Bergische Universität Wuppertal

Fachbereich Mathematik und Naturwissenschaften

Institute of Mathematical Modelling, Analysis and
Computational Mathematics (IMACM)

Preprint BUW-IMACM 13/29

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May 2013
http://www.math.uni-wuppertal.de

# Sensitivity analysis of linear dynamical systems in uncertainty quantification

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We consider linear dynamical systems including random parameters for uncertainty quantification. A sensitivity analysis of the stochastic model is applied to the input-output behaviour of the systems. Thus the parameters that contribute most to the variance are detected. Both intrusive and non-intrusive methods based on the polynomial chaos yield the required sensitivity coefficients. We use this approach to analyse a test example from electrical engineering.



## 1 Problem definition

Let the linear dynamical system

$$\begin{array}{rcl} C(p)\frac{\mathrm{d}}{\mathrm{d}t}x(t,p) + G(p)x(t,p) & = & Bu(t) \\ y(t,p) & = & Lx(t,p) \end{array} \qquad (1)$$

be given with inputs $u \in \mathbb{R}^k$, outputs $y \in \mathbb{R}^m$ and state variables $x \in \mathbb{R}^n$. The matrices $C, G \in \mathbb{R}^{n\times n}$ include physical parameters $p \in \Pi \subseteq \mathbb{R}^q$, whereas $B \in \mathbb{R}^{n\times k}$ and $L \in \mathbb{R}^{m\times n}$ are constant. Thus the state variables as well as the outputs depend on the parameters. We assume that the matrix pencil of $C, G$ is regular for each $p \in \Pi$. The input-output behaviour of the system (1) is described by a transfer function $H(\mathrm{i}\omega, p) \in \mathbb{C}^{m\times k}$ with $\mathrm{i} := \sqrt{-1}$ and $\omega \in \mathbb{R}$ in frequency domain, see [1].

To quantify uncertainties, we replace the physical parameters by random variables $p : \Omega \to \Pi$ on an event space $\Omega$. We assume independent random distributions. Let an associated probability density function $\rho : \Pi \to \mathbb{R}$ be available. Consequently, the state variables as well as the outputs become random processes.

## 2 Sensitivity analysis

Our aim is to analyse the sensitivity of the transfer function $H$ with respect to the random parameters $p$. For uniform random distributions, the Sobol decomposition yields variance-based sensitivity coefficients. For general stochastic models, we determine variance-based sensitivity coefficients by the expansions of the polynomial chaos (PC), see [5]. The expansion of a function $f : \Pi \to \mathbb{R}$ reads

$$f(p) = \sum_{l=0}^{\infty} f_l \Phi_l(p) \qquad \text{with} \qquad f_l = \int_{\Pi} f(p)\Phi_l(p)\rho(p)\;\mathrm{d}p \qquad (2)$$

using a system of orthonormal basis polynomials $(\Phi_l)_{l\in\mathbb{N}}$ with $\Phi_l : \Pi \to \mathbb{R}$. Now the sensitivity coefficients are defined by

$$S_j := \frac{V_j}{\mathrm{Var}(f(p))} \quad \text{and} \quad V_j := \sum_{l\in I_j} f_l^2 \quad \text{for} \quad j = 1,\ldots,q \qquad (3)$$

with index sets $I_j := \{l \in \mathbb{N} : \Phi_l(p_1,\ldots,p_q) \text{ is not constant in } p_j\}$. To achieve a finite approximation, the index sets $I_j$ in (3) are replaced by $I_j^d := \{l \in I_j : \mathrm{degree}(\Phi_l) \leq d\}$. As function $f$, we investigate the real part, the imaginary part and the absolute value of each component of the transfer function $H(\mathrm{i}\omega, p) \in \mathbb{C}^{m\times k}$ pointwise for each frequency $\omega$.

There are two types of numerical techniques to compute the sensitivity coefficients $S_1, \ldots, S_q$ :

i) *non-intrusive methods*: The coefficients $f_l$ in (2) are given in form of probabilistic integrals. A quadrature rule or sampling method implies a finite sum as an approximation. It follows that we have to evaluate the transfer function $H$ of the deterministic system (1) for each node of the quadrature scheme.

ii) *intrusive methods*: The stochastic Galerkin approach yields a large linear dynamical system (1) with constant matrices $\hat{C}, \hat{G} \in \mathbb{R}^{rn\times rn}$, where $r$ is the number of basis polynomials up to degree $d$, see [3]. We obtain an approximation of the PC expansion of the transfer function $H$ directly by the components of the transfer function of this larger system.

More details can be found in [4].

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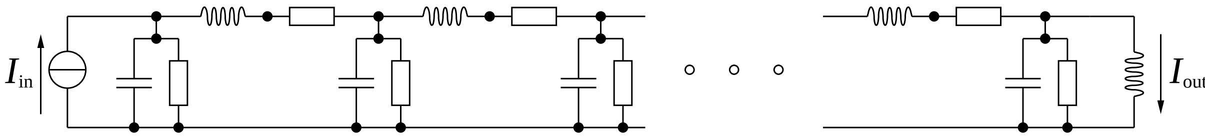


**Fig. 1:** Linear electric network.

## 3 Simulation of a test example

We investigate a linear electric circuit from [2], depicted in Fig. 1. A single input current and a single output current appear ($k = m = 1$), i.e., the transfer function becomes a scalar $H \in \mathbb{C}$. The circuit consists of a repetition of $c$ cells and an additional inductance. As physical parameters, we obtain $c + 1$ capacitances, $c + 1$ inductances, $c + 1$ conductances parallel to the capacitances (group I) and $c$ conductances in line with the inductances (group II). In the stochastic model, we arrange $q = 4c + 3$ random variables for all physical parameters. We choose independent uniform distributions with variations of 20% around the following mean values: $10^{-9}$ for capacitances, $10^{-6}$ for inductances, $10^{-3}$ and $10^{-2}$ for conductances of group I and group II, respectively. A mathematical modelling of the circuit yields a system (1) of differential-algebraic equations of index one including $n = 3c + 3$ state variables. The number $c = 10$ is applied in our simulation.

We employ a non-intrusive technique with the Stroud quadrature formula of order 5, which exhibits $3,699$ nodes within the $43$-dimensional space of the random parameters. The frequency domain $\omega \in [10^2, 10^{10}]$ is observed only, since the transfer function becomes constant outside this interval for each realisation of the parameters. Fig. 2 shows the approximations of the expected values and the standard deviations of the transfer functions separately for the real part and the imaginary part. To compute the sensitivity coefficients (3), we use the PC expansion with all polynomials up to degree $d = 2$, i.e., $r = 946$ basis functions appear. For brevity, just the absolute value $|H|$ of the transfer function is analysed. Fig. 3 illustrates the sensitivities obtained by the non-intrusive technique. On the one hand, the capacitances and inductances are inessential for low frequencies $\omega < 10^6$. Furthermore, the sensitivities of all inductances coincide except for the additional inductance. On the other hand, the sensitivities of the conductance groups are small and agree for high frequencies $\omega > 10^8$. Since some sensitivities are tiny in respective frequency intervals, a model order reduction can be performed in the random space, cf. [4].

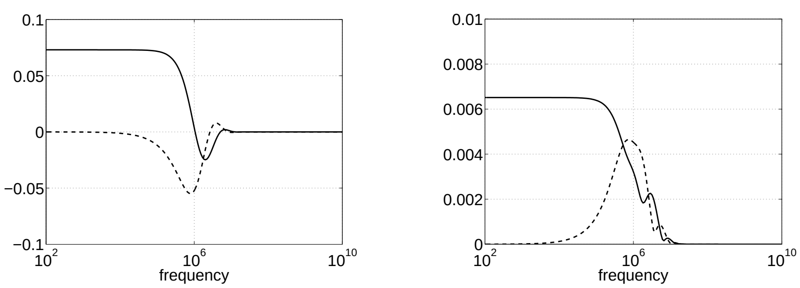


**Fig. 2:** Expected values (left) and st. deviations (right) for real part (solid line) and imaginary part (dashed line) of the transfer function.

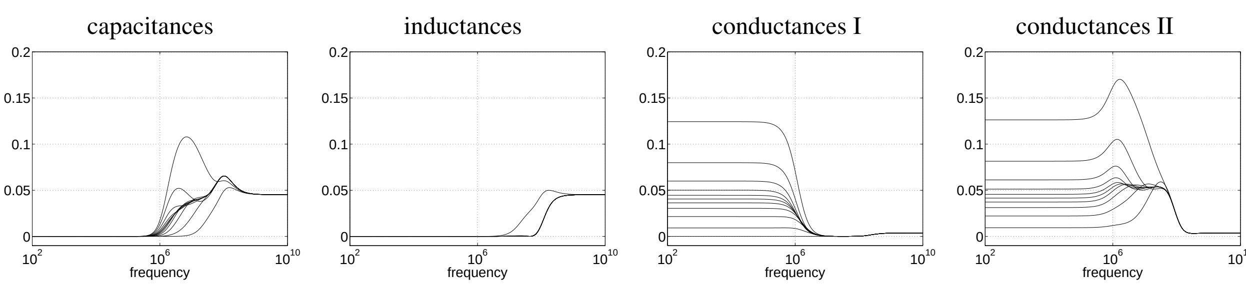


**Fig. 3:** Sensitivity coefficients of absolute value of the transfer function for the four groups of physical parameters.

## References


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