivative of the relative Cauchy evolution given in (3.12), does not change this fact. Repeating the arguments given in [BFV03, §4], the non-conservation law (3.16) (up to constant functionals) can also be derived directly from the relative Cauchy evolution. This perhaps unpleasant feature can be understood as follows: diffeomorphism invariance of the action derived from (1.1) (with $\lambda = 1$) entails the identity

$$\frac{\delta S}{\delta \boldsymbol{g}}(\pounds_X \boldsymbol{g}) + \frac{\delta S}{\delta \boldsymbol{J}}(\pounds_X \boldsymbol{J}) + \frac{\delta S}{\delta \phi}(\pounds_X \phi) = 0 \tag{3.17}$$

for all compactly supported vector fields X. When ϕ is on-shell, the last term vanishes and the identity implies (3.16). We cannot expect conservation of the stress-energy tensor in our theory, because $\boldsymbol{J}$ is non-dynamical; indeed (3.16) is the correct generalized conservation law in this case. (Were we to modify the theory, so that $\boldsymbol{J}$ became dynamical, the additional Euler–Lagrange equation $\phi = 0$ would rather trivially restore conservation of the stress-energy tensor.)

4 Automorphism group of the functor $\mathfrak{PS}_p$

Given any covariant functor from LocSrc_p to $\mathsf{PreSymp}$ it is interesting to study its endomorphisms and automorphisms [Few13]. The latter typically sheds light on possible symmetries of the theory at the functorial level, which is comparable to the global gauge group of Minkowski algebraic quantum field theory. In [Few13], the automorphism group of a theory describing a multiplet of $p \in \mathbb{N}$ classical real scalar fields satisfying the minimally coupled Klein–Gordon equation was found to be the orthogonal group $O(p)$ if all masses coincide and are nonzero, or $O(p) \ltimes \mathbb{R}^p$ if they all vanish. As mentioned in the Introduction, we expect the source terms in the inhomogeneous Klein–Gordon theory to break (at least for the massive case $m \neq 0$) all the symmetries of the homogeneous Klein–Gordon theory. It therefore comes as a surprise that the functor $\mathfrak{PS}_p$ turns out to have a nontrivial automorphism group for any mass m.

We shall briefly fix some notation. Given any covariant functor $\mathfrak{F} : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$, an endomorphism of $\mathfrak{F}$ is a natural transformation $\eta : \mathfrak{F} \Rightarrow \mathfrak{F}$, i.e. a collection of morphisms $\eta_{(\boldsymbol{M},\boldsymbol{J})} : \mathfrak{F}(\boldsymbol{M}, \boldsymbol{J}) \to \mathfrak{F}(\boldsymbol{M}, \boldsymbol{J})$ in $\mathsf{PreSymp}$, such that for any morphism $f : (\boldsymbol{M}_1, \boldsymbol{J}_1) \to (\boldsymbol{M}_2, \boldsymbol{J}_2)$ in LocSrc_p the following diagram commutes

$$\begin{tikzcd}
\mathfrak{F}(\boldsymbol{M}_1, \boldsymbol{J}_1) \arrow[r, "\mathfrak{F}(f)"] \arrow[d, "\eta_{(\boldsymbol{M}_1,\boldsymbol{J}_1)}"'] & \mathfrak{F}(\boldsymbol{M}_2, \boldsymbol{J}_2) \arrow[d, "\eta_{(\boldsymbol{M}_2,\boldsymbol{J}_2)}"] \\
\mathfrak{F}(\boldsymbol{M}_1, \boldsymbol{J}_1) \arrow[r, "\mathfrak{F}(f)"] & \mathfrak{F}(\boldsymbol{M}_2, \boldsymbol{J}_2)
\end{tikzcd} \tag{4.1}$$

We denote the collection of all endomorphisms of $\mathfrak{F}$ by $\mathrm{End}(\mathfrak{F})$. An automorphism of $\mathfrak{F}$ is an endomorphism $\eta \in \mathrm{End}(\mathfrak{F})$, such that all $\eta_{(\boldsymbol{M},\boldsymbol{J})}$ are isomorphisms. Under composition, the automorphisms of $\mathfrak{F}$ form a group denoted by $\mathrm{Aut}(\mathfrak{F})$.

The goal of this section is to characterize the automorphism group of the functor $\mathfrak{PS}_p : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ introduced in Section 2. Due to the following general statement, $\mathrm{Aut}(\mathfrak{PS}_p)$ is nontrivial.

Proposition 4.1. *Let $\mathfrak{F} : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ be any covariant functor. Then there exists a faithful homomorphism $\eta : \mathbb{Z}_2 \to \mathrm{Aut}(\mathfrak{F})$ given by $\eta(\sigma) = \{\sigma\, \mathrm{id}_{\mathfrak{F}(\boldsymbol{M},\boldsymbol{J})}\}$, where $\sigma \in \mathbb{Z}_2 = \{-1, +1\}$.*

Proof. Injectivity of η and the group law $\eta(\sigma) \circ \eta(\sigma') = \eta(\sigma\,\sigma')$ are obvious. All $\eta(\sigma)_{(\boldsymbol{M},\boldsymbol{J})}$ are clearly linear automorphisms and since $\sigma^2 = 1$ they preserve the presymplectic structure in $\mathfrak{F}(\boldsymbol{M}, \boldsymbol{J})$ (this follows from bilinearity of any presymplectic structure). For any morphism f in LocSrc_p, $\eta(\sigma)$ satisfies the diagram in (4.1), since $\mathfrak{F}(f)$ are in particular linear maps and hence commute with the multiplication by σ. □

The previous proposition in particular shows that $\mathrm{Aut}(\mathfrak{PS}_p)$ contains a $\mathbb{Z}_2$ subgroup for all values of m. In the massless case we can say more.

Proposition 4.2. *If $m = 0$ there exists a faithful homomorphism $\eta : \mathbb{Z}_2 \times \mathbb{R}^p \to \mathrm{Aut}(\mathfrak{PS}_p)$ given by $\eta(\sigma,\mu) = \{\eta(\sigma,\mu)_{(\boldsymbol{M},\boldsymbol{J})}\}$, where, for all $[(\varphi,\alpha)] \in \mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$,*

$$\eta(\sigma,\mu)_{(\boldsymbol{M},\boldsymbol{J})}\big([(\varphi,\alpha)]\big) = \left[\left(\sigma\,\varphi, \sigma\,\alpha + \sigma \int_M \langle\varphi,\mu\rangle \ \mathrm{vol}_{\boldsymbol{M}}\right)\right] . \tag{4.2}$$

Here we have identified $\mu \in \mathbb{R}^p$ with the corresponding constant function in $C^\infty(M,\mathbb{R}^p)$.

Proof. The main burden is to show that (4.2) does define a natural $\eta(\sigma,\mu) \in \mathrm{End}(\mathfrak{PS}_p)$ for each $(\sigma,\mu) \in \mathbb{Z}_2 \times \mathbb{R}^p$, because injectivity of η is obvious and it is easy to establish the group law $\eta(\sigma,\mu)\circ\eta(\sigma',\mu') = \eta(\sigma\,\sigma',\mu+\mu')$, whereupon it is clear that each $\eta(\sigma,\mu)$ is a linear automorphism. We notice that $\eta(\sigma,\mu)_{(\boldsymbol{M},\boldsymbol{J})}$ is compatible with the quotient in $\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$, since, for all $h \in C^\infty_0(M,\mathbb{R}^p)$,

$$\begin{aligned}
\eta(\sigma,\mu)_{(\boldsymbol{M},\boldsymbol{J})}\big(\mathrm{P}^*_{(\boldsymbol{M},\boldsymbol{J})}(h)\big) &= \eta(\sigma,\mu)_{(\boldsymbol{M},\boldsymbol{J})}\left(\mathrm{KG}_{\boldsymbol{M}}(h), \int_M \langle\boldsymbol{J},h\rangle \ \mathrm{vol}_{\boldsymbol{M}}\right)\\
&= \left(\sigma\,\mathrm{KG}_{\boldsymbol{M}}(h), \sigma\int_M \langle\boldsymbol{J},h\rangle \ \mathrm{vol}_{\boldsymbol{M}} + \sigma\int_M \langle\mathrm{KG}_{\boldsymbol{M}}(h),\mu\rangle \ \mathrm{vol}_{\boldsymbol{M}}\right)\\
&= \mathrm{P}^*_{(\boldsymbol{M},\boldsymbol{J})}(\sigma\,h) ,
\end{aligned} \tag{4.3}$$

where in the last equality we have used that $\int_M \langle\mathrm{KG}_{\boldsymbol{M}}(h),\mu\rangle \ \mathrm{vol}_{\boldsymbol{M}} = \int_M \langle h,\mathrm{KG}_{\boldsymbol{M}}(\mu)\rangle \ \mathrm{vol}_{\boldsymbol{M}} = 0$ for the massless Klein–Gordon operator. It is easily seen that the linear map $\eta(\sigma,\mu)_{(\boldsymbol{M},\boldsymbol{J})}$ preserves the presymplectic structure in $\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$ and that it is injective (indeed, invertible, as already mentioned). Thus $\eta(\sigma,\mu)_{(\boldsymbol{M},\boldsymbol{J})} \in \mathrm{Aut}(\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J}))$.

Now suppose that $f : (\boldsymbol{M}_1,\boldsymbol{J}_1) \to (\boldsymbol{M}_2,\boldsymbol{J}_2)$ is a morphism in LocSrc_p. Then, for all $[(\varphi,\alpha)] \in \mathfrak{PS}_p(\boldsymbol{M}_1,\boldsymbol{J}_1)$,

$$\begin{aligned}
\eta(\sigma,\mu)_{(\boldsymbol{M}_2,\boldsymbol{J}_2)}\circ\mathfrak{PS}_p(f)\big([(\varphi,\alpha)]\big) &= \left[\left(\sigma\,f_*(\varphi), \sigma\,\alpha + \sigma\int_{M_2} \langle f_*(\varphi),\mu\rangle \ \mathrm{vol}_{\boldsymbol{M}_2}\right)\right]\\
&= \mathfrak{PS}_p(f)\circ\eta(\sigma,\mu)_{(\boldsymbol{M}_1,\boldsymbol{J}_1)}\big([(\varphi,\alpha)]\big) ,
\end{aligned} \tag{4.4}$$

because $\int_{M_2} \langle f_*(\varphi),\mu\rangle \ \mathrm{vol}_{\boldsymbol{M}_2} = \int_{M_1} \langle\varphi,f^*(\mu)\rangle \ \mathrm{vol}_{\boldsymbol{M}_1} = \int_{M_1} \langle\varphi,\mu\rangle \ \mathrm{vol}_{\boldsymbol{M}_1}$. Hence, naturality is proved. □

Our aim is now to prove that we actually have an isomorphism $\mathrm{Aut}(\mathfrak{PS}_p) \simeq \mathbb{Z}_2$ in the massive case and an isomorphism $\mathrm{Aut}(\mathfrak{PS}_p) \simeq \mathbb{Z}_2 \times \mathbb{R}^p$ for $m = 0$. For this endeavor we generalize the results of [Few13, §2.2.], which shall allow us to prove that every endomorphism $\eta \in \mathrm{End}(\mathfrak{PS}_p)$ is uniquely determined by its component $\eta_{(\boldsymbol{M},\boldsymbol{J})}$ on any given object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p.

We start by collecting some useful lemmas, which were obtained in [Few13, §2.2.] for the category Loc.

Lemma 4.3. *Let $\eta \in \mathrm{End}(\mathfrak{PS}_p)$ be any endomorphism and $(\boldsymbol{M},\boldsymbol{J})$ any object in LocSrc_p. Then for all globally hyperbolic perturbations $(\boldsymbol{h},\boldsymbol{j}) \in H(\boldsymbol{M},\boldsymbol{J})$ we have that*

$$\mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}]\circ\eta_{(\boldsymbol{M},\boldsymbol{J})} = \eta_{(\boldsymbol{M},\boldsymbol{J})}\circ\mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}] . \tag{4.5}$$

Proof. The essential idea is naturality of η. The detailed steps can be found in [FV12a, Proposition 3.8.]. □

Lemma 4.4. *Let $\eta,\eta' \in \mathrm{End}(\mathfrak{PS}_p)$ and suppose that $\eta_{(\boldsymbol{M},\boldsymbol{J})} = \eta'_{(\boldsymbol{M},\boldsymbol{J})}$ for some object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p. Then the following hold true:*

(i) If $f : (\boldsymbol{L},\boldsymbol{J}_{\boldsymbol{L}}) \to (\boldsymbol{M},\boldsymbol{J})$ is a morphism in LocSrc_p, then $\eta_{(\boldsymbol{L},\boldsymbol{J}_{\boldsymbol{L}})} = \eta'_{(\boldsymbol{L},\boldsymbol{J}_{\boldsymbol{L}})}$.

(ii) If $f : (\boldsymbol{M},\boldsymbol{J}) \to (\boldsymbol{N},\boldsymbol{J}_{\boldsymbol{N}})$ is a Cauchy morphism in LocSrc_p, then $\eta_{(\boldsymbol{N},\boldsymbol{J}_{\boldsymbol{N}})} = \eta'_{(\boldsymbol{N},\boldsymbol{J}_{\boldsymbol{N}})}$.

(iii) $\eta_{(\boldsymbol{L},\boldsymbol{J_L})} = \eta'_{(\boldsymbol{L},\boldsymbol{J_L})}$ *for any object* $(\boldsymbol{L},\boldsymbol{J_L})$ *in* LocSrc_p*, such that the Cauchy surfaces of* $\boldsymbol{L}$ *are oriented diffeomorphic to those of* $\boldsymbol{M}|_O$*, for some* $O \in \mathscr{O}(\boldsymbol{M})$*. Here* $\mathscr{O}(\boldsymbol{M})$ *is the set of all causally compatible, open and globally hyperbolic subsets of* $\boldsymbol{M}$ *with finitely many connected components all of which are mutually causally disjoint.*

Proof. Item (i) and (ii) are simple consequences of naturality of η, η' and the fact that $\mathfrak{PS}_p(f)$ is injective for (i) and even an isomorphism for (ii), cf. [Few13, Proof of Lemma 2.2.]. Item (iii) follows from a generalization of the "Cauchy wedge connectedness" property to the category LocSrc_p that we shall discuss now in detail. Forgetting the source terms, more precisely, applying the forgetful functor from LocSrc_p to Loc, it was shown in [FV12a, Proposition 2.4.] that under our hypotheses there is a chain of morphisms in Loc

$$\boldsymbol{L} \xleftarrow{\;c\;} \boldsymbol{L}' \xrightarrow{\;c\;} \boldsymbol{L}'' \xleftarrow{\;c\;} \boldsymbol{L}''' \xrightarrow{\;c\;} \boldsymbol{M}|_O \xrightarrow{\iota_{\boldsymbol{M};O}} \boldsymbol{M} \quad , \tag{4.6}$$

where $\iota_{\boldsymbol{M};O}$ denotes the canonical inclusion and by 'c' we indicate Cauchy morphisms. If we could construct from this chain a chain of morphisms in LocSrc_p of the form

$$(\boldsymbol{L},\boldsymbol{J_L}) \xleftarrow{\;c\;} (\boldsymbol{L}',\boldsymbol{J_{L'}}) \xrightarrow{\;c\;} (\boldsymbol{L}'',\boldsymbol{J_{L''}}) \xleftarrow{\;c\;} (\boldsymbol{L}''',\boldsymbol{J_{L'''}}) \xrightarrow{\;c\;} (\boldsymbol{M}|_O,\boldsymbol{J}|_O) \xrightarrow{\iota_{\boldsymbol{M};O}} (\boldsymbol{M},\boldsymbol{J}) \quad , \tag{4.7}$$

then the same argument as in [Few13, Proof of Lemma 2.2.] (combining results (i) and (ii)) would prove (iii). Since the morphisms in (4.6) uniquely fix $\boldsymbol{J_{L'}}$ and $\boldsymbol{J_{L'''}}$ via pulling back, respectively, $\boldsymbol{J_L}$ and $\boldsymbol{J}|_O$, the remaining step is to show that we can construct a $\boldsymbol{J_{L''}}$ completing the chain (4.7). This is indeed possible if we equip the spacetime $\boldsymbol{L}''$ constructed in [FV12a, Proof of Proposition 2.4.] with a $\boldsymbol{J_{L''}}$ that is obtained by gluing $\boldsymbol{J_{L'}}$ and $\boldsymbol{J_{L'''}}$ via a suitable cutoff function (e.g. the one used to construct the metric in $\boldsymbol{L}''$). □

We are now ready to prove the analog of [Few13, Theorem 2.6.] for our specific functor $\mathfrak{PS}_p : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$. Since we are working with a specific model we can give a direct proof and we do not have to dwell on the technicalities concerning abstract categorical unions and equalizers appearing in [Few13].

Theorem 4.5. *Every* $\eta \in \mathrm{End}(\mathfrak{PS}_p)$ *is uniquely determined by its component* $\eta_{(\boldsymbol{M},\boldsymbol{J})}$ *on any object* $(\boldsymbol{M},\boldsymbol{J})$ *in* LocSrc_p*.*

Proof. Suppose that $\eta' \in \mathrm{End}(\mathfrak{PS}_p)$ agrees with η on the object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p, i.e. $\eta_{(\boldsymbol{M},\boldsymbol{J})} = \eta'_{(\boldsymbol{M},\boldsymbol{J})}$. Let $(\boldsymbol{N},\boldsymbol{J_N})$ be any object in LocSrc_p and let D be any diamond in $\boldsymbol{N}$. Then $\boldsymbol{N}|_D$ has Cauchy surfaces which are oriented diffeomorphic to any diamond in $\boldsymbol{M}$. Then $\eta_{(\boldsymbol{N}|_D,\boldsymbol{J_N}|_D)} = \eta'_{(\boldsymbol{N}|_D,\boldsymbol{J_N}|_D)}$ by Lemma 4.4 (iii). Using the canonical inclusion $\iota_{\boldsymbol{N};D} : (\boldsymbol{N}|_D,\boldsymbol{J_N}|_D) \to (\boldsymbol{N},\boldsymbol{J_N})$ we obtain by naturality

$$\begin{aligned}\eta_{(\boldsymbol{N},\boldsymbol{J_N})} \circ \mathfrak{PS}_p(\iota_{\boldsymbol{N};D}) = \mathfrak{PS}_p(\iota_{\boldsymbol{N};D}) \circ \eta_{(\boldsymbol{N}|_D,\boldsymbol{J_N}|_D)} &= \mathfrak{PS}_p(\iota_{\boldsymbol{N};D}) \circ \eta'_{(\boldsymbol{N}|_D,\boldsymbol{J_N}|_D)} \\ &= \eta'_{(\boldsymbol{N},\boldsymbol{J_N})} \circ \mathfrak{PS}_p(\iota_{\boldsymbol{N};D}) \ .\end{aligned} \tag{4.8}$$

This equation implies that $\eta_{(\boldsymbol{N},\boldsymbol{J_N})}\big([(\varphi,\alpha)]\big) = \eta'_{(\boldsymbol{N},\boldsymbol{J_N})}\big([(\varphi,\alpha)]\big)$, for all $[(\varphi,\alpha)] \in \mathfrak{PS}_p(\boldsymbol{N},\boldsymbol{J_N})$ for which there exists a representative (φ,α) such that φ has compact support in D. We shall now prove that $\eta_{(\boldsymbol{N},\boldsymbol{J_N})}\big([(\varphi,\alpha)]\big) = \eta'_{(\boldsymbol{N},\boldsymbol{J_N})}\big([(\varphi,\alpha)]\big)$, for all $[(\varphi,\alpha)] \in \mathfrak{PS}_p(\boldsymbol{N},\boldsymbol{J_N})$, and hence that $\eta' = \eta$ since $(\boldsymbol{N},\boldsymbol{J_N})$ was arbitrary. This proof follows from a partition of unity argument: Given any $[(\varphi,\alpha)] \in \mathfrak{PS}_p(\boldsymbol{N},\boldsymbol{J_N})$ we take some representative $(\varphi,\alpha) \in C^\infty_0(N,\mathbb{R}^p) \oplus \mathbb{R}$. Since $\mathrm{supp}(\varphi)$ is compact, the open cover of N given by the set of all diamonds in N has a finite subcover of $\mathrm{supp}(\varphi)$, which we denote by $\{D_i\}_{i=1,\dots,n}$. Using a partition of unity subordinated to $\{D_i\}_{i=1,\dots,n}$ we can write $(\varphi,\alpha) = \sum_{i=1}^n (\varphi_i, \alpha/n)$, where φ_i has compact support in D_i. Hence,

$$\begin{aligned}\eta_{(\boldsymbol{N},\boldsymbol{J_N})}\big([(\varphi,\alpha)]\big) &= \sum_{i=1}^n \eta_{(\boldsymbol{N},\boldsymbol{J_N})}\big([(\varphi_i,\alpha/n)]\big) = \sum_{i=1}^n \eta'_{(\boldsymbol{N},\boldsymbol{J_N})}\big([(\varphi_i,\alpha/n)]\big) \\ &= \eta'_{(\boldsymbol{N},\boldsymbol{J_N})}\big([(\varphi,\alpha)]\big) \ .\end{aligned} \tag{4.9}$$

□

We now come to the main statement of this section.

Theorem 4.6. *Every endomorphism of the functor $\mathfrak{PS}_p$ is an automorphism and*

$$\mathrm{End}(\mathfrak{PS}_p) = \mathrm{Aut}(\mathfrak{PS}_p) \simeq \begin{cases} \mathbb{Z}_2 & , \ \textit{for } m \neq 0 \,, \\ \mathbb{Z}_2 \times \mathbb{R}^p & , \ \textit{for } m = 0 \,, \end{cases} \tag{4.10}$$

where the action is given for $m \neq 0$ by Proposition 4.1, and for $m = 0$ by Proposition 4.2.

Proof. Due to Theorem 4.5, any $\eta \in \mathrm{End}(\mathfrak{PS}_p)$ is uniquely determined by its component $\eta_{(\boldsymbol{M}_0,0)}$, where $\boldsymbol{M}_0$ is Minkowski spacetime and we have chosen $\boldsymbol{J}_0 = 0$. The presymplectic vector space $\mathfrak{PS}_p(\boldsymbol{M}_0, 0)$ can be expressed more conveniently using the following linear isomorphism for the underlying vector space

$$\begin{aligned} \big(C^\infty_0(M_0,\mathbb{R}^p)\oplus\mathbb{R}\big)/\mathrm{P}^\ast_{(\boldsymbol{M}_0,0)}\big[C^\infty_0(M_0,\mathbb{R}^p)\big] &= \big(C^\infty_0(M_0,\mathbb{R}^p)\oplus\mathbb{R}\big)/\big(\mathrm{KG}_{\boldsymbol{M}_0}\big[C^\infty_0(M_0,\mathbb{R}^p)\big]\oplus\{0\}\big) \\ &= \big(C^\infty_0(M_0,\mathbb{R}^p)/\mathrm{KG}_{\boldsymbol{M}_0}\big[C^\infty_0(M_0,\mathbb{R}^p)\big]\big)\oplus\mathbb{R} \\ &\simeq \mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)\oplus\mathbb{R}\,, \end{aligned} \tag{4.11}$$

where $\mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0) := \{\phi \in C^\infty_{\mathrm{sc}}(M_0,\mathbb{R}^p) : \mathrm{KG}_{\boldsymbol{M}_0}(\phi) = 0\}$ is the space of solutions of the Klein–Gordon equation with spacelike compact support. The isomorphism in the last line of (4.11) is the usual one given by the advanced-minus-retarded Green's operator $\mathrm{E}_{\boldsymbol{M}_0}$. The induced presymplectic structure on $\mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)\oplus\mathbb{R}$ is given by, for all $(\phi,\alpha),(\psi,\beta)\in\mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)\oplus\mathbb{R}$,

$$\sigma_{(\boldsymbol{M}_0,0)}\big((\phi,\alpha),(\psi,\beta)\big) = \widetilde{\sigma}_{\boldsymbol{M}_0}(\phi,\psi)\,, \tag{4.12}$$

where $\widetilde{\sigma}_{\boldsymbol{M}_0}$ is the usual symplectic structure on $\mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)$.

Via the isomorphism (4.11), $\eta_{(\boldsymbol{M}_0,0)}$ induces an endomorphism $\widetilde{\eta}$ of $\mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)\oplus\mathbb{R}$ which, by naturality, commutes with the action of all Poincaré transformations. By Lemma 4.3, $\widetilde{\eta}$ commutes with the relative Cauchy evolution, and therefore with its derivatives given in (3.11a) and (3.11b). Taking into account the isomorphism (4.11) and our specific choice of object $(\boldsymbol{M}_0,0)$ they read, for all $(\phi,\alpha)\in\mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)\oplus\mathbb{R}$,

$$\mathcal{T}_{(\boldsymbol{M}_0,0)}[\boldsymbol{h}](\phi,\alpha) = \Big(\mathrm{E}_{\boldsymbol{M}_0}\big(\mathrm{KG}'_{\boldsymbol{M}_0[\boldsymbol{h}]}(\phi)\big),0\Big)\,, \tag{4.13a}$$

$$\mathcal{J}_{(\boldsymbol{M}_0,0)}[\boldsymbol{j}](\phi,\alpha) = \Big(0,\int_{M_0}\langle\boldsymbol{j},\phi\rangle\ \mathrm{vol}_{\boldsymbol{M}_0}\Big)\,. \tag{4.13b}$$

Since $\widetilde{\eta} : \mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)\oplus\mathbb{R}\to\mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)\oplus\mathbb{R}$ is a linear map, it decomposes into linear maps $L_{11} : \mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)\to\mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)$, $L_{12} : \mathbb{R}\to\mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)$, $L_{21} : \mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)\to\mathbb{R}$ and $L_{22} : \mathbb{R}\to\mathbb{R}$. As $\widetilde{\eta}$ commutes with the maps in (4.13), we obtain the following conditions on the L_{ij}, for all $(\phi,\alpha)\in\mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)\oplus\mathbb{R}$ and $(\boldsymbol{h},\boldsymbol{j})\in H(\boldsymbol{M}_0,0)$,

$$\Big(\mathrm{E}_{\boldsymbol{M}_0}\big(\mathrm{KG}'_{\boldsymbol{M}_0[\boldsymbol{h}]}(L_{11}(\phi)+L_{12}(\alpha))\big),0\Big) = \Big(L_{11}\big(\mathrm{E}_{\boldsymbol{M}_0}\big(\mathrm{KG}'_{\boldsymbol{M}[\boldsymbol{h}]}(\phi)\big)\big),L_{21}\big(\mathrm{E}_{\boldsymbol{M}_0}\big(\mathrm{KG}'_{\boldsymbol{M}[\boldsymbol{h}]}(\phi)\big)\big)\Big) \tag{4.14a}$$

and

$$\Big(0,\int_{M_0}\langle\boldsymbol{j},L_{11}(\phi)+L_{12}(\alpha)\rangle\ \mathrm{vol}_{\boldsymbol{M}_0}\Big) = \Big(L_{12}\Big(\int_{M_0}\langle\boldsymbol{j},\phi\rangle\ \mathrm{vol}_{\boldsymbol{M}_0}\Big),L_{22}\Big(\int_{M_0}\langle\boldsymbol{j},\phi\rangle\ \mathrm{vol}_{\boldsymbol{M}_0}\Big)\Big)\,. \tag{4.14b}$$

From the first component of (4.14b) we obtain that $L_{12} = 0$. Substituting into (4.14a) we obtain from the first component the same condition that is present for multiplets of homogeneous Klein–Gordon fields. The only solution of this condition (supplemented by additional conditions stemming from the Poincaré invariance of Minkowski spacetime) is that L_{11} is an $O(p)$ transformation acting in the obvious way on the components of ϕ, cf. [Few13, Theorem 5.2.]. Using this information the second component of (4.14b) implies that $L_{11} = \sigma\,\mathrm{id}_{\mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)}$ and $L_{22} = \sigma\,\mathrm{id}_{\mathbb{R}}$, with $\sigma\in\mathbb{Z}_2 = \{-1,+1\}$. Finally, the fact that endomorphisms commute with the Poincaré transformations entails that $L_{21} : \mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)\to\mathbb{R}$ is Poincaré invariant. By

Lemma 4.7 below, we deduce that $L_{21} = 0$ for $m \neq 0$, or that $L_{21}(\,\cdot\,) = \widetilde{\sigma}_{\boldsymbol{M}_0}(\mu, \cdot\,)$ for some $\mu \in \mathbb{R}^p$ if $m = 0$ (here μ denotes a constant solution). Combining these facts, we have

$$\widetilde{\eta}(\phi, \alpha) = \begin{cases} (\sigma\,\phi, \sigma\,\alpha) & , \ \text{for } m \neq 0\,, \\ (\sigma\,\phi, \sigma\,(\alpha + \widetilde{\sigma}_{\boldsymbol{M}_0}(\mu, \phi))) & , \ \text{for } m = 0\,, \end{cases} \tag{4.15}$$

for some $\sigma \in \mathbb{Z}_2$ (and $\mu \in \mathbb{R}^p$ if $m = 0$). Undoing the isomorphism (4.11), this means that for $m = 0$ we have $\eta_{(\boldsymbol{M}_0,0)} = \eta(\sigma, \mu)_{(\boldsymbol{M}_0,0)}$ and hence $\eta = \eta(\sigma, \mu)$ by Theorem 4.5. Similarly, $\eta = \eta(\sigma) = \{\sigma\, \mathrm{id}_{\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})}\}$ if $m \neq 0$. This proves the result. □

It remains to prove the following

Lemma 4.7. *Suppose $L : \mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0) \to \mathbb{R}$ is linear and translationally invariant. If $m \neq 0$ then $L = 0$. If $m = 0$ then there exists $\mu \in \mathbb{R}^p$ such that $L(\phi) = \widetilde{\sigma}_{\boldsymbol{M}_0}(\mu, \phi)$.*

Proof. We use the automatic continuity result of Meisters [Mei71] that the translationally invariant linear functionals on $C_0^\infty(\mathbb{R}^k, \mathbb{R}^p)$ are precisely the scalar multiples of the integral. Passing to Cauchy data on any surface $t = \text{const.}$, L decomposes into two linear functionals on $C_0^\infty(\mathbb{R}^k, \mathbb{R}^p)$ that are each translationally invariant, owing to spatial translational invariance of L. Hence, for each t there are $\alpha_t, \beta_t \in \mathbb{R}^p$ such that, for any $\phi \in \mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)$,

$$L(\phi) = \int_{\mathbb{R}^k} d^k\boldsymbol{x} \left(\langle \alpha_t, \phi(t, \boldsymbol{x})\rangle + \left\langle \beta_t, \frac{\partial \phi}{\partial t}(t, \boldsymbol{x}) \right\rangle \right) . \tag{4.16}$$

By time-translational invariance of L, $\alpha_t \equiv \alpha$ and $\beta_t \equiv \beta$ are independent of t. Further, differentiating with respect to t, using the Klein–Gordon equation and Gauss' theorem, we obtain, for all $\phi \in \mathfrak{Sol}_{\mathrm{sc}}(\boldsymbol{M}_0)$,

$$0 = \int_{\mathbb{R}^k} d^k\boldsymbol{x} \left(\left\langle \alpha, \frac{\partial \phi}{\partial t}(t, \boldsymbol{x}) \right\rangle - \langle \beta m^2, \phi(t, \boldsymbol{x})\rangle \right) \tag{4.17}$$

and hence that $\alpha = 0$, $\beta m^2 = 0$. The result follows (with $\mu = \beta$). □

The results of this section reveal that the functor $\mathfrak{PS}_p$ has a larger automorphism group than one would expect for the global gauge group of the inhomogeneous theory. To close the section, we mention that the $\mathbb{Z}_2$ factor of $\mathrm{Aut}(\mathfrak{PS}_p)$ can be removed if we change the category in which $\mathfrak{PS}_p$ takes values to reflect the fact that the underlying vector space of $\mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J})$ is a linear subspace of the affine dual of the space of solutions $\mathfrak{Sol}_p(\boldsymbol{M}, \boldsymbol{J})$. Were we to restrict to morphisms arising as restrictions of duals to affine maps, we would be left with a trivial automorphism group for $m \neq 0$ and $\mathbb{R}^p$ for $m = 0$. We do not develop this line of thought in detail, because the next section shows that $\mathfrak{PS}_p$ has further pathologies, which would not be eliminated by this device. Nevertheless, it is worth noting that the unexpected symmetries appear because we have discarded information about the action of the observables in $\mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J})$ on the solution space.

5 Violation of the composition property of the functor $\mathfrak{PS}_p$

The theory we aim to construct consists of p inhomogeneous Klein–Gordon fields without mutual interactions. One would expect that an equivalent formulation would be produced if the multiplet were decomposed into independent submultiplets of $0 < q < p$ and $p - q$ fields, which are treated separately according to the general prescription and then recombined. In this section, we describe how the splitting and recombination may be formalized and then show that the functor $\mathfrak{PS}_p$ fails to respect this composition property.

Let $p \geq 2$ and let $\Pi_q : \mathbb{R}^p \to \mathbb{R}^p\,,\ (a_1, \dots, a_p) \mapsto (a_1, \dots, a_q, 0, \dots, 0)$ be the projection onto the first q dimensions, where $0 < q < p$. Given any object $(\boldsymbol{M}, \boldsymbol{J})$ of LocSrc_p, we can split $\boldsymbol{J} = \Pi_q(\boldsymbol{J}) + (\mathrm{id}_{\mathbb{R}^p} - \Pi_q)(\boldsymbol{J}) =: \boldsymbol{J}^q + \boldsymbol{J}^{p-q}$ and identify $\boldsymbol{J}^q$ as an element in $C^\infty(M, \mathbb{R}^q)$ and $\boldsymbol{J}^{p-q}$ as an element in $C^\infty(M, \mathbb{R}^{p-q})$. Hence, we can associate to $(\boldsymbol{M}, \boldsymbol{J})$ the object $\mathfrak{Split}_{p,q}(\boldsymbol{M}, \boldsymbol{J}) := \big((\boldsymbol{M}, \boldsymbol{J}^q), (\boldsymbol{M}, \boldsymbol{J}^{p-q})\big)$ in the product category $\mathsf{LocSrc}_q \times \mathsf{LocSrc}_{p-q}$. Moreover, given any morphism $f : (\boldsymbol{M}_1, \boldsymbol{J}_1) \to (\boldsymbol{M}_2, \boldsymbol{J}_2)$ in LocSrc_p we associate a morphism in $\mathsf{LocSrc}_q \times \mathsf{LocSrc}_{p-q}$ via $\mathfrak{Split}_{p,q}(f) := (f, f) : \mathfrak{Split}_{p,q}(\boldsymbol{M}_1, \boldsymbol{J}_1) \to$

$\mathfrak{Split}_{p,q}(\boldsymbol{M}_2, \boldsymbol{J}_2)$, where with a slight abuse of notation we have denoted the smooth map underlying the morphism by the same symbol $f : M_1 \to M_2$. In this way we obtain a covariant functor $\mathfrak{Split}_{p,q} : \mathsf{LocSrc}_p \to \mathsf{LocSrc}_q \times \mathsf{LocSrc}_{p-q}$ representing the decomposition into submultiplets.

Treating each submultiplet according to the prescription of Section 2.2, we compose $\mathfrak{Split}_{p,q}$ with the covariant functor $\mathfrak{PS}_q \times \mathfrak{PS}_{p-q}$ to obtain $\big(\mathfrak{PS}_q \times \mathfrak{PS}_{p-q}\big) \circ \mathfrak{Split}_{p,q} : \mathsf{LocSrc}_p \to \mathsf{PreSymp} \times \mathsf{PreSymp}$. Finally, we recombine the resulting theories by composing with the covariant functor $\oplus : \mathsf{PreSymp} \times \mathsf{PreSymp} \to \mathsf{PreSymp}$ that forms the direct sum of two presymplectic vector spaces – the monoidal structure in this category. Explicitly, for any object $((V, \sigma_V), (W, \sigma_W))$ in $\mathsf{PreSymp} \times \mathsf{PreSymp}$ we define $\oplus\big((V, \sigma_V), (W, \sigma_W)\big) := (V \oplus W, \sigma_{V\oplus W})$, where $V \oplus W$ is the direct sum of vector spaces and, for all $(v, w), (v', w') \in V \oplus W$,

$$\sigma_{V\oplus W}\big((v, w), (v', w')\big) := \sigma_V(v, v') + \sigma_W(w, w') \; . \tag{5.1}$$

On morphisms, $\oplus(L, K) := L \oplus K$. The resulting covariant functor is

$$\mathfrak{PS}_{p,q} := \oplus \circ \big(\mathfrak{PS}_q \times \mathfrak{PS}_{p-q}\big) \circ \mathfrak{Split}_{p,q} : \mathsf{LocSrc}_p \to \mathsf{PreSymp} \; . \tag{5.2}$$

Since the covariant functor $\mathfrak{PS}_p : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ satisfies the causality property and the time-slice axiom for all $p \in \mathbb{N}$ (cf. Proposition 2.3), it is not hard to see that the same holds true for the covariant functor $\mathfrak{PS}_{p,q} : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$.

Proposition 5.1. *The covariant functor $\mathfrak{PS}_{p,q} : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ satisfies the causality property and the time-slice axiom for all $2 \leq p \in \mathbb{N}$ and $0 < q < p$.*

Proof. Causality holds owing to causality of $\mathfrak{PS}_{p-q}$ and $\mathfrak{PS}_q$ and the following property of the direct sum: if $\mathsf{PreSymp}$-morphisms $L_i : (V_i, \sigma_{V_i}) \to (V, \sigma_V)$, $i = 1, 2$, have symplectically orthogonal images, and so do $K_i : (W_i, \sigma_{W_i}) \to (W, \sigma_W)$, $i = 1, 2$, then $L_1 \oplus K_1$ and $L_2 \oplus K_2$ have symplectically orthogonal images in $(V \oplus W, \sigma_{V\oplus W})$ because $\sigma_{V\oplus W} = \sigma_V \oplus \sigma_W$ by (5.1). The time-slice axiom holds simply because the direct sum of two presymplectic isomorphisms is itself a presymplectic isomorphism. □

We shall now prove that the theories $\mathfrak{PS}_{p,q}$ and $\mathfrak{PS}_p$ are inequivalent for $0 < q < p$; that is, the functors are not naturally isomorphic. This will be a consequence of the following simple

Lemma 5.2. *Let $L : (V, \sigma_V) \to (W, \sigma_W)$ be an isomorphism in $\mathsf{PreSymp}$. Then L induces a linear isomorphism between the null spaces $N(V, \sigma_V)$ and $N(W, \sigma_W)$.*

Proof. The $\mathsf{PreSymp}$-isomorphism L induces an injective linear map $L : N(V, \sigma_V) \to W$. The image $L[N(V, \sigma_V)]$ is contained in $N(W, \sigma_W)$, since for all $v \in N(V, \sigma_V)$ and $w \in W$,

$$\sigma_W\big(L(v), w\big) = \sigma_W\big(L(v), L(L^{-1}(w))\big) = \sigma_V\big(v, L^{-1}(w)\big) = 0 \; . \tag{5.3}$$

Hence, $L : N(V, \sigma_V) \to N(W, \sigma_W)$ is an injective linear map which is invertible via $L^{-1} : N(W, \sigma_W) \to N(V, \sigma_V)$. □

Proposition 5.3. *For any $2 \leq p \in \mathbb{N}$ and $0 < q < p$, the covariant functors $\mathfrak{PS}_{p,q} : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ and $\mathfrak{PS}_p : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ are not naturally isomorphic. Indeed, there is no object $(\boldsymbol{M}, \boldsymbol{J})$ in LocSrc_p for which the presymplectic vector spaces $\mathfrak{PS}_{p,q}(\boldsymbol{M}, \boldsymbol{J})$ and $\mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J})$ are isomorphic.*

Proof. We argue by contradiction. Suppose there were a $\mathsf{PreSymp}$-isomorphism $L : \mathfrak{PS}_{p,q}(\boldsymbol{M}, \boldsymbol{J}) \to \mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J})$ for some object $(\boldsymbol{M}, \boldsymbol{J})$ in LocSrc_p. Then L induces a linear isomorphism between the null spaces of $\mathfrak{PS}_{p,q}(\boldsymbol{M}, \boldsymbol{J})$ and $\mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J})$ by Lemma 5.2. However, the latter is isomorphic to $\mathbb{R}$ (cf. Proposition 2.3), while the former is easily seen to be isomorphic to $\mathbb{R}^2$. Hence, no such isomorphism exists and consequently the functors $\mathfrak{PS}_{p,q}$ and $\mathfrak{PS}_p$ are not naturally isomorphic. □

6 The Poisson algebra functor

In order to resolve the pathological composition property of the functor $\mathfrak{PS}_p$ obtained in Section 5, as well as the mysterious automorphism group of Theorem 4.6 (cf. also the text below Lemma 4.7), we introduce further structures. Naively speaking, we aim to make the theory given by $\mathfrak{PS}_p$ remember that it describes the affine functionals on the affine space of solutions of the inhomogeneous Klein–Gordon equation. To realize this idea, we first construct from $\mathfrak{PS}_p$ a functor describing canonical Poisson algebras of observables, which are then represented *non-faithfully* on the solution space. The kernel of this representation is then identified and it is shown that the quotients of the Poisson algebras by these kernels are described by a covariant functor which has the desired automorphism group and the composition property. In Appendix B we present an alternative strategy for improving the classical theory of the inhomogeneous multiplet of Klein–Gordon fields by using pointed presymplectic spaces.

6.1 Canonical Poisson algebras

Let $\mathsf{PoisAlg}$ denote the category of unital Poisson algebras over $\mathbb{R}$, with injective unit-preserving Poisson algebra homomorphisms as morphisms. We first construct a covariant functor $\mathfrak{CanPois} : \mathsf{PreSymp} \to \mathsf{PoisAlg}$ that associates canonical Poisson algebras to presymplectic vector spaces: Given any object (V, σ_V) in $\mathsf{PreSymp}$, let us consider the symmetric tensor algebra $S(V) := \bigoplus_{k=0}^{\infty} S^k(V)$, where $S^0(V) = \mathbb{R}$, $S^1(V) = V$ and $S^k(V) := \bigvee^k V$, for $k \geq 2$, is the k-th symmetric power of V. The product in $S(V)$ is denoted by juxtaposition and turns $S(V)$ into an associative and commutative algebra over $\mathbb{R}$ with unit $1 \in S^0(V) \subset S(V)$. We define a Poisson bracket $\{\cdot,\cdot\}_{\sigma_V} : S(V) \times S(V) \to S(V)$ by, for all $\alpha \in S^0(V)$ and $v_1, \dots, v_k, v'_1, \dots, v'_l \in V$,

$$\{\alpha, v_1 \cdots v_k\}_{\sigma_V} = \{v_1 \cdots v_k, \alpha\}_{\sigma_V} = 0 \,, \tag{6.1a}$$

$$\{v_1 \cdots v_k, v'_1 \cdots v'_l\}_{\sigma_V} = \sum_{i=1}^{k} \sum_{j=1}^{l} v_1 \cdots \overset{i}{\vee} \cdots v_k \, v'_1 \cdots \overset{j}{\vee} \cdots v'_l \; \sigma_V(v_i, v'_j) \,. \tag{6.1b}$$

The symbols $\overset{i}{\vee}$ mean the omission of the i-th element. We denote the resulting Poisson algebra by $\mathfrak{CanPois}(V, \sigma_V) := \big(S(V), \{\cdot,\cdot\}_{\sigma_V}\big)$. Given any morphism $L : (V, \sigma_V) \to (W, \sigma_W)$ in $\mathsf{PreSymp}$ we associate a map $\mathfrak{CanPois}(L) : \mathfrak{CanPois}(V, \sigma_V) \to \mathfrak{CanPois}(W, \sigma_W)$ via $\mathfrak{CanPois}(L)(\alpha) = \alpha$, for all $\alpha \in S^0(V)$, and $\mathfrak{CanPois}(L)(v_1 \cdots v_k) = L(v_1) \cdots L(v_k)$, for all $v_1, \dots, v_k \in V$. It is easy to see that $\mathfrak{CanPois}(L)$ is an injective Poisson algebra homomorphism. Thus, we have shown the following

Proposition 6.1. *The association* $\mathfrak{CanPois} : \mathsf{PreSymp} \to \mathsf{PoisAlg}$ *constructed above is a covariant functor.*

Remark 6.2. Notice that for any object (V, σ_V) in $\mathsf{PreSymp}$ the Poisson algebra $\mathfrak{CanPois}(V, \sigma_V)$ is $\mathbb{N}^0$-graded. Furthermore, for any morphism $L : (V, \sigma_V) \to (W, \sigma_W)$ in $\mathsf{PreSymp}$ the morphism $\mathfrak{CanPois}(L) : \mathfrak{CanPois}(V, \sigma_V) \to \mathfrak{CanPois}(W, \sigma_W)$ is a graded Poisson algebra morphism. Hence, $\mathfrak{CanPois}$ is also a covariant functor to the category of $\mathbb{N}^0$-graded Poisson algebras. As will become clear in the next subsection, the latter category is too restrictive for our purposes, hence we shall usually disregard this natural grading.

We can compose our functor $\mathfrak{PS}_p : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ with $\mathfrak{CanPois} : \mathsf{PreSymp} \to \mathsf{PoisAlg}$ to obtain the covariant functor $\mathfrak{CPA}_p := \mathfrak{CanPois} \circ \mathfrak{PS}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$. The functor $\mathfrak{CPA}_p$ describes the association of *canonical* Poisson algebras of observables for a multiplet of $p \in \mathbb{N}$ inhomogeneous Klein–Gordon fields. We immediately observe the following

Proposition 6.3. *The covariant functor* $\mathfrak{CPA}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ *satisfies the causality property and the time-slice axiom. Moreover,* $\mathrm{Aut}(\mathfrak{CPA}_p)$ *contains a* $\mathbb{Z}_2$ *subgroup for* $m \neq 0$ *and a* $\mathbb{Z}_2 \times \mathbb{R}^p$ *subgroup for* $m = 0$.

Proof. By Proposition 2.3 c) the functor $\mathfrak{PS}_p$ satisfies these properties. Thus $\mathfrak{CanPois}$ automatically obeys the time-slice property because functors preserve isomorphisms. The causality property is seen as follows: Given any two $\mathsf{PreSymp}$-morphisms $L_i : (V_i, \sigma_{V_i}) \to (V, \sigma_V)$, $i = 1, 2$, such that $\sigma_V\big(L_1[V_1], L_2[V_2]\big) = \{0\}$

in (V,σ_V), then the explicit expression for the Poisson bracket (6.1) implies that $\{\cdot,\cdot\}_{\sigma_V}$ acts trivially between $\mathfrak{CanPois}(L_1)\big[\mathfrak{CanPois}(V_1,\sigma_1)\big]$ and $\mathfrak{CanPois}(L_2)\big[\mathfrak{CanPois}(V_2,\sigma_2)\big]$. The statement on the automorphism group follows from Theorem 4.6 and the fact that every automorphism $\eta = \{\eta_{(\boldsymbol{M},\boldsymbol{J})}\}$ of $\mathfrak{PS}_p$ lifts to an automorphism $\{\mathfrak{CanPois}(\eta_{(\boldsymbol{M},\boldsymbol{J})})\}$ of $\mathfrak{CPA}_p$. □

Next, we shall show that the functor $\mathfrak{CPA}_p$ violates the analog of the composition property for $\mathfrak{PS}_p$ discussed in Section 5, cf. Proposition 5.3. For $2\leq p\in\mathbb{N}$ and $0<q<p$, we define the covariant functor

$$\mathfrak{CPA}_{p,q} := \otimes\circ\big(\mathfrak{CPA}_q\times\mathfrak{CPA}_{p-q}\big)\circ\mathfrak{Split}_{p,q} : \mathsf{LocSrc}_p\to\mathsf{PoisAlg}\ , \tag{6.2}$$

where $\otimes : \mathsf{PoisAlg}\times\mathsf{PoisAlg}\to\mathsf{PoisAlg}$ is the covariant functor that takes (algebraic) tensor products of Poisson algebras. Explicitly, to any object $(\mathcal{A},\mathcal{B})$ in $\mathsf{PoisAlg}\times\mathsf{PoisAlg}$ we associate the object $\otimes(\mathcal{A},\mathcal{B}) := \mathcal{A}\otimes\mathcal{B}$ in $\mathsf{PoisAlg}$, which is the tensor product of the underlying commutative and associative unital algebras, equipped with the Poisson bracket specified by linearity and, for all $a,a'\in\mathcal{A}$ and $b,b'\in\mathcal{B}$,

$$\{a\otimes b,a'\otimes b'\}_{\mathcal{A}\otimes\mathcal{B}} := \{a,a'\}_{\mathcal{A}}\otimes(b\,b') + (a\,a')\otimes\{b,b'\}_{\mathcal{B}}\ . \tag{6.3}$$

To any morphism $(\kappa,\lambda) : (\mathcal{A}_1,\mathcal{B}_1)\to(\mathcal{A}_2,\mathcal{B}_2)$ in $\mathsf{PoisAlg}\times\mathsf{PoisAlg}$ we associate the morphism $\otimes(\kappa,\lambda) := \kappa\otimes\lambda : \mathcal{A}_1\otimes\mathcal{B}_1\to\mathcal{A}_2\otimes\mathcal{B}_2$ in $\mathsf{PoisAlg}$ specified by linearity and, for all $a\in\mathcal{A}_1$ and $b\in\mathcal{B}_1$, $(\kappa\otimes\lambda)\big(a\otimes b\big) = \kappa(a)\otimes\lambda(b)$. To show that the covariant functors $\mathfrak{CPA}_{p,q}$ and $\mathfrak{CPA}_p$ are inequivalent, we require two lemmas.

Lemma 6.4. *Let (V,σ_V) and (W,σ_W) be two objects in* $\mathsf{PreSymp}$. *Then there exists an isomorphism $L : (V,\sigma_V)\to(W,\sigma_W)$ in* $\mathsf{PreSymp}$ *if and only if there exists an isomorphism $\kappa : \mathfrak{CanPois}(V,\sigma_V)\to\mathfrak{CanPois}(W,\sigma_W)$ in* $\mathsf{PoisAlg}$. *Moreover, the isomorphisms κ and L are related by $L = \pi_{S^1(W)}\circ\kappa|_{S^1(V)}$, where $\kappa|_{S^1(V)}$ is the restriction of κ to the vector subspace $S^1(V)\subseteq\mathfrak{CanPois}(V,\sigma_V)$ and $\pi_{S^1(W)} : \mathfrak{CanPois}(W,\sigma_W)\to S^1(W)$ is the projection to the degree one vector subspace $S^1(W)$. L is uniquely determined by κ, but L does not determine κ.*

Proof. The direction "⇒" is a consequence of functoriality, because $\mathfrak{CanPois}$ preserves isomorphisms. To show the reverse direction, suppose that $\kappa : \mathfrak{CanPois}(V,\sigma_V)\to\mathfrak{CanPois}(W,\sigma_W)$ is a $\mathsf{PoisAlg}$-isomorphism. In particular, κ is a unital algebra isomorphism $\kappa : S(V)\to S(W)$ between the symmetric tensor algebras of V and W. This algebra isomorphism is uniquely specified by its action on arbitrary $v\in S^1(V)=V$, so let us write $\kappa(v)=\kappa_0(v)+\kappa_1(v)+\kappa_{\geq 2}(v)$, where $\kappa_0 : V\to\mathbb{R}$, $\kappa_1 : V\to W$ and $\kappa_{\geq 2} : V\to S^{\geq 2}(W)$ are the projections of $\kappa|_{S^1(V)}$ to the subspaces $S^0(W)$, $S^1(W)$ and $S^{\geq 2}(W) := \bigoplus_{k=2}^{\infty}S^k(W)$.

We now will show that, given any $\mathsf{PoisAlg}$-isomorphism κ, there exists a $\mathsf{PoisAlg}$-isomorphism $\widetilde{\kappa} : \mathfrak{CanPois}(V,\sigma_V)\to\mathfrak{CanPois}(W,\sigma_W)$, with $\widetilde{\kappa}_0 = 0$ and $\widetilde{\kappa}_1=\kappa_1$. Consider the $\mathsf{PoisAlg}$-automorphism $\chi : \mathfrak{CanPois}(V,\sigma_V)\to\mathfrak{CanPois}(V,\sigma_V)$ defined by, for all $v\in V$, $\chi(v) = v-\kappa_0(v)$. Define $\widetilde{\kappa} := \kappa\circ\chi : \mathfrak{CanPois}(V,\sigma_V)\to\mathfrak{CanPois}(W,\sigma_W)$, which is as a composition of $\mathsf{PoisAlg}$-isomorphisms again a $\mathsf{PoisAlg}$-isomorphism and notice that $\widetilde{\kappa}(v) = \kappa_1(v)+\kappa_{\geq 2}(v)$, for any $v\in V$. As $\widetilde{\kappa}$ is an algebra homomorphism, it is therefore lower-triangular with respect to the gradings of $\mathfrak{CanPois}(V,\sigma_V)$ and $\mathfrak{CanPois}(W,\sigma_W)$: the degree-$k$ component of any $\widetilde{\kappa}(a)$ depends only on the components of a with degree k or less. Accordingly, $\widetilde{\kappa}^{-1}$ is also lower-triangular, and all diagonal entries $\pi_{S^k(W)}\circ\widetilde{\kappa}|_{S^k(V)}$ $(k\in\mathbb{N}^0)$ are vector space isomorphisms. In particular, $\kappa_1 : V\to W$ is a vector space isomorphism. The claim that $\kappa_1 : (V,\sigma_V)\to(W,\sigma_W)$ is a $\mathsf{PreSymp}$-isomorphism follows by evaluating both sides of the condition, for all $v,v'\in V$, $\widetilde{\kappa}\big(\{v,v'\}_{\sigma_V}\big) = \{\widetilde{\kappa}(v),\widetilde{\kappa}(v')\}_{\sigma_W}$. □

We next show that the covariant functor $\mathfrak{CPA}_{p,q}$ defined in (6.2) is naturally isomorphic to the covariant functor $\mathfrak{CanPois}\circ\mathfrak{PS}_{p,q}$, where $\mathfrak{PS}_{p,q}$ is defined in (5.2). This follows from the more general

Lemma 6.5. *The covariant functors $\mathfrak{CanPois}\circ\oplus : \mathsf{PreSymp}\times\mathsf{PreSymp}\to\mathsf{PoisAlg}$ and $\otimes\circ(\mathfrak{CanPois}\times\mathfrak{CanPois}) : \mathsf{PreSymp}\times\mathsf{PreSymp}\to\mathsf{PoisAlg}$ are naturally isomorphic. Specifically, the* $\mathsf{PoisAlg}$*-morphisms*

$$\eta_{((V,\sigma_V),(W,\sigma_W))} : \mathfrak{CanPois}(V\oplus W,\sigma_{V\oplus W})\to\mathfrak{CanPois}(V,\sigma_V)\otimes\mathfrak{CanPois}(W,\sigma_W) \tag{6.4}$$

specified by, for all $(v,w)\in V\oplus W$, $\eta_{((V,\sigma_V),(W,\sigma_W))}(v,w) = v\otimes 1+1\otimes w$ define a natural isomorphism $\{\eta_{((V,\sigma_V),(W,\sigma_W))}\} : \mathfrak{CanPois}\circ\oplus\Rightarrow\otimes\circ(\mathfrak{CanPois}\times\mathfrak{CanPois})$.

Proof. $\eta := \eta_{((V,\sigma_V),(W,\sigma_W))}$ is clearly a linear map (of $\mathbb{N}^0$-degree zero) and therefore induces a unital algebra homomorphism $S(V \oplus W) \to S(V) \otimes S(W)$, which preserves the natural $\mathbb{N}^0$-gradings (with a slight abuse of notation we use for the graded tensor product the same symbol $\otimes$). A simple computation shows that η preserves the Poisson bracket of elements in $S^1(V \oplus W)$ and is therefore a Poisson morphism by (6.1). The inverse to η is given by $\eta^{-1}(v \otimes 1) = (v, 0)$ and $\eta^{-1}(1 \otimes w) = (0, w)$ on the generators $v \otimes 1$ and $1 \otimes w$, with $v \in V$ and $w \in W$, of $S(V) \otimes S(W)$, and extended as a unital algebra homomorphism.

It remains to show naturality. Let $(L, K) : ((V_1, \sigma_{V_1}), (W_1, \sigma_{W_1})) \to ((V_2, \sigma_{V_2}), (W_2, \sigma_{W_2}))$ be a morphism in $\mathsf{PreSymp} \times \mathsf{PreSymp}$. Let us denote $\eta_1 := \eta_{((V_1,\sigma_{V_1}),(W_1,\sigma_{W_1}))}$ and $\eta_2 := \eta_{((V_2,\sigma_{V_2}),(W_2,\sigma_{W_2}))}$. Then, for all $(v, w) \in V_1 \oplus W_1$,

$$\begin{aligned} \eta_2\big(L \oplus K(v, w)\big) &= \eta_2(L(v), K(w)) = L(v) \otimes 1 + 1 \otimes K(w) \\ &= (L \otimes K)(v \otimes 1) + (L \otimes K)(1 \otimes w) = (L \otimes K)(\eta_1(v, w)), \end{aligned} \tag{6.5}$$

which proves naturality since $(v, w) \in V_1 \oplus W_1$ are the generators of $S(V_1 \oplus W_1)$. □

We now can prove the violation of the composition property.

Proposition 6.6. *For any $2 \leq p \in \mathbb{N}$ and $0 < q < p$, the covariant functors $\mathfrak{CPA}_{p,q} : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ and $\mathfrak{CPA}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ are not naturally isomorphic. Indeed, there is no object $(\boldsymbol{M}, \boldsymbol{J})$ in LocSrc_p for which the Poisson algebras $\mathfrak{CPA}_{p,q}(\boldsymbol{M}, \boldsymbol{J})$ and $\mathfrak{CPA}_p(\boldsymbol{M}, \boldsymbol{J})$ are isomorphic.*

Proof. We argue by contradiction: suppose $\mathfrak{CPA}_{p,q}(\boldsymbol{M}, \boldsymbol{J})$ and $\mathfrak{CPA}_p(\boldsymbol{M}, \boldsymbol{J})$ are $\mathsf{PoisAlg}$-isomorphic for some $(\boldsymbol{M}, \boldsymbol{J})$ in LocSrc_p. Then $\mathfrak{CanPois}(\mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J}))$ and $\mathfrak{CanPois}\big(\mathfrak{PS}_q(\boldsymbol{M}, \boldsymbol{J}^q) \oplus \mathfrak{PS}_{p-q}(\boldsymbol{M}, \boldsymbol{J}^{p-q})\big)$ are also $\mathsf{PoisAlg}$-isomorphic by Lemma 6.5, and hence $\mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J})$ and $\mathfrak{PS}_q(\boldsymbol{M}, \boldsymbol{J}^q) \oplus \mathfrak{PS}_{p-q}(\boldsymbol{M}, \boldsymbol{J}^{p-q})$ are $\mathsf{PreSymp}$-isomorphic by Lemma 6.4. But this is excluded by Proposition 5.3. □

6.2 Improved Poisson algebras

In this subsection we will modify the canonical Poisson algebras constructed in Subsection 6.1 in order to address the problems concerning the unexpectedly large automorphism group and the violation of the composition property. As already mentioned, the key point is to represent the algebras given by the functor $\mathfrak{CPA}_p$ as functionals on the affine space of solutions to the inhomogeneous Klein–Gordon equation. When this is done, a degeneracy becomes apparent which was not visible in the description available in the category of presymplectic vector spaces. However, we will show that the degeneracy may be described and also removed in the category of Poisson algebras, thereby resolving the problems discussed above (cf. Subsections 6.4 and 6.5, and Appendix B for another approach).

The abstract Poisson algebras $\mathfrak{CPA}_p$ are represented as functionals on the solution spaces $\mathfrak{Sol}_p : \mathsf{LocSrc}_p \to \mathsf{Aff}$ by extending the pairing (2.9) of $\mathfrak{PS}_p$ and $\mathfrak{Sol}_p$ as follows: For any object $(\boldsymbol{M}, \boldsymbol{J})$ in LocSrc_p we extend $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})}$ to a map $\mathfrak{CPA}_p(\boldsymbol{M}, \boldsymbol{J}) \times \mathfrak{Sol}_p(\boldsymbol{M}, \boldsymbol{J}) \to \mathbb{R}$ (denoted with a slight abuse of notation by the same symbol) in such a way that it is a unital algebra homomorphism in the left entry. Explicitly, we set, for all $\phi \in \mathfrak{Sol}_p(\boldsymbol{M}, \boldsymbol{J})$ and $\alpha \in \mathbb{R}$,

$$\langle\langle \alpha, \phi \rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} = \alpha\,, \tag{6.6a}$$

and, for all $\phi \in \mathfrak{Sol}_p(\boldsymbol{M}, \boldsymbol{J})$ and $[(\varphi_1, \alpha_1)], \dots, [(\varphi_k, \alpha_k)] \in \mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J})$,

$$\langle\langle [(\varphi_1, \alpha_1)] \cdots [(\varphi_k, \alpha_k)], \phi \rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} = \langle\langle [(\varphi_1, \alpha_1)], \phi \rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} \cdots \langle\langle [(\varphi_k, \alpha_k)], \phi \rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})}\ . \tag{6.6b}$$

It is useful to note that the pairing also induces a pairing $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle^{\mathrm{lin}}_{\boldsymbol{M}} : \mathfrak{PS}^{\mathrm{lin}}_p(\boldsymbol{M}) \times \mathfrak{Sol}^{\mathrm{lin}}_p(\boldsymbol{M}) \to \mathbb{R}$ between the linearized (pre)symplectic vector space and solution space, which describe a multiplet of $p \in \mathbb{N}$ homogeneous Klein–Gordon fields. Explicitly, we have $\mathfrak{PS}^{\mathrm{lin}}_p(\boldsymbol{M}) := \big(C^\infty_0(M, \mathbb{R}^p)/\mathrm{KG}_{\boldsymbol{M}}[C^\infty_0(M, \mathbb{R}^p)], \sigma^{\mathrm{lin}}_{\boldsymbol{M}}\big)$, where, for all $[\varphi]^{\mathrm{lin}}, [\psi]^{\mathrm{lin}} \in C^\infty_0(M, \mathbb{R}^p)/\mathrm{KG}_{\boldsymbol{M}}[C^\infty_0(M, \mathbb{R}^p)]$, $\sigma^{\mathrm{lin}}_{\boldsymbol{M}}([\varphi]^{\mathrm{lin}}, [\psi]^{\mathrm{lin}}) := \int_M \langle \varphi, \mathrm{E}_{\boldsymbol{M}}(\psi) \rangle\ \mathrm{vol}_{\boldsymbol{M}}$. The linearized pairing is given by, for all $[\varphi]^{\mathrm{lin}} \in \mathfrak{PS}^{\mathrm{lin}}_p(\boldsymbol{M})$ and $\underline{\phi} \in \mathfrak{Sol}^{\mathrm{lin}}_p(\boldsymbol{M})$,

$$\Big\langle\Big\langle [\varphi]^{\mathrm{lin}}, \underline{\phi} \Big\rangle\Big\rangle^{\mathrm{lin}}_{\boldsymbol{M}} = \int_M \langle \varphi, \underline{\phi} \rangle\ \mathrm{vol}_{\boldsymbol{M}} \tag{6.7}$$

and is related to $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})}$ via, for all $[(\varphi,\alpha)] \in \mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$, $\phi \in \mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})$ and $\underline{\phi} \in \mathfrak{Sol}_p^{\mathrm{lin}}(\boldsymbol{M})$,

$$\langle\langle[(\varphi,\alpha)], \phi + \underline{\phi}\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} = \langle\langle[(\varphi,\alpha)], \phi\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} + \Big\langle\Big\langle[\varphi]^{\mathrm{lin}}, \underline{\phi}\Big\rangle\Big\rangle_{\boldsymbol{M}}^{\mathrm{lin}} . \tag{6.8}$$

Notice that for the linearized setting the analog of the diagram in (2.10) holds true. Moreover, we can extend $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_{\boldsymbol{M}}^{\mathrm{lin}}$ to a map $\mathfrak{CPA}_p^{\mathrm{lin}}(\boldsymbol{M}) \times \mathfrak{Sol}_p^{\mathrm{lin}}(\boldsymbol{M}) \to \mathbb{R}$, where $\mathfrak{CPA}_p^{\mathrm{lin}} := \mathfrak{CanPois} \circ \mathfrak{PS}_p^{\mathrm{lin}}$. These extended pairings are also natural, i.e. the analog of the diagram in (2.10) holds true.

Remark 6.7. The pairing $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})}$ provides us with a representation of the canonical (abstract) Poisson algebra $\mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})$ as a polynomial algebra of functionals on the affine space $\mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})$. Analogously, the pairing $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_{\boldsymbol{M}}^{\mathrm{lin}}$ leads to a representation of $\mathfrak{CPA}_p^{\mathrm{lin}}(\boldsymbol{M})$ as a polynomial algebra of functionals on the vector space $\mathfrak{Sol}_p^{\mathrm{lin}}(\boldsymbol{M})$. The Poisson bracket (6.1) can be expressed in this representation as follows, for all $a,b \in \mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})$ and $\phi \in \mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})$,

$$\Big\langle\Big\langle\{a,b\}_{\sigma_{(\boldsymbol{M},\boldsymbol{J})}}, \phi\Big\rangle\Big\rangle_{(\boldsymbol{M},\boldsymbol{J})} = \int_M \Big\langle \Big\langle\Big\langle a^{(1)}, \phi\Big\rangle\Big\rangle_{(\boldsymbol{M},\boldsymbol{J})}, \mathrm{E}_{\boldsymbol{M}}\Big(\Big\langle\Big\langle b^{(1)}, \phi\Big\rangle\Big\rangle_{(\boldsymbol{M},\boldsymbol{J})}\Big)\Big\rangle \;\mathrm{vol}_{\boldsymbol{M}} , \tag{6.9}$$

where $a^{(1)}$ and $b^{(1)}$ are the first functional derivatives of a and b, respectively, defined uniquely so that

$$\int_M \Big\langle \Big\langle\Big\langle a^{(1)}, \phi\Big\rangle\Big\rangle_{(\boldsymbol{M},\boldsymbol{J})}, \underline{\phi}\Big\rangle \;\mathrm{vol}_{\boldsymbol{M}} := \frac{d}{d\epsilon}\,\langle\langle a, \phi + \epsilon\,\underline{\phi}\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})}\,|_{\epsilon=0} , \tag{6.10}$$

for all $a \in \mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})$, $\phi \in \mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})$ and $\underline{\phi} \in \mathfrak{Sol}_p^{\mathrm{lin}}(\boldsymbol{M})$.

We notice that the pairing $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})}$ is non-degenerate when acting on $\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$. This means that $\langle\langle[(\varphi,\alpha)],\phi\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} = 0$, for all $\phi \in \mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})$, implies that $[(\varphi,\alpha)] = 0$, and, vice versa, that $\langle\langle[(\varphi,\alpha)],\phi\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} = \langle\langle[(\varphi,\alpha)],\phi'\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})}$, for all $[(\varphi,\alpha)] \in \mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$, implies that $\phi = \phi'$. However, the extended pairing on $\mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J}) \times \mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})$ turns out to be degenerate in the left entry and non-degenerate in the right entry. For example, taking $[(0,\alpha)] \in \mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$ with $\alpha \in \mathbb{R}$ we obtain

$$\langle\langle[(0,\alpha)] - \alpha, \phi\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} = \Big(\int_M \langle 0,\phi\rangle \;\mathrm{vol}_{\boldsymbol{M}}\Big) + \alpha - \alpha = 0 , \tag{6.11}$$

for all $\phi \in \mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})$. Hence, the extension of $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ from $\mathfrak{PS}_p$ to $\mathfrak{CPA}_p$ has introduced a new degeneracy, which can not be seen at the level of presymplectic vector spaces as it mixes different $\mathbb{N}^0$-degrees in $\mathfrak{CPA}_p$. This degeneracy is removed precisely by taking the quotient via a suitable Poisson ideal, namely the vanishing ideal

$$\mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J}) := \Big\{a \in \mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J}) : \langle\langle a,\phi\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} = 0\,, \;\; \text{for all } \phi \in \mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})\Big\} \tag{6.12}$$

of the pairing $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ (we will check that it is indeed a Poisson ideal below). The corresponding theory will turn out to have the correct automorphism group and composition property. At this point we would like to note that the pairing $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_{\boldsymbol{M}}^{\mathrm{lin}}$ is non-degenerate when acting on both $\mathfrak{PS}_p^{\mathrm{lin}}(\boldsymbol{M})$ and $\mathfrak{CPA}_p^{\mathrm{lin}}(\boldsymbol{M})$.

Lemma 6.8. *For any object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p the vanishing ideal $\mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$ is a proper Poisson ideal of $\mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})$. Hence the quotient $\mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})/\mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$ is a nontrivial unital Poisson algebra.*

Proof. Given any element $a \in \mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$ it is easy to see that all its functional derivatives vanish, in particular $\langle\langle a^{(1)},\phi\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} = 0$ for all $\phi \in \mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})$. Thus $\{a,b\}_{\sigma_{(\boldsymbol{M},\boldsymbol{J})}} \in \mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$ for any $b \in \mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})$ by (6.9). Since $\mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$ is certainly an ideal, it is a Poisson ideal, and (6.6a) shows that it is proper. □

The quotient $\mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})/\mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$ gives our improved Poisson algebra for the multiplet of $p \in \mathbb{N}$ inhomogeneous Klein–Gordon fields. It is of course free of the redundancy discussed above. However, it is sometimes cumbersome to do explicit calculations involving $\mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$. In order to simplify the following constructions, we shall provide an equivalent characterization of $\mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$ in terms of an algebraically generated ideal. Let us define the following ideal (generated by a set) of $\mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})$

$$\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J}) := \Big\langle \big\{[(0,\alpha)] - \alpha \in \mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J}) : \alpha \in \mathbb{R}\big\} \Big\rangle \, . \tag{6.13}$$

Since $\big\{[(0,\alpha)] - \alpha, a\big\}_{\sigma_{(\boldsymbol{M},\boldsymbol{J})}} = 0$, for all $\alpha \in \mathbb{R}$ and $a \in \mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})$, the ideal $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J})$ is a Poisson ideal of $\mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})$. Furthermore, (6.11) and (6.6b) implies that

$$\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J}) \subseteq \mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J}) \, . \tag{6.14}$$

We now prove that $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J}) = \mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$, which will allow us to work with the easy-to-use algebraically generated ideal $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J})$ whenever it is suitable.

Lemma 6.9. *Let $(\boldsymbol{M},\boldsymbol{J})$ be any object in LocSrc_p.*

a) The Poisson algebra $\mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})/\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J})$ is (noncanonically) isomorphic to $\mathfrak{CPA}_p^{\mathrm{lin}}(\boldsymbol{M})$.

b) $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J}) = \mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$.

Proof. Proof of a): Let us define a Poisson algebra homomorphism $\kappa : \mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J}) \to \mathfrak{CPA}_p^{\mathrm{lin}}(\boldsymbol{M})$ by setting, for all $[(\varphi,\alpha)] \in \mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})$,

$$\kappa\big([(\varphi,\alpha)]\big) = \langle\langle [(\varphi,\alpha)], \phi_0\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} + [\varphi]^{\mathrm{lin}} \, , \tag{6.15}$$

where $\phi_0 \in \mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})$ is any fixed solution. As $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J})$ clearly lies in the kernel of κ, we can induce a Poisson algebra homomorphism (denoted by the same symbol) $\kappa : \mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})/\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J}) \to \mathfrak{CPA}_p^{\mathrm{lin}}(\boldsymbol{M})$. To show that the induced κ is a $\mathsf{PoisAlg}$-isomorphism, we notice that setting, for all $[\varphi]^{\mathrm{lin}} \in \mathfrak{CPA}_p^{\mathrm{lin}}(\boldsymbol{M})$,

$$\kappa^{-1}\big([\varphi]^{\mathrm{lin}}\big) := \big[[(\varphi,0)] - \langle\langle [(\varphi,0)], \phi_0\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})}\,\big] \in \mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})/\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J}) \tag{6.16}$$

is well-defined and defines the inverse of κ.

Proof of b): By a), $\mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})/\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J})$ is a simple (and nontrivial) Poisson algebra. Hence $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J})$ is a maximal proper ideal. In view of (6.14), this shows that $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J}) = \mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$ $\square$

These results now allow us to construct our improved functor for the classical theory of a multiplet of $p \in \mathbb{N}$ inhomogeneous Klein–Gordon fields.

Proposition 6.10. *The following rules define a covariant functor $\mathfrak{PA}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$: To any object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p we associate the Poisson algebra $\mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J}) := \mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})/\mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$. To any morphism $f : (\boldsymbol{M}_1,\boldsymbol{J}_1) \to (\boldsymbol{M}_2,\boldsymbol{J}_2)$ in LocSrc_p we associate the map $\mathfrak{PA}_p(f) : \mathfrak{PA}_p(\boldsymbol{M}_1,\boldsymbol{J}_1) \to \mathfrak{PA}_p(\boldsymbol{M}_2,\boldsymbol{J}_2)$ that is canonically induced from $\mathfrak{CPA}_p(f) : \mathfrak{CPA}_p(\boldsymbol{M}_1,\boldsymbol{J}_1) \to \mathfrak{CPA}_p(\boldsymbol{M}_2,\boldsymbol{J}_2)$.*

Proof. Lemma 6.8 has established that the quotients are nontrivial unital Poisson algebras. Next, let $f : (\boldsymbol{M}_1,\boldsymbol{J}_1) \to (\boldsymbol{M}_2,\boldsymbol{J}_2)$ be any morphism in LocSrc_p. Then $\mathfrak{CPA}_p(f)$ induces a Poisson algebra homomorphism $\mathfrak{PA}_p(f) : \mathfrak{PA}_p(\boldsymbol{M}_1,\boldsymbol{J}_1) \to \mathfrak{PA}_p(\boldsymbol{M}_2,\boldsymbol{J}_2)$ because it restricts to a map $\mathfrak{CPA}_p(f) : \mathfrak{I}_p(\boldsymbol{M}_1,\boldsymbol{J}_1) \to \mathfrak{I}_p(\boldsymbol{M}_2,\boldsymbol{J}_2)$, as is obvious from Lemma 6.9 b), the explicit characterization of the algebraic Poisson ideal (6.13) and the fact that $\mathfrak{CPA}_p(f)([(0,\alpha)] - \alpha) = [(0,\alpha)] - \alpha$, for any $\alpha \in \mathbb{R}$. It is clear that $\mathfrak{PA}_p(f)$ is unit-preserving, because $\mathfrak{CPA}_p(f)$ is; moreover, it is injective, since $\mathfrak{PA}_p(\boldsymbol{M}_1,\boldsymbol{J}_1)$ is simple by Lemma 6.9 a), and hence it does not have any nontrivial proper Poisson ideals (and $\mathfrak{PA}_p(f)$ is a unit-preserving map to a nontrivial unital Poisson algebra, so it is not the zero map). The composition and identity properties of $\mathfrak{PA}_p$ are inherited from $\mathfrak{CPA}_p$, hence $\mathfrak{PA}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ is a covariant functor. $\square$

The covariant functor $\mathfrak{PA}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ is a locally covariant classical field theory:

Proposition 6.11. *The covariant functor* $\mathfrak{PA}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ *satisfies the causality property and the time-slice axiom.*

Proof. By Proposition 6.3 the covariant functor $\mathfrak{CPA}_p$ satisfies these properties. The quotients by Poisson ideals used in the definition of the functor $\mathfrak{PA}_p$ preserve these properties due to the following arguments: For the time-slice axiom we just have to notice that any PoisAlg-isomorphism which preserves the Poisson ideals induces a PoisAlg-isomorphism on the quotients (simply induce the morphism and its inverse to the quotients). Causality holds because if two subalgebras of a Poisson algebra $\mathcal{A}$ Poisson-commute, then the same is true of the corresponding subalgebras of any quotient of $\mathcal{A}$ by a Poisson ideal. □

6.3 Relative Cauchy evolution of the functor $\mathfrak{PA}_p$

The relative Cauchy evolution of the functor $\mathfrak{PS}_p : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ induces that of the functor $\mathfrak{PA}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ as follows: Let $(\boldsymbol{M},\boldsymbol{J})$ be any object in LocSrc_p and let $(\boldsymbol{h},\boldsymbol{j}) \in H(\boldsymbol{M},\boldsymbol{J})$ be any globally hyperbolic perturbation. From the explicit expression for $\mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}] \in \mathrm{Aut}(\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J}))$ given in (3.9) we observe that the relative Cauchy evolution $\mathrm{rce}^{(\mathfrak{PA}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}] \in \mathrm{Aut}(\mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J}))$ of $\mathfrak{PA}_p$ is uniquely specified by, for all $[(\varphi,\alpha)] \in \mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J})$,

$$\begin{aligned}\mathrm{rce}^{(\mathfrak{PA}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}]\big([(\varphi,\alpha)]\big) &= \big[\big(\varphi + (\mathrm{KG}_{\boldsymbol{M}} - \mathrm{KG}_{\boldsymbol{M}[\boldsymbol{h}]})\big(\mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\big), 0\big)\big] + \alpha \\ &\quad + \int_M \big(\big\langle -\boldsymbol{j}, \mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\big\rangle + \big\langle (1-\rho_{\boldsymbol{h}})\,(\boldsymbol{J}+\boldsymbol{j}), \mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\big\rangle\big)\ \mathrm{vol}_{\boldsymbol{M}}\ , \end{aligned} \tag{6.17}$$

where on the right hand side we have used the equivalence relation entering the definition of $\mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J})$ (cf. Proposition 6.10) and we have chosen as in (3.9) a representative φ with compact support in M^+. With the techniques presented in Appendix A one can differentiate this expression to yield

$$\begin{aligned}\frac{d}{ds}\mathrm{rce}^{(\mathfrak{PA}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[s\boldsymbol{h},s\boldsymbol{j}]\big([(\varphi,\alpha)]\big)\Big|_{s=0} &= -\Big[\Big(\mathrm{KG}'_{\boldsymbol{M}[\boldsymbol{h}]}\big(\mathrm{E}_{\boldsymbol{M}}(\varphi)\big),0\Big)\Big] \\ &\quad - \int_M \Big\langle \frac{1}{2} g^{ab}\, h_{ab}\,\boldsymbol{J} + \boldsymbol{j}, \mathrm{E}_{\boldsymbol{M}}(\varphi)\Big\rangle\ \mathrm{vol}_{\boldsymbol{M}} \\ &= -\Big\{\frac{1}{2}T_{(\boldsymbol{M},\boldsymbol{J})}(\boldsymbol{h}) + [(\boldsymbol{j},0)], [(\varphi,\alpha)]\Big\}_{\sigma_{(\boldsymbol{M},\boldsymbol{J})}}\ , \end{aligned} \tag{6.18}$$

where $T^{ab}_{(\boldsymbol{M},\boldsymbol{J})}$ is the stress-energy tensor (3.13). Although (6.17) was derived under an assumption on the support of the representative φ, the formulae in (6.18) do not require to choose a suitable representative as they depend only on the equivalence class of (φ,α).

6.4 Automorphism group of the functor $\mathfrak{PA}_p$

We study the automorphism group of the covariant functor $\mathfrak{PA}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ defined in Proposition 6.10. We shall obtain that it is, as expected, the trivial group for $m \neq 0$ and isomorphic to $\mathbb{R}^p$ for $m = 0$.

We first show that for $m = 0$ the automorphism group of $\mathfrak{PA}_p$ contains $\mathbb{R}^p$ as a subgroup.

Proposition 6.12. *If* $m = 0$ *there exists a faithful homomorphism* $\eta : \mathbb{R}^p \to \mathrm{Aut}(\mathfrak{PA}_p)$ *induced by the one in Proposition 4.2 restricted to* $\{+1\} \times \mathbb{R}^p \subseteq \mathbb{Z}_2 \times \mathbb{R}^p$*. Explicitly, for any object* $(\boldsymbol{M},\boldsymbol{J})$ *in* LocSrc_p *the automorphism* $\eta(\mu)_{(\boldsymbol{M},\boldsymbol{J})}$ *is specified by, for all* $[(\varphi,\alpha)] \in \mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J})$,

$$\eta(\mu)_{(\boldsymbol{M},\boldsymbol{J})}\big([(\varphi,\alpha)]\big) = \Big[\Big(\varphi, \alpha + \int_M \langle\varphi,\mu\rangle\ \mathrm{vol}_{\boldsymbol{M}}\Big)\Big]\ . \tag{6.19}$$

Proof. Applying the functor $\mathfrak{CanPois}$, the automorphism $\eta(\sigma,\mu) \in \mathrm{Aut}(\mathfrak{PS}_p)$ of Proposition 4.2 induces an element in $\mathrm{Aut}(\mathfrak{CPA}_p)$, which we denote with a slight abuse of notation by the same symbol $\eta(\sigma,\mu)$. For $\sigma = -1$ this automorphism does not preserve the Poisson ideals $\mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$: Indeed, for $[(0,\alpha)] - \alpha \in$

$\mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$ we find $\eta(-1,\mu)_{(\boldsymbol{M},\boldsymbol{J})}([(0,\alpha)]-\alpha) = [(0,-\alpha)]-\alpha \notin \mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$. For $\sigma=+1$ and $\mu\in\mathbb{R}^p$ arbitrary the Poisson ideals are preserved, hence $\eta(+1,\mu)$ induces the automorphism $\eta(\mu)\in\mathrm{Aut}(\mathfrak{PA}_p)$ given by (6.19). The group law $\eta(\mu)\circ\eta(\mu') = \eta(\mu+\mu')$ is a consequence of the group law for $\eta(\sigma,\mu)$, cf. Proposition 4.2. □

Remark 6.13. The above argument shows that the nontrivial $\mathbb{Z}_2$-automorphism in the massless case (cf. Proposition 4.2) does not induce an automorphism of $\mathfrak{PA}_p$. The same holds true for the nontrivial $\mathbb{Z}_2$-automorphism in the massive case (cf. Proposition 4.1).

We now prove that the automorphisms found in Proposition 6.12 exhaust the automorphism group of the covariant functor $\mathfrak{PA}_p$. For the proof we require the analog of Theorem 4.5, stating that an endomorphism is uniquely determined by its component on one object, for the functor $\mathfrak{PA}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$. This follows by a similar proof as for Theorem 4.5 and we omit the details.

Theorem 6.14. *Every endomorphism of the functor $\mathfrak{PA}_p$ is an automorphism and*

$$\mathrm{End}(\mathfrak{PA}_p) = \mathrm{Aut}(\mathfrak{PA}_p) \simeq \begin{cases} \{\mathrm{id}_{\mathfrak{PA}_p}\} & , \ \textit{for } m\neq 0\,,\\ \mathbb{R}^p & , \ \textit{for } m=0\,, \end{cases} \tag{6.20}$$

where the action for $m=0$ is given by Proposition 6.12.

Proof. Let $\eta\in\mathrm{End}(\mathfrak{PA}_p)$ and consider its component $\eta_{(\boldsymbol{M}_0,0)}\in\mathrm{End}(\mathfrak{PA}_p(\boldsymbol{M}_0,0))$ on the Minkowski spacetime $\boldsymbol{M}_0$ with $\boldsymbol{J}_0=0$. Now, $\eta_{(\boldsymbol{M}_0,0)}$ must commute with the relative Cauchy evolution (6.17) and its derivative (6.18); considering the $\boldsymbol{h}=0$ case of (6.18), we obtain the requirement

$$\eta_{(\boldsymbol{M}_0,0)}\Big(\big\{[(\boldsymbol{j},0)],[(\varphi,0)]\big\}_{\sigma_{(\boldsymbol{M}_0,0)}}\Big) = \big\{[(\boldsymbol{j},0)],\eta_{(\boldsymbol{M}_0,0)}\big([(\varphi,0)]\big)\big\}_{\sigma_{(\boldsymbol{M}_0,0)}} \tag{6.21}$$

for all $\boldsymbol{j},\varphi\in C^\infty_0(M_0,\mathbb{R}^p)$. Next, we exploit the fact (cf. Lemma 6.9) that there is a preferred isomorphism $\kappa:\mathfrak{PA}_p(\boldsymbol{M}_0,0)\to\mathfrak{CPA}^{\mathrm{lin}}_p(\boldsymbol{M}_0)$ given by $\kappa\big([(\varphi,\alpha)]\big)=\alpha+[\varphi]^{\mathrm{lin}}$, i.e. the $\phi_0=0$ case of (6.15), which intertwines the natural actions of the Poincaré transformations on $\mathfrak{PA}_p(\boldsymbol{M}_0,0)$ and $\mathfrak{CPA}^{\mathrm{lin}}_p(\boldsymbol{M}_0)$. Then the induced endomorphism $\widetilde{\eta}=\kappa\circ\eta_{(\boldsymbol{M}_0,0)}\circ\kappa^{-1}$ of $\mathfrak{CPA}^{\mathrm{lin}}_p(\boldsymbol{M}_0)$ must commute with all Poincaré transformations, because $\eta_{(\boldsymbol{M}_0,0)}$ does. Owing to (6.21), $\widetilde{\eta}$ also satisfies, for all generators $[\varphi]^{\mathrm{lin}}\in\mathfrak{CPA}^{\mathrm{lin}}_p(\boldsymbol{M}_0)$ and all $\boldsymbol{j}\in C^\infty_0(M_0,\mathbb{R}^p)$,

$$\widetilde{\eta}\Big(\big\{[\boldsymbol{j}]^{\mathrm{lin}},[\varphi]^{\mathrm{lin}}\big\}_{\sigma^{\mathrm{lin}}_{\boldsymbol{M}_0}}\Big) = \big\{[\boldsymbol{j}]^{\mathrm{lin}},\widetilde{\eta}\big([\varphi]^{\mathrm{lin}}\big)\big\}_{\sigma^{\mathrm{lin}}_{\boldsymbol{M}_0}}\ . \tag{6.22}$$

The left hand side of this equation is simply $\sigma^{\mathrm{lin}}_{\boldsymbol{M}_0}([\boldsymbol{j}]^{\mathrm{lin}},[\varphi]^{\mathrm{lin}})$ and the right hand side can be simplified as follows: We write $\widetilde{\eta}([\varphi]^{\mathrm{lin}})=\widetilde{\eta}_0([\varphi]^{\mathrm{lin}})+\widetilde{\eta}_1([\varphi]^{\mathrm{lin}})+\widetilde{\eta}_{\geq 2}([\varphi]^{\mathrm{lin}})$, where the index labels the $\mathbb{N}^0$-degree of $\widetilde{\eta}([\varphi]^{\mathrm{lin}})$ in $\mathfrak{CPA}^{\mathrm{lin}}_p(\boldsymbol{M}_0)$. This yields the condition, for all $[\varphi]^{\mathrm{lin}}\in\mathfrak{PS}^{\mathrm{lin}}_p(\boldsymbol{M}_0)$ and all $\boldsymbol{j}\in C^\infty_0(M_0,\mathbb{R}^p)$,

$$\sigma^{\mathrm{lin}}_{\boldsymbol{M}_0}([\boldsymbol{j}]^{\mathrm{lin}},[\varphi]^{\mathrm{lin}}) = \sigma^{\mathrm{lin}}_{\boldsymbol{M}_0}\big([\boldsymbol{j}]^{\mathrm{lin}},\widetilde{\eta}_1\big([\varphi]^{\mathrm{lin}}\big)\big) + \big\{[\boldsymbol{j}]^{\mathrm{lin}},\widetilde{\eta}_{\geq 2}\big([\varphi]^{\mathrm{lin}}\big)\big\}_{\sigma^{\mathrm{lin}}_{\boldsymbol{M}_0}}\ . \tag{6.23}$$

Counting the $\mathbb{N}^0$-degree of the individual terms and using the fact that the Poisson bracket in $\mathfrak{CPA}^{\mathrm{lin}}_p(\boldsymbol{M}_0)$ is non-degenerate we obtain that $\widetilde{\eta}_{\geq 2}=0$ and $\widetilde{\eta}_1=\mathrm{id}_{\mathfrak{CPA}^{\mathrm{lin}}_p(\boldsymbol{M}_0)}$. Hence, $\widetilde{\eta}([\varphi]^{\mathrm{lin}})=\widetilde{\eta}_0([\varphi]^{\mathrm{lin}})+[\varphi]^{\mathrm{lin}}$, for all $[\varphi]^{\mathrm{lin}}\in\mathfrak{PS}^{\mathrm{lin}}_p(\boldsymbol{M}_0)$, and the remaining freedom in $\widetilde{\eta}$ is a linear map $\widetilde{\eta}_0:\mathfrak{PS}^{\mathrm{lin}}_p(\boldsymbol{M}_0)\to\mathbb{R}$, which also has to be Poincaré invariant. By Lemma 4.7, $\widetilde{\eta}_0\equiv 0$ in the case of $m\neq 0$ and $\widetilde{\eta}_0([\varphi]^{\mathrm{lin}})=\int_{M_0}\langle\varphi,\mu\rangle\,\mathrm{vol}_{\boldsymbol{M}_0}$ for some $\mu\in\mathbb{R}^p$ in the massless case. Hence, there are no nontrivial endomorphisms of $\mathfrak{PA}_p(\boldsymbol{M}_0,0)$ in the massive case. For $m=0$ the endomorphisms of $\mathfrak{PA}_p(\boldsymbol{M}_0,0)$ coincide with the Minkowski space components of the functor automorphisms found in Proposition 6.12. Since any endomorphism $\eta\in\mathrm{End}(\mathfrak{PA}_p)$ is uniquely determined by its component on one object, this proves our claim. □

6.5 Composition property of the functor $\mathfrak{PA}_p$

It remains to prove the validity of the composition property of the covariant functor $\mathfrak{PA}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$. Explicitly, we define for $p \geq 2$ and $0 < q < p$ the covariant functor

$$\mathfrak{PA}_{p,q} := \otimes \circ \left(\mathfrak{PA}_q \times \mathfrak{PA}_{p-q}\right) \circ \mathfrak{Split}_{p,q} : \mathsf{LocSrc}_p \to \mathsf{PoisAlg} \tag{6.24}$$

and we will prove that $\mathfrak{PA}_{p,q}$ and $\mathfrak{PA}_p$ are naturally isomorphic.

Theorem 6.15. *For any $2 \leq p \in \mathbb{N}$ and $0 < q < p$, the covariant functors $\mathfrak{PA}_{p,q} : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ and $\mathfrak{PA}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ are naturally isomorphic. The natural isomorphism $\eta = \{\eta_{(\boldsymbol{M},\boldsymbol{J})}\} : \mathfrak{PA}_{p,q} \Rightarrow \mathfrak{PA}_p$ is specified by, for all $[(\varphi,\alpha)] \in \mathfrak{PA}_q(\boldsymbol{M},\boldsymbol{J}^q)$ and $[(\psi,\beta)] \in \mathfrak{PA}_{p-q}(\boldsymbol{M},\boldsymbol{J}^{p-q})$,*

$$\eta_{(\boldsymbol{M},\boldsymbol{J})}\big([(\varphi,\alpha)] \otimes 1\big) = \big[(\varphi,\alpha)\big] \ , \ \ \eta_{(\boldsymbol{M},\boldsymbol{J})}\big(1 \otimes [(\psi,\beta)]\big) = \big[(\psi,\beta)\big] \ , \tag{6.25}$$

where on the right hand sides we have identified $\varphi \in C_0^\infty(M,\mathbb{R}^q)$ and $\psi \in C_0^\infty(M,\mathbb{R}^{p-q})$ as elements in $C_0^\infty(M,\mathbb{R}^p)$ (φ is placed in the first q and ψ in the last $p-q$ components of $\mathbb{R}^p$) .

Proof. We first notice that (6.25) actually defines a Poisson algebra homomorphism $\mathfrak{CPA}_q(\boldsymbol{M},\boldsymbol{J}^q) \otimes \mathfrak{CPA}_{p-q}(\boldsymbol{M},\boldsymbol{J}^{p-q}) \to \mathfrak{CPA}_p(\boldsymbol{M},\boldsymbol{J})$. It induces a unital Poisson algebra homomorphism between the quotients, $\mathfrak{PA}_q(\boldsymbol{M},\boldsymbol{J}^q) \otimes \mathfrak{PA}_{p-q}(\boldsymbol{M},\boldsymbol{J}^{p-q}) \to \mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J})$, since, for all $\alpha \in \mathbb{R}$,

$$\eta_{(\boldsymbol{M},\boldsymbol{J})}\big(\big([(0,\alpha)] - \alpha\big) \otimes 1\big) = \eta_{(\boldsymbol{M},\boldsymbol{J})}\big(1 \otimes \big([(0,\alpha)] - \alpha\big)\big) = [(0,\alpha)] - \alpha \ . \tag{6.26}$$

Naturality of the $\eta_{(\boldsymbol{M},\boldsymbol{J})}$ is also a straightforward check. We next show that $\eta_{(\boldsymbol{M},\boldsymbol{J})}$ is invertible, hence a $\mathsf{PoisAlg}$-isomorphism. Notice that setting, for any $[(\varphi,\alpha)] \in \mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J})$,

$$\eta^{-1}_{(\boldsymbol{M},\boldsymbol{J})}\big([(\varphi,\alpha)]\big) = \Big[[(\varphi^q,\alpha)] \otimes 1 + 1 \otimes [(\varphi^{p-q},0)]\Big] \in \mathfrak{PA}_q(\boldsymbol{M},\boldsymbol{J}^q) \otimes \mathfrak{PA}_{p-q}(\boldsymbol{M},\boldsymbol{J}^{p-q}) \ , \tag{6.27}$$

where $\varphi = \varphi^q + \varphi^{p-q}$ is the split of φ into the first q and last $p-q$ components, is well-defined and extends to a unital Poisson algebra homomorphism $\eta^{-1}_{(\boldsymbol{M},\boldsymbol{J})} : \mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J}) \to \mathfrak{PA}_q(\boldsymbol{M},\boldsymbol{J}^q) \otimes \mathfrak{PA}_{p-q}(\boldsymbol{M},\boldsymbol{J}^{p-q})$. One checks directly that $\eta^{-1}_{(\boldsymbol{M},\boldsymbol{J})}$ is the inverse of $\eta_{(\boldsymbol{M},\boldsymbol{J})}$. □

7 Quantization

We shall now turn to the quantization of our model. As a first step, we are going to use the CCR-functor (in polynomial form) in order to construct a covariant functor $\mathfrak{CQA}_p := \mathfrak{CCR} \circ \mathfrak{PS}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$, where ${}^*\mathsf{Alg}$ is the category of unital $*$-algebras over $\mathbb{C}$ with injective unital $*$-algebra homomorphisms as morphisms. As $\mathfrak{CQA}_p$ is a deformation quantization of the Poisson algebra functor $\mathfrak{CPA}_p$, it is not surprising to find that its automorphism group is too large and that it violates the composition property. We then improve this functor following a strategy similar to that of Subsection 6.2 for the Poisson algebras. The essential step is to specify a suitable state space for $\mathfrak{CQA}_p$. The kernel corresponding to this state space forms a two-sided $*$-ideal in the algebras described by $\mathfrak{CQA}_p$, which when quotiented out leads to a covariant functor $\mathfrak{QA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ that has the correct automorphism group and satisfies the composition property. Accordingly, we find that $\mathfrak{QA}_p$ is the correct description of the quantum field theory of a multiplet of $p \in \mathbb{N}$ inhomogeneous Klein–Gordon fields and not the functor $\mathfrak{CQA}_p$, which was used in [BDS12]. An alternative construction of $\mathfrak{QA}_p$ via the quantization of pointed presymplectic spaces is presented in Appendix B.

7.1 Canonical algebras

We briefly review the CCR-functor $\mathfrak{CCR} : \mathsf{PreSymp} \to {}^*\mathsf{Alg}$ in polynomial form, following the slightly non-standard approach taken in [BSZ92] and [FV12b, §5], which is equivalent to the standard presentation in terms of generators and relations. To any object (V,σ_V) in $\mathsf{PreSymp}$ we associate the following unital $*$-algebra $\mathfrak{CCR}(V,\sigma_V)$: The vector space underlying $\mathfrak{CCR}(V,\sigma_V)$ is the complexification of the vector space underlying the symmetric tensor algebra $S(V) := \bigoplus_{k=0}^\infty S^k(V)$. The involution $*$ is defined by $\mathbb{C}$-antilinearity and

$(v_1 \cdots v_k)^* = v_1 \cdots v_k$, for all $v_1, \dots, v_k \in V$, where juxtaposition denotes a symmetric product. The product $\star$ in $\mathfrak{CCR}(V, \sigma_V)$ is specified (uniquely) by demanding, for all $v_1, v_2 \in V$ and $n, m \in \mathbb{N}_0$,

$$v_1^m \star v_2^n = \sum_{r=0}^{\min\{m,n\}} \left(\frac{i\sigma_V(v_1, v_2)}{2}\right)^r \frac{m!\,n!}{r!\,(m-r)!\,(n-r)!}\, v_1^{m-r}\, v_2^{n-r} \,. \tag{7.1}$$

To any morphism $L : (V, \sigma_V) \to (W, \sigma_W)$ in $\mathsf{PreSymp}$ we associate the injective unital $*$-algebra homomorphism $\mathfrak{CCR}(L) : \mathfrak{CCR}(V, \sigma_V) \to \mathfrak{CCR}(W, \sigma_W)$, which is specified by $\mathfrak{CCR}(L)(v_1\, v_2 \cdots v_k) = L(v_1)\, L(v_2) \cdots L(v_k)$, for all $v_1, \dots, v_k \in V$, and $\mathbb{C}$-linearity.

Composing the covariant functor $\mathfrak{PS}_p : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ with $\mathfrak{CCR}$ yields the covariant functor $\mathfrak{CQA}_p := \mathfrak{CCR} \circ \mathfrak{PS}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$. It is standard that $\mathfrak{CCR}$ preserves the time-slice axiom and the causality property; as $\mathfrak{PS}_p$ satisfies these conditions by Proposition 2.3, $\mathfrak{CQA}_p$ is a locally covariant quantum field theory. In [BDS12] $\mathfrak{CQA}_p$ was taken to describe the quantized field polynomial algebras of a multiplet of $p \in \mathbb{N}$ inhomogeneous Klein–Gordon fields.

It is easy to see that the automorphism group $\mathrm{Aut}(\mathfrak{CQA}_p)$ contains a $\mathbb{Z}_2$-subgroup for the massive case and a $\mathbb{Z}_2 \times \mathbb{R}^p$-subgroup for $m = 0$. This is an immediate consequence of Theorem 4.6 and the fact that any automorphism $\eta = \{\eta_{(\boldsymbol{M},\boldsymbol{J})}\}$ of $\mathfrak{PS}_p$ lifts to an automorphism of $\mathfrak{CQA}_p$ with components $\{\mathfrak{CCR}(\eta_{(\boldsymbol{M},\boldsymbol{J})})\}$.

To show that $\mathfrak{CQA}_p$ violates the composition property, we define, for all $2 \leq p \in \mathbb{N}$ and $0 < q < p$, the covariant functor

$$\mathfrak{CQA}_{p,q} := \otimes \circ (\mathfrak{CQA}_q \times \mathfrak{CQA}_{p-q}) \circ \mathfrak{Split}_{p,q} : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg} \,, \tag{7.2}$$

where $\otimes : {}^*\mathsf{Alg} \times {}^*\mathsf{Alg} \to {}^*\mathsf{Alg}$ now denotes the covariant functor that takes the algebraic tensor product of unital $*$-algebras. Adapting the proof of Lemma 6.5, one observes that the two covariant functors $\mathfrak{CCR} \circ \oplus : \mathsf{PreSymp} \times \mathsf{PreSymp} \to {}^*\mathsf{Alg}$ and $\otimes \circ (\mathfrak{CCR} \times \mathfrak{CCR}) : \mathsf{PreSymp} \times \mathsf{PreSymp} \to {}^*\mathsf{Alg}$ are naturally isomorphic. Furthermore, the result of Lemma 6.4 also extends to our present setting: two unital $*$-algebras $\mathfrak{CCR}(V, \sigma_V)$ and $\mathfrak{CCR}(W, \sigma_W)$ are isomorphic if and only if (V, σ_V) and (W, σ_W) are isomorphic as presymplectic vector spaces. Then by an argument similar to that of Proposition 6.6 we obtain

Proposition 7.1. *For any $2 \leq p \in \mathbb{N}$ and $0 < q < p$, the covariant functors $\mathfrak{CQA}_{p,q} : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ and $\mathfrak{CQA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ are not naturally isomorphic. Indeed, there is no object $(\boldsymbol{M}, \boldsymbol{J})$ in LocSrc_p for which the unital $*$-algebras $\mathfrak{CQA}_{p,q}(\boldsymbol{M}, \boldsymbol{J})$ and $\mathfrak{CQA}_p(\boldsymbol{M}, \boldsymbol{J})$ are isomorphic.*

7.2 Improved algebras

Employing a strategy similar to the one in Subsection 6.2, we now modify the canonical algebras of Subsection 7.1 in order to obtain the correct automorphism group and satisfy the composition property. The essential idea is again to make our theory remember that it came from affine functionals acting on the affine space of solutions of the inhomogeneous Klein–Gordon equation.

We implement this idea mathematically by introducing suitable state spaces. Recall a state space for a unital $*$-algebra A is a subset S of the set of normalized and positive linear functionals on A that is closed under convex linear combinations and operations induced by A. The latter property means that, given any state $\omega \in S$ and $b \in A$ such that $\omega(b^* b) > 0$, then the state $\omega_b(a) := \omega(b^* a b)/\omega(b^* b)$, for all $a \in A$, is also contained in S. To promote the concept of state spaces to the categorical setting, we define the category State as follows: The objects in State are all possible state spaces for objects in ${}^*\mathsf{Alg}$, with affine maps as morphisms. A state space for our covariant functor $\mathfrak{CQA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ is a contravariant functor $\mathfrak{S}_p : \mathsf{LocSrc}_p \to \mathsf{State}$, such that $\mathfrak{S}_p(\boldsymbol{M}, \boldsymbol{J})$ is a state space for $\mathfrak{CQA}_p(\boldsymbol{M}, \boldsymbol{J})$ for each object $(\boldsymbol{M}, \boldsymbol{J})$ and $\mathfrak{S}_p(f) = \mathfrak{CQA}_p(f)^*|_{\mathfrak{S}_p(\boldsymbol{M}_2, \boldsymbol{J}_2)}$ for every morphism $f : (\boldsymbol{M}_1, \boldsymbol{J}_1) \to (\boldsymbol{M}_2, \boldsymbol{J}_2)$ in LocSrc_p (it is a necessary condition that $\mathfrak{CQA}_p(f)^*[\mathfrak{S}_p(\boldsymbol{M}_2, \boldsymbol{J}_2)] \subseteq \mathfrak{S}_p(\boldsymbol{M}_1, \boldsymbol{J}_1)$).

Definition 7.2. An **admissible state space** $\mathfrak{S}_p : \mathsf{LocSrc}_p \to \mathsf{State}$ for the covariant functor $\mathfrak{CQA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ is a state space $\mathfrak{S}_p$, such that for each object $(\boldsymbol{M}, \boldsymbol{J})$ in LocSrc_p, and for all $\omega \in \mathfrak{S}_p(\boldsymbol{M}, \boldsymbol{J})$ and $[(0, \alpha)], [(0, \beta)] \in \mathfrak{CQA}_p(\boldsymbol{M}, \boldsymbol{J})$,

$$\omega\big([(0, \alpha)]\big) = \alpha \quad , \qquad \omega\big([(0, \alpha)] \star [(0, \beta)]\big) = \alpha\, \beta \,. \tag{7.3}$$

Remark 7.3. The first condition in (7.3) demands that the expectation values of the quantum observables corresponding to $[(0,\alpha)]$ agree with the classical result (2.9). The second condition in (7.3) sets the fluctuations around this classical result to zero, cf. the lemma below. This behavior of states for the quantum theory is motivated by the fact that $[(0,\alpha)]$ corresponds in the classical theory to a constant functional.

Lemma 7.4. *Let $\mathfrak{S}_p$ be any admissible state space for $\mathfrak{CQA}_p$. Then for any object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p, and for all $\omega\in\mathfrak{S}_p(\boldsymbol{M},\boldsymbol{J})$ and $[(0,\alpha_1)],\dots,[(0,\alpha_n)]\in\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})$,*

$$\omega\big([(0,\alpha_1)]\star\cdots\star[(0,\alpha_n)]\big)=\alpha_1\,\cdots\,\alpha_n\;. \tag{7.4}$$

Proof. This is a straightforward consequence of the Cauchy-Schwarz inequality and a simple proof by induction. Using the short notation $\widehat{\alpha}:=[(0,\alpha)]$ we obtain

$$\begin{aligned}\big|\omega\big(\widehat{\alpha_1}\star\cdots\star\widehat{\alpha_n}\big)-\alpha_1\cdots\alpha_n\big|^2&=\big|\omega\big(\widehat{\alpha_1}\star\cdots\star\widehat{\alpha_n}\big)-\alpha_1\,\omega\big(\widehat{\alpha_2}\star\cdots\star\widehat{\alpha_n}\big)\big|^2\\&=\big|\omega\big((\widehat{\alpha_1}-\alpha_1)\star\widehat{\alpha_2}\star\cdots\star\widehat{\alpha_n}\big)\big|^2\\&\leq\omega\big((\widehat{\alpha_1}-\alpha_1)^2\big)\,\omega\big((\widehat{\alpha_2}\star\cdots\star\widehat{\alpha_n})^2\big)=0\;,\end{aligned} \tag{7.5}$$

where the last equality follows from the admissibility condition (7.3). □

Lemma 7.5. *There exists a non-empty admissible state space $\mathfrak{S}_p$ for $\mathfrak{CQA}_p$, i.e. $\mathfrak{S}_p(\boldsymbol{M},\boldsymbol{J})$ is non-empty for all objects $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p.*

Proof. Let $\mathfrak{S}_p^{\max}(\boldsymbol{M},\boldsymbol{J})$ be the set of all states on $\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})$ satisfying (7.3). This is non-empty, since it was shown in [BDS12, §8] that any state of the homogeneous Klein–Gordon theory induces a state in $\mathfrak{S}_p^{\max}(\boldsymbol{M},\boldsymbol{J})$. The admissibility condition of states in $\mathfrak{S}_p^{\max}(\boldsymbol{M},\boldsymbol{J})$ is met by construction and it is preserved under convex linear combinations and operations induced by $\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})$ (to prove the latter statement, use the Cauchy-Schwarz inequality and the fact that $[(0,\alpha)]-\alpha$ lies in the center of $\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})$). Thus, it remains to show that

$$\mathfrak{S}_p^{\max}(f):\mathfrak{S}_p^{\max}(\boldsymbol{M}_2,\boldsymbol{J}_2)\to\mathfrak{S}_p^{\max}(\boldsymbol{M}_1,\boldsymbol{J}_1)\;,\;\;\omega\mapsto\mathfrak{S}_p^{\max}(f)(\omega)=\omega\circ\mathfrak{CQA}_p(f) \tag{7.6}$$

is a morphism in State, i.e. that $\mathfrak{S}_p^{\max}(f)(\omega)\in\mathfrak{S}_p^{\max}(\boldsymbol{M}_1,\boldsymbol{J}_1)$, for all $\omega\in\mathfrak{S}_p^{\max}(\boldsymbol{M}_2,\boldsymbol{J}_2)$. This holds because $\mathfrak{S}_p^{\max}(f)(\omega)$ is clearly a state, and obeys (7.3) because $\mathfrak{CQA}_p(f)\big([(0,\alpha)]\big)=[(0,\alpha)]$. □

Given any non-empty admissible state space $\mathfrak{S}_p$ for $\mathfrak{CQA}_p$, we define for every object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p

$$\mathfrak{I}^{\mathfrak{S}_p}(\boldsymbol{M},\boldsymbol{J}):=\bigcap_{\omega\in\mathfrak{S}_p(\boldsymbol{M},\boldsymbol{J})}\ker(\pi_\omega)\subseteq\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})\;, \tag{7.7}$$

where π_ω denotes the GNS-representation of $\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})$ induced by $\omega\in\mathfrak{S}_p(\boldsymbol{M},\boldsymbol{J})$. The subset (7.7) of $\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})$ is clearly a two-sided $*$-ideal, and it must be proper because Gelfand ideals necessarily exclude the unit. Hence $\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})/\mathfrak{I}^{\mathfrak{S}_p}(\boldsymbol{M},\boldsymbol{J})$ is a nontrivial unital $*$-algebra.

It will again be convenient to express $\mathfrak{I}^{\mathfrak{S}_p}(\boldsymbol{M},\boldsymbol{J})$ in terms of an algebraically generated ideal. Let us consider the following two-sided $*$-ideal (generated by a set) of $\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})$

$$\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J}):=\Big\langle\big\{[(0,\alpha)]-\alpha\in\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J}):\alpha\in\mathbb{R}\big\}\Big\rangle\;. \tag{7.8}$$

It is easy to see that $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J})\subseteq\mathfrak{I}^{\mathfrak{S}_p}(\boldsymbol{M},\boldsymbol{J})$: Let $\omega\in\mathfrak{S}_p(\boldsymbol{M},\boldsymbol{J})$ be arbitrary. Then, for all $b,c\in\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})$ and all $\alpha\in\mathbb{R}$,

$$\begin{aligned}\Big|\omega\big(b\star\big([(0,\alpha)]-\alpha\big)\star c\big)\Big|^2&=\Big|\omega\big(b\star c\star\big([(0,\alpha)]-\alpha\big)\big)\Big|^2\\&\leq\omega\big(b\star c\star(b\star c)^*\big)\,\omega\big(\big([(0,\alpha)]-\alpha\big)^2\big)=0\;,\end{aligned} \tag{7.9}$$

where in the first step we have used that $[(0,\alpha)]-\alpha$ lies in the center of $\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})$, in the second step the Cauchy-Schwarz inequality and in the last one the admissibility condition (7.3). Hence, $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J})\subseteq\ker(\pi_\omega)$ and since ω was arbitrary we have $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J})\subseteq\mathfrak{I}^{\mathfrak{S}_p}(\boldsymbol{M},\boldsymbol{J})$, for any non-empty admissible state space $\mathfrak{S}_p$.

Lemma 7.6. *Let $(\boldsymbol{M},\boldsymbol{J})$ be any object in* LocSrc_p.

a) The unital $*$-algebra $\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})/\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J})$ is (noncanonically) isomorphic to $\mathfrak{CQA}_p^{\mathrm{lin}}(\boldsymbol{M}) := \mathfrak{CCR}(\mathfrak{PS}_p^{\mathrm{lin}}(\boldsymbol{M}))$.

b) $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J}) = \mathfrak{I}^{\mathfrak{S}_p}(\boldsymbol{M},\boldsymbol{J})$ whenever $\mathfrak{S}_p$ is a non-empty admissible state space.

Proof. Proof of a): We define a unital $*$-algebra homomorphism $\kappa : \mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J}) \to \mathfrak{CQA}_p^{\mathrm{lin}}(\boldsymbol{M})$ by setting, for all $[(\varphi,\alpha)] \in \mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})$,

$$\kappa\big([(\varphi,\alpha)]\big) = \omega_0\big([(\varphi,\alpha)]\big) + [\varphi]^{\mathrm{lin}} , \tag{7.10}$$

where ω_0 is any choice of admissible state. As $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J})$ clearly lies in the kernel of κ, we can induce a unital $*$-algebra homomorphism $\kappa : \mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})/\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J}) \to \mathfrak{CQA}_p^{\mathrm{lin}}(\boldsymbol{M})$. To show that the induced κ is a $^*\mathsf{Alg}$-isomorphism we notice that setting, for all $[\varphi]^{\mathrm{lin}} \in \mathfrak{CQA}_p^{\mathrm{lin}}(\boldsymbol{M})$,

$$\kappa^{-1}\big([\varphi]^{\mathrm{lin}}\big) := \Big[[(\varphi,0)] - \omega_0\big([\varphi,0]\big)\Big] \in \mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})/\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J}) \tag{7.11}$$

is well-defined and defines the inverse of κ.

Proof of b): By a), $\mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})/\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J})$ is a simple nontrivial unital $*$-algebra. Hence $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J})$ is a maximal proper ideal. But $\widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J}) \subseteq \mathfrak{I}^{\mathfrak{S}_p}(\boldsymbol{M},\boldsymbol{J})$ and $\mathfrak{I}^{\mathfrak{S}_p}(\boldsymbol{M},\boldsymbol{J})$ is proper so the ideals are equal. □

Remark 7.7. As a consequence of this lemma, the two-sided $*$-ideals $\mathfrak{I}^{\mathfrak{S}_p}(\boldsymbol{M},\boldsymbol{J})$ do not depend on which (non-empty) admissible state space $\mathfrak{S}_p$ for $\mathfrak{CQA}_p$ we use in the construction. We therefore introduce a simpler notation and set for any object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p

$$\mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J}) := \mathfrak{I}^{\mathfrak{S}_p}(\boldsymbol{M},\boldsymbol{J}) = \widetilde{\mathfrak{I}}_p(\boldsymbol{M},\boldsymbol{J}) . \tag{7.12}$$

These studies now allow us to construct our improved functor for the quantum theory of a multiplet of $p \in \mathbb{N}$ inhomogeneous Klein–Gordon fields.

Proposition 7.8. *The following rules define a covariant functor $\mathfrak{QA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$: To any object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p we associate $\mathfrak{QA}_p(\boldsymbol{M},\boldsymbol{J}) := \mathfrak{CQA}_p(\boldsymbol{M},\boldsymbol{J})/\mathfrak{I}_p(\boldsymbol{M},\boldsymbol{J})$. To any morphism $f : (\boldsymbol{M}_1,\boldsymbol{J}_1) \to (\boldsymbol{M}_2,\boldsymbol{J}_2)$ in LocSrc_p we associate the map $\mathfrak{QA}_p(f) : \mathfrak{QA}_p(\boldsymbol{M}_1,\boldsymbol{J}_1) \to \mathfrak{QA}_p(\boldsymbol{M}_2,\boldsymbol{J}_2)$ that is canonically induced from $\mathfrak{CQA}_p(f) : \mathfrak{CQA}_p(\boldsymbol{M}_1,\boldsymbol{J}_1) \to \mathfrak{CQA}_p(\boldsymbol{M}_2,\boldsymbol{J}_2)$.*

Proof. Lemma 7.6 has established that the quotients are nontrivial unital $*$-algebras. Next, let $f : (\boldsymbol{M}_1,\boldsymbol{J}_1) \to (\boldsymbol{M}_2,\boldsymbol{J}_2)$ be any morphism in LocSrc_p. Then $\mathfrak{CQA}_p(f)$ induces a unital $*$-homomorphism $\mathfrak{QA}_p(f) : \mathfrak{QA}_p(\boldsymbol{M}_1,\boldsymbol{J}_1) \to \mathfrak{QA}_p(\boldsymbol{M}_2,\boldsymbol{J}_2)$ because it restricts to a map $\mathfrak{CQA}_p(f) : \mathfrak{I}_p(\boldsymbol{M}_1,\boldsymbol{J}_1) \to \mathfrak{I}_p(\boldsymbol{M}_2,\boldsymbol{J}_2)$. This is clear from the fact that $\mathfrak{CQA}_p(f)\big([(0,\alpha)] - \alpha\big) = [(0,\alpha)] - \alpha$, for any $\alpha \in \mathbb{R}$. The induced unital $*$-algebra homomorphism $\mathfrak{QA}_p(f) : \mathfrak{QA}_p(\boldsymbol{M}_1,\boldsymbol{J}_1) \to \mathfrak{QA}_p(\boldsymbol{M}_2,\boldsymbol{J}_2)$ is injective (i.e. a morphism in $^*\mathsf{Alg}$), since $\mathfrak{QA}_p(\boldsymbol{M}_1,\boldsymbol{J}_1)$ is simple, cf. Lemma 7.6. The composition and identity properties of the association $\mathfrak{QA}_p$ are consequences of the same properties of $\mathfrak{CQA}_p$, hence $\mathfrak{QA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ is a covariant functor. □

The following statement may be proved in complete analogy with Proposition 6.11:

Proposition 7.9. *The covariant functor $\mathfrak{QA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ satisfies the causality property and the time-slice axiom and is therefore a locally covariant quantum field theory.*

7.3 Relative Cauchy evolution of the functor $\mathfrak{Q}\mathfrak{A}_p$

The relative Cauchy evolution of the functor $\mathfrak{P}\mathfrak{S}_p : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ induces that of the functor $\mathfrak{Q}\mathfrak{A}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ as follows: Let $(\boldsymbol{M},\boldsymbol{J})$ be any object in LocSrc_p and let $(\boldsymbol{h},\boldsymbol{j}) \in H(\boldsymbol{M},\boldsymbol{J})$ be any globally hyperbolic perturbation. From the explicit expression for $\mathrm{rce}^{(\mathfrak{P}\mathfrak{S}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}] \in \mathrm{Aut}(\mathfrak{P}\mathfrak{S}_p(\boldsymbol{M},\boldsymbol{J}))$ given in (3.9) we observe that the relative Cauchy evolution $\mathrm{rce}^{(\mathfrak{Q}\mathfrak{A}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}] \in \mathrm{Aut}(\mathfrak{Q}\mathfrak{A}_p(\boldsymbol{M},\boldsymbol{J}))$ of $\mathfrak{Q}\mathfrak{A}_p$ is uniquely specified by, for all $[(\varphi,\alpha)] \in \mathfrak{Q}\mathfrak{A}_p(\boldsymbol{M},\boldsymbol{J})$,

$$\begin{aligned}\mathrm{rce}^{(\mathfrak{Q}\mathfrak{A}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}]\big([(\varphi,\alpha)]\big) &= \big[\big(\varphi + (\mathrm{KG}_{\boldsymbol{M}} - \mathrm{KG}_{\boldsymbol{M}[\boldsymbol{h}]})\big(\mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\big), 0\big)\big] + \alpha \\ &\quad + \int_M \big(\big\langle -\boldsymbol{j}, \mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\big\rangle + \big\langle (1-\rho_{\boldsymbol{h}})\,(\boldsymbol{J}+\boldsymbol{j}), \mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\big\rangle\big)\ \mathrm{vol}_{\boldsymbol{M}}\ , \end{aligned} \tag{7.13}$$

where on the right hand side we have used the equivalence relation entering the definition of $\mathfrak{Q}\mathfrak{A}_p(\boldsymbol{M},\boldsymbol{J})$ (cf. Proposition 7.8) and we have chosen as in (3.9) a representative φ with compact support in M^+. In sufficiently regular representations of the algebra $\mathfrak{Q}\mathfrak{A}_p(\boldsymbol{M},\boldsymbol{J})$ one can differentiate this expression, yielding

$$\begin{aligned}\frac{d}{ds}\mathrm{rce}^{(\mathfrak{Q}\mathfrak{A}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[s\boldsymbol{h},s\boldsymbol{j}]\big([(\varphi,\alpha)]\big)\Big|_{s=0} &= -\Big[\Big(\mathrm{KG}'_{\boldsymbol{M}[\boldsymbol{h}]}\big(\mathrm{E}_{\boldsymbol{M}}(\varphi)\big),0\Big)\Big] \\ &\quad - \int_M \Big\langle \frac{1}{2} g^{ab}\, h_{ab}\, \boldsymbol{J} + \boldsymbol{j}, \mathrm{E}_{\boldsymbol{M}}(\varphi)\Big\rangle\ \mathrm{vol}_{\boldsymbol{M}} \\ &= i\,\Big[\frac{1}{2}T_{(\boldsymbol{M},\boldsymbol{J})}(\boldsymbol{h}) + [(\boldsymbol{j},0)] \stackrel{*}{,} [(\varphi,\alpha)]\Big]\ , \end{aligned} \tag{7.14}$$

where $T^{ab}_{(\boldsymbol{M},\boldsymbol{J})}$ is the quantization of the stress-energy tensor (3.13), with regularization by point-splitting (as emphasized in [BFV03], the precise nature of the c-number subtraction is irrelevant owing to the commutator). Although (7.13) was derived under an assumption on the support of the representative φ, the formulae in (7.14) are valid for any representative (φ,α) of its equivalence class. Of course, (7.14) is the Dirac quantization of (6.18). Finally, we note the special case of (7.13) for vanishing metric perturbation $\boldsymbol{h} = 0$, namely

$$\mathrm{rce}^{(\mathfrak{Q}\mathfrak{A}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[0,\boldsymbol{j}]\big([(\varphi,\alpha)]\big) = [(\varphi,\alpha)] - \int_M \langle \boldsymbol{j}, \mathrm{E}_{\boldsymbol{M}}(\varphi)\rangle = [(\varphi,\alpha)] + i\,\Big[[(\boldsymbol{j},0)] \stackrel{*}{,} [(\varphi,\alpha)]\Big]\ . \tag{7.15}$$

7.4 Automorphism group of the functor $\mathfrak{Q}\mathfrak{A}_p$

We study the automorphism group of the covariant functor $\mathfrak{Q}\mathfrak{A}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ defined in Proposition 7.8. For this we first notice that in the massless case $m = 0$ the automorphism group contains a $\mathbb{R}^p$ subgroup.

Proposition 7.10. *If $m = 0$ there exists a faithful homomorphism $\eta : \mathbb{R}^p \to \mathrm{Aut}(\mathfrak{Q}\mathfrak{A}_p)$ induced by the one in Proposition 4.2 restricted to $\{+1\} \times \mathbb{R}^p \subseteq \mathbb{Z}_2 \times \mathbb{R}^p$. Explicitly, for any object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p the automorphism $\eta(\mu)_{(\boldsymbol{M},\boldsymbol{J})}$ is specified by, for all $[(\varphi,\alpha)] \in \mathfrak{Q}\mathfrak{A}_p(\boldsymbol{M},\boldsymbol{J})$,*

$$\eta(\mu)_{(\boldsymbol{M},\boldsymbol{J})}\big([(\varphi,\alpha)]\big) = \Big[\Big(\varphi, \alpha + \int_M \langle \varphi, \mu\rangle\ \mathrm{vol}_{\boldsymbol{M}}\Big)\Big]\ . \tag{7.16}$$

Proof. Applying the functor $\mathfrak{CCR}$, the automorphism $\eta(\sigma,\mu) \in \mathrm{Aut}(\mathfrak{P}\mathfrak{S}_p)$ of Proposition 4.2 induces an element in $\mathrm{Aut}(\mathfrak{C}\mathfrak{Q}\mathfrak{A}_p)$ (denoted with a slight abuse of notation by the same symbol). For $\sigma = -1$ this automorphism does not preserve the two-sided $*$-ideals $\mathfrak{J}_p(\boldsymbol{M},\boldsymbol{J})$, since $\eta(-1,\mu)_{(\boldsymbol{M},\boldsymbol{J})}\big([(0,\alpha)] - \alpha\big) = [(0,-\alpha)] - \alpha \notin \mathfrak{J}_p(\boldsymbol{M},\boldsymbol{J})$. For $\sigma = +1$ and $\mu \in \mathbb{R}^p$ arbitrary the two-sided $*$-ideals are preserved, hence $\eta(+1,\mu)$ induces the automorphism $\eta(\mu) \in \mathrm{Aut}(\mathfrak{Q}\mathfrak{A}_p)$ which is claimed in this proposition. The group law is an obvious consequence of the group law of the automorphisms $\eta(\sigma,\mu)$ of Proposition 4.2. □

Remark 7.11. In the same way, one may also show for $m \neq 0$ that the nontrivial $\mathbb{Z}_2$-automorphism of $\mathfrak{P}\mathfrak{S}_p$ does not lift to an automorphism of $\mathfrak{Q}\mathfrak{A}_p$.

We may now prove that the automorphisms found in Proposition 7.10 exhaust $\mathrm{Aut}(\mathfrak{QA}_p)$. We require the analog of Theorem 4.5 for the functor $\mathfrak{QA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$, which can be obtained by a similar proof as in Theorem 4.5 and hence can be omitted.

Theorem 7.12. *Every endomorphism of the functor $\mathfrak{QA}_p$ is an automorphism and*

$$\mathrm{End}(\mathfrak{QA}_p) = \mathrm{Aut}(\mathfrak{QA}_p) \simeq \begin{cases} \{\mathrm{id}_{\mathfrak{QA}_p}\} & , \text{ for } m \neq 0 \,, \\ \mathbb{R}^p & , \text{ for } m = 0 \,, \end{cases} \tag{7.17}$$

where the action for $m = 0$ is given by Proposition 7.10.

Proof. The steps in this proof are similar to the ones in Theorem 6.14. Let $\eta \in \mathrm{End}(\mathfrak{QA}_p)$ be any endomorphism and let us consider its component $\eta_{(\boldsymbol{M}_0,0)}$, where $\boldsymbol{M}_0$ is Minkowski spacetime. For this particular object, the ${}^*\mathsf{Alg}$-isomorphism $\kappa : \mathfrak{QA}_p(\boldsymbol{M}_0, 0) \to \mathfrak{CQA}_p^{\mathrm{lin}}(\boldsymbol{M}_0)$ defined by $\kappa\big([(\varphi,\alpha)]\big) = \alpha + [\varphi]^{\mathrm{lin}}$ intertwines the natural action of the Poincaré transformations on $\mathfrak{QA}_p(\boldsymbol{M}_0, 0)$ and $\mathfrak{CQA}_p^{\mathrm{lin}}(\boldsymbol{M}_0)$. Consequently, the endomorphism $\widetilde{\eta} := \kappa \circ \eta_{(\boldsymbol{M}_0,0)} \circ \kappa^{-1}$ of $\mathfrak{CQA}_p^{\mathrm{lin}}(\boldsymbol{M}_0)$ has to commute with all Poincaré transformations. Furthermore, because $\eta_{(\boldsymbol{M}_0,0)}$ commutes with (derivatives of) the relative Cauchy evolution on $\mathfrak{QA}_p(\boldsymbol{M}_0, 0)$ – in particular those with $\boldsymbol{h} = 0$ – we obtain the condition, for all $\boldsymbol{j} \in C_0^\infty(M_0, \mathbb{R}^p)$ and $[\varphi]^{\mathrm{lin}} \in \mathfrak{CQA}_p^{\mathrm{lin}}(\boldsymbol{M}_0)$,

$$\widetilde{\eta}\Big(\Big[[\boldsymbol{j}]^{\mathrm{lin}} \stackrel{\star}{,} [\varphi]^{\mathrm{lin}}\Big]\Big) = \Big[[\boldsymbol{j}]^{\mathrm{lin}} \stackrel{\star}{,} \widetilde{\eta}\big([\varphi]^{\mathrm{lin}}\big)\Big] \tag{7.18}$$

on $\widetilde{\eta}$, where we have used (7.14) with $\boldsymbol{h} = 0$. The left hand side of (7.18), which is analogous to (6.22) in the proof of Theorem 6.14, is simply $i\,\sigma_{\boldsymbol{M}_0}^{\mathrm{lin}}\big([\boldsymbol{j}]^{\mathrm{lin}}, [\varphi]^{\mathrm{lin}}\big)$. Using the explicit expression (7.1) for the $\star$-product in $\mathfrak{CQA}_p^{\mathrm{lin}}(\boldsymbol{M}_0)$, we find that the right hand side of this equality is equal to the Poisson bracket $i\,\big\{[\boldsymbol{j}]^{\mathrm{lin}}, \widetilde{\eta}\big([\varphi]^{\mathrm{lin}}\big)\big\}_{\sigma_{\boldsymbol{M}_0}^{\mathrm{lin}}}$. The remainder of the proof runs in complete analogy with that of Theorem 6.14. □

7.5 Composition property of the functor $\mathfrak{QA}_p$

It remains to prove that the covariant functor $\mathfrak{QA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ satisfies the composition property. We define for $p \geq 2$ and $0 < q < p$ the covariant functor

$$\mathfrak{QA}_{p,q} := \otimes \circ \big(\mathfrak{QA}_q \times \mathfrak{QA}_{p-q}\big) \circ \mathfrak{Split}_{p,q} : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg} \tag{7.19}$$

and we obtain the following, in complete analogy with the proof of Theorem 6.15

Theorem 7.13. *For any $p \geq 2$ and $0 < q < p$, the covariant functors $\mathfrak{QA}_{p,q} : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ and $\mathfrak{QA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ are naturally isomorphic. The natural isomorphism $\eta = \{\eta_{(\boldsymbol{M},\boldsymbol{J})}\} : \mathfrak{QA}_{p,q} \Rightarrow \mathfrak{QA}_p$ is specified by, for all $[(\varphi,\alpha)] \in \mathfrak{QA}_q(\boldsymbol{M}, \boldsymbol{J}^q)$ and $[(\psi,\beta)] \in \mathfrak{QA}_{p-q}(\boldsymbol{M}, \boldsymbol{J}^{p-q})$,*

$$\eta_{(\boldsymbol{M},\boldsymbol{J})}\big([(\varphi,\alpha)] \otimes 1\big) = \big[(\varphi,\alpha)\big] \ , \ \ \eta_{(\boldsymbol{M},\boldsymbol{J})}\big(1 \otimes [(\psi,\beta)]\big) = \big[(\psi,\beta)\big] \ , \tag{7.20}$$

where on the right hand sides we have identified $\varphi \in C_0^\infty(M, \mathbb{R}^q)$ and $\psi \in C_0^\infty(M, \mathbb{R}^{p-q})$ as elements in $C_0^\infty(M, \mathbb{R}^p)$ (φ is placed in the first q and ψ in the last $p-q$ components of $\mathbb{R}^p$).

7.6 Dynamical locality

To conclude, we shall study whether or not our improved functor $\mathfrak{QA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ satisfies the dynamical locality property, which was introduced in [FV12a] as part of an investigation into question of what it means for a theory to describe the same physics in all spacetimes (SPASs). The dynamical locality property has been proven previously for the homogeneous Klein–Gordon theory with non-vanishing mass $m \neq 0$ in [FV12b] and for extended algebras of Wick polynomials in [Fer13].

We start by formulating the content of the dynamical locality property, essentially following [FV12a, FV12b], suitably adapted to theories on the category LocSrc_p. Let $(\boldsymbol{M}, \boldsymbol{J})$ be any object in LocSrc_p. As above, we shall denote by $\mathscr{O}(\boldsymbol{M})$ the set of all causally compatible, open and globally hyperbolic subsets of

$\boldsymbol{M}$ with finitely many connected components all of which are mutually causally disjoint. To each non-empty $O \in \mathscr{O}(\boldsymbol{M})$, there is an object $(\boldsymbol{M}, \boldsymbol{J})|_O$ in LocSrc_p obtained by restricting all the geometric data (including the source $\boldsymbol{J}$) to the subset O of $\boldsymbol{M}$. Moreover, there is a canonical inclusion $\iota_{(\boldsymbol{M},\boldsymbol{J});O} : (\boldsymbol{M}, \boldsymbol{J})|_O \to (\boldsymbol{M}, \boldsymbol{J})$ which is a morphism in LocSrc_p. Adapting an idea from [BFV03], we may construct from $\mathfrak{QA}_p$ a net of unital $*$-algebras as follows: Given any non-empty $O \in \mathscr{O}(\boldsymbol{M})$, we denote by $\mathfrak{QA}_p^{\mathrm{kin}}((\boldsymbol{M}, \boldsymbol{J}); O)$ the image of $\mathfrak{QA}_p((\boldsymbol{M}, \boldsymbol{J})|_O)$ in $\mathfrak{QA}_p(\boldsymbol{M}, \boldsymbol{J})$ under the ${}^*\mathsf{Alg}$-morphism $\mathfrak{QA}_p(\iota_{(\boldsymbol{M},\boldsymbol{J});O})$. The assignment

$$O \mapsto \mathfrak{QA}_p^{\mathrm{kin}}((\boldsymbol{M}, \boldsymbol{J}); O) \subseteq \mathfrak{QA}_p(\boldsymbol{M}, \boldsymbol{J}) \tag{7.21}$$

is called the *kinematic net*, and is one way of describing the local physics of the theory $\mathfrak{QA}_p$ in a region O in the spacetime $\boldsymbol{M}$ underlying the object $(\boldsymbol{M}, \boldsymbol{J})$. It is easily seen that $\mathfrak{QA}_p^{\mathrm{kin}}((\boldsymbol{M}, \boldsymbol{J}); O)$ is generated by the unit together with all $[(\varphi, 0)] \in \mathfrak{QA}_p(\boldsymbol{M}, \boldsymbol{J})$ such that $\mathrm{supp}(\varphi) \subseteq O$.

Another description of the local physics of the theory $\mathfrak{QA}_p$ in a region O in the spacetime $\boldsymbol{M}$ underlying the object $(\boldsymbol{M}, \boldsymbol{J})$ can be obtained by using the relative Cauchy evolution and was introduced in [FV12a]. For $K \subseteq M$ compact, let us denote by $H((\boldsymbol{M}, \boldsymbol{J}); K^\perp)$ the set of all globally hyperbolic perturbations $(\boldsymbol{h}, \boldsymbol{j})$ of $(\boldsymbol{M}, \boldsymbol{J})$, such that $\mathrm{supp}(\boldsymbol{h}) \cup \mathrm{supp}(\boldsymbol{j}) \subseteq K^\perp$, with $K^\perp := M \setminus J_{\boldsymbol{M}}(K)$ the causal complement of K. We define $\mathfrak{QA}_p^\bullet((\boldsymbol{M}, \boldsymbol{J}); K)$ to be the subalgebra of $\mathfrak{QA}_p(\boldsymbol{M}, \boldsymbol{J})$ consisting of fixed points under arbitrary relative Cauchy evolutions $\mathrm{rce}^{(\mathfrak{QA}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h}, \boldsymbol{j}]$ with $(\boldsymbol{h}, \boldsymbol{j}) \in H((\boldsymbol{M}, \boldsymbol{J}); K^\perp)$. The idea behind this definition is that the elements in $\mathfrak{QA}_p^\bullet((\boldsymbol{M}, \boldsymbol{J}); K)$ can be regarded as localized in K because they are insensitive to perturbations $(\boldsymbol{h}, \boldsymbol{j})$ of the background localized in the causal complement $K^\perp$. By taking the subalgebra of $\mathfrak{QA}_p(\boldsymbol{M}, \boldsymbol{J})$ that is generated by the $\mathfrak{QA}_p^\bullet((\boldsymbol{M}, \boldsymbol{J}); K)$ as K ranges over suitable compact subsets of $O \in \mathscr{O}(\boldsymbol{M})$ we obtain the *dynamical algebras* $\mathfrak{QA}_p^{\mathrm{dyn}}((\boldsymbol{M}, \boldsymbol{J}); O)$, which can be compared with the kinematic ones $\mathfrak{QA}_p^{\mathrm{kin}}((\boldsymbol{M}, \boldsymbol{J}); O)$. More precisely, let us denote by $\mathscr{K}_b(\boldsymbol{M}; O)$ the set of finite unions of causally disjoint subsets of $O \in \mathscr{O}(\boldsymbol{M})$, each of which is the closure of a Cauchy ball B with a relatively compact Cauchy development $D_{\boldsymbol{M}}(B)$. Here, a Cauchy ball B is a subset of a Cauchy surface, for which there is a chart containing the closure of B, and in which B is a non-empty open ball. With these definitions in place, we set for any non-empty $O \in \mathscr{O}(\boldsymbol{M})$

$$\mathfrak{QA}_p^{\mathrm{dyn}}((\boldsymbol{M}, \boldsymbol{J}); O) := \bigvee_{K \in \mathscr{K}_b(\boldsymbol{M}; O)} \mathfrak{QA}_p^\bullet((\boldsymbol{M}, \boldsymbol{J}); K) \subseteq \mathfrak{QA}_p(\boldsymbol{M}, \boldsymbol{J}) \;. \tag{7.22}$$

Definition 7.14. The functor $\mathfrak{QA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ satisfies the *dynamical locality property* if, for all objects $(\boldsymbol{M}, \boldsymbol{J})$ in LocSrc_p and all non-empty $O \in \mathscr{O}(\boldsymbol{M})$, we have

$$\mathfrak{QA}_p^{\mathrm{kin}}((\boldsymbol{M}, \boldsymbol{J}); O) = \mathfrak{QA}_p^{\mathrm{dyn}}((\boldsymbol{M}, \boldsymbol{J}); O) \;. \tag{7.23}$$

Remark 7.15. In its original formulation [FV12a], dynamical locality was defined using relative Cauchy evolution induced by metric perturbations, because only theories defined on Loc were considered. In generalizing to theories on LocSrc_p, one has a choice as to whether to consider perturbations in both the metric and the external source, or just the metric, or potentially something intermediate. We have adopted the first of these possibilities as being the most natural – it would indeed appear strange to regard as local an observable that was sensitive to perturbations in the external source located in the causal complement of the localization region. However, our consideration of massless inhomogeneous theories will suggest a more nuanced view, which will be discussed below.

Using the relative Cauchy evolution of the functor $\mathfrak{QA}_p$ derived in Subsection 7.3, we can characterize the fixed point subalgebras $\mathfrak{QA}_p^\bullet((\boldsymbol{M}, \boldsymbol{J}); K)$ of $\mathrm{rce}^{(\mathfrak{QA}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h}, \boldsymbol{j}]$ with $(\boldsymbol{h}, \boldsymbol{j}) \in H((\boldsymbol{M}, \boldsymbol{J}); K^\perp)$.

Lemma 7.16. *Let $(\boldsymbol{M}, \boldsymbol{J})$ be any object in LocSrc_p and let K be any compact subset of $\boldsymbol{M}$. Then $\mathfrak{QA}_p^\bullet((\boldsymbol{M}, \boldsymbol{J}); K)$ is the subalgebra of $\mathfrak{QA}_p(\boldsymbol{M}, \boldsymbol{J})$ generated by the unit together with all $[(\varphi, 0)] \in \mathfrak{QA}_p(\boldsymbol{M}, \boldsymbol{J})$ such that $\mathrm{supp}\big(\mathrm{E}_{\boldsymbol{M}}(\varphi)\big) \subseteq J_{\boldsymbol{M}}(K)$.*

Proof. The stated subalgebra of $\mathfrak{QA}_p(\boldsymbol{M}, \boldsymbol{J})$ is clearly a subalgebra of $\mathfrak{QA}_p^\bullet((\boldsymbol{M}, \boldsymbol{J}); K)$ for the following reason: If $[(\varphi, 0)]$ obeys $\mathrm{supp}\big(\mathrm{E}_{\boldsymbol{M}}(\varphi)\big) \subseteq J_{\boldsymbol{M}}(K)$ then, for any $(\boldsymbol{h}, \boldsymbol{j}) \in H((\boldsymbol{M}, \boldsymbol{J}); K^\perp)$, we have

$\mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi) = \mathrm{E}_{\boldsymbol{M}}(\varphi)$ and hence $\mathrm{rce}^{(\mathfrak{Q}\mathfrak{A}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}]\big([(\varphi,0)]\big) = [(\varphi,0)]$ by (7.13). Thus $\mathfrak{Q}\mathfrak{A}^{\bullet}_p((\boldsymbol{M},\boldsymbol{J});K)$ contains the subalgebra generated by (finite sums of finite products of) such elements and the unit.

To show the reverse inclusion, let us take any element $a \in \mathfrak{Q}\mathfrak{A}^{\bullet}_p((\boldsymbol{M},\boldsymbol{J});K)$. In particular, using (7.15), we find the condition that, for all $\boldsymbol{j} \in C^\infty_0(K^\perp,\mathbb{R}^p)$, $\big[[(\boldsymbol{j},0)] \stackrel{\star}{,} a\big] = 0$. Evaluating the $\star$-product (7.1) in the commutator, this condition reduces to the vanishing Poisson bracket condition $\big\{[(\boldsymbol{j},0)],a\big\}_{\sigma_{(\boldsymbol{M},\boldsymbol{J})}} = 0$, for all $\boldsymbol{j} \in C^\infty_0(K^\perp,\mathbb{R}^p)$. We can now express a as a finite sum of finite *symmetric* products of the unit and the elements $[(\varphi,0)]$ with $\varphi \in C^\infty_0(M,\mathbb{R}^p)$. Notice that if a is one of the generators $[(\varphi,0)]$, with $\varphi \in C^\infty_0(M,\mathbb{R}^p)$, then the vanishing Poisson bracket condition implies that $\mathrm{supp}\big(\mathrm{E}_{\boldsymbol{M}}(\varphi)\big) \subseteq J_{\boldsymbol{M}}(K)$. For generic $a \in \mathfrak{Q}\mathfrak{A}^{\bullet}_p((\boldsymbol{M},\boldsymbol{J});K)$ we follow the strategy in [FV12b, Lemma 5.2. and Appendix A] and associate to a its *support subspace* Y_a, which is a finite dimensional vector subspace of the complex vector space spanned by the $[(\varphi,0)]$, $\varphi \in C^\infty_0(M,\mathbb{R}^p)$, such that the element a lies in the subalgebra generated by Y_a together with the unit. If $a \in \mathfrak{Q}\mathfrak{A}^{\bullet}_p((\boldsymbol{M},\boldsymbol{J});K)$ then Y_a is invariant under the relative Cauchy evolution corresponding to perturbations supported in $K^\perp$; considering relative Cauchy evolutions of the form (7.15), we see that all $[(\varphi,0)]$ in the support subspace must satisfy $\mathrm{supp}\big(\mathrm{E}_{\boldsymbol{M}}(\varphi)\big) \subseteq J_{\boldsymbol{M}}(K)$. Hence, a is generated only by (finite sums of finite *symmetric* products of) the unit and those generators $[(\varphi,0)]$ satisfying $\mathrm{supp}\big(\mathrm{E}_{\boldsymbol{M}}(\varphi)\big) \subseteq J_{\boldsymbol{M}}(K)$. As one can invert the formula (7.1) for the $\star$-product (leading to an expression for the symmetric product in terms of $\star$-products) this implies that a is also generated by finite sums of finite $\star$-products of the unit and the elements $[(\varphi,0)]$ with $\mathrm{supp}\big(\mathrm{E}_{\boldsymbol{M}}(\varphi)\big) \subseteq J_{\boldsymbol{M}}(K)$. $\square$

With this preparation we can prove the main statement of this subsection.

Theorem 7.17. *The functor $\mathfrak{Q}\mathfrak{A}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ satisfies the dynamical locality property.*

Proof. We must show that (7.23) holds for all objects $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p and all non-empty $O \in \mathscr{O}(\boldsymbol{M})$. Notice that the unit is contained in both $\mathfrak{Q}\mathfrak{A}^{\mathrm{kin}}_p((\boldsymbol{M},\boldsymbol{J});O)$ and $\mathfrak{Q}\mathfrak{A}^{\mathrm{dyn}}_p((\boldsymbol{M},\boldsymbol{J});O)$. To show the inclusion "$\subseteq$", note that any $a \in \mathfrak{Q}\mathfrak{A}^{\mathrm{kin}}_p((\boldsymbol{M},\boldsymbol{J});O)$ is generated by finite sums of finite products of the unit and the elements $[(\varphi,0)]$ with $\mathrm{supp}(\varphi) \subseteq O$, all of which may be shown to lie in $\mathfrak{Q}\mathfrak{A}^{\mathrm{dyn}}_p((\boldsymbol{M},\boldsymbol{J});O)$ by an argument similar to [FV12b, Lemma 3.3.]: We can decompose $\varphi \in C^\infty_0(O,\mathbb{R}^p)$ into a finite sum $\varphi = \sum_{i=1}^n \varphi_i$, such that $\mathrm{supp}(\varphi_i) \in \mathscr{K}_b(\boldsymbol{M};O)$. (Take for example an open cover of $\mathrm{supp}(\varphi)$ by diamonds, pass to a finite subcover and then use a partition of unity.) For each φ_i we have $\mathrm{supp}\big(\mathrm{E}_{\boldsymbol{M}}(\varphi_i)\big) \subseteq J_{\boldsymbol{M}}(\mathrm{supp}(\varphi_i))$, which shows that $[(\varphi,0)] = \sum_{i=1}^n [(\varphi_i,0)] \in \mathfrak{Q}\mathfrak{A}^{\mathrm{dyn}}_p((\boldsymbol{M},\boldsymbol{J});O)$.

To show the other inclusion "$\supseteq$", it is by Lemma 7.16 sufficient to prove that, for any $K \in \mathscr{K}_b(\boldsymbol{M};O)$, all elements $[(\varphi,0)] \in \mathfrak{Q}\mathfrak{A}_p(\boldsymbol{M},\boldsymbol{J})$ with $\mathrm{supp}\big(\mathrm{E}_{\boldsymbol{M}}(\varphi)\big) \subseteq J_{\boldsymbol{M}}(K)$ are contained in $\mathfrak{Q}\mathfrak{A}^{\mathrm{kin}}_p((\boldsymbol{M},\boldsymbol{J});O)$. This is a simple consequence of the following argument: Since $\mathrm{E}_{\boldsymbol{M}}(\varphi)$ has support in $J_{\boldsymbol{M}}(K)$ and $K \subseteq O$ is a compact subset, there exists a $\varphi' \in C^\infty_0(O,\mathbb{R}^p)$, such that $\mathrm{E}_{\boldsymbol{M}}(\varphi') = \mathrm{E}_{\boldsymbol{M}}(\varphi)$, see e.g. [FV12b, Lemma 3.1. (i)]. As the Klein–Gordon operator is normally hyperbolic, we have $\varphi' = \varphi + \mathrm{KG}_{\boldsymbol{M}}(h)$, for some $h \in C^\infty_0(M,\mathbb{R}^p)$. Thus, $[(\varphi,0)] = [(\varphi + \mathrm{KG}_{\boldsymbol{M}}(h), \int_M \langle \boldsymbol{J},h\rangle\ \mathrm{vol}_{\boldsymbol{M}})] = [(\varphi',0)] + \int_M \langle \boldsymbol{J},h\rangle\ \mathrm{vol}_{\boldsymbol{M}}$ lies in $\mathfrak{Q}\mathfrak{A}^{\mathrm{kin}}_p((\boldsymbol{M},\boldsymbol{J});O)$. $\square$

Remark 7.18. The proofs of Lemma 7.16 and Theorem 7.17 do not distinguish between the massless and the massive case. In contrast, this distinction was essential for the *homogeneous* Klein–Gordon theory studied in [FV12b]; indeed only the massive homogeneous Klein–Gordon field satisfies the dynamical locality property. At first sight this looks like a discrepancy, because the homogeneous Klein–Gordon theory seems to be contained as a special case of our inhomogeneous model by setting all source terms to zero. However, the inhomogeneous theory is formulated as a functor $\mathfrak{Q}\mathfrak{A}_p$ from the category LocSrc_p to ${}^*\mathsf{Alg}$ and the relative Cauchy evolution $\mathrm{rce}^{(\mathfrak{Q}\mathfrak{A}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}]$ depends on both a metric perturbation $\boldsymbol{h}$ and a source term perturbation $\boldsymbol{j}$. Even restricting to (the full subcategory of) objects with zero source term $\boldsymbol{J} = 0$, we still can study the response (via the relative Cauchy evolution) of the restricted theory to source term perturbations $\boldsymbol{j}$, as well as the response to metric perturbations $\boldsymbol{h}$, thus obtaining stronger restrictions on the fixed point subalgebras $\mathfrak{Q}\mathfrak{A}^{\bullet}_p((\boldsymbol{M},\boldsymbol{J});K)$ than those arising from metric perturbations $\boldsymbol{h}$ alone as in [FV12b]. To conclude, we point out that if we *forbade* nontrivial source term perturbations $\boldsymbol{j}$ in our proofs above, i.e. making only use of the relative Cauchy evolutions $\mathrm{rce}^{(\mathfrak{Q}\mathfrak{A}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},0]$ depending on $\boldsymbol{h}$, we would obtain as in [FV12b] (and by similar arguments) that the massive theory satisfies (this restricted form of) the dynamical locality property and that the massless theory does not. We finally remark that if we were to restrict to coexact source perturbations $\boldsymbol{j} = \delta\alpha$ (for compactly supported one-forms α) and traceless metric perturbations $\boldsymbol{h}$ we would also lose dynamical locality in the massless case. This will be discussed further below.

8 Gauge theory interpretation in the massless case

In this section we shall briefly point out and discuss some features of the classical and quantum theory of a massless multiplet of $p \in \mathbb{N}$ inhomogeneous Klein–Gordon fields.

Let us start with the automorphisms of this theory. As was shown in Theorem 6.14 for the classical and in Theorem 7.12 for the quantized case, this theory has a nontrivial automorphism group isomorphic to $\mathbb{R}^p$. These symmetries can also be understood from the Lagrangian of this model (see equation (1.1) with $\lambda = 1$ and $m = 0$) as they correspond to shifts of the classical field ϕ, i.e. transformations $\phi \mapsto \phi + \mu$ with $\mu \in \mathbb{R}^p$. According to [Few13] one should regard the massless Klein–Gordon theory as a gauge theory of the first kind with ϕ playing the role of a zero-form gauge field. This is supported by the fact that the Lagrangian can also be written as

$$\mathcal{L} = \frac{1}{2} \langle \mathrm{d}\phi, *\mathrm{d}\phi \rangle - \langle \phi, *\boldsymbol{J} \rangle \ , \tag{8.1}$$

where $*$ denotes the Hodge operator corresponding to $\boldsymbol{M}$. The differentials $\mathrm{d}\phi$ play the same role as the field strength $F = \mathrm{d}A$ in electromagnetism, just one differential form degree lower. Under gauge transformations $\phi \mapsto \phi' = \phi + \mu$, $\mu \in \mathbb{R}^p$, the Lagrangian transforms as

$$\mathcal{L} \mapsto \mathcal{L}' = \mathcal{L} - \langle \mu, *\boldsymbol{J} \rangle \ , \tag{8.2}$$

thus it is gauge invariant up to a ϕ-independent term $-\langle \mu, *\boldsymbol{J} \rangle$, which however depends on the metric via the Hodge operator. In particular, the gauge transformations map the solution space of the inhomogeneous massless Klein–Gordon equation to itself. This global gauge invariance is exactly the one described by the automorphism groups characterized in Theorem 6.14 and Theorem 7.12. With this interpretation in mind, the observables of the theory should be identified with those elements of the Poisson or quantized algebras that are fixed under the action of the automorphism group. As described in [Few13, §3.3] this would lead in a natural way to subfunctors of $\mathfrak{PA}_p$ and $\mathfrak{QA}_p$ that can be interpreted as the 'theories of observables'. This strategy was implemented for the massless homogeneous Klein–Gordon theory in [Few13, §5.3] and, while we have not worked through the analogue for the present models, our expectation is that it would result in the theories obtained by the following construction: For any object $(\boldsymbol{M}, \boldsymbol{J})$ in LocSrc_p we take the vector subspace $\mathfrak{PS}_p^{\mathrm{inv}}(\boldsymbol{M}, \boldsymbol{J}) \subseteq \mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J})$ consisting of all $[(\varphi, \alpha)]$, such that $\int_M \langle \varphi, \mu \rangle \ \mathrm{vol}_{\boldsymbol{M}} = 0$ for all $\mu \in \mathbb{R}^p$. It is easy to see that $\mathfrak{PS}_p^{\mathrm{inv}} : \mathsf{LocSrc}_p \to \mathsf{PreSymp}$ is a subfunctor of $\mathfrak{PS}_p$ and that, by the same construction as in Section 6 and Section 7, we arrive at subfunctors $\mathfrak{PA}_p^{\mathrm{inv}} : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ and $\mathfrak{QA}_p^{\mathrm{inv}} : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ of, respectively, $\mathfrak{PA}_p$ and $\mathfrak{QA}_p$, which are gauge-invariant. (The remaining issue is whether they coincide with the fixed-point subtheories of $\mathfrak{PA}_p$ and $\mathfrak{QA}_p$, but this is our expectation.)

The role of this gauge invariance is obscured when we study globally hyperbolic perturbations $(\boldsymbol{h}, \boldsymbol{j})$ of the background $(\boldsymbol{M}, \boldsymbol{J})$ via the relative Cauchy evolution. The stress-energy tensor (see (3.13) and set $m = 0$) obtained by the $\boldsymbol{h}$-derivative of the relative Cauchy evolution is *not* gauge invariant under $\phi \mapsto \phi' = \phi + \mu$, $\mu \in \mathbb{R}^p$; it transforms as

$$T^{ab}_{(\boldsymbol{M},\boldsymbol{J})}[\phi] \mapsto T^{ab}_{(\boldsymbol{M},\boldsymbol{J})}[\phi'] = T^{ab}_{(\boldsymbol{M},\boldsymbol{J})}[\phi] + g^{ab} \langle \mu, \boldsymbol{J} \rangle \ , \tag{8.3}$$

and therefore is not an observable according to our discussion above.[6] This feature becomes again clear by looking at the transformation property of the Lagrangian (8.2): In fact, the stress-energy tensor derived from the transformed Lagrangian $\mathcal{L}'$ via taking the functional derivative along g_{ab} does not coincide with the one

[6]The stress-energy tensor does not actually belong to the algebras we have considered; here we have in mind an extended (quantum) algebra containing (Wick) products.

obtained from the untransformed Lagrangian $\mathcal{L}$ due to the metric-dependent extra term in the transformation law (8.2). Note, however, that smearings of the stress-energy tensor by *traceless* tensor fields are gauge-invariant and thus qualify as observables. Likewise, smearings of the field against test functions that are derivatives of compactly supported 1-forms also give observables. This gives an interesting perspective on some of the points made in Remark 7.18: the massless inhomogeneous theory fails to be dynamically local if one restricts to variations of background structures with relative Cauchy evolution generated by observable fields, but is dynamically local if one also allows variations generated by unobservable fields.

In order to obtain a gauge invariant stress-energy tensor we might proceed as follows: If we replace the source terms $\boldsymbol{J} \in C^\infty(M, \mathbb{R}^p)$ by source terms that are top-form valued $\widetilde{\boldsymbol{J}} \in \Omega^{\dim(M)}(M, \mathbb{R}^p)$, the Lagrangian (8.1) can be written as

$$\widetilde{\mathcal{L}} = \frac{1}{2} \langle \mathrm{d}\phi, *\mathrm{d}\phi \rangle - \left\langle \phi, \widetilde{\boldsymbol{J}} \right\rangle . \tag{8.4}$$

Under gauge transformations $\phi \mapsto \phi + \mu$, $\mu \in \mathbb{R}^p$, the Lagrangian transforms as

$$\widetilde{\mathcal{L}} \mapsto \widetilde{\mathcal{L}}' = \widetilde{\mathcal{L}} - \left\langle \mu, \widetilde{\boldsymbol{J}} \right\rangle , \tag{8.5}$$

where now the ϕ-independent additional term does not depend on the metric. As a consequence, the stress-energy tensor obtained by the functional derivative of the Lagrangian along g_{ab} reads

$$\widetilde{T}^{ab}_{(\boldsymbol{M},\widetilde{\boldsymbol{J}})}[\phi] = \left\langle \nabla^a \phi, \nabla^b \phi \right\rangle - \frac{1}{2} g^{ab} \langle \nabla_c \phi, \nabla^c \phi \rangle \tag{8.6}$$

and is gauge invariant. The functorial theory with top-form valued source terms can be obtained from our functors $\mathfrak{PA}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$ and $\mathfrak{QA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ by noticing the following equivalence of categories: Let us define in analogy to LocSrc_p (see Definition 2.1) the category LocTop_p, where objects are tuples $(\boldsymbol{M}, \widetilde{\boldsymbol{J}})$ with $\widetilde{\boldsymbol{J}} \in \Omega^{\dim(M)}(M, \mathbb{R}^p)$ a top-form source term. The categories LocSrc_p and LocTop_p are equivalent via the Hodge operator. Explicitly, we define the covariant functor $\mathfrak{Hodge} : \mathsf{LocTop}_p \to \mathsf{LocSrc}_p$ on objects by $\mathfrak{Hodge}(\boldsymbol{M}, \widetilde{\boldsymbol{J}}) = (\boldsymbol{M}, *\widetilde{\boldsymbol{J}})$ and on morphisms by $\mathfrak{Hodge}(f) = f$ (with a slight abuse of notation we denote both a morphism and its underlying smooth map by the same symbol). The inverse Hodge operator provides us with the inverse functor $\mathfrak{Hodge}^{-1} : \mathsf{LocSrc}_p \to \mathsf{LocTop}_p$. Hence, LocSrc_p and LocTop_p are equivalent categories. Composing the functor $\mathfrak{Hodge}$ with our functors $\mathfrak{PA}_p$ and $\mathfrak{QA}_p$ we obtain the covariant functors

$$\widetilde{\mathfrak{PA}_p} := \mathfrak{PA}_p \circ \mathfrak{Hodge} : \mathsf{LocTop}_p \to \mathsf{PoisAlg} , \tag{8.7a}$$

$$\widetilde{\mathfrak{QA}_p} := \mathfrak{QA}_p \circ \mathfrak{Hodge} : \mathsf{LocTop}_p \to \mathsf{PoisAlg} , \tag{8.7b}$$

which are respectively a locally covariant classical and quantum field theory. The endomorphisms of $\widetilde{\mathfrak{PA}_p}$ and $\widetilde{\mathfrak{QA}_p}$ of course coincide with the ones of $\mathfrak{PA}_p$ and $\mathfrak{QA}_p$. However, as the functor $\mathfrak{Hodge}$ mixes between the metric and the external source terms, the relative Cauchy evolution of the new theories differs from the that of the original theories. Indeed, from (3.7) and (3.9) one easily observes that, for all $[(\varphi, \alpha)] \in \widetilde{\mathfrak{PA}_p}(\boldsymbol{M}, \widetilde{\boldsymbol{J}})$,

$$\mathrm{rce}^{(\widetilde{\mathfrak{PA}_p})}_{(\boldsymbol{M},\widetilde{\boldsymbol{J}})}[\boldsymbol{h}, \widetilde{\boldsymbol{j}}]\big([(\varphi, \alpha)]\big) = \left[\left(\varphi + (\mathrm{KG}_{\boldsymbol{M}} - \mathrm{KG}_{\boldsymbol{M}[\boldsymbol{h}]})\big(\mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\big), \alpha - \int_M \left\langle \widetilde{\boldsymbol{j}}, \mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi) \right\rangle \mathrm{vol}_{\boldsymbol{M}} \right)\right] , \tag{8.8}$$

where now $\widetilde{\boldsymbol{j}} \in \Omega^{\dim(M)}_0(M, \mathbb{R}^p)$ is a compactly supported perturbation of the top-form source term $\widetilde{\boldsymbol{J}}$. A similar formula holds true for the relative Cauchy evolution of $\widetilde{\mathfrak{QA}_p}$. Following the same steps as in Appendix A, we can extract the stress-energy tensor (up to a constant functional) from the derivative of this relative Cauchy evolution along $\boldsymbol{h}$. In the massless case, we find exactly the one obtained from the Lagrangian, see (8.6). As already mentioned, this stress-energy tensor is gauge invariant under the gauge transformations $\phi \mapsto \phi + \mu$, $\mu \in \mathbb{R}^p$.

9 Concluding remarks

Our original aim was to understand how the methods of [Few13] could be extended to the setting of locally covariant theories with external sources in order to compute the automorphism group (which should be the global gauge group) of the inhomogeneous Klein–Gordon theory. This has brought to light various shortcomings in the formulation of the theory according to the prescription of [BDS12]: it has automorphisms that are not gauge symmetries of the original theory (cf. Theorem 4.6); furthermore, it violates a natural composition property that expresses the lack of interaction between fields in the multiplet (cf. Proposition 5.3). To remedy these problems, we have proposed an improved formulation of such theories at the classical and quantum level. We have traced the source of the pathological behavior to a failure of [BDS12] to adequately capture the interplay between the observables spaces (described by the presymplectic vector spaces) with the solution spaces. We have reintroduced this information by studying the representation of the abstract Poisson algebras derived from these presymplectic vector spaces on the solution spaces. These representations of the Poisson algebras have a kernel, which has no corresponding analog in the category of presymplectic vector spaces. Performing the quotient by these kernels, we have obtained a functor to the category of Poisson algebras which gives an appropriate description of the classical theory of a multiplet of inhomogeneous Klein–Gordon fields. We have substantiated this claim by proving that the theory has the correct automorphism group and satisfies the composition property. In the quantized setting, we have replaced the pairing between observables and solutions with carefully defined state spaces on the CCR-algebras derived from our presymplectic vector spaces. The GNS representation of our CCR-algebras in these state spaces has a kernel, and we have shown that the quantum algebras obtained by quotienting out these kernels are given functorially. Again, we have justified our constructions by showing that the automorphism group of this improved functor is the correct one and that the composition property holds true.

In this paper we have restricted ourselves to the simplest case given by a multiplet of inhomogeneous Klein–Gordon fields, as this choice made it possible to characterize explicitly the relative Cauchy evolution and the automorphism groups, which were important tools in unraveling the pathological features of the earlier approach to affine field theories [BDS12]. However, our insights concerning the improved functors describing the classical and quantum theory of this model remain valid for generic affine field theories as described in [BDS12]. In particular, the general presymplectic vector space functor in [BDS12] can be promoted via $\mathfrak{CanPois}$ to a covariant functor with values in the category of Poisson algebras. This functor can be paired with the solution space functor corresponding to the equation of motion operators, which are part of the source category in [BDS12], and the corresponding kernel forms a Poisson ideal. The improved classical functor for generic affine field theories is then given by taking the quotient of the canonical Poisson algebras by these Poisson ideals. In the quantized setting one proceeds analogously to Section 7, i.e. one defines suitable state spaces and studies the kernels of the corresponding GNS representation. The same techniques apply to Abelian gauge theories [BDS13, BDHS13], where however the following remark is in order: The kernel of the presymplectic spaces in [BDS13, BDHS13] does *not* only consist of constant affine functionals, but also topological observables ('electric charges') depending on the topology of spacetime. This implies that quotienting out the kernel of the pairing between the abstract Poisson algebras and the solution spaces does not generally yield simple Poisson algebras. The same holds true for quotienting out the kernel of the GNS representation of the CCR-algebras given by suitable state spaces. This additional degeneracy in the improved Poisson algebras (or the center in the improved quantum algebras) is the source of the violation of the injectivity property in Abelian gauge theories. As is clear from the general no-go theorem in [BDHS13], our approach does not lead to a solution of this problem and hence a complete understanding of Abelian gauge theories remains to be achieved in future work.

We end this section by commenting on the relation between our improved quantum algebras and the algebras for the inhomogeneous Klein–Gordon theory used by Hollands and Wald [HW05]. In our notation, what Hollands and Wald propose is the following construction: Consider the *off-shell* presymplectic vector space for a multiplet of $p \in \mathbb{N}$ Klein–Gordon fields $\mathfrak{PS}_p^{\mathrm{HW}}(\boldsymbol{M},\boldsymbol{J}) := \big(C_0^\infty(M,\mathbb{R}^p),\sigma_{\boldsymbol{M}}\big)$, where for all $\varphi,\psi \in C_0^\infty(M,\mathbb{R}^p)$, $\sigma_{\boldsymbol{M}}(\varphi,\psi) = \int_M \langle \varphi, \mathrm{E}_{\boldsymbol{M}}(\psi)\rangle \;\mathrm{vol}_{\boldsymbol{M}}$. Apply the CCR-functor $\mathfrak{CCR}(\mathfrak{PS}_p^{\mathrm{HW}}(\boldsymbol{M},\boldsymbol{J}))$

and consider the two-sided $*$-ideal

$$\mathfrak{I}_p^{\mathrm{HW}}(\boldsymbol{M},\boldsymbol{J}) := \left\langle \left\{ \mathrm{KG}_{\boldsymbol{M}}(\varphi) + \int_M \langle \varphi, \boldsymbol{J}\rangle \ \mathrm{vol}_{\boldsymbol{M}} : \varphi \in C_0^\infty(M,\mathbb{R}^p) \right\} \right\rangle , \tag{9.1}$$

which is supposed to describe the inhomogeneous Klein–Gordon equation. The algebras of Hollands and Wald are then defined by the quotient $\mathfrak{QA}_p^{\mathrm{HW}}(\boldsymbol{M},\boldsymbol{J}) := \mathfrak{CCR}(\mathfrak{PS}_p^{\mathrm{HW}}(\boldsymbol{M},\boldsymbol{J}))/\mathfrak{I}_p^{\mathrm{HW}}(\boldsymbol{M},\boldsymbol{J})$ and it is easy to see that they are functorial, i.e. that we have a covariant functor $\mathfrak{QA}_p^{\mathrm{HW}} : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$. The covariant functor $\mathfrak{QA}_p^{\mathrm{HW}}$ turns out to be naturally isomorphic to our functor $\mathfrak{QA}_p$ given in Proposition 7.8. Explicitly, the natural isomorphism $\kappa : \mathfrak{QA}_p^{\mathrm{HW}} \Rightarrow \mathfrak{QA}_p$ is given by setting, for all $\varphi \in C_0^\infty(M,\mathbb{R}^p)$,

$$\kappa_{(\boldsymbol{M},\boldsymbol{J})}(\varphi) = \big[[(\varphi,0)]\big] , \tag{9.2}$$

where the outer brackets denote the equivalence relation used in defining $\mathfrak{QA}_p$ and the inner square brackets that used in defining $\mathfrak{PS}_p$. The only nontrivial property to show is that $\kappa_{(\boldsymbol{M},\boldsymbol{J})}\big(\mathrm{KG}_{\boldsymbol{M}}(\varphi) + \int_M \langle \varphi, \boldsymbol{J}\rangle \ \mathrm{vol}_{\boldsymbol{M}}\big) = 0$, which is easily seen by the following calculation

$$\begin{aligned}
\kappa_{(\boldsymbol{M},\boldsymbol{J})}\Big(\mathrm{KG}_{\boldsymbol{M}}(\varphi) + \int_M \langle \varphi, \boldsymbol{J}\rangle \ \mathrm{vol}_{\boldsymbol{M}}\Big) &= \Big[\big[(\mathrm{KG}_{\boldsymbol{M}}(\varphi),0)\big]\Big] + \Big[\int_M \langle \varphi, \boldsymbol{J}\rangle \ \mathrm{vol}_{\boldsymbol{M}}\Big]\\
&= \Big[\big[(\mathrm{KG}_{\boldsymbol{M}}(\varphi),0)\big]\Big] + \Big[\big[(0,\int_M \langle \varphi, \boldsymbol{J}\rangle \ \mathrm{vol}_{\boldsymbol{M}})\big]\Big]\\
&= \Big[\big[(\mathrm{KG}_{\boldsymbol{M}}(\varphi),\int_M \langle \varphi, \boldsymbol{J}\rangle \ \mathrm{vol}_{\boldsymbol{M}})\big]\Big] = 0 ,
\end{aligned} \tag{9.3}$$

where we have used (7.8) in the second equality and (2.5) in the last one. Even though our improved quantum algebras reproduce the earlier constructions by Hollands and Wald, we believe that our strategy for obtaining these algebras is conceptually better motivated than the slightly ad-hoc ideal (9.1) used in [HW05]. Moreover, we have determined a number of detailed properties of these models and exemplified the opportunities for analyzing and distinguishing locally covariant theories by functorial invariants opened up by [Few13].

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A Differentiation of the relative Cauchy evolution and stress-energy tensor

We discuss the sense in which the relative Cauchy evolution of $\mathfrak{PS}_p$ can be differentiated, and show that it has the derivative stated in the text. Our method follows that of [FV12b, Appendix A]; however, additional care must be taken when defining a suitable topology. In [FV12b], the relative Cauchy evolution for the (homogeneous) real scalar field was differentiated using the weak symplectic topology on the solution space induced by seminorms of the form $|\sigma_{\boldsymbol{M}}(\,\cdot\,,\phi)|$, where $\sigma_{\boldsymbol{M}}$ is the symplectic structure and ϕ ranges over the symplectic vector space. An obvious generalization to our present context is to induce a topology from the presymplectic structure on $\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$ in a similar way. However, the resulting topology does not separate points, because the presymplectic structure is degenerate. In particular, it is clear from (3.7) that the presymplectic structure of any element with an element $\mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h},\boldsymbol{j}]\big([(\varphi,\alpha)]\big)$ is independent of $\boldsymbol{j}$ – it would therefore be impossible to obtain the 'obviously correct' derivative given in (3.11b).

The solution to this problem is, as at various other points in this paper, to recall that $\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$ acts as a space of functionals on the affine solution space $\mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})$ of the inhomogeneous theory. Dually, therefore, each solution $\phi \in \mathfrak{Sol}_p(\boldsymbol{M}, \boldsymbol{J})$ defines a seminorm $|\langle\langle \,\cdot\,, \phi\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})}|$, where the pairing $\langle\langle \,\cdot\,, \,\cdot\,\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})}$ was defined in (2.9). The resulting weak-$*$ topology does separate points and is the appropriate generalization of the weak symplectic topology used in [FV12b]. With the topology fixed, differentiability of the relative Cauchy evolution may be established as follows. From (3.9) we have

$$\Big\langle\Big\langle \Big(\mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h}, \boldsymbol{j}] - \mathrm{id}_{\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})}\Big)\big([(\varphi, \alpha)]\big), \phi\Big\rangle\Big\rangle_{(\boldsymbol{M},\boldsymbol{J})}$$
$$= \int_M \Big(\big\langle (\mathrm{KG}_{\boldsymbol{M}} - \mathrm{KG}_{\boldsymbol{M}[\boldsymbol{h}]})\big(\mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\big), \phi\big\rangle - \big\langle \boldsymbol{j}, \mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\big\rangle + (1-\rho_{\boldsymbol{h}}) \big\langle \boldsymbol{J} + \boldsymbol{j}, \mathrm{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\varphi)\big\rangle \Big)\, \mathrm{vol}_{\boldsymbol{M}}\,, \tag{A.1}$$

and the integration region may be restricted, without loss, to any strip S of $\boldsymbol{M}$ containing the support of $\boldsymbol{j}$ and $\boldsymbol{h}$. As in [FV12b, Appendix B], energy estimates entail that $s \mapsto \mathrm{E}_{\boldsymbol{M}[s\boldsymbol{h}]}(\varphi)$ is differentiable in $L^2(S, \mathrm{vol}_{\boldsymbol{M}})$; the same is true for $s \mapsto 1 - \rho_{s\boldsymbol{h}}$, with derivative $-\frac{1}{2}g^{ab}h_{ab}$ at $s = 0$. It follows that

$$\int_M \big\langle s\boldsymbol{j}, \mathrm{E}_{\boldsymbol{M}[s\boldsymbol{h}]}(\varphi)\big\rangle \;\mathrm{vol}_{\boldsymbol{M}} = s \int_M \langle \boldsymbol{j}, \mathrm{E}_{\boldsymbol{M}}(\varphi)\rangle \;\mathrm{vol}_{\boldsymbol{M}} + O(s^2) \tag{A.2a}$$

and

$$\int_M (1 - \rho_{s\boldsymbol{h}}) \,\big\langle \boldsymbol{J} + s\boldsymbol{j}, \mathrm{E}_{\boldsymbol{M}[s\boldsymbol{h}]}(\varphi)\big\rangle \;\mathrm{vol}_{\boldsymbol{M}} = -\frac{s}{2} \int_M g^{ab}h_{ab} \,\langle \boldsymbol{J}, \mathrm{E}_{\boldsymbol{M}}(\varphi)\rangle \;\mathrm{vol}_{\boldsymbol{M}} + O(s^2)\,. \tag{A.2b}$$

Moreover, the formula

$$\int_M \big\langle (\mathrm{KG}_{\boldsymbol{M}} - \mathrm{KG}_{\boldsymbol{M}[s\boldsymbol{h}]})\big(\mathrm{E}_{\boldsymbol{M}[s\boldsymbol{h}]}(\varphi)\big), \phi\big\rangle \;\mathrm{vol}_{\boldsymbol{M}} = -s \int_M \Big\langle \mathrm{KG}'_{\boldsymbol{M}[\boldsymbol{h}]}\big(\mathrm{E}_{\boldsymbol{M}}(\varphi)\big), \phi\Big\rangle \;\mathrm{vol}_{\boldsymbol{M}} + O(s^2) \tag{A.3}$$

was established in [FV12b, Appendix A]. Note that here ϕ solves the inhomogeneous equation, while its analogue in the cited reference solved the homogeneous equation. However the difference is inessential, because the only property of ϕ used is that it is square-integrable on the strip S. Assembling these observations,

$$\frac{d}{ds} \Big\langle\Big\langle \mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[s\boldsymbol{h}, s\boldsymbol{j}]\big([(\varphi, \alpha)]\big), \phi\Big\rangle\Big\rangle_{(\boldsymbol{M},\boldsymbol{J})}\bigg|_{s=0} = -\big\langle\big\langle \big(\mathcal{T}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h}] + \mathcal{J}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{j}]\big)\,\big([(\varphi, \alpha)]\big), \phi\big\rangle\big\rangle_{(\boldsymbol{M},\boldsymbol{J})} \tag{A.4}$$

for every $\phi \in \mathfrak{Sol}_p(\boldsymbol{M}, \boldsymbol{J})$, where $\mathcal{T}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h}]$ and $\mathcal{J}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{j}]$ were given in (3.11a) and (3.11b). Accordingly, (3.10) holds in the weak-$*$ topology on $\mathfrak{PS}_p(\boldsymbol{M}, \boldsymbol{J})$.

To relate these formulae with the classical stress-energy tensor and action, we note that

$$\mathrm{KG}'_{\boldsymbol{M}[\boldsymbol{h}]} = \frac{d}{ds}\mathrm{KG}_{\boldsymbol{M}[s\boldsymbol{h}]}\Big|_{s=0} = -\nabla_a h^{ab}\nabla_b + \frac{1}{2}\left(\nabla^a {h^b}_b\right)\nabla_a \tag{A.5}$$

(unfortunately, a sign error appears in the analogous step in [FV12b]: see the second line of the central displayed formula on p.1706 of that reference). Inserting this formula and integrating by parts we obtain

$$\begin{aligned}
\langle\langle \mathcal{T}_{(\boldsymbol{M},\boldsymbol{J})}[\boldsymbol{h}]\big([(\varphi, \alpha)]\big), \phi\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} &= \int_M \left(h_{ab} \left\langle \nabla^b \mathrm{E}_{\boldsymbol{M}}(\varphi), \nabla^a\phi\right\rangle - \frac{1}{2} {h^b}_b \nabla^a \left\langle \nabla_a \mathrm{E}_{\boldsymbol{M}}(\varphi), \phi\right\rangle \right) \mathrm{vol}_{\boldsymbol{M}} \\
&\qquad + \int_M \frac{1}{2} g^{ab}\, h_{ab}\, \langle \boldsymbol{J}, \mathrm{E}_{\boldsymbol{M}}(\varphi)\rangle \;\mathrm{vol}_{\boldsymbol{M}} \\
&= \int_M h_{ab} \left(\left\langle \nabla^b \mathrm{E}_{\boldsymbol{M}}(\varphi), \nabla^a\phi\right\rangle - \frac{1}{2} g^{ab}\, \langle \nabla_c \mathrm{E}_{\boldsymbol{M}}(\varphi), \nabla^c\phi\rangle \right. \\
&\qquad \left. + \frac{1}{2} m^2 g^{ab}\, \langle \mathrm{E}_{\boldsymbol{M}}(\varphi), \phi\rangle + \frac{1}{2} g^{ab}\, \langle \boldsymbol{J}, \mathrm{E}_{\boldsymbol{M}}(\varphi)\rangle \right) \mathrm{vol}_{\boldsymbol{M}} \\
&= \frac{1}{2}\, \frac{d}{ds} \int_M h_{ab}\, T^{ab}_{(\boldsymbol{M},\boldsymbol{J})}[\phi + s\, \mathrm{E}_{\boldsymbol{M}}(\varphi)]\; \mathrm{vol}_{\boldsymbol{M}} \bigg|_{s=0}\,,
\end{aligned} \tag{A.6}$$

thus establishing (3.12), where the stress-energy tensor is given by (3.13) and we have used the fact that

$$\begin{aligned}\nabla^a \langle \nabla_a \mathrm{E}_{\boldsymbol{M}}(\varphi), \phi\rangle &= \langle \Box_{\boldsymbol{M}} \big(\mathrm{E}_{\boldsymbol{M}}(\varphi)\big), \phi\rangle + \langle \nabla_a \mathrm{E}_{\boldsymbol{M}}(\varphi), \nabla^a \phi\rangle \\ &= -m^2 \langle \mathrm{E}_{\boldsymbol{M}}(\varphi), \phi\rangle + \langle \nabla_a \mathrm{E}_{\boldsymbol{M}}(\varphi), \nabla^a \phi\rangle \ . \end{aligned} \tag{A.7}$$

To conclude, we will express the derivative of the relative Cauchy evolution using the Poisson bracket. Identifying the space of solutions $\mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})$ with phase space, and identifying an element $[(\varphi,\alpha)] \in \mathfrak{PS}_p(\boldsymbol{M})$ with the functional

$$\phi \mapsto \langle\langle [(\varphi,\alpha)], \phi\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} = \left(\int_M \langle \phi, \varphi\rangle \ \mathrm{vol}_{\boldsymbol{M}}\right) + \alpha \tag{A.8}$$

on phase space (cf. (2.9)), we already have the Poisson bracket

$$\big\{[(\varphi,\alpha)], [(\varphi',\alpha')]\big\}_{\sigma_{(\boldsymbol{M},\boldsymbol{J})}}(\phi) := \Big\langle\Big\langle \big\{[(\varphi,\alpha)], [(\varphi',\alpha')]\big\}_{\sigma_{(\boldsymbol{M},\boldsymbol{J})}}, \phi \Big\rangle\Big\rangle_{(\boldsymbol{M},\boldsymbol{J})} = \mathrm{E}_{\boldsymbol{M}}(\varphi,\varphi') \ . \tag{A.9}$$

Unsmearing the second slot, this gives

$$\big\{[(\varphi,\alpha)], \phi(x)\big\}_{\sigma_{(\boldsymbol{M},\boldsymbol{J})}}(\phi) = -\big(\mathrm{E}_{\boldsymbol{M}}(\varphi)\big)(x) \ , \tag{A.10}$$

where, in an obvious way, $\phi(x)$, $x \in M$, stands for the functional $\phi \mapsto \phi(x)$ on phase space. Thus, we also have for the functional $\phi \mapsto \int_M f \, \langle \phi, \phi\rangle \ \mathrm{vol}_{\boldsymbol{M}}$

$$\left\{[(\varphi,\alpha)], \phi \mapsto \int_M f \, \langle \phi, \phi\rangle \ \mathrm{vol}_{\boldsymbol{M}}\right\}_{\sigma_{(\boldsymbol{M},\boldsymbol{J})}}(\phi) = 2\,\mathrm{E}_{\boldsymbol{M}}(\varphi, f\phi) \ . \tag{A.11}$$

Proceeding in this way, one easily obtains the formula

$$\begin{aligned}&\frac{d}{ds}\Big\langle\Big\langle \mathrm{rce}^{(\mathfrak{PS}_p)}_{(\boldsymbol{M},\boldsymbol{J})}[s\boldsymbol{h}, s\boldsymbol{j}]\big([(\varphi,\alpha)]\big), \phi\Big\rangle\Big\rangle_{(\boldsymbol{M},\boldsymbol{J})}\bigg|_{s=0} \\ &\qquad = \left\{[(\varphi,\alpha)], \phi \mapsto \int_M \left(\frac{1}{2} h_{ab}\, T^{ab}_{(\boldsymbol{M},\boldsymbol{J})}[\phi] + \langle \boldsymbol{j}, \phi\rangle\right) \ \mathrm{vol}_{\boldsymbol{M}}\right\}_{\sigma_{(\boldsymbol{M},\boldsymbol{J})}}(\phi) \ , \end{aligned} \tag{A.12}$$

which can also be written in the form (6.18). In this form, it is clear that, just as the relative Cauchy evolution correctly identifies the current coupling to the metric as the stress-energy tensor, so it also correctly identifies the 'current' coupling to the external source $\boldsymbol{J}$ as the field ϕ itself.

B Pointed presymplectic spaces

We briefly discuss an alternative method for obtaining a good classical and quantum theory of the inhomogeneous models. The idea is to modify the functor $\mathfrak{PS}_p$, so that it takes values in the category of pointed presymplectic spaces $\bullet\mathsf{PreSymp}$ defined as follows: Objects of $\bullet\mathsf{PreSymp}$ are pairs $((V,\sigma_V), 1_V)$, where (V,σ_V) is an object in $\mathsf{PreSymp}$ (so that σ_V has nontrivial null space) and 1_V is a distinguished nonzero vector in the null space of σ_V. A morphism $L : ((V,\sigma_V),1_V) \to ((W,\sigma_W),1_W)$ is a $\mathsf{PreSymp}$ morphism $L : (V,\sigma_V) \to (W,\sigma_W)$, such that $L(1_V) = 1_W$. For example, in the one-dimensional vector space with trivial presymplectic structure $\mathcal{I} = (\mathbb{R},0)$ we may single out the unit of $\mathbb{R}$ to obtain a pointed presymplectic space $(\mathcal{I},1)$, which is then an initial object of $\bullet\mathsf{PreSymp}$ – in fact, we may also regard $\bullet\mathsf{PreSymp}$ as the category of arrows in $\mathsf{PreSymp}$ with domain $\mathcal{I}$.

Our modified functor $\bullet\mathfrak{PS}_p : \mathsf{LocSrc}_p \to \bullet\mathsf{PreSymp}$ is defined on objects by

$$\bullet\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J}) = \big(\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J}), [(0,1)]\big) \tag{B.1}$$

and on morphisms by $\bullet\mathfrak{PS}_p(f) = \mathfrak{PS}_p(f)$, noting that the latter indeed preserve the distinguished elements $[(0,1)] \in \mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$, which are naturally distinguished by their action on solutions: $\langle\langle [(0,1)], \phi\rangle\rangle_{(\boldsymbol{M},\boldsymbol{J})} = 1$ for all $\phi \in \mathfrak{Sol}_p(\boldsymbol{M},\boldsymbol{J})$.

The resulting theory has the expected symmetries.

Theorem B.1. *Every endomorphism of the covariant functor* $\bullet\mathfrak{PS}_p : \mathsf{LocSrc}_p \to \bullet\mathsf{PreSymp}$ *is an automorphism and*

$$\mathrm{End}(\bullet\mathfrak{PS}_p) = \mathrm{Aut}(\bullet\mathfrak{PS}_p) \simeq \begin{cases} \{\mathrm{id}_{\bullet\mathfrak{PS}_p}\} & , \; \textit{for } m \neq 0 \,, \\ \mathbb{R}^p & , \; \textit{for } m = 0 \,, \end{cases} \tag{B.2}$$

where for $m = 0$ *the action is given by, for all* $[(\varphi,\alpha)] \in \bullet\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$ *and* $\mu \in \mathbb{R}^p$*,*

$$\bullet\eta(\mu)_{(\boldsymbol{M},\boldsymbol{J})}\big([(\varphi,\alpha)]\big) = \left[\left(\varphi, \alpha + \int_M \langle \varphi, \mu\rangle \;\mathrm{vol}_{\boldsymbol{M}}\right)\right] \,. \tag{B.3}$$

Proof. The forgetful functor $\bullet\mathsf{PreSymp} \to \mathsf{PreSymp}$ induces a faithful homomorphism $\mathrm{End}(\bullet\mathfrak{PS}_p) \to \mathrm{End}(\mathfrak{PS}_p)$ of monoids. Thus, it suffices to determine which endomorphisms of $\mathfrak{PS}_p$ lift to $\bullet\mathfrak{PS}_p$. By Proposition 4.1 and Theorem 4.6, we see that $\eta(-1)_{(\boldsymbol{M},\boldsymbol{J})}([(0,1)]) = [(0,-1)]$ so $\eta(-1)$ does not lift, leaving only the trivial group for $m \neq 0$. Similarly, by Proposition 4.2 and Theorem 4.6, we see in the massless case that $\eta(\sigma,\mu)_{(\boldsymbol{M},\boldsymbol{J})}([(0,1)]) = [(0,\sigma)]$ so $\eta(\sigma,\mu)$ lifts if and only if $\sigma = 1$. □

The second problem identified with $\mathfrak{PS}_p$ was its failure to behave correctly with respect to the composition of systems, see Section 5. In our present context this can be remedied in the following way: The natural composition of pointed presymplectic spaces $((V,\sigma_V),1_V)$ and $((W,\sigma_W),1_W)$ is not the direct sum, but rather the direct sum with amalgamation of distinguished points

$$((V,\sigma_V),1_V) \circledast ((W,\sigma_W),1_W) = \big(((V\oplus W)/\sim, \sigma_{V\oplus W}), [\![(1_V,0)]\!]\big) \,, \tag{B.4}$$

where the equivalence relation implements a quotient by the subspace $\{(\lambda 1_V, -\lambda 1_W) : \lambda \in \mathbb{R}\}$. Note this is well-defined due to our assumption that the distinguished elements lie in the null space of the presymplectic structures. The operation $\circledast$ gives a monoidal structure on $\bullet\mathsf{PreSymp}$, with monoidal unit $(\mathcal{I},1)$, just as the direct sum $\oplus$ does for $\mathsf{PreSymp}$ (with the zero-dimensional presymplectic vector space as the monoidal unit).

Theorem B.2. *The theory* $\bullet\mathfrak{PS}_p$ *obeys the composition property, i.e., there is a natural isomorphism*

$$\bullet\mathfrak{PS}_p \cong \bullet\mathfrak{PS}_{p,q} := \circledast \circ \big(\bullet\mathfrak{PS}_q \times \bullet\mathfrak{PS}_{p-q}\big) \circ \mathfrak{Split}_{p,q} \,. \tag{B.5}$$

Proof. For any object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p, define $(\eta_{p,q})_{(\boldsymbol{M},\boldsymbol{J})} : \bullet\mathfrak{PS}_{p,q}(\boldsymbol{M},\boldsymbol{J}) \to \bullet\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J})$ by, for all $[(\varphi,\alpha)] \in \bullet\mathfrak{PS}_q(\boldsymbol{M},\boldsymbol{J}^q)$ and $[(\psi,\beta)] \in \bullet\mathfrak{PS}_{p-q}(\boldsymbol{M},\boldsymbol{J}^{p-q})$,

$$(\eta_{p,q})_{(\boldsymbol{M},\boldsymbol{J})}\left([\![([(\varphi,\alpha)],[(\psi,\beta)])]\!]\right) = \big[(\varphi+\psi, \alpha+\beta)\big] \,, \tag{B.6}$$

where on the right hand side we have identified $\varphi \in C^\infty_0(M,\mathbb{R}^q)$ and $\psi \in C^\infty_0(M,\mathbb{R}^{p-q})$ as elements in $C^\infty_0(M,\mathbb{R}^p)$ (φ is placed in the first q and ψ in the last $p-q$ components of $\mathbb{R}^p$). This map is well-defined (because α and β are summed on the right-hand side of (B.6)), linear and it preserves the distinguished element and the presymplectic structure. Furthermore, it is invertible via $\bullet\mathfrak{PS}_p(\boldsymbol{M},\boldsymbol{J}) \ni [(\varphi,\alpha)] \mapsto [\![([(\varphi^q,\alpha)],[(\varphi^{p-q},0)])]\!]$, where $\varphi = \varphi^q + \varphi^{p-q}$ denotes the split of $\varphi \in C^\infty_0(M,\mathbb{R}^p)$ into the first q and last $p-q$ components. Hence, (B.6) is an isomorphism in $\bullet\mathsf{PreSymp}$.

To establish naturality, consider any morphism $f : (\boldsymbol{M}_1,\boldsymbol{J}_1) \to (\boldsymbol{M}_2,\boldsymbol{J}_2)$ in LocSrc_p. It is straightforward to check commutativity of

$$\begin{tikzcd}
{[\![([(\varphi,\alpha)],[(\psi,\beta)])]\!]} \arrow[r, mapsto, "(\eta_{p,q})_{(\boldsymbol{M}_1,\boldsymbol{J}_1)}"] \arrow[d, mapsto, "\bullet\mathfrak{PS}_{p,q}(f)"'] & {[(\varphi+\psi,\alpha+\beta)]} \arrow[d, mapsto, "\bullet\mathfrak{PS}_p(f)"] \\
{[\![([(f_*(\varphi),\alpha)],[(f_*(\psi),\beta)])]\!]} \arrow[r, mapsto, "(\eta_{p,q})_{(\boldsymbol{M}_2,\boldsymbol{J}_2)}"] & {[(f_*(\varphi+\psi),\alpha+\beta)]}
\end{tikzcd} \tag{B.7}$$

for all $[(\varphi,\alpha)] \in \bullet\mathfrak{PS}_q(\boldsymbol{M}_1,\boldsymbol{J}_1^q)$ and $[(\psi,\beta)] \in \bullet\mathfrak{PS}_{p-q}(\boldsymbol{M}_1,\boldsymbol{J}_1^{p-q})$. Hence, the $(\eta_{p,q})_{(\boldsymbol{M},\boldsymbol{J})}$ form the components of a natural isomorphism. □

Finally, we consider the quantization of pointed presymplectic spaces via a suitable covariant functor $\bullet\mathfrak{CCR} : \bullet\mathsf{PreSymp} \to {}^*\mathsf{Alg}$. On objects we set $\bullet\mathfrak{CCR}((V,\sigma_V),1_V) = \mathfrak{CCR}(V,\sigma_V)/\mathfrak{I}(V,\sigma_V)$, where $\mathfrak{I}(V,\sigma_V)$ is the two-sided $*$-ideal generated by $1_V - 1$. On morphisms $L : ((V,\sigma_V),1_V) \to ((W,\sigma_W),1_W)$ we have $\mathfrak{CCR}(L)(1_V) = 1_W$ and hence there is a uniquely defined injective $*$-algebra homomorphism $\bullet\mathfrak{CCR}(L) : \bullet\mathfrak{CCR}((V,\sigma_V),1_V) \to \bullet\mathfrak{CCR}((W,\sigma_W),1_W)$ making the following diagram commute

$$\begin{tikzcd}
\mathfrak{CCR}(V,\sigma_V) \arrow[r, "\mathfrak{CCR}(L)"] \arrow[d] & \mathfrak{CCR}(W,\sigma_W) \arrow[d] \\
\bullet\mathfrak{CCR}((V,\sigma_V),1_V) \arrow[r, "\bullet\mathfrak{CCR}(L)"] & \bullet\mathfrak{CCR}((W,\sigma_W),1_W)
\end{tikzcd} \tag{B.8}$$

where the vertical morphisms are the quotient maps (and the diagram is drawn in the category of unital $*$-algebras without the requirement that morphisms be monic). Our last result is simply a restatement of the definition of the improved functor $\mathfrak{QA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ in Subsection 7.2.

Theorem B.3. $\mathfrak{QA}_p = \bullet\mathfrak{CCR} \circ \bullet\mathfrak{PS}_p$.

C Deformation quantization

As a further alternative construction of the improved functor $\mathfrak{QA}_p : \mathsf{LocSrc}_p \to {}^*\mathsf{Alg}$ we focus on deformation quantization. We show that, starting from the *improved* Poisson algebra functor $\mathfrak{PA}_p : \mathsf{LocSrc}_p \to \mathsf{PoisAlg}$, we can obtain for each object $(\boldsymbol{M},\boldsymbol{J})$ in LocSrc_p the unital $*$-algebra $\mathfrak{QA}_p(\boldsymbol{M},\boldsymbol{J})$ by deformation quantization of (the complexification of) the Poisson algebra $\mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J})$. After this, we make some remarks on the application of Fedosov quantization [Fed94], which has been studied recently in [SDH12] (although not adhering strictly to [Fed94], as we will explain) for the inhomogeneous Maxwell field.

Let $(\boldsymbol{M},\boldsymbol{J})$ be any object in LocSrc_p and consider the Poisson algebra $\mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J})$ constructed in Subsection 6.2, which we shall denote also by $A := \mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J})$ in order to simplify the notation. We define a *differential graded algebra* over the unital algebra A as follows: Consider the graded commutative algebra $\Omega^\bullet := \mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J}) \otimes \bigwedge^\bullet \mathfrak{PS}_p^{\mathrm{lin}}(\boldsymbol{M})$ with product defined by linearity and, for all $a\otimes\lambda, a'\otimes\lambda' \in \Omega^\bullet$,

$$(a\otimes\lambda)\,(a'\otimes\lambda') := (a\,a')\otimes(\lambda\wedge\lambda')~. \tag{C.1}$$

We define a differential $\mathrm{d} : \Omega^\bullet \to \Omega^{\bullet+1}$ by linearity, the graded Leibniz rule and setting, for all $1\otimes\lambda\in\Omega^\bullet$ and $[(\varphi,\alpha)]\otimes\lambda \in \Omega^\bullet$,

$$\mathrm{d}\big(1\otimes\lambda\big) = 0 \quad,\qquad \mathrm{d}\big([(\varphi,\alpha)]\otimes\lambda\big) = 1\otimes\big([\varphi]^{\mathrm{lin}}\wedge\lambda\big)~. \tag{C.2}$$

Using the differential graded algebra $(\Omega^\bullet,\mathrm{d})$ over A, the Poisson bracket in $\mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J})$ can be reformulated as, for all $a,a'\in A$,

$$\{a,a'\}_{\sigma_{(\boldsymbol{M},\boldsymbol{J})}} = \Pi\big(\mathrm{d}a,\mathrm{d}a'\big)~, \tag{C.3}$$

where the *Poisson tensor* $\Pi : \Omega^1\times\Omega^1 \to A$ is the A-bilinear map that is defined by the following extension of $\sigma_{\boldsymbol{M}}^{\mathrm{lin}} : \mathfrak{PS}_p^{\mathrm{lin}}(\boldsymbol{M})\times\mathfrak{PS}_p^{\mathrm{lin}}(\boldsymbol{M}) \to \mathbb{R}$, for all $a\otimes\lambda, a'\otimes\lambda'\in\Omega^1$,

$$\Pi\big(a\otimes\lambda, a'\otimes\lambda'\big) := \sigma_{\boldsymbol{M}}^{\mathrm{lin}}(\lambda,\lambda')\,a\,a'~. \tag{C.4}$$

On the A-module Ω^1 there is a *canonical connection* $\nabla : \Omega^1 \to \Omega^1\otimes_A\Omega^1$ specified by linearity and, for all $a\otimes\lambda\in\Omega^1$,

$$\nabla(a\otimes\lambda) := (1\otimes\lambda)\otimes_A(\mathrm{d}a)~. \tag{C.5}$$

The Leibniz rule $\nabla(\omega\,a) = \nabla(\omega)\,a + \omega\otimes_A\mathrm{d}a$, for all $\omega\in\Omega^1$ and $a\in A$, is obviously satisfied. The *torsion* of ∇ is the A-module homomorphism $\mathrm{T} : \Omega^1\to\Omega^2$ defined by, for all $\omega\in\Omega^1$,

$$\mathrm{T}(\omega) := \wedge\big(\nabla(\omega)\big) + \mathrm{d}\omega~. \tag{C.6}$$

One easily checks that the canonical connection ∇ is torsion free. The *curvature* of ∇ is the A-module homomorphism $\mathrm{R} : \Omega^1 \to \Omega^1 \otimes_A \Omega^2$ defined by, for all $\omega \in \Omega^1$,

$$\mathrm{R}(\omega) := \nabla^2(\omega) := \nabla\nabla(\omega)\ . \tag{C.7}$$

When applying ∇ the second time, the usual extension $\nabla : \Omega^1 \otimes_A \Omega^\bullet \to \Omega^1 \otimes_A \Omega^{\bullet+1}$ defined by linearity and, for all $\omega \otimes_A \omega' \in \Omega^1 \otimes_A \Omega^\bullet$,

$$\nabla(\omega \otimes_A \omega') := (\mathrm{id}_{\Omega^1} \otimes \wedge)\big(\nabla(\omega) \otimes_A \omega'\big) + \omega \otimes_A \mathrm{d}\omega' \tag{C.8}$$

is implicitly understood. One easily checks that the canonical connection ∇ is flat, i.e., $\mathrm{R} = 0$. As a last property, notice the canonical connection preserves the Poisson tensor Π, i.e. the A-bilinear map $\mathrm{Q} : \Omega^1 \times \Omega^1 \to \Omega^1$ defined by, for all $\omega, \omega' \in \Omega^1$,

$$\mathrm{Q}(\omega, \omega') := \mathrm{d}\big(\Pi\big(\omega, \omega'\big)\big) - \Pi\big(\nabla(\omega), \omega'\big) - \Pi\big(\omega, \nabla(\omega')\big) \tag{C.9}$$

vanishes. To sum up, we have shown that the canonical connection $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ is a flat and torsion free Poisson connection.

As the next step, we consider the tensor module $\mathcal{E} := \bigoplus_{n=0}^\infty \big(\Omega^1\big)^{\otimes_A^n}$, which is an $\mathbb{N}_0$-graded A-module that describes *tensor fields* on A. The connection ∇ on Ω^1 lifts to a connection on $\mathcal{E} = \bigoplus_{n=0}^\infty \mathcal{E}^n$ via a recursive construction: On $\mathcal{E}^0 \simeq A$ we choose the connection $\nabla_0 : A \to A \otimes_A \Omega^1 \simeq \Omega^1$ given by the differential d. On $\mathcal{E}^1 = \Omega^1$ we take the canonical connection $\nabla_1 = \nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$. Given a connection ∇_n on $\mathcal{E}^n = \Omega^1 \otimes_A \cdots \otimes_A \Omega^1$ (n-times) we construct a connection $\nabla_{n+1} : \mathcal{E}^{n+1} \to \mathcal{E}^{n+1} \otimes_A \Omega^1$ on $\mathcal{E}^{n+1} = \mathcal{E}^n \otimes_A \Omega^1$ by linearity and setting, for all $\omega \otimes_A \omega' \in \mathcal{E}^n \otimes_A \Omega^1$,

$$\nabla_{n+1}(\omega \otimes_A \omega') := (\mathrm{id}_{\mathcal{E}^n} \otimes \tau)\big(\nabla_n(\omega) \otimes_A \omega'\big) + \omega \otimes_A \nabla_1(\omega')\ , \tag{C.10}$$

where $\tau : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ is the flip map, for all $\omega \otimes_A \omega'$, $\tau(\omega \otimes_A \omega') = \omega' \otimes_A \omega$. To simplify the notation, we shall denote the resulting connection on $\mathcal{E}$ by $\nabla_{\mathcal{E}}$ and notice that we can regard it as a linear map (of degree 1) $\nabla_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$.

Since ∇ is a flat and torsion free Poisson connection, we can define an associative $\star$-product on the complexification of A (denoted with abuse of notation also by A), for all $a, a' \in A$,

$$a \star_{(\Pi,\nabla)} a' := \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\right)^n \Pi^n\big(\nabla_{\mathcal{E}}^n(a), \nabla_{\mathcal{E}}^n(a')\big)\ . \tag{C.11}$$

Here $\nabla_{\mathcal{E}}^n$ denotes the n-times iterated application of $\nabla_{\mathcal{E}}$ and $\Pi^n : \mathcal{E}^n \times \mathcal{E}^n \to A$ is the A-bilinear map specified by, for all $\omega_1, \dots, \omega_n, \omega_1', \dots, \omega_n' \in \Omega^1$,

$$\Pi^n\big(\omega_1 \otimes_A \cdots \otimes_A \omega_n, \omega_1' \otimes_A \cdots, \otimes_A \omega_n'\big) := \Pi(\omega_1, \omega_1') \cdots \Pi(\omega_n, \omega_n') \tag{C.12}$$

and $\Pi^0(a, a') = a\, a'$, for all $a, a' \in A$. Notice that the sum in (C.11) terminates, since $a, a' \in A$ are polynomials, hence $\nabla_{\mathcal{E}}^m(a) = 0$ and $\nabla_{\mathcal{E}}^{m'}(a') = 0$ for sufficiently large $m, m' \in \mathbb{N}$ (see also (C.13) below). Furthermore, the $\star$-product $\star_{(\Pi,\nabla)}$ is hermitian if we equip A with the involution $*$ defined by $\big([(\varphi_1, \alpha_1)] \cdots [(\varphi_n, \alpha_n)]\big)^* = [(\varphi_1, \alpha_1)] \cdots [(\varphi_n, \alpha_n)]$ and $\mathbb{C}$-antilinear extension.

It remains to show that the $\star$-product (C.11) coincides with the product in $\mathfrak{QA}_p(\boldsymbol{M}, \boldsymbol{J})$, which is defined in (7.1). For the elements $[(\varphi, \alpha)]^m \in A$ we find, for all $k \leq m$,

$$\nabla_{\mathcal{E}}^k\big([(\varphi, \alpha)]^m\big) = \frac{m!}{(m-k)!}\, [(\varphi, \alpha)]^{m-k} \otimes [\varphi]^{\mathrm{lin}\otimes k}\ , \tag{C.13}$$

and for $k > m$, $\nabla_{\mathcal{E}}^k\big([(\varphi, \alpha)]^m\big) = 0$. Plugging this into (C.11) we obtain, for all $[(\varphi, \alpha)]^m, [(\varphi', \alpha')]^n \in A$,

$$\begin{aligned}[(\varphi, \alpha)]^m &\star_{(\Pi,\nabla)} [(\varphi', \alpha')]^n \\ &= \sum_{k=0}^{\min(m,n)} \left(\frac{i\sigma_{\boldsymbol{M}}^{\mathrm{lin}}([\varphi]^{\mathrm{lin}}, [\varphi']^{\mathrm{lin}})}{2}\right)^k \frac{m!\, n!}{k!\,(m-k)!\,(n-k)!} [(\varphi, \alpha)]^{m-k}\, [(\varphi', \alpha')]^{n-k}\ .\end{aligned} \tag{C.14}$$

This shows that the products (C.11) and (7.1) coincide.

In view of this direct construction of the $\star$-product (C.11) given above, the application of full-fledged Fedosov quantization [Fed94] to our model is not required. However, we shall now focus on this quantization method using our algebraic approach developed in this appendix, as this will clarify certain issues in an earlier treatment of this subject [SDH12]. The basic structure entering Fedosov's approach is a bundle of CCR-algebras over a symplectic (or regular Poisson) manifold. In our algebraic approach this bundle is given by the A-module $\mathcal{W} := S^{\otimes_A}(\Omega^1)$, where $S^{\otimes_A}(\Omega^1)$ is the (complexified) symmetric tensor algebra with respect to the tensor product $\otimes_A$ of the A-module of one-forms Ω^1 on A. Using that $\Omega^1 = \mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J}) \otimes \mathfrak{PS}_p^{\mathrm{lin}}(\boldsymbol{M})$, $\mathcal{W}$ is isomorphic to $\mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J}) \otimes S\big(\mathfrak{PS}_p^{\mathrm{lin}}(\boldsymbol{M})\big)$. We shall suppress this isomorphism. Notice that $S\big(\mathfrak{PS}_p^{\mathrm{lin}}(\boldsymbol{M})\big)$ is the vector space underlying the CCR-algebra $B := \mathfrak{CQA}_p^{\mathrm{lin}}(\boldsymbol{M}) = \mathfrak{CCR}\big(\mathfrak{PS}_p^{\mathrm{lin}}(\boldsymbol{M})\big)$ – the quantum algebra of observables of the homogeneous theory – hence, we can equip the A-module $\mathcal{W} = A \otimes B$ with a (noncommutative) product, for all $a \otimes b, a' \otimes b' \in A \otimes B$,

$$(a \otimes b)\,(a' \otimes b') := (a\,a') \otimes (b \star b')\,. \tag{C.15}$$

Notice that A can be identified with a commutative subalgebra of $\mathcal{W}$ via $A \to \mathcal{W}\,,~ a \mapsto a \otimes 1$. Fedosov's idea is to characterize a subalgebra $A_\star$ of $\mathcal{W}$ in such a way that $A_\star$ is isomorphic to A as a vector space and that the induced product on $A_\star$ is a deformation quantization of the Poisson structure on A. To achieve this goal, the first step is to construct a suitable connection on $\mathcal{W}$ as follows: Given the canonical connection ∇ on Ω^1, we can induce a tensor product connection $\nabla_{\mathcal{W}}$ on $\mathcal{W}$ via the prescription outlined in (C.10). This connection extends analogously to (C.8) to a linear map $\nabla_{\mathcal{W}} : \mathcal{W} \otimes_A \Omega^\bullet \to \mathcal{W} \otimes_A \Omega^{\bullet+1}$. Notice that $\mathcal{W} \otimes_A \Omega^\bullet$ is an $\mathbb{N}_0$-graded algebra (the grading is inherited from that of differential forms $\Omega^\bullet$) by setting, for all $\mathrm{w} \otimes_A \omega, \mathrm{w}' \otimes_A \omega' \in \mathcal{W} \otimes_A \Omega^\bullet$, $(\mathrm{w} \otimes_A \omega)\,(\mathrm{w}' \otimes_A \omega') = (\mathrm{w}\,\mathrm{w}') \otimes_A (\omega\,\omega')$. It is easy to check that $\nabla_{\mathcal{W}}$ is flat, i.e. $\mathrm{R}_{\mathcal{W}} = \nabla_{\mathcal{W}}^2 = 0$, since it is the tensor product connection of our flat canonical connection ∇ on Ω^1. Furthermore, $\nabla_{\mathcal{W}}$ satisfies the graded Leibniz rule on the $\mathbb{N}_0$-graded algebra, for all homogeneous elements $\mathrm{w}^\bullet, \mathrm{w}^{\bullet\prime} \in \mathcal{W} \otimes_A \Omega^\bullet$,

$$\nabla_{\mathcal{W}}\big(\mathrm{w}^\bullet\,\mathrm{w}^{\bullet\prime}\big) = \big(\nabla_{\mathcal{W}}(\mathrm{w}^\bullet)\big)\,\mathrm{w}^{\bullet\prime} + (-1)^{|\mathrm{w}^\bullet|}\,\mathrm{w}^\bullet\,\nabla_{\mathcal{W}}(\mathrm{w}^{\bullet\prime})\,. \tag{C.16}$$

Thus, $\nabla_{\mathcal{W}}$ structures $\mathcal{W} \otimes_A \Omega^\bullet$ as a differential graded algebra. Fedosov's idea [Fed94] is now to modify $\nabla_{\mathcal{W}}$ into a differential $\mathsf{D} : \mathcal{W} \otimes_A \Omega^\bullet \to \mathcal{W} \otimes_A \Omega^{\bullet+1}$, such that the kernel $\ker(\mathsf{D}) \cap \mathcal{W}$, which is a unital $*$-algebra under the product inherited from $\mathcal{W} \otimes_A \Omega^\bullet$, gives the desired deformation quantization of A. Because $\nabla_{\mathcal{W}}$ is flat, this construction drastically simplifies and we do not have to take into account the corrections by curvature dependent terms as in [Fed94]. The Fedosov differential for our model is given by

$$\mathsf{D} := -\delta + \nabla_{\mathcal{W}}\,, \tag{C.17}$$

where $\delta : \mathcal{W} \otimes_A \Omega^\bullet \to \mathcal{W} \otimes_A \Omega^{\bullet+1}$ is the A-module homomorphism defined by linearity and, for all $a \otimes \big([\varphi_1]^{\mathrm{lin}} \cdots [\varphi_n]^{\mathrm{lin}}\big) \otimes \lambda \in A \otimes B \otimes \bigwedge^\bullet \mathfrak{PS}_p^{\mathrm{lin}}(\boldsymbol{M}) \simeq \mathcal{W} \otimes_A \Omega^\bullet$,

$$\delta\Big(a \otimes \big([\varphi_1]^{\mathrm{lin}} \cdots [\varphi_n]^{\mathrm{lin}}\big) \otimes \lambda\Big) = \sum_{j=1}^{n} a \otimes \big([\varphi_1]^{\mathrm{lin}} \cdots \overset{j}{\vee} \cdots [\varphi_n]^{\mathrm{lin}}\big) \otimes \big([\varphi_j]^{\mathrm{lin}} \wedge \lambda\big)\,. \tag{C.18}$$

It is easy to check that δ satisfies the graded Leibniz rule, $\delta^2 = 0$ and $\delta \circ \nabla_{\mathcal{W}} + \nabla_{\mathcal{W}} \circ \delta = 0$, from which it follows that D is a differential on $\mathcal{W} \otimes_A \Omega^\bullet$.

We now come to the characterization of the kernel $\ker(\mathsf{D}) \cap \mathcal{W}$. Like in [Fed94], we are making use of the A-module homomorphism $\delta^* : \mathcal{W} \otimes_A \Omega^\bullet \to \mathcal{W} \otimes_A \Omega^{\bullet-1}$ defined by linearity and, for all $a \otimes b \otimes \big([\varphi_1]^{\mathrm{lin}} \wedge \cdots \wedge [\varphi_n]^{\mathrm{lin}}\big) \in A \otimes B \otimes \bigwedge^\bullet \mathfrak{PS}_p^{\mathrm{lin}}(\boldsymbol{M}) \simeq \mathcal{W} \otimes_A \Omega^\bullet$,

$$\delta^*\Big(a \otimes b \otimes \big([\varphi_1]^{\mathrm{lin}} \wedge \cdots \wedge [\varphi_n]^{\mathrm{lin}}\big)\Big) = \sum_{j=1}^{n} (-1)^{j+1}\, a \otimes \big(b\,[\varphi_j]^{\mathrm{lin}}\big) \otimes \big([\varphi_1]^{\mathrm{lin}} \wedge \cdots \overset{j}{\vee} \cdots \wedge [\varphi_n]^{\mathrm{lin}}\big)\,. \tag{C.19}$$

It is easy to check that $\delta^{*2} = 0$ and that $\delta^* \circ \delta + \delta \circ \delta^* = (n+m)\,\mathrm{id}$, when acting on homogeneous elements $a \otimes \big([\varphi_1]^{\mathrm{lin}} \cdots [\varphi_n]^{\mathrm{lin}}\big) \otimes \big([\varphi_{n+1}]^{\mathrm{lin}} \wedge \cdots \wedge [\varphi_{n+m}]^{\mathrm{lin}}\big)$. The latter property implies that

$$\delta^{-1} \circ \delta + \delta \circ \delta^{-1} + \sigma = \mathrm{id}_{\mathcal{W} \otimes_A \Omega^\bullet} \tag{C.20}$$

on all of $\mathcal{W} \otimes_A \Omega^\bullet$, where $\delta^{-1} : \mathcal{W} \otimes_A \Omega^\bullet \to \mathcal{W} \otimes_A \Omega^{\bullet - 1}$ is defined on homogeneous elements by $\delta^{-1} = \delta^*/(n+m)$ for $n + m \neq 0$ and $\delta^{-1} = 0$ for $n + m = 0$. The linear map $\sigma : \mathcal{W} \otimes_A \Omega^\bullet \to A$ is the projection defined by $\sigma(a \otimes 1 \otimes 1) = a$ and $\sigma(a \otimes b \otimes \lambda) = 0$ if the degree of b or λ is not equal to zero. Following the proof of Fedosov [Fed94], we can show that the map $\sigma : \ker(\mathsf{D}) \cap \mathcal{W} \to A$ is bijective, i.e. that for any $a \in A$ there exists a unique $\mathrm{w} \in \mathcal{W}$, such that $\mathsf{D}(\mathrm{w}) = 0$ and $\sigma(\mathrm{w}) = a$. We briefly sketch the relevant steps: Let $\mathrm{w} \in \mathcal{W}$ be such that $0 = \mathsf{D}(\mathrm{w}) = -\delta \mathrm{w} + \nabla_{\mathcal{W}}(\mathrm{w})$. Applying δ^{-1} and using (C.20) this yields the equation

$$\mathrm{w} = \sigma(\mathrm{w}) + \delta^{-1}\big(\nabla_{\mathcal{W}}(\mathrm{w})\big) . \tag{C.21}$$

Notice that $\sigma(\mathrm{w})$ has degree $(0,0)$ according to the natural grading (n,m) on $\mathcal{W} \otimes_A \Omega^\bullet$ discussed above. The map $\nabla_{\mathcal{W}}$ increases the form-degree by one $(n,m) \mapsto (n, m+1)$, while the map δ^{-1} decreases the form-degree by one and increases the B-degree by one $(n,m) \mapsto (n+1, m-1)$. Hence, $\delta^{-1} \circ \nabla_{\mathcal{W}}$ increases the B-degree by one $(n,m) \mapsto (n+1, m)$ and equation (C.21) can be solved uniquely by iteration for any initial condition $\sigma(\mathrm{w}) = a$ (since A is a polynomial algebra this requires just a finite number of iterations). For any initial condition $\sigma(\mathrm{w}) = a \in A$, the solution w to (C.21) satisfies $\mathsf{D}(\mathrm{w}) = 0$ as a consequence of D being a differential, cf. [Fed94]. This establishes the bijection $\sigma : \ker(\mathsf{D}) \cap \mathcal{W} \to A$ and we define a $\star$-product on A by setting, for all $a, a' \in A$,

$$a \star_{\mathrm{F}} a' := \sigma\big(\sigma^{-1}(a)\, \sigma^{-1}(a')\big) , \tag{C.22}$$

where the product between $\sigma^{-1}(a)$ and $\sigma^{-1}(a')$ is of course taken in $\mathcal{W}$.

It remains to show that (C.22) coincides with the product (7.1). For this let us take $[(\varphi, \alpha)]^m \in A$ and notice that

$$\sigma^{-1}\big([(\varphi,\alpha)]^m\big) = \sum_{j=0}^{m} \binom{m}{j} [(\varphi,\alpha)]^{m-j} \otimes \Big([\varphi]^{\mathrm{lin}}\Big)^j . \tag{C.23}$$

From this expression and a slightly tedious calculation one obtains that

$$[(\varphi,\alpha)]^m \star_{\mathrm{F}} [(\varphi',\alpha')]^n = \sigma\Big(\sigma^{-1}\big([(\varphi,\alpha)]^m\big)\, \sigma^{-1}\big([(\varphi',\alpha')]^n\big)\Big) = [(\varphi,\alpha)]^m \star [(\varphi',\alpha')]^n , \tag{C.24}$$

where the product on the right hand side is (7.1). So the three products (7.1), (C.11) and (C.22) all coincide on the complexification of $\mathfrak{PA}_p(\boldsymbol{M},\boldsymbol{J})$ and hence give the same quantum algebra $\mathfrak{QA}_p(\boldsymbol{M},\boldsymbol{J})$.

To conclude, we make some remarks on the quantization prescription pursued in [SDH12], which is described as being a Fedosov quantization, but in fact differs in essential respects from the Fedosov method [Fed94]. Briefly, the method of [SDH12] is to construct a bundle $\mathcal{A}$ of infinitesimal Weyl algebras over an affine space V, equipped with a Poisson structure, and then to define the quantized algebra as the algebra of flat sections in $\mathcal{A}$ with respect to a certain connection. While this basic idea matches exactly with the Fedosov construction, the starting point chosen in [SDH12] is an *affine* connection on the tangent bundle of V with prescribed affine parallel transport maps between all fibres. The problem with this choice is that the affine connection does not dualize to the cotangent bundle and in particular not to the bundle of infinitesimal Weyl algebras $\mathcal{A}$.[7] This issue is sidestepped in [SDH12], by regarding elements of the infinitesimal Weyl algebras as symmetric polynomials acting on the vector space V_0 on which V is modeled, which permits a unique parallel transport between fibres in $\mathcal{A}$ to be defined. This could be regarded as a slightly ad hoc

[7] The correct dualization would be to the vector dual bundle of the tangent bundle regarded in the category of affine bundles, but this would lead to the following logical problem: Instead of replacing the problem of quantizing affine Poisson spaces by quantizing linear Poisson spaces (which Fedosov's method does, as explained above), the choice of affine connection in [SDH12] replaces the problem of quantizing affine Poisson spaces by quantizing affine Poisson spaces in the fibres.

mixture of the quantized and classical theories, because V_0 is analogous to the classical solution space of the homogeneous theory. By contrast, our prescriptions for improved theories are based on classical structures in the classical case and quantum structures in the quantum case. The connection thus employed in [SDH12] is much more rigid than that of [Fed94], which does not integrate to a unique parallel transport between fibres (see [Fed94, p. 222]).[8] With these thoughts in mind, the approach in [SDH12] might be better described as 'Fedosov-inspired' rather than an application of Fedosov's method as such; nonetheless, this procedure does lead to the correct 'improved algebra'. We note, however, that the summary [SDH12, Definition 4.3.] does not adequately capture the result of the 'Fedosov-inspired' procedure, because the algebraic relations posed do not ensure that the generator corresponding to the unit constant functional on V is a unit in the algebra [as needed in the 'improved' algebra]; as it stands, the algebra therefore has an obvious $\mathbb{Z}_2$ automorphism by negating the generators, which becomes a natural automorphism when the prescription is regarded as a functor from affine Poisson spaces to unital $*$-algebras (as in [SDH12, Lemma 4.11.]) and recreates one of the pathologies of the unimproved theory. The quantizations of specific theories given in [SDH12] are free of this problem, because they are not strict applications of [SDH12, Definition 4.3.], but add a further axiom implementing the inhomogeneous field equation after the manner of [HW05].

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[8] Unique parallel transport depends on theorems on existence and uniqueness of solutions to differential equations that do not necessarily apply to bundles with infinite-dimensional fibres (beyond the Banach case). A related point is that [Fed94] obtains a deformation of the algebra of all smooth functions on the classical phase space, while the result of [SDH12] is more analogous to a deformation of an algebra of certain polynomial functions. In our algebraic version of Fedosov's procedure, the restriction to polynomial functions appears naturally; it should also be possible to extend our construction to cover Wick polynomials by modifying the underlying Poisson algebra, but without modifying the fibre algebras, whereas the corresponding extension of [SDH12] would require modification of the fibres.

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