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A CLASS OF GENERALIZED ADDITIVE RUNGE-KUTTA METHODS*

ADRIAN SANDU[†] AND MICHAEL GÜNTHER[‡]

Abstract. This work generalizes the additively partitioned Runge-Kutta methods by allowing for different stage values as arguments of different components of the right hand side. An order conditions theory is developed for the new family of generalized additive methods, and stability and monotonicity investigations are carried out. The paper discusses the construction and properties of implicit-explicit and implicit-implicit methods in the new framework. The new family, named GARK, introduces additional flexibility when compared to traditional partitioned Runge-Kutta methods, and therefore offers additional opportunities for the development of flexible solvers for systems with multiple scales, or driven by multiple physical processes.

Key words. Partitioned Runge-Kutta methods, NB-series, algebraic stability, absolute monotonicity, implicit-explicit, implicit-implicit methods

AMS subject classifications. 65L05, 65L06, 65L07, 65L020.

1. Introduction. In many applications, initial value problems of ordinary differential equations are given as *additively* partitioned systems

$$(1.1) \quad y' = f(y) = \sum_{m=1}^N f^{\{m\}}(y), \quad y(t_0) = y_0,$$

where the right-hand side $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is split into N different parts with respect to, for example, stiffness, nonlinearity, dynamical behavior, and evaluation cost. Additive partitioning also includes the special case of *component* partitioning where the solution vector is split into N disjoint sets, $\{1, 2, \dots, d\} = \cup_{m=1}^N \mathcal{I}^{\{m\}}$, with the m -th set containing the components y^i with indices $i \in \mathcal{I}^{\{m\}}$. One defines a corresponding partitioning of the right hand side

$$(1.2) \quad f^{\{m\}}(y) := \sum_{i \in \mathcal{I}^{\{m\}}} \mathbf{e}_i \mathbf{e}_i^T f(y), \quad f^{\{m\}i}(y) = \begin{cases} f^i(y), & i \in \mathcal{I}^{\{m\}} \\ 0, & i \notin \mathcal{I}^{\{m\}} \end{cases},$$

where \mathbf{e}_i is the i -th column of the identity matrix, and superscripts (without parentheses) represent vector components. A particular case is *coordinate* partitioning where $d = N$ and $\mathcal{I}^{\{m\}} = \{m\}$, $m = 1, \dots, d$:

$$(1.3) \quad y' = \begin{bmatrix} y^1 \\ \vdots \\ y^N \end{bmatrix}' = \sum_{m=1}^d \begin{bmatrix} 0 \\ y^m \\ 0 \end{bmatrix}' = \sum_{m=1}^d \begin{bmatrix} 0 \\ f^m(y) \\ 0 \end{bmatrix}.$$

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The development of Runge-Kutta (RK) methods that are tailored to the partitioned system (1.1) started with the early work of Rice [21]. Hairer [7] developed the concept of P-trees and laid the foundation for the modern order conditions theory for partitioned Runge-Kutta methods. The investigation of practical partitioned Runge-Kutta methods with implicit and explicit components [20] was revitalized by the work of Ascher, Ruuth, and Spiteri [23]. Additive Runge-Kutta methods have been investigated by Cooper and Sayfy [2] and Kennedy and Carpenter [16], and partitioning strategies have been discussed by Weiner [24].

This work focuses on specialized schemes which exploit the different dynamics in the right hand sides $f^{\{m\}}$ (e.g., stiff and non-stiff), allow for arbitrarily high orders of accuracy, and possess good stability properties with respect to dissipative systems. The approach taken herein generalizes the additively partitioned Runge-Kutta family of methods [16] by allowing for different stage values as arguments of different components of the right hand side.

The paper is organized as follows. Section 2 introduces the new family of generalized additively partitioned Runge-Kutta schemes. Generalized implicit-explicit Runge-Kutta schemes are discussed in Section 3. A stability and monotonicity analysis is performed in Section 4. Section 5 builds implicit-implicit generalized additively partitioned Runge-Kutta methods. Conclusions are drawn in Section 6.

2. Generalized additively partitioned Runge-Kutta schemes. In this section we extend additive Runge-Kutta to generalized additively partitioned Runge-Kutta schemes. The order conditions for this new class are derived using the N-tree theory of Sanz-Serna [1].

2.1. Traditional additive Runge-Kutta methods. Kennedy and Carpenter [16] developed additive Runge-Kutta (ARK) methods for systems with partitioned right hand sides (1.1). One step of an ARK scheme reads

$$(2.1a) \quad Y_i = y_n + h \sum_{m=1}^N \sum_{j=1}^s a_{i,j}^{\{m\}} f^{\{m\}}(Y_j), \quad i = 1, \dots, s,$$

$$(2.1b) \quad y_{n+1} = y_n + h \sum_{q=1}^N \sum_{i=1}^s b_i^{\{q\}} f^{\{q\}}(Y_i).$$

The corresponding Butcher tableau is

$$(2.2) \quad \begin{array}{cccc} \mathbf{A}^{\{1\}} & \mathbf{A}^{\{2\}} & \dots & \mathbf{A}^{\{N\}} \\ \hline \mathbf{b}^{\{1\}} & \mathbf{b}^{\{2\}} & \dots & \mathbf{b}^{\{N\}} \end{array}.$$

2.2. Generalized additively partitioned Runge-Kutta schemes. The *generalized additively partitioned Runge-Kutta* (GARK) family of methods extends the traditional approach (2.1) by allowing for different stage values with different compo-

with the exact solution $y(t) = [t, t]^T$. The GARK stages (2.3a) are

$$Y_i^{\{q\}} = \begin{bmatrix} t_n + h \sum_{m=1}^N \alpha^{\{m\}} \sum_{j=1}^{s^{\{m\}}} a_{i,j}^{\{q,m\}} \\ t_n + h \sum_{m=1}^N \beta^{\{m\}} \sum_{j=1}^{s^{\{m\}}} a_{i,j}^{\{q,m\}} \end{bmatrix} = \begin{bmatrix} t_n + h c_i^{\{q\}} \\ t_n + h c_i^{\{q\}} \end{bmatrix}.$$

The internal consistency condition (2.6) ensures that all components of the stage vectors are calculated at the same internal approximation times.

2.3. Order conditions. We derive the GARK order conditions by applying Araujo, Murua, and Sanz-Serna's N-tree theory [1] to (2.3), while taking into account the fact that the internal stages $Y_i^{\{m\}}$ and the stage numbers $s^{\{m\}}$ depend on the partition $f^{\{m\}}$.

N-trees [1] are a generalization of P-trees from the case of component partitioning (1.3) to the general case of right-hand side partitioning (1.2). The set NT of N-trees consists of all Butcher trees with colored vertices; each vertex is assigned one of N different colors corresponding to the N components of the partition. Similar to regular Butcher trees each vertex is also assigned a label. The order $\rho(u)$ is the number of nodes of $u \in \text{NT}$.

The empty N-tree is denoted by \emptyset . The N-tree with a single vertex of color ν is denoted by $\tau^{[\nu]}$. The N-tree $u \in \text{NT}$ with $\rho(u) > 1$ and a root of color ν can be represented as $u = [u_1, \dots, u_m]^{[\nu]}$, where $\{u_1, \dots, u_m\}$ are the non-empty subtrees (N-trees) arising from removing the root of u .

Similar to regular Butcher trees one denotes by $\sigma(u)$ the number of symmetries of $u \in \text{NT}$, and by $\gamma(u)$ the density defined recursively by

$$\begin{aligned} \gamma(\emptyset) &= 1, \\ \gamma(\tau^{[\nu]}) &= 1, \quad \nu = 1, \dots, N, \\ \gamma(u) &= \rho(u) \gamma(u_1) \cdots \gamma(u_m) \text{ for } u = [u_1, \dots, u_m]^{[\nu]}. \end{aligned}$$

An elementary differential $F(u)(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is associated to each N-tree $u \in \text{NT}$. The elementary differentials are defined recursively for each component $i = 1, \dots, d$ as follows:

$$\begin{aligned} F^i(\emptyset)(y) &= y^i, \\ F^i(\tau^{[\nu]})(y) &= f^{[\nu]i}(y), \quad \nu = 1, \dots, N, \\ F^i(u)(y) &= \sum_{i_1, \dots, i_m=1}^d \frac{\partial^m f^{[\nu]i}}{\partial y^{i_1} \cdots \partial y^{i_m}}(y) F^{i_1}(u_1)(y) \cdots F^{i_m}(u_m)(y) \\ &\quad \text{for } u = [u_1, \dots, u_m]^{[\nu]}. \end{aligned}$$

An NB-series is a formal power expansion

$$\text{NB}(\mathbf{c}, y(t)) := \sum_{u \in \text{NT}} \frac{h^{\rho(u)}}{\sigma(u)} \mathbf{c}(u) F(u)(y(t)),$$

where $\mathbf{c} : \text{NT} \rightarrow \mathbb{R}$ is a mapping that assigns a real number to each N-tree. For example, the exact solution of (1.1) can be written as the following NB-series [1]

$$(2.7) \quad y(t+h) = \text{NB}(\mathbf{c}, y(t)) \quad \text{with} \quad \mathbf{c}(u) = \frac{1}{\gamma(u)}.$$

THEOREM 2.3 (GARK order conditions). *The order conditions for a GARK method (2.3) are obtained from the order conditions of ordinary Runge-Kutta methods. The usual labeling of the Runge-Kutta coefficients (subscripts i, j, k, \dots) is accompanied by a corresponding labeling of the different partitions for the N -tree (superscripts σ, ν, μ, \dots).*

Let $\mathbf{1}^{\{\nu\}}$ be a vector of ones of dimension $s^{\{\nu\}}$, and $\mathbf{c}^{\{\sigma, \nu\}} = \mathbf{A}^{\{\sigma, \nu\}} \cdot \mathbf{1}^{\{\nu\}}$. The specific conditions for orders one to four are as follows.

$$(2.8a) \quad \mathbf{b}^{\{\sigma\}^T} \cdot \mathbf{1}^{\{\sigma\}} = 1, \quad \forall \sigma. \quad (\text{order } 1)$$

$$(2.8b) \quad \mathbf{b}^{\{\sigma\}^T} \cdot \mathbf{c}^{\{\sigma, \nu\}} = \frac{1}{2}, \quad \forall \sigma, \nu, \quad (\text{order } 2)$$

$$(2.8c) \quad \mathbf{b}^{\{\sigma\}^T} \cdot (\mathbf{c}^{\{\sigma, \nu\}} \mathbf{c}^{\{\sigma, \mu\}}) = \frac{1}{3}, \quad \forall \sigma, \nu, \mu, \quad (\text{order } 3)$$

$$(2.8d) \quad \mathbf{b}^{\{\sigma\}^T} \cdot \mathbf{A}^{\{\sigma, \nu\}} \cdot \mathbf{c}^{\{\nu, \mu\}} = \frac{1}{6}, \quad \forall \sigma, \nu, \mu, \quad (\text{order } 3)$$

$$(2.8e) \quad \mathbf{b}^{\{\sigma\}^T} \cdot (\mathbf{c}^{\{\sigma, \nu\}} \mathbf{c}^{\{\sigma, \lambda\}} \mathbf{c}^{\{\sigma, \mu\}}) = \frac{1}{4}, \quad \forall \sigma, \nu, \lambda, \mu, \quad (\text{order } 4)$$

$$(2.8f) \quad (\mathbf{b}^{\{\sigma\}} \mathbf{c}^{\{\sigma, \mu\}})^T \cdot \mathbf{A}^{\{\sigma, \nu\}} \cdot \mathbf{c}^{\{\nu, \lambda\}} = \frac{1}{8}, \quad \forall \sigma, \nu, \lambda, \mu, \quad (\text{order } 4)$$

$$(2.8g) \quad \mathbf{b}^{\{\sigma\}^T} \cdot \mathbf{A}^{\{\sigma, \nu\}} \cdot (\mathbf{c}^{\{\nu, \lambda\}} \mathbf{c}^{\{\nu, \mu\}}) = \frac{1}{12}, \quad \forall \sigma, \nu, \lambda, \mu, \quad (\text{order } 4)$$

$$(2.8h) \quad \mathbf{b}^{\{\sigma\}^T} \cdot \mathbf{A}^{\{\sigma, \nu\}} \cdot \mathbf{A}^{\{\lambda, \mu\}} \cdot \mathbf{c}^{\{\lambda, \mu\}} = \frac{1}{24}, \quad \forall \sigma, \nu, \lambda, \mu. \quad (\text{order } 4)$$

Here, and throughout this paper, the matrix and vector multiplication is denoted by dot (e.g., $\mathbf{b}^T \cdot \mathbf{c}$ is a dot product), while two adjacent vectors denote component-wise multiplication (e.g., $\mathbf{b} \mathbf{c}$ is a vector of element-wise products). These order conditions are given in element-wise form in Appendix A.

Proof. We derive the NB-series of the solution of the GARK scheme (2.3) following the approach in [1] for the NB-series of ARK schemes. The GARK solution, the stage vectors, and the stage function values are expanded in NB-series

$$\begin{aligned} y_{n+1} &= \text{NB}(\mathbf{d}, y_n), \\ Y_i^{\{q\}} &= \text{NB}(\mathbf{d}_i^{\{q\}}, y_n), \\ hf^{\{m\}}(Y_i^{\{m\}}) &= \text{NB}(\mathbf{g}_i^{\{m\}}, y_n). \end{aligned}$$

From (2.7) we have that the method GARK is of order p iff

$$(2.9) \quad \mathbf{d}(u) = \frac{1}{\gamma(u)}, \quad \forall u \in \text{NT} \quad \text{with} \quad 1 \leq \rho(u) \leq p.$$

The coefficients \mathbf{d} , $\mathbf{d}_i^{\{q\}}$ and $\mathbf{g}_i^{\{m\}}$ are related through the numerical equations (2.3). For $u \in \text{NT} \setminus \{\emptyset\}$ relation (2.3b) yields

$$(2.10a) \quad \mathbf{d}(u) = \sum_{q=1}^N \sum_{i=1}^{s^{\{q\}}} b_i^{\{q\}} \mathbf{g}_i^{\{q\}}(u),$$

and relation (2.3a) gives

$$(2.10b) \quad \mathbf{d}_i^{\{q\}}(u) = \sum_{m=1}^N \sum_{j=1}^{s^{\{m\}}} a_{i,j}^{\{q,m\}} \mathbf{g}_j^{\{m\}}(u),$$

$$i = 1, \dots, s^{\{q\}}, \quad q = 1, \dots, N.$$

From the properties of a derivative of a B-series on has

$$(2.10c) \quad \mathbf{g}_i^{\{m\}}(u) = \begin{cases} 0 & \text{for } u = \emptyset, \\ \delta_{q,m} & \text{for } u = \tau^{\{q\}}, \\ \delta_{q,m} \prod_{k=1}^{\ell} \mathbf{d}_i^{\{m\}}(u_k) & \text{for } u = [u_1, \dots, u_\ell]^{\{q\}}, \end{cases}$$

$$i = 1, \dots, s^{\{m\}}, \quad m, q = 1, \dots, N.$$

Using equations (2.10b)–(2.10c) recursively, we have that for $u = [u_1, \dots, u_\ell]^{\{q\}}$

$$\mathbf{g}_i^{\{m\}}(u) = \delta_{q,m} \sum_{n_1, \dots, n_\ell} \sum_{j_1, \dots, j_\ell} a_{i,j_1}^{\{m,n_1\}} \dots a_{i,j_\ell}^{\{m,n_\ell\}} \mathbf{g}_{j_1}^{\{n_1\}}(u_1) \dots \mathbf{g}_{j_\ell}^{\{n_\ell\}}(u_\ell).$$

Equation (2.10c) becomes

$$(2.11) \quad \mathbf{g}_i^{\{m\}}(u) = \begin{cases} 0 & \text{for } u = \emptyset, \\ 0 & \text{for } u = [u_1, \dots, u_\ell]^{\{q\}} \text{ and } q \neq m, \\ 1 & \text{for } u = \tau^{\{m\}}, \\ \sum_{j_1, \dots, j_\ell=1}^{s^{\{m\}}} a_{i,j_1}^{\{m,\nu_1\}} \dots a_{i,j_\ell}^{\{m,\nu_\ell\}} \mathbf{g}_{j_1}^{\{\nu_1\}}(u_1) \dots \mathbf{g}_{j_\ell}^{\{\nu_\ell\}}(u_\ell) & \text{for } u = [u_1, \dots, u_\ell]^{\{m\}} \text{ and } \text{color}(\text{root}(u_j)) = \nu_j, \end{cases}$$

$$i = 1, \dots, s^{\{m\}}, \quad m = 1, \dots, N.$$

From (2.10a) we have

$$(2.12) \quad \mathbf{d}(u) = \sum_{i=1}^{s^{\{m\}}} b_i^{\{m\}} \mathbf{g}_i^{\{m\}}(u) \quad \text{for } u = [u_1, \dots, u_\ell]^{\{m\}}.$$

For a regular Runge-Kutta method we ignore the coloring of the tree nodes and consider $u \in T$, where T are the regular Butcher trees. The NB-series expansions are regular B-series expansions

$$y_{n+1} = B(\mathbf{c}, y_n), \quad Y_i = B(\mathbf{c}_i, y_n), \quad hf(Y_i) = B(\mathbf{f}_i, y_n)$$

whose coefficients are given by

$$(2.13) \quad \mathbf{f}_i(u) = \begin{cases} 0 & \text{for } u = \emptyset, \\ 1 & \text{for } u = \tau, \\ \sum_{j_1, \dots, j_\ell=1}^s a_{i,j_1} \dots a_{i,j_\ell} \mathbf{f}_{j_1}(u) \dots \mathbf{f}_{j_\ell}(u) & \text{for } u = [u_1, \dots, u_\ell]^{\{\bullet\}}, \end{cases}$$

$$i = 1, \dots, s,$$

$$(2.14) \quad \mathbf{c}(u) = \sum_{i=1}^s b_i \mathbf{f}_i(u).$$

The RK method has order p iff

$$(2.15) \quad \mathbf{c}(u) = \frac{1}{\gamma(u)}, \quad \forall u \in \mathbf{T} \quad \text{with} \quad 1 \leq \rho(u) \leq p.$$

For any N-tree $u \in \mathbf{NT}$ the recurrences (2.11) and (2.12) mimic the B-series relations (2.13) and (2.14), respectively. Each coefficient subscript in (2.13), (2.14) is paired with a unique color superscript in (2.12) and (2.13), for example $i \leftrightarrow \{m\}$, $j_1 \leftrightarrow \{\nu_1\}$, and $j_\ell \leftrightarrow \{\nu_\ell\}$. The NB-series coefficient $\mathbf{d}(u)$ written in terms of the GARK scheme coefficients $b_i^{\{m\}}$, $a_{i,j}^{\{m,\nu\}}$ has precisely the same form as the B-series coefficient $\mathbf{c}(u)$ written in terms of the RK scheme coefficients b_i , $a_{i,j}$, with color superscripts added to match the corresponding subscripts. (The calculation of $\mathbf{c}(u)$ ignores the vertex coloring of u .) Since (2.9) has to hold for any coloring of the vertices of $u \in \mathbf{NT}$, the corresponding RK condition (2.15) has to hold for arbitrary sets of superscripts, provided that they are paired correctly with the coefficient subscripts. \square

REMARK 1. For N-trees $u \in \mathbf{NT}$ with all vertices of the same color q one obtains the traditional RK order conditions for the individual method $(\mathbf{A}^{\{q,q\}}, \mathbf{b}^{\{q\}})$. As an example consider the order conditions (2.8) for equal superscripts $\sigma = \nu = \lambda = \mu = q$. Therefore a necessary condition for a GARK method to have order p is that each individual component method has order p . In addition, the GARK method needs to satisfy the coupling conditions resulting from trees with vertices of different colors.

REMARK 2. For internally consistent GARK methods (2.6) the order conditions simplify considerably. Conditions (2.8a), (2.8b), and (2.8c) become

$$\mathbf{b}^{\{\sigma\}^T} \cdot \mathbf{1}^{\{\sigma\}} = 1, \quad \mathbf{b}^{\{\sigma\}^T} \cdot \mathbf{c}^{\{\sigma\}} = \frac{1}{2}, \quad \mathbf{b}^{\{\sigma\}^T} \cdot (\mathbf{c}^{\{\sigma\}} \mathbf{c}^{\{\sigma\}}) = \frac{1}{3}, \quad \forall \sigma,$$

and correspond to order two conditions, and to the first order three condition of each individual component method $(\mathbf{A}^{\{\sigma,\sigma\}}, \mathbf{b}^{\{\sigma\}})$, $\sigma = 1, \dots, N$.

Condition (2.8d) becomes

$$\mathbf{b}^{\{\sigma\}^T} \cdot \mathbf{A}^{\{\sigma,\nu\}} \cdot \mathbf{c}^{\{\nu\}} = \frac{1}{6}, \quad \forall \sigma, \nu,$$

and gives rise to N order three conditions for each component method, and $N^2 - N$ coupling conditions. Compare this with the $N^3 - N$ coupling conditions in the absence of internal consistency.

The $4N^4$ order four conditions (2.8e)–(2.8h) reduce to the following $N + 2N^2 + N^3$ equations

$$\begin{aligned} \mathbf{b}^{\{\sigma\}^T} \cdot (\mathbf{c}^{\{\sigma\}} \mathbf{c}^{\{\sigma\}} \mathbf{c}^{\{\sigma\}}) &= \frac{1}{4}, \quad \forall \sigma, \nu, \lambda, \\ (\mathbf{b}^{\{\sigma\}} \mathbf{c}^{\{\sigma\}})^T \cdot \mathbf{A}^{\{\sigma,\nu\}} \cdot \mathbf{c}^{\{\nu\}} &= \frac{1}{8}, \quad \forall \sigma, \nu, \lambda, \\ \mathbf{b}^{\{\sigma\}^T} \cdot \mathbf{A}^{\{\sigma,\nu\}} \cdot (\mathbf{c}^{\{\nu\}} \mathbf{c}^{\{\nu\}}) &= \frac{1}{12}, \quad \forall \sigma, \nu, \lambda, \\ \mathbf{b}^{\{\sigma\}^T} \cdot \mathbf{A}^{\{\sigma,\nu\}} \cdot \mathbf{A}^{\{\lambda,\mu\}} \cdot \mathbf{c}^{\{\lambda\}} &= \frac{1}{24}, \quad \forall \sigma, \nu, \lambda. \end{aligned}$$

3. Implicit-explicit GARK schemes. We now focus on systems (1.1) with a two-way partitioned right hand side

$$(3.1) \quad y' = f(y) + g(y), \quad y(t_0) = y_0,$$

where f is non-stiff and g is stiff.

3.1. Formulation of IMEX-GARK schemes. An implicit-explicit (IMEX) GARK scheme is a two way partitioned method (2.3) where one component method ($\mathbf{A}^{\{E,E\}}, \mathbf{b}^{\{E\}}$) is explicit, and the other one ($\mathbf{A}^{\{I,I\}}, \mathbf{b}^{\{I\}}$) implicit:

$$(3.2a) \quad Y_i = y_n + h \sum_{j=1}^{s^{\{E\}}} a_{i,j}^{\{E,E\}} f(Y_j) + h \sum_{j=1}^{s^{\{I\}}} a_{i,j}^{\{E,I\}} g(Z_j), \quad i = 1, \dots, s^{\{E\}},$$

$$(3.2b) \quad Z_i = y_n + h \sum_{j=1}^{s^{\{E\}}} a_{i,j}^{\{I,E\}} f(Y_j) + h \sum_{j=1}^{s^{\{I\}}} a_{i,j}^{\{I,I\}} g(Z_j), \quad i = 1, \dots, s^{\{I\}},$$

$$(3.2c) \quad y_{n+1} = y_n + h \sum_{i=1}^{s^{\{E\}}} b_i^{\{E\}} f(Y_i) + h \sum_{i=1}^{s^{\{I\}}} b_i^{\{I\}} g(Z_i).$$

The corresponding generalized Butcher tableau is

$$(3.3) \quad \begin{array}{c|c|c|c} \mathbf{c}^{\{E,E\}} & \mathbf{A}^{\{E,E\}} & \mathbf{A}^{\{E,I\}} & \mathbf{c}^{\{E,I\}} \\ \mathbf{c}^{\{I,E\}} & \mathbf{A}^{\{I,E\}} & \mathbf{A}^{\{I,I\}} & \mathbf{c}^{\{I,I\}} \\ \hline & \mathbf{b}^{\{E\}} & \mathbf{b}^{\{I\}} & \end{array},$$

with

$$\mathbf{A}^{\{\sigma,\nu\}} \in \mathbb{R}^{s^{\{\sigma\}} \times s^{\{\nu\}}}, \quad \mathbf{b}^{\{\sigma\}} \in \mathbb{R}^{s^{\{\sigma\}}}, \quad \mathbf{c}^{\{\sigma,\nu\}} = \mathbf{A}^{\{\sigma,\nu\}} \cdot \mathbf{1}^{\{\nu\}} \in \mathbb{R}^{s^{\{\sigma\}}},$$

for all $\sigma, \nu \in \{I, E\}$, where $\mathbf{1}^{\{\nu\}} = [1 \dots 1]^T \in \mathbb{R}^{s^{\{\nu\}}}$.

EXAMPLE 1 (Classical IMEX RK methods).

In (3.2) we use

$$\mathbf{A}^{\{I,E\}} = \mathbf{A}^{\{E,E\}} \equiv \mathbf{A}^{\{E\}}, \quad \mathbf{A}^{\{E,I\}} = \mathbf{A}^{\{I,I\}} \equiv \mathbf{A}^{\{I\}},$$

with $\mathbf{A}^{\{I\}}$ lower triangular and $\mathbf{A}^{\{E\}}$ strictly lower triangular. We obtain $s^{\{E\}} = s^{\{I\}} = s$, $Y_i = Z_i$, and recover the classical IMEX RK of Ascher, Ruuth, and Spiteri [23]:

$$(3.4a) \quad Y_i = y_n + h \sum_{j=1}^{i-1} a_{i,j}^{\{E\}} f(Y_j) + h \sum_{j=1}^i a_{i,j}^{\{I\}} g(Y_j), \quad i = 1, \dots, s,$$

$$(3.4b) \quad y_{n+1} = y_n + h \sum_{i=1}^s b_i^{\{E\}} f(Y_i) + h \sum_{i=1}^s b_i^{\{I\}} g(Y_i).$$

EXAMPLE 2 (Classical-transposed IMEX RK methods).

In (3.2) we use

$$\mathbf{A}^{\{E,I\}} = \mathbf{A}^{\{E,E\}} \equiv \mathbf{A}^{\{E\}}, \quad \mathbf{A}^{\{I,E\}} = \mathbf{A}^{\{I,I\}} \equiv \mathbf{A}^{\{I\}},$$

with $\mathbf{A}^{\{I\}}$ lower triangular and $\mathbf{A}^{\{E\}}$ strictly lower triangular, to obtain $s^{\{E\}} = s^{\{I\}} = s$, and the interesting family of schemes:

$$(3.5a) \quad Y_i = y_n + h \sum_{j=1}^{i-1} a_{i,j}^{\{E\}} (f(Y_j) + g(Z_j)), \quad i = 1, \dots, s,$$

$$(3.5b) \quad Z_i = y_n + h \sum_{j=1}^i a_{i,j}^{\{I\}} (f(Y_j) + g(Z_j)), \quad i = 1, \dots, s,$$

$$(3.5c) \quad y_{n+1} = y_n + h \sum_{i=1}^s b_i^{\{E\}} f(Y_i) + h \sum_{i=1}^s b_i^{\{I\}} g(Z_i),$$

The low storage IMEX methods recently presented by Higuera and Roldan [14] can also be formulated as IMEX-GARK schemes (3.2).

3.2. Order conditions. The GARK order conditions in matrix form are given in (2.8). Each of the implicit and explicit methods $(\mathbf{A}^{\{\sigma,\sigma\}}, \mathbf{b}^{\{\sigma\}})$ and $\mathbf{c}^{\{\sigma,\sigma\}} = \mathbf{A}^{\{\sigma,\sigma\}} \cdot \mathbf{1}$ has to satisfy the corresponding order conditions for $\sigma \in \{E, I\}$. In addition, two coupling conditions (B.1) are required for second order, and 12 coupling conditions (B.2) are required for third order accuracy. These conditions are listed in Appendix B.

The GARK internal consistency condition (2.6) reads

$$(3.6) \quad \mathbf{c}^{\{I,E\}} = \mathbf{c}^{\{I,I\}} = \mathbf{c}^{\{I\}}, \quad \mathbf{c}^{\{E,I\}} = \mathbf{c}^{\{E,E\}} = \mathbf{c}^{\{E\}}.$$

These conditions are automatically satisfied in the case of classical-transposed IMEX RK methods. In case of classical IMEX RK equations (3.6) are equivalent to $\mathbf{c}^{\{I\}} = \mathbf{c}^{\{E\}}$.

Assuming that both the implicit and the explicit methods have order at least three, (3.6) implies that the second order coupling conditions (B.1) are automatically satisfied. The third order coupling conditions (B.2) reduce to:

$$\mathbf{b}^{\{E\}^T} \cdot \mathbf{A}^{\{E,I\}} \cdot \mathbf{c}^{\{I\}} = \frac{1}{6}, \quad \mathbf{b}^{\{I\}^T} \cdot \mathbf{A}^{\{I,E\}} \cdot \mathbf{c}^{\{E\}} = \frac{1}{6}.$$

Assume that the implicit and explicit methods have order at least four. With (3.6) the order conditions (2.8e)–(2.8h) reduce to the following order four coupling relations:

$$(3.7) \quad \begin{aligned} & \left(\mathbf{b}^{\{\sigma\}} \mathbf{c}^{\{\sigma\}} \right)^T \cdot \mathbf{A}^{\{\sigma,\nu\}} \cdot \mathbf{c}^{\{\nu\}} = \frac{1}{8}, \quad \forall \sigma, \nu \in \{E, I\}, \sigma \neq \nu, \\ & \mathbf{b}^{\{\sigma\}^T} \cdot \mathbf{A}^{\{\sigma,\nu\}} \cdot \left(\mathbf{c}^{\{\nu\}} \mathbf{c}^{\{\nu\}} \right) = \frac{1}{12}, \quad \forall \sigma, \nu \in \{E, I\}, \sigma \neq \nu, \\ & \mathbf{b}^{\{\sigma\}^T} \cdot \mathbf{A}^{\{\sigma,\nu\}} \cdot \mathbf{A}^{\{\lambda,\mu\}} \cdot \mathbf{c}^{\{\lambda\}} = \frac{1}{24}, \quad \forall \sigma, \nu, \lambda, \mu \in \{E, I\}. \end{aligned}$$

These order conditions are listed explicitly in Appendix C.

Additional simplifying assumptions

$$(3.8) \quad \mathbf{s}^{\{E\}} = \mathbf{s}^{\{I\}}, \quad \mathbf{b}^{\{E\}} = \mathbf{b}^{\{I\}} = \mathbf{b}, \quad \mathbf{c}^{\{E\}} = \mathbf{c}^{\{I\}} = \mathbf{c}.$$

further reduce the number of order conditions; these are listed in Appendix C.

3.3. Construction of classical-transposed IMEX RK. Consider now the classical-transposed methods (3.5) and assume that both the explicit and the implicit method have order three. Since (3.6) holds, the remaining third order coupling conditions read:

$$(3.9) \quad \mathbf{b}^{\{E\}^T} \cdot \mathbf{A}^{\{E\}} \cdot \mathbf{c}^{\{I\}} = \frac{1}{6}, \quad \mathbf{b}^{\{I\}^T} \cdot \mathbf{A}^{\{I\}} \cdot \mathbf{c}^{\{E\}} = \frac{1}{6}.$$

Assuming that the explicit and implicit methods have each order at least four,

the order four coupling conditions are:

$$(3.10) \quad \begin{aligned} (\mathbf{b}^{\{E\}} \mathbf{c}^{\{E\}})^T \cdot \mathbf{A}^{\{E\}} \cdot \mathbf{c}^{\{I\}} &= \frac{1}{8} \\ \mathbf{b}^{\{E\}T} \cdot \mathbf{A}^{\{E\}} \cdot (\mathbf{c}^{\{I\}} \mathbf{c}^{\{I\}}) &= \frac{1}{12} \\ \mathbf{b}^{\{E\}T} \cdot \mathbf{A}^{\{E\}} \cdot \mathbf{A}^{\{I\}} \cdot \mathbf{c}^{\{I\}} &= \frac{1}{24} \\ (\mathbf{b}^{\{I\}} \mathbf{c}^{\{I\}})^T \cdot \mathbf{A}^{\{I\}} \cdot \mathbf{c}^{\{E\}} &= \frac{1}{8} \\ \mathbf{b}^{\{I\}T} \cdot \mathbf{A}^{\{I\}} \cdot (\mathbf{c}^{\{E\}} \mathbf{c}^{\{E\}}) &= \frac{1}{12} \\ \mathbf{b}^{\{I\}T} \cdot \mathbf{A}^{\{I\}} \cdot \mathbf{A}^{\{E\}} \cdot \mathbf{c}^{\{E\}} &= \frac{1}{24} \end{aligned}$$

If, in addition, $\mathbf{b}^{\{E\}} = \mathbf{b}^{\{I\}} = \mathbf{b}$ and $\mathbf{c}^{\{E\}} = \mathbf{c}^{\{I\}} = \mathbf{c}$, then the third order coupling conditions (3.9) are automatically satisfied. The remaining order four coupling conditions (3.10) read:

$$\mathbf{b}^T \cdot \mathbf{A}^{\{E\}} \cdot \mathbf{A}^{\{I\}} \cdot \mathbf{c} = \frac{1}{24}, \quad \mathbf{b}^T \cdot \mathbf{A}^{\{I\}} \cdot \mathbf{A}^{\{E\}} \cdot \mathbf{c} = \frac{1}{24}.$$

Example: an order three IMEX method. The implicit part is Kvaerno's four stages, order three method [18, ESDIRK 3/2]:

$$\gamma = 0.435866521508459,$$

$$\mathbf{A}^{\{I\}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \gamma & \gamma & 0 & 0 \\ 0.490563388421781 & 0.073570090069760 & \gamma & 0 \\ 0.308809969976747 & 1.490563388421781 & -1.235239879906987 & \gamma \end{bmatrix},$$

$$\mathbf{b}^{\{I\}} = \begin{bmatrix} 0.308809969976747 \\ 1.490563388421781 \\ -1.235239879906987 \\ 0.435866521508459 \end{bmatrix}, \quad \mathbf{c}^{\{I\}} = \begin{bmatrix} 0 \\ 0.871733043016918 \\ 1 \\ 1 \end{bmatrix}.$$

The explicit method has $\mathbf{b}^{\{E\}} = \mathbf{b}^{\{I\}}$, $\mathbf{c}^{\{E\}} = \mathbf{c}^{\{I\}}$, and

$$\mathbf{A}^{\{E\}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.871733043016918 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0.5 & 0.916993298352020 & -0.416993298352020 & 0 \end{bmatrix}.$$

Example: an order four IMEX method. The implicit part is Kvaerno's five stages, order four method [18, ESDIRK 4/3]:

$$\begin{aligned} \mathbf{A}_{2,2}^{\{I\}} &= \mathbf{A}_{3,3}^{\{I\}} = \mathbf{A}_{4,4}^{\{I\}} = \mathbf{A}_{5,5}^{\{I\}} = 0.572816062482134 \\ \mathbf{A}_{2,1}^{\{I\}} &= 0.572816062482134, \quad \mathbf{A}_{3,1}^{\{I\}} = 0.167235462027210, \\ \mathbf{A}_{3,2}^{\{I\}} &= -0.142946536857034, \quad \mathbf{A}_{4,1}^{\{I\}} = 0.262603290252694, \\ \mathbf{A}_{4,2}^{\{I\}} &= -0.311904327420564, \quad \mathbf{A}_{4,3}^{\{I\}} = 0.476484974685735, \\ \mathbf{A}_{5,1}^{\{I\}} &= 0.197216548312835, \quad \mathbf{A}_{5,2}^{\{I\}} = 0.176843783906372, \\ \mathbf{A}_{5,3}^{\{I\}} &= 0.815442181350836, \quad \mathbf{A}_{5,4}^{\{I\}} = -0.762318576052177, \end{aligned}$$

with

$$\mathbf{b} = \begin{bmatrix} 0.197216548312835 \\ 0.176843783906372 \\ 0.815442181350836 \\ -0.762318576052177 \\ 0.572816062482134 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 1.145632124964268 \\ 0.597104987652310 \\ 1 \\ 1 \end{bmatrix},$$

together with the explicit method

$$\begin{aligned} \mathbf{A}_{2,1}^{\{E\}} &= 1.145632124964268, & \mathbf{A}_{3,1}^{\{E\}} &= 0.486402211775915, \\ \mathbf{A}_{3,2}^{\{E\}} &= 0.110702775876395, & \mathbf{A}_{4,1}^{\{E\}} &= 0.527357281908146, \\ \mathbf{A}_{4,2}^{\{E\}} &= -0.234882275336215, & \mathbf{A}_{4,3}^{\{E\}} &= 0.707524993428070, \\ \mathbf{A}_{5,1}^{\{E\}} &= 0, & \mathbf{A}_{5,2}^{\{E\}} &= -0.515140880433405, \\ \mathbf{A}_{5,3}^{\{E\}} &= 1.515140880433405, & \mathbf{A}_{5,4}^{\{E\}} &= 0. \end{aligned}$$

All other coefficients are zero. The above form a transposed-classical IMEX RK method of order four.

3.4. Prothero-Robinson analysis. We consider the Prothero-Robinson (PR) [19] test problem written as a split system (3.1)

$$(3.11) \quad \begin{bmatrix} y \\ t \end{bmatrix}' = \underbrace{\begin{bmatrix} \mu(y - \phi(t)) \\ 0 \end{bmatrix}}_{g(t,y)} + \underbrace{\begin{bmatrix} \phi'(t) \\ 1 \end{bmatrix}}_{f(t,y)}, \quad \mu < 0, \quad y(0) = \phi(0),$$

where the exact solution is $y(t) = \phi(t)$. An IMEX-GARK method (3.2) is PR-convergent with order p if its application to (3.11) gives a solution whose global error decreases as $\mathcal{O}(h^p)$ for $h \rightarrow 0$ and $h\mu \rightarrow -\infty$.

THEOREM 3.1 (PR convergence of IMEX-GARK methods). *Consider the IMEX-GARK method (3.2) of order p . Assume that the implicit component has a nonsingular coefficient matrix $\mathbf{A}^{\{I,I\}}$, and an implicit transfer function stable at infinity,*

$$R^{\{I,I\}}(h\mu) = \left(1 + h\mu \mathbf{b}^{\{I\}T} \left(\mathbf{I} - h\mu \mathbf{A}^{\{I,I\}} \right)^{-1} \mathbf{1} \right), \quad |R^{\{I,I\}}(-\infty)| \leq 1.$$

Assume also that the additional order conditions hold:

$$(3.12) \quad k \mathbf{b}^{\{I\}T} \left(\mathbf{A}^{\{I,I\}} \right)^{-1} \mathbf{A}^{\{I,E\}} \left(\mathbf{c}^{\{E,E\}} \right)^{k-1} = \mathbf{b}^{\{I\}T} \left(\mathbf{A}^{\{I,I\}} \right)^{-1} \left(\mathbf{c}^{\{I,E\}} \right)^k$$

for $k = 1, \dots, q$, $q \leq p$. Then the IMEX-GARK method is PR-convergent with order:

- $q + 1$ if $-1 \leq R^{\{I,I\}}(-\infty) \leq 1$, and
- q if $R^{\{I,I\}}(-\infty) = 1$.

Proof.

The method (3.2) applied to the scalar equation (3.11) reads

$$(3.13a) \quad Y = y_n \mathbf{1} + h \mathbf{A}^{\{E,E\}} \phi'^{\{E\}} + h\mu \mathbf{A}^{\{E,I\}} \left(Z - \phi^{\{I\}} \right),$$

$$(3.13b) \quad Z = y_n \mathbf{1} + h \mathbf{A}^{\{I,E\}} \phi'^{\{E\}} + h\mu \mathbf{A}^{\{I,I\}} \left(Z - \phi^{\{I\}} \right),$$

$$(3.13c) \quad y_{n+1} = y_n + h \mathbf{b}^{\{E\}T} \phi'^{\{E\}} + h\mu \mathbf{b}^{\{I\}T} \left(Z - \phi^{\{I\}} \right).$$

Here

$$\begin{aligned}\phi^{\{\mathbf{E}\}} &= \phi \left(t_{n-1} + \mathbf{d}^{\{\mathbf{E}\}} h \right) = \left[\phi(t_{n-1} + d_1^{\{\mathbf{E}\}} h), \dots, \phi(t_{n-1} + d_{s^{\{\mathbf{E}\}}}^{\{\mathbf{E}\}} h) \right]^T, \\ \phi^{\{\mathbf{I}\}} &= \phi \left(t_{n-1} + \mathbf{d}^{\{\mathbf{I}\}} h \right) = \left[\phi(t_{n-1} + d_1^{\{\mathbf{I}\}} h), \dots, \phi(t_{n-1} + d_{s^{\{\mathbf{I}\}}}^{\{\mathbf{I}\}} h) \right]^T,\end{aligned}$$

where $\mathbf{d}^{\{\mathbf{E}\}}, \mathbf{d}^{\{\mathbf{I}\}}$ are the stage approximation times. Due to the structure of the test problem (3.11), the method (3.2) uses an explicit approach for the time variable, therefore

$$(3.14) \quad \mathbf{d}^{\{\mathbf{E}\}} = \mathbf{c}^{\{\mathbf{E}, \mathbf{E}\}}, \quad \mathbf{d}^{\{\mathbf{I}\}} = \mathbf{c}^{\{\mathbf{I}, \mathbf{E}\}}.$$

The exact solution is expanded in Taylor series about t_n :

$$(3.15) \quad \begin{aligned}\phi(t_n + \mathbf{d} h) - \mathbf{1} \phi(t_n) &= \sum_{k=1}^{\infty} \frac{h^k \mathbf{d}^k}{k!} \phi^{(k)}(t_n), \\ h \phi'(t_n + \mathbf{d} h) &= \sum_{k=1}^{\infty} \frac{k h^k \mathbf{d}^{k-1}}{k!} \phi^{(k)}(t_n),\end{aligned}$$

where the vector power \mathbf{d}^k is taken componentwise.

Consider the global errors

$$e_n = y_n - \phi(t_n), \quad E_Y = Y - \phi^{\{\mathbf{E}\}}, \quad E_Z = Z - \phi^{\{\mathbf{I}\}}.$$

Write the stage equation (3.13b) in terms of the exact solution and global errors, and use the Taylor expansions (3.15) to obtain

$$\begin{aligned}(\mathbf{I} - h \mu \mathbf{A}^{\{\mathbf{I}, \mathbf{I}\}}) E_Z &= e_n \mathbf{1} + \phi(t_n) \mathbf{1} - \phi^{\{\mathbf{I}\}} + h \mathbf{A}^{\{\mathbf{I}, \mathbf{E}\}} \phi'^{\{\mathbf{E}\}} \\ &= e_n \mathbf{1} + \sum_{k=1}^{\infty} \left(k \mathbf{A}^{\{\mathbf{I}, \mathbf{E}\}} \left(\mathbf{d}^{\{\mathbf{E}\}} \right)^{k-1} - \left(\mathbf{d}^{\{\mathbf{I}\}} \right)^k \right) \frac{h^k}{k!} \phi^{(k)}(t_n).\end{aligned}$$

Similarly, write the solution equation (3.13c) in terms of the exact solution and global errors:

$$\begin{aligned}e_{n+1} &= e_n + \phi(t_n) - \phi(t_{n+1}) + h \mathbf{b}^{\{\mathbf{E}\} T} \phi'^{\{\mathbf{E}\}} + h \mu \mathbf{b}^{\{\mathbf{I}\} T} E_Z \\ &= R^{\{\mathbf{I}, \mathbf{I}\}}(h \mu) e_n + \sum_{k=1}^{\infty} \left(k \mathbf{b}^{\{\mathbf{E}\} T} \left(\mathbf{d}^{\{\mathbf{E}\}} \right)^{k-1} - 1 \right) \frac{h^k}{k!} \phi^{(k)}(t_n) \\ &\quad + h \mu \mathbf{b}^{\{\mathbf{I}\} T} \left(\mathbf{I} - h \mu \mathbf{A}^{\{\mathbf{I}, \mathbf{I}\}} \right)^{-1} \cdot \sum_{k=1}^{\infty} \left(k \mathbf{A}^{\{\mathbf{I}, \mathbf{E}\}} \left(\mathbf{d}^{\{\mathbf{E}\}} \right)^{k-1} - \left(\mathbf{d}^{\{\mathbf{I}\}} \right)^k \right) \frac{h^k}{k!} \phi^{(k)}(t_n),\end{aligned}$$

where the stability function of the implicit component method is

$$R^{\{\mathbf{I}, \mathbf{I}\}}(h \mu) = \left(1 + h \mu \mathbf{b}^{\{\mathbf{I}\} T} \left(\mathbf{I} - h \mu \mathbf{A}^{\{\mathbf{I}, \mathbf{I}\}} \right)^{-1} \mathbf{1} \right).$$

Since the explicit component method (by itself) has at least order p , it follows from (3.14) and the explicit order conditions that

$$(3.16) \quad k \cdot \left(\mathbf{b}^{\{E\}} \right)^T \left(\mathbf{d}^{\{E\}} \right)^{k-1} - 1 = 0 \quad \text{for } k = 1, \dots, p.$$

Consequently, the global error recurrence reads

$$e_{n+1} = R^{\{I,I\}}(h\mu) e_n + \mathcal{O}(h^{p+1}) + \sum_{k=1}^{\infty} r_k(h\mu) \frac{h^k}{k!} \phi^{(k)}(t_n),$$

$$r_k(h\mu) = h\mu \mathbf{b}^{\{I\}}{}^T \left(\mathbf{I} - h\mu \mathbf{A}^{\{I,I\}} \right)^{-1} \left(k \mathbf{A}^{\{I,E\}} \left(\mathbf{d}^{\{E\}} \right)^{k-1} - \left(\mathbf{d}^{\{I\}} \right)^k \right).$$

For $h\mu \rightarrow -\infty$

$$r_k(\infty) = -\mathbf{b}^{\{I\}}{}^T \left(\mathbf{A}^{\{I,I\}} \right)^{-1} \left(k \mathbf{A}^{\{I,E\}} \left(\mathbf{d}^{\{E\}} \right)^{k-1} - \left(\mathbf{d}^{\{I\}} \right)^k \right).$$

If conditions (3.12) hold we infer from (3.14) that $r_k(\infty) = 0$ for $k = 1, \dots, q$. The global error recurrence reads

$$e_{n+1} = R^{\{I,I\}}(\infty) e_n + \mathcal{O}\left(h^{\min\{q+1, p+1\}}\right).$$

This gives the desired result. \square

REMARK 3. *If one defines the test problem (3.11) such that the time is an implicit variable then (3.14) becomes*

$$(3.17) \quad \mathbf{d}^{\{E\}} = \mathbf{c}^{\{E,I\}}, \quad \mathbf{d}^{\{I\}} = \mathbf{c}^{\{I,I\}}.$$

Equation (3.16) is satisfied automatically by the IMEX coupling conditions of order p . The additional order conditions (3.12) become

$$(3.18) \quad k \mathbf{b}^{\{I\}}{}^T \left(\mathbf{A}^{\{I,I\}} \right)^{-1} \mathbf{A}^{\{I,E\}} \left(\mathbf{c}^{\{E,I\}} \right)^{k-1} = \mathbf{b}^{\{I\}}{}^T \left(\mathbf{A}^{\{I,I\}} \right)^{-1} \left(\mathbf{c}^{\{I,I\}} \right)^k.$$

3.5. Stiff semi-linear analysis. Consider now the semi-linear (SL) problem

$$(3.19) \quad y' = \underbrace{\mu y}_{g(y)} + f(y), \quad \mu < 0, \quad y(0) = y_0,$$

where $f(y(t))$ is smooth and non-stiff. An IMEX-GARK method (3.2) is SL-convergent with order p if its application to (3.19) gives a solution whose global error decreases as $\mathcal{O}(h^p)$ for $h \rightarrow 0$ and $h\mu \rightarrow -\infty$.

THEOREM 3.2 (SL convergence of IMEX-GARK methods). *Consider an internally consistent IMEX-GARK method (3.2) of order p . Assume that the implicit component has a nonsingular coefficient matrix $\mathbf{A}^{\{I,I\}}$, and an implicit transfer function strictly stable at infinity,*

$$R^{\{I,I\}}(h\mu) = \left(1 + h\mu \mathbf{b}^{\{I\}}{}^T \left(\mathbf{I} - h\mu \mathbf{A}^{\{I,I\}} \right)^{-1} \right), \quad \left| R^{\{I,I\}}(-\infty) \right| < 1.$$

Assume also that the additional order conditions hold:

$$(3.20a) \quad \mathbf{b}^{\{1\}^T} \mathbf{A}^{\{1,1\}} \mathbf{c}^{\{1\}^k} = 1, \quad k = 1, \dots, q,$$

$$(3.20b) \quad \mathbf{b}^{\{1\}^T} \mathbf{A}^{\{1,1\}} \mathbf{A}^{\{1,E\}} \mathbf{c}^{\{E\}^{k-1}} = \frac{1}{k}, \quad k = 1, \dots, q,$$

for $q \leq p$. Then the IMEX-GARK method is SL-convergent with order $q + 1$.

Proof. The exact solution is expanded in Taylor series about t_n :

$$\begin{aligned} y(t_n + \mathbf{c}h) - \mathbf{1}y(t_n) &= \sum_{k=1}^{\infty} \frac{h^k \mathbf{c}^k}{k!} y^{(k)}(t_n), \\ h y'(t_n + \mathbf{c}h) &= \sum_{k=1}^{\infty} \frac{k h^k \mathbf{c}^{k-1}}{k!} y^{(k)}(t_n). \end{aligned}$$

Application of the method (3.2) to problem (3.19) gives

$$\begin{aligned} (3.21) \quad Y &= y_n \mathbf{1} + h \mathbf{A}^{\{E,E\}} f(Y) + h \mu \mathbf{A}^{\{E,1\}} Z, \\ Z &= y_n \mathbf{1} + h \mathbf{A}^{\{1,E\}} f(Y) + h \mu \mathbf{A}^{\{1,1\}} Z, \\ y_{n+1} &= y_n + h \mathbf{b}^{\{E\}^T} f(Y) + h \mu \mathbf{b}^{\{1\}^T} Z. \end{aligned}$$

Insert the exact solutions in the numerical scheme (3.21) to obtain

$$\begin{aligned} (3.22) \quad y(t_n^{\{E\}}) &= y(t_n) \mathbf{1} + h \mathbf{A}^{\{E,E\}} f(y(t_n^{\{E\}})) + h \mu \mathbf{A}^{\{E,1\}} y(t_n^{\{1\}}) + \delta^{\{E\}}, \\ y(t_n^{\{1\}}) &= y(t_n) \mathbf{1} + h \mathbf{A}^{\{1,E\}} f(y(t_n^{\{E\}})) + h \mu \mathbf{A}^{\{1,1\}} y(t_n^{\{1\}}) + \delta^{\{1\}}, \\ y(t_{n+1}) &= y(t_n) + h \mathbf{b}^{\{E\}^T} f(y(t_n^{\{E\}})) + h \mu \mathbf{b}^{\{1\}^T} y(t_n^{\{1\}}) + \delta. \end{aligned}$$

where $t_n^{\{E\}} = t_n + \mathbf{c}^{\{E\}}h$ and $t_n^{\{1\}} = t_n + \mathbf{c}^{\{1\}}h$. The exact solutions satisfy the numerical scheme (3.21) only approximately. The residuals are as follows:

$$\begin{aligned} \delta^{\{1\}} &= y(t + \mathbf{c}^{\{1\}}h) - y(t_n) \mathbf{1} - h \mathbf{A}^{\{1,E\}} f(y(t + \mathbf{c}^{\{E\}}h)) \\ &\quad + h \mathbf{A}^{\{1,1\}} f(y(t + \mathbf{c}^{\{1\}}h)) - \mathbf{A}^{\{1,1\}} h y'(t + \mathbf{c}^{\{1\}}h) \\ &= \sum_{k=1}^{\infty} \left(\mathbf{c}^{\{1\}^k} - k \mathbf{A}^{\{1,1\}} \mathbf{c}^{\{1\}^{k-1}} \right) \frac{h^k}{k!} y^{(k)}(t_n) \\ &\quad + h \sum_{k=0}^{\infty} \left(\mathbf{A}^{\{1,1\}} \mathbf{c}^{\{1\}^k} - \mathbf{A}^{\{1,E\}} \mathbf{c}^{\{E\}^k} \right) \frac{h^k}{k!} f^{(k)}(t_n) \end{aligned}$$

For the solution equation we have that

$$\begin{aligned}
\delta &= y(t_{n+1}) - y(t_n) - h \mathbf{b}^{\{E\}T} f(y(t + \mathbf{c}^{\{E\}}h)) \\
&\quad + \mathbf{b}^{\{I\}T} h f(y(t + \mathbf{c}^{\{I\}}h)) - \mathbf{b}^{\{I\}T} h y'(t + \mathbf{c}^{\{I\}}h) \\
&= \sum_{k=1}^{\infty} \left(1 - k \mathbf{b}^{\{I\}T} (\mathbf{c}^{\{I\}})^{k-1}\right) \frac{h^k}{k!} y^{(k)}(t_n) \\
&\quad - h \mathbf{b}^{\{E\}T} \mathbf{1} f(t_n) - h \sum_{k=1}^{\infty} \frac{h^k \mathbf{b}^{\{E\}T} \mathbf{c}^{\{E\}}^k}{k!} f^{(k)}(t_n) \\
&\quad + h \mathbf{b}^{\{I\}T} \mathbf{1} f(t_n) + h \sum_{k=1}^{\infty} \frac{h^k \mathbf{b}^{\{I\}T} \mathbf{c}^{\{I\}}^k}{k!} f^{(k)}(t_n) \\
&= \sum_{k=1}^{\infty} \left(1 - k \mathbf{b}^{\{I\}T} (\mathbf{c}^{\{I\}})^{k-1}\right) \frac{h^k}{k!} y^{(k)}(t_n) \\
&\quad + h \sum_{k=1}^{\infty} \left(\mathbf{b}^{\{I\}T} \mathbf{c}^{\{I\}}^k - \mathbf{b}^{\{E\}T} \mathbf{c}^{\{E\}}^k\right) \frac{h^k}{k!} f^{(k)}(t_n) \\
&= \mathcal{O}(h^{p+1})
\end{aligned}$$

where the last equality follows from the order p conditions for the implicit and explicit components.

Consider the global errors

$$e_n = y_n - y(t_n), \quad E_Y = Y - y(t_n + \mathbf{c}^{\{E\}}h), \quad E_Z = Z - y(t_n + \mathbf{c}^{\{I\}}h).$$

Relations for these errors are obtained by subtracting (3.22) from (3.21):

$$\begin{aligned}
(3.23) \quad E_Y &= e_n \mathbf{1} + h \mathbf{A}^{\{E,E\}} \Delta f^{\{E\}} + h \mu \mathbf{A}^{\{E,I\}} E_Z + \delta^{\{E\}}, \\
E_Z &= e_n \mathbf{1} + h \mathbf{A}^{\{I,E\}} \Delta f^{\{E\}} + h \mu \mathbf{A}^{\{I,I\}} E_Z + \delta^{\{I\}}, \\
e_{n+1} &= e_n + h \mathbf{b}^{\{E\}T} \Delta f^{\{E\}} + h \mu \mathbf{b}^{\{I\}T} E_Z + \delta.
\end{aligned}$$

Using the mean function theorem,

$$\Delta f^{\{E\}} = f(Y) - f(y(t + \mathbf{c}^{\{E\}}h)) = \lambda E_Y \quad \text{where } \lambda = f_y(\cdot)$$

the error equations (3.23) become

$$(3.24a) \quad \begin{bmatrix} \mathbf{I} - h \lambda \mathbf{A}^{\{E,E\}} & -h \mu \mathbf{A}^{\{E,I\}} \\ -h \lambda \mathbf{A}^{\{I,E\}} & \mathbf{I} - h \mu \mathbf{A}^{\{I,I\}} \end{bmatrix} \begin{bmatrix} E_Y \\ E_Z \end{bmatrix} = e_n \mathbf{1} + \begin{bmatrix} \delta^{\{E\}} \\ \delta^{\{I\}} \end{bmatrix},$$

$$\begin{aligned}
(3.24b) \quad e_{n+1} &= \left(1 + [h \lambda \mathbf{b}^{\{E\}T} \quad h \mu \mathbf{b}^{\{I\}T}] \begin{bmatrix} \mathbf{I} - h \lambda \mathbf{A}^{\{E,E\}} & -h \mu \mathbf{A}^{\{E,I\}} \\ -h \lambda \mathbf{A}^{\{I,E\}} & \mathbf{I} - h \mu \mathbf{A}^{\{I,I\}} \end{bmatrix}^{-1} \mathbf{1}\right) e_n \\
&\quad + [h \lambda \mathbf{b}^{\{E\}T} \quad h \mu \mathbf{b}^{\{I\}T}] \begin{bmatrix} \mathbf{I} - h \lambda \mathbf{A}^{\{E,E\}} & -h \mu \mathbf{A}^{\{E,I\}} \\ -h \lambda \mathbf{A}^{\{I,E\}} & \mathbf{I} - h \mu \mathbf{A}^{\{I,I\}} \end{bmatrix}^{-1} \begin{bmatrix} \delta^{\{E\}} \\ \delta^{\{I\}} \end{bmatrix} + \delta.
\end{aligned}$$

Inserting (3.24b) into (3.24b) and rescaling gives

$$\begin{aligned}
e_{n+1} &= \left(1 + [h \lambda \mathbf{b}^{\{E\}T} \quad \mathbf{b}^{\{I\}T}] \begin{bmatrix} \mathbf{I} - h \lambda \mathbf{A}^{\{E,E\}} & -\mathbf{A}^{\{E,I\}} \\ -h \lambda \mathbf{A}^{\{I,E\}} & \frac{1}{h \mu} \mathbf{I} - \mathbf{A}^{\{I,I\}} \end{bmatrix}^{-1} \mathbf{1}\right) e_n \\
&\quad + [h \lambda \mathbf{b}^{\{E\}T} \quad \mathbf{b}^{\{I\}T}] \begin{bmatrix} \mathbf{I} - h \lambda \mathbf{A}^{\{E,E\}} & -\mathbf{A}^{\{E,I\}} \\ -h \lambda \mathbf{A}^{\{I,E\}} & \frac{1}{h \mu} \mathbf{I} - \mathbf{A}^{\{I,I\}} \end{bmatrix}^{-1} \begin{bmatrix} \delta^{\{E\}} \\ \delta^{\{I\}} \end{bmatrix} + \delta,
\end{aligned}$$

which for $h\mu \rightarrow -\infty$ gives the following recurrence for the global error:

$$(3.25) \quad e_{n+1} = \left(1 + [h\lambda \mathbf{b}^{\{E\}T} \quad \mathbf{b}^{\{I\}T}] \begin{bmatrix} \mathbf{I} - h\lambda \mathbf{A}^{\{E,E\}} & -\mathbf{A}^{\{E,I\}} \\ -h\lambda \mathbf{A}^{\{I,E\}} & -\mathbf{A}^{\{I,I\}} \end{bmatrix}^{-1} \mathbf{1} \right) e_n \\ + [h\lambda \mathbf{b}^{\{E\}T} \quad \mathbf{b}^{\{I\}T}] \begin{bmatrix} \mathbf{I} - h\lambda \mathbf{A}^{\{E,E\}} & -\mathbf{A}^{\{E,I\}} \\ -h\lambda \mathbf{A}^{\{I,E\}} & -\mathbf{A}^{\{I,I\}} \end{bmatrix}^{-1} \begin{bmatrix} \delta^{\{E\}} \\ \delta^{\{I\}} \end{bmatrix} + \delta.$$

We have that

$$\begin{aligned} & \mathbf{b}^{\{I\}T} \left(\mathbf{A}^{\{I,I\}} \right)^{-1} \delta^{\{I\}} \\ &= \sum_{k=1}^{\infty} \left(\mathbf{b}^{\{I\}T} \mathbf{A}^{\{I,I\}^{-1}} \mathbf{c}^{\{I\}k} - k \mathbf{b}^{\{I\}T} \mathbf{c}^{\{I\}k-1} \right) \frac{h^k}{k!} y^{(k)}(t_n) \\ & \quad + \sum_{k=0}^{\infty} \left(\mathbf{b}^{\{I\}T} \mathbf{c}^{\{I\}k} - \mathbf{b}^{\{I\}T} \mathbf{A}^{\{I,I\}^{-1}} \mathbf{A}^{\{I,E\}} \mathbf{c}^{\{E\}k} \right) \frac{h^{k+1}}{k!} f^{(k)}(t_n) \\ &= \sum_{k=1}^p \left(\mathbf{b}^{\{I\}T} \mathbf{A}^{\{I,I\}^{-1}} \mathbf{c}^{\{I\}k} - 1 \right) \frac{h^k}{k!} y^{(k)}(t_n) \\ & \quad + \sum_{k=1}^p \left(\frac{1}{k} - \mathbf{b}^{\{I\}T} \mathbf{A}^{\{I,I\}^{-1}} \mathbf{A}^{\{I,E\}} \mathbf{c}^{\{E\}k-1} \right) \frac{h^k}{(k-1)!} f^{(k-1)}(t_n) + \mathcal{O}(h^{p+1}) \\ &= \mathcal{O}\left(h^{\min(p+1, q+1)}\right) \end{aligned}$$

where the last equality follows from the additional order conditions (3.20).

Since

$$\begin{aligned} & [h\lambda \mathbf{b}^{\{E\}T} \quad \mathbf{b}^{\{I\}T}] \cdot \begin{bmatrix} \mathbf{I} - h\lambda \mathbf{A}^{\{E,E\}} & -\mathbf{A}^{\{E,I\}} \\ -h\lambda \mathbf{A}^{\{I,E\}} & -\mathbf{A}^{\{I,I\}} \end{bmatrix}^{-1} \\ &= (\mathbf{I} + \mathcal{O}(h)) \begin{bmatrix} 0 & -\mathbf{b}^{\{I\}T} \mathbf{A}^{\{I,I\}^{-1}} \end{bmatrix}, \end{aligned}$$

the global error recurrence (3.25) becomes

$$e_{n+1} = \left(R^{\{I,I\}}(-\infty) + \mathcal{O}(h) \right) e_n + \mathcal{O}\left(h^{\min(p+1, q+1)}\right)$$

which gives the desired result in the limit $h \rightarrow 0$. \square

3.6. Stiffly accurate GARK methods. The following extension of the stiff accuracy concept [8] offers a convenient way to satisfy the additional order conditions (3.12) and (3.20).

DEFINITION 3.3 (Stiffly accurate GARK methods.). *A GARK method (2.4) is stiffly accurate if*

$$(3.26) \quad \mathbf{e}_{s\{N\}}^T \mathbf{A}^{\{N,m\}} = \left(\mathbf{b}^{\{m\}} \right)^T, \quad m = 1, \dots, N.$$

Note that stiff stability can be formulated with respect to any component method q by replacing N with q in (3.26).

A stiffly accurate IMEX-GARK satisfies

$$\mathbf{e}_{s\{I\}}^T \mathbf{A}^{\{I,E\}} = \left(\mathbf{b}^{\{E\}} \right)^T, \quad \mathbf{e}_{s\{I\}}^T \mathbf{A}^{\{I,I\}} = \left(\mathbf{b}^{\{I\}} \right)^T, \quad R^{\{I,I\}}(\infty) = 0,$$

and consequently

$$\mathbf{b}^{\{I\}}{}^T \left(\mathbf{A}^{\{I,I\}} \right)^{-1} = \mathbf{e}_{s^{\{I\}}}^T, \quad \mathbf{c}_{s^{\{I\}}}^{\{I,E\}} = 1.$$

The Prothero-Robinson order conditions (3.12) are equivalent to

$$k \mathbf{e}_{s^{\{I\}}}^T \mathbf{A}^{\{I,E\}} \left(\mathbf{c}^{\{E,E\}} \right)^{k-1} = \mathbf{e}_{s^{\{I\}}}^T \left(\mathbf{c}^{\{I,E\}} \right)^k, \quad k = 1, \dots, q,$$

and therefore to

$$\left(\mathbf{b}^{\{E\}} \right)^T \left(\mathbf{c}^{\{E,E\}} \right)^{k-1} = \frac{1}{k}, \quad k = 1, \dots, q.$$

The conditions are automatically satisfied for $q = p$ as they are part of the explicit component order conditions.

For a stiffly accurate IMEX method applied to Prothero-Robinson with implicit time (3.17) the order conditions (3.18) are equivalent to

$$\mathbf{b}^{\{E\}}{}^T \left(\mathbf{c}^{\{E,I\}} \right)^{k-1} = \frac{1}{k},$$

and is satisfied automatically for $k = 1, \dots, p$ due to the IMEX coupling conditions of order p . Thus a stiffly accurate method is PR-convergent with order $p + 1$ regardless of the form of the test problem (3.11).

For a stiffly accurate IMEX-GARK the semi-linear order conditions (3.20) read

$$\begin{aligned} \mathbf{e}_{s^{\{I\}}}^T \mathbf{c}^{\{I\}}{}^k &= 1, \\ \mathbf{e}_{s^{\{I\}}}^T \mathbf{A}^{\{I,E\}} \mathbf{c}^{\{E\}}{}^{k-1} &= \mathbf{b}^{\{E\}}{}^T \mathbf{c}^{\{E\}}{}^{k-1} = \frac{1}{k}, \end{aligned}$$

and are satisfied automatically through the explicit order conditions.

4. Stability and monotonicity. In this section a stability and monotonicity analysis is performed. We derive a linear stability theory, as well as nonlinear stability theories for both dispersive and coercive problems.

4.1. Linear stability analysis. We apply the GARK scheme (2.3) to the linear scalar test problem

$$y' = \sum_{m=1}^N \lambda^{\{m\}} y.$$

With $z^{\{m\}} = h \lambda^{\{m\}}$ one obtains

$$\begin{aligned} Y_i^{\{q\}} &= y_n + \sum_{m=1}^N \sum_{j=1}^{s^{\{m\}}} a_{i,j}^{\{q,m\}} z^{\{m\}} Y_j^{\{m\}}, \quad q = 1, \dots, N \\ y_{n+1} &= y_n + \sum_{q=1}^N \sum_{i=1}^{s^{\{q\}}} b_i^{\{q\}} z^{\{q\}} Y_i^{\{q\}}. \end{aligned}$$

Denote

$$(4.1) \quad s = \sum_{q=1}^N s^{\{q\}}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}^{\{1,1\}} & \dots & \mathbf{A}^{\{1,N\}} \\ & \ddots & \\ \mathbf{A}^{\{N,1\}} & \dots & \mathbf{A}^{\{N,N\}} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}^{\{1\}} \\ \vdots \\ \mathbf{b}^{\{N\}} \end{bmatrix},$$

and

$$z = \begin{bmatrix} z^{\{1\}} \\ \vdots \\ z^{\{N\}} \end{bmatrix}, \quad Z = \begin{bmatrix} z^{\{1\}} \mathbf{I}_{s^{\{1\}} \times s^{\{1\}}} & \cdots & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & \cdots & z^{\{N\}} \mathbf{I}_{s^{\{N\}} \times s^{\{N\}}} \end{bmatrix}.$$

We have that

$$y_{n+1} = R(z^{\{1\}}, \dots, z^{\{N\}}) y_n,$$

where the stability function can be written compactly as

$$(4.2) \quad R(z) = 1 + \mathbf{b}^T \cdot Z \cdot (\mathbf{I}_{s \times s} - \mathbf{A} Z)^{-1} \cdot \mathbf{1}_{s \times 1}.$$

EXAMPLE 3 (Linear stability of stiffly accurate GARK methods). *Consider a stiffly accurate GARK method (3.26) and assume that $\text{Re}(z_i) < 0$ for $i = 1, \dots, N$. We have*

$$\begin{aligned} R(z) &= 1 + \mathbf{e}_s^T \cdot \mathbf{A} Z \cdot (\mathbf{I}_{s \times s} - \mathbf{A} Z)^{-1} \cdot \mathbf{1}_{s \times 1} \\ &= \mathbf{e}_s^T \cdot (\mathbf{I}_{s \times s} - \mathbf{A} Z)^{-1} \cdot \mathbf{1}_{s \times 1} \\ &= \mathbf{e}_s^T Z^{-1} \cdot (Z^{-1} - \mathbf{A})^{-1} \cdot \mathbf{1}_{s \times 1} \\ &= \frac{1}{z^{\{N\}}} \mathbf{e}_s^T \cdot (Z^{-1} - \mathbf{A})^{-1} \cdot \mathbf{1}_{s \times 1}. \end{aligned}$$

Assume that the system integrated with the stiffly accurate component is very stiff, $z^{\{N\}} \rightarrow -\infty$. Then the entire GARK stability function becomes zero, $R(z) \rightarrow 0$.

4.2. Nonlinear stability analysis. We now study the nonlinear stability of GARK methods (2.3) applied to partitioned systems (1.1) where each of the component functions is dispersive with respect to the same scalar product $\langle \cdot, \cdot \rangle$:

$$(4.3) \quad \langle f^{\{m\}}(y) - f^{\{m\}}(z), y - z \rangle \leq \nu^{\{m\}} \|y - z\|^2, \quad \nu^{\{m\}} < 0.$$

Consider two solutions $y(t)$ and $\tilde{y}(t)$ of (1.1), each starting from a different initial condition. Equation (4.3) implies that

$$\langle f(y) - f(z), y - z \rangle \leq \left(\sum_{m=1}^N \nu^{\{m\}} \right) \|y - z\|^2, \quad \sum_{m=1}^N \nu^{\{m\}} < 0,$$

and consequently the norm of the solution difference $\Delta y(t) = \tilde{y}(t) - y(t)$ is non-increasing, $\lim_{\varepsilon \rightarrow 0} \|\Delta y(t + \varepsilon)\| \leq \|\Delta y(t)\|$. It is desirable that the difference of the corresponding numerical solutions is also non-increasing, $\|\Delta y_{n+1}\| \leq \|\Delta y_n\|$. The analysis is carried out in the norm associated with the scalar product in (4.3).

Several matrices are defined from the coefficients of (2.3) for $m, \ell = 1, \dots, N$:

$$(4.4) \quad \mathbf{B}^{\{m\}} = \text{diag}(\mathbf{b}^{\{m\}}),$$

$$(4.5) \quad \mathbf{P}^{\{m, \ell\}} = \left(\mathbf{A}^{\{\ell, m\}} \right)^T \mathbf{B}^{\{\ell\}} + \mathbf{B}^{\{m\}} \mathbf{A}^{\{m, \ell\}} - \mathbf{b}^{\{m\}} \left(\mathbf{b}^{\{\ell\}} \right)^T,$$

with $\mathbf{B}^{\{m\}} \in \mathbb{R}^{s^{\{m\}} \times s^{\{m\}}}$, $\mathbf{P}^{\{m, \ell\}} \in \mathbb{R}^{s^{\{m\}} \times s^{\{\ell\}}}$, and

$$\mathbf{P}^{\{\ell, m\}} = \left(\mathbf{P}^{\{m, \ell\}} \right)^T.$$

The following definition and analysis generalize the ones in [10].

DEFINITION 4.1 (Algebraically stable GARK methods). *A generalized additive Runge-Kutta method (2.3) is algebraically stable if the weight vectors are non-negative*

$$(4.6a) \quad b_i^{\{m\}} \geq 0 \quad \text{for all } i = 1, \dots, s^{\{m\}}, \quad m = 1, \dots, N,$$

and the following matrix is non-negative definite:

$$(4.6b) \quad \mathbf{P} = [\mathbf{P}^{\{\ell, m\}}]_{1 \leq \ell, m \leq N} \in \mathbb{R}^{s \times s}, \quad \mathbf{P} \geq 0.$$

We have the following result.

THEOREM 4.2 (Algebraic stability of GARK methods). *An algebraically stable GARK method (2.3) applied to a partitioned system (1.1) with dispersive component functions (4.3) is unconditionally nonlinearly stable, in the sense that the difference of any two numerical solutions is non-increasing*

$$\|\Delta y_{n+1}\| \leq \|\Delta y_n\|$$

for any step size $h > 0$.

Proof. The difference between solutions $\Delta y_n = \tilde{y}_n - y_n$ advances in time as follows:

$$(4.7a) \quad \Delta Y_i^{\{q\}} = \Delta y_n + h \sum_{m=1}^N \sum_{j=1}^{s^{\{m\}}} a_{i,j}^{\{q,m\}} \Delta f_j^{\{m\}}, \quad q = 1, \dots, N,$$

$$(4.7b) \quad \Delta y_{n+1} = \Delta y_n + h \sum_{q=1}^N \sum_{i=1}^{s^{\{q\}}} b_i^{\{q\}} \Delta f_i^{\{q\}}.$$

where

$$\Delta Y_j^{\{m\}} = \tilde{Y}_j^{\{m\}} - Y_j^{\{m\}}, \quad \Delta f_j^{\{m\}} = f^{\{m\}}(\tilde{Y}_j^{\{m\}}) - f^{\{m\}}(Y_j^{\{m\}}).$$

From (4.7b) we get

$$(4.8) \quad \begin{aligned} \|\Delta y_{n+1}\|^2 &= \|\Delta y_n\|^2 \\ &+ 2h \sum_{m=1}^N \sum_{i=1}^{s^{\{m\}}} b_i^{\{m\}} \langle \Delta f_i^{\{m\}}, \Delta y_n \rangle \\ &+ h^2 \sum_{m,q=1}^N \sum_{i=1}^{s^{\{m\}}} \sum_{j=1}^{s^{\{q\}}} b_i^{\{m\}} b_j^{\{q\}} \langle \Delta f_i^{\{m\}}, \Delta f_j^{\{q\}} \rangle. \end{aligned}$$

From (4.7a) it follows that

$$(4.9) \quad \Delta y_n = \Delta Y_i^{\{q\}} - h \sum_{m=1}^N \sum_{j=1}^{s^{\{m\}}} a_{i,j}^{\{q,m\}} \Delta f_j^{\{m\}}.$$

Substituting (4.9) into (4.8) leads to

$$\begin{aligned}
(4.10) \quad & \|\Delta y_{n+1}\|^2 = \|\Delta y_n\|^2 \\
& + 2h \sum_{m=1}^N \sum_{i=1}^{s^{\{m\}}} b_i^{\{m\}} \left\langle \Delta f_i^{\{m\}}, \Delta Y_i^{\{m\}} \right\rangle \\
& - 2h^2 \sum_{m,l=1}^N \sum_{i=1}^{s^{\{m\}}} \sum_{j=1}^{s^{\{l\}}} a_{i,j}^{\{m,l\}} b_i^{\{m\}} \left\langle \Delta f_i^{\{m\}}, \Delta f_j^{\{l\}} \right\rangle \\
& + h^2 \sum_{m,l=1}^N \sum_{i=1}^{s^{\{m\}}} \sum_{j=1}^{s^{\{l\}}} b_i^{\{m\}} b_j^{\{l\}} \left\langle \Delta f_i^{\{m\}}, \Delta f_j^{\{l\}} \right\rangle \\
(4.11) \quad & = \|\Delta y_n\|^2 \\
& + 2h \sum_{m=1}^N \sum_{i=1}^{s^{\{m\}}} b_i^{\{m\}} \left\langle \Delta f_i^{\{m\}}, \Delta Y_i^{\{m\}} \right\rangle \\
& - h^2 \sum_{m,l=1}^N \sum_{i=1}^{s^{\{m\}}} \sum_{j=1}^{s^{\{l\}}} a_{i,j}^{\{m,l\}} b_i^{\{m\}} \left\langle \Delta f_i^{\{m\}}, \Delta f_j^{\{l\}} \right\rangle \\
& - h^2 \sum_{m,l=1}^N \sum_{i=1}^{s^{\{m\}}} \sum_{j=1}^{s^{\{l\}}} a_{j,i}^{\{l,m\}} b_j^{\{l\}} \left\langle \Delta f_i^{\{m\}}, \Delta f_j^{\{l\}} \right\rangle \\
& + h^2 \sum_{m,l=1}^N \sum_{i=1}^{s^{\{m\}}} \sum_{j=1}^{s^{\{l\}}} b_i^{\{m\}} b_j^{\{l\}} \left\langle \Delta f_i^{\{m\}}, \Delta f_j^{\{l\}} \right\rangle.
\end{aligned}$$

Equation (4.11) can be written in the equivalent form

$$\begin{aligned}
\|\Delta y_{n+1}\|^2 &= \|\Delta y_n\|^2 + 2h \sum_{m=1}^N \sum_{i=1}^{s^{\{m\}}} b_i^{\{m\}} \left\langle \Delta f_i^{\{m\}}, \Delta Y_i^{\{m\}} \right\rangle \\
&\quad - h^2 \sum_{m,l=1}^N \sum_{i=1}^{s^{\{m\}}} \sum_{j=1}^{s^{\{l\}}} \left(b_i^{\{m\}} a_{i,j}^{\{m,l\}} + b_j^{\{l\}} a_{j,i}^{\{l,m\}} - b_i^{\{m\}} b_j^{\{l\}} \right) \left\langle \Delta f_i^{\{m\}}, \Delta f_j^{\{l\}} \right\rangle \\
(4.12) \quad &= \|\Delta y_n\|^2 + 2h \sum_{m=1}^N \sum_{i=1}^{s^{\{m\}}} b_i^{\{m\}} \left\langle \Delta f_i^{\{m\}}, \Delta Y_i^{\{m\}} \right\rangle \\
&\quad - h^2 \sum_{m,l=1}^N (\Delta f^{\{m\}})^T \left(\mathbf{P}^{\{m,l\}} \otimes \mathbf{I}_{d \times d} \right) \Delta f^{\{l\}}
\end{aligned}$$

where

$$\Delta f^{\{m\}} = \left[(\Delta f_1^{\{m\}})^T, \dots, (\Delta f_{s^{\{m\}}}^{\{m\}})^T \right]^T.$$

From (4.12) and the positive definiteness of \mathbf{P} (4.6b) we have that

$$\|\Delta y_{n+1}\|^2 \leq \|\Delta y_n\|^2 + 2h \sum_{m=1}^N \sum_{i=1}^s b_i^{\{m\}} \left\langle \Delta f_i^{\{m\}}, \Delta Y_i^{\{m\}} \right\rangle.$$

The positivity of the weights (4.6a) and dispersion condition (4.3) give the desired result:

$$\|\Delta y_{n+1}\|^2 \leq \|\Delta y_n\|^2 + 2h \sum_{m=1}^N \nu^{\{m\}} \sum_{i=1}^s b_i^{\{m\}} \left\| \Delta Y_i^{\{m\}} \right\|^2 \leq \|\Delta y_n\|^2.$$

□

DEFINITION 4.3 (Stability-decoupled GARK schemes). *A GARK method (2.3) is stability-decoupled if*

$$(4.13) \quad \mathbf{P}^{\{m,\ell\}} = \mathbf{0} \quad \text{for } m \neq \ell.$$

For stability decoupled GARK methods the interaction of different components does not influence the overall nonlinear stability. If each of the component methods is nonlinearly stable, perhaps under a suitable step size restriction, then the overall method is nonlinearly stable (under a step size restriction that satisfies each of the components). In particular, if each of the component Runge-Kutta scheme is algebraically stable

$$b_i^{\{m\}} > 0, \quad \mathbf{B}^{\{m\}} = \text{diag}(\mathbf{b}^{\{m\}}), \quad \mathbf{P}^{\{m,m\}} \geq 0 \quad m = 1, \dots, N,$$

then equation (4.12) shows that (4.13) is a sufficient condition for the algebraic stability of the GARK scheme.

REMARK 4. *Equation (4.12) shows that the nonlinear stability of each of the component methods is sufficient to obtain a nonlinear stable GARK scheme in the case of component partitioning (1.2).*

4.3. Conditional stability for coercive problems. Next, consider partitioned systems (1.1) where each of the component functions is coercive [10]:

$$(4.14) \quad \left\langle f^{\{m\}}(y) - f^{\{m\}}(z), y - z \right\rangle \leq \mu^{\{m\}} \left\| f^{\{m\}}(y) - f^{\{m\}}(z) \right\|^2, \quad \mu^{\{m\}} < 0.$$

THEOREM 4.4 (Conditional stability of GARK methods). *Consider a partitioned system (1.1) with coercive component functions (4.14) solved by a GARK method (2.3). Assume that there exist $r^{\{m\}} \geq 0$ such that the following matrix is positive definite*

$$(4.15) \quad \tilde{\mathbf{P}} = \mathbf{P} + \text{diag}_{m=1,\dots,N} \left\{ r^{\{m\}} \mathbf{B}^{\{m\}} \right\} \geq 0,$$

where \mathbf{P} was defined in (4.6b). Then the solution is conditionally nonlinearly stable, in the sense that $\|\Delta y_{n+1}\| \leq \|\Delta y_n\|$, under the step size restriction

$$h \leq \min_{m=1,\dots,N} \frac{-2\mu^{\{m\}}}{r^{\{m\}}}.$$

Proof. Equation (4.12) and condition (4.14) yield

$$\begin{aligned}
\|\Delta y_{n+1}\|^2 &\leq \|\Delta y_n\|^2 + 2h \sum_{m=1}^N \sum_{i=1}^s b_i^{\{m\}} \mu^{\{m\}} \left\| \Delta f_i^{\{m\}} \right\|^2 \\
&\quad - h^2 \sum_{m,l=1}^N (\Delta f^{\{m\}})^T \left(\tilde{\mathbf{P}}^{\{m,l\}} \otimes \mathbf{I}_{d \times d} \right) \Delta f^{\{l\}} \\
&\quad + h^2 \sum_{m=1}^N (\Delta f^{\{m\}})^T \left(r^{\{m\}} \mathbf{B}^{\{m\}} \otimes \mathbf{I}_{d \times d} \right) \Delta f^{\{m\}} \\
&= \|\Delta y_n\|^2 + 2h \sum_{m=1}^N \sum_{i=1}^s b_i^{\{m\}} \mu^{\{m\}} \left\| \Delta f_i^{\{m\}} \right\|^2 \\
&\quad - h^2 \sum_{m,l=1}^N (\Delta f^{\{m\}})^T \left(\tilde{\mathbf{P}}^{\{m,l\}} \otimes \mathbf{I}_{d \times d} \right) \Delta f^{\{l\}} \\
&\quad + h^2 \sum_{m=1}^N \sum_{i=1}^s r^{\{m\}} b_i^{\{m\}} \left\| \Delta f_i^{\{m\}} \right\|^2.
\end{aligned}$$

From the positive definiteness assumption (4.15)

$$\|\Delta y_{n+1}\|^2 \leq \|\Delta y_n\|^2 + h \sum_{m=1}^N \sum_{i=1}^s b_i^{\{m\}} \left(2\mu^{\{m\}} + hr^{\{m\}} \right) \left\| \Delta f_i^{\{m\}} \right\|^2.$$

We see that if $2\mu^{\{m\}} + hr^{\{m\}} \leq 0$ for all $m = 1, \dots, N$ then $\|\Delta y_{n+1}\|^2 \leq \|\Delta y_n\|^2$. This proves the desired result, which extends the one given in [10] for classical additive Runge-Kutta methods. \square

REMARK 5. If the GARK method is stability decoupled (4.13) then the weights $r^{\{m\}}$ in (4.15) are chosen independently for each component. In this case each component method, applied to the corresponding subsystem, is conditionally stable under a step restriction $h \leq h_{\max}^{\{m\}}$. The GARK method's step size restriction is given by the bounds for individual components, $h \leq \min_{m=1, \dots, N} h_{\max}^{\{m\}}$, i.e., no additional stability restrictions are imposed on the step size.

EXAMPLE 4 (A second order, stability-decoupled IMEX-GARK scheme).

We construct a second order IMEX-GARK method where the implicit and explicit parts have different numbers of stages. The method has a free parameter denoted β . The implicit method

$$\mathbf{A}^{\{1,1\}} = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}, \quad \mathbf{b}^{\{1\}} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad \mathbf{c}^{\{1,1\}} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix},$$

is second order accurate and algebraically stable since

$$\mathbf{P}^{\{1,1\}} = \mathbf{A}^{\{1,1\}T} \mathbf{B}^{\{1\}} + \mathbf{B}^{\{1\}} \mathbf{A}^{\{1,1\}} - \mathbf{b}^{\{1\}} \mathbf{b}^{\{1\}T} = 0.$$

The explicit method is:

$$\mathbf{A}^{\{E,E\}} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 1 - \beta & \beta & 0 \end{bmatrix}, \quad \mathbf{b}^{\{E\}} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}, \quad \mathbf{c}^{\{E,E\}} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

The explicit method is conditionally stable for coercive problems. A good value of the free parameter for stability is $\beta = -1/4$ for which (4.15) holds with $r \approx 2.6$. The coupling coefficients are

$$\mathbf{A}^{\{E,I\}} = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \mathbf{c}^{\{E,I\}} = \mathbf{c}^{\{E,E\}} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix},$$

and

$$\mathbf{A}^{\{I,E\}} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \end{bmatrix}, \quad \mathbf{c}^{\{I,E\}} = \mathbf{c}^{\{I,I\}} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}.$$

The IMEX-GARK method is stability-decoupled

$$\mathbf{P}^{\{E,I\}} = \left[\mathbf{A}^{\{I,E\}} \right]^T \mathbf{B}^{\{I\}} + \mathbf{B}^{\{E\}} \mathbf{A}^{\{E,I\}} - \mathbf{b}^{\{E\}} \left[\mathbf{b}^{\{I\}} \right]^T = \mathbf{0}.$$

This property, and the algebraic stability of the implicit part, imply that the GARK method is nonlinearly stable under the same step size restriction for which the explicit component is nonlinearly stable (e.g., for $\beta = -1/4$ we have $h \leq -2\mu^{\{E\}}/2.6$).

This IMEX-GARK scheme is represented compactly by its generalized Butcher tableau (3.3) as:

$$\begin{array}{c|c|c|c} \mathbf{c}^{\{E,E\}} & \mathbf{A}^{\{E,E\}} & \mathbf{A}^{\{E,I\}} & \mathbf{c}^{\{E,I\}} \\ \hline \mathbf{c}^{\{I,E\}} & \mathbf{A}^{\{I,E\}} & \mathbf{A}^{\{I,I\}} & \mathbf{c}^{\{I,I\}} \\ \hline & \mathbf{b}^{\{E\}} & \mathbf{b}^{\{I\}} & \end{array} := \begin{array}{c|c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 1 - \beta & \beta & 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ \hline \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ \hline & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \end{array}.$$

4.4. Monotonicity analysis. This section studies the contractivity and monotonicity of the generalized additively partitioned Runge-Kutta methods. We are concerned with partitioned systems (1.1) where there exist $\rho^{\{1\}}, \dots, \rho^{\{N\}} > 0$ such that

$$(4.16) \quad \forall y : \left\| y + \rho^{\{m\}} f^{\{m\}}(y) \right\| \leq \|y\|, \quad m = 1, \dots, N.$$

This implies that condition (4.16) holds for any $0 \leq \tau^{\{m\}} \leq \rho^{\{m\}}$, i.e., for each individual subsystem m the solution of one forward Euler step is monotone under this step size restriction. The condition (4.16) also implies that the system (1.1) has a solution of non increasing norm. To see this write an Euler step with the full system as a convex combination

$$\left\| y + \theta \sum_{m=1}^N f^{\{m\}}(y) \right\| = \left\| \sum_{m=1}^N \frac{1}{N} \left(y + N\theta f^{\{m\}}(y) \right) \right\| \leq \sum_{m=1}^N \frac{1}{N} \|y\| = \|y\|$$

if $0 < \theta \leq \min\{\rho^{\{m\}}\}/N$, and consequently $\lim_{\varepsilon \rightarrow 0} \|y(t + \varepsilon)\| \leq \|y(t)\|$ [12].

We seek to construct GARK schemes which guarantee a monotone numerical solution $\|y_{n+1}\| \leq \|y_n\|$ for (4.16) under suitable step size restrictions.

A comprehensive study of contractivity of Runge-Kutta methods is given in [17]. Step size conditions for monotonicity are discussed in [22]. Strong stability preserving methods suitable for hyperbolic PDEs are reviewed in [4, 9, 11]. Monotonicity for Runge-Kutta methods in inner product norms is discussed in [10]. This study follows the approach of Higueras and co-workers, who have extended the monotonicity theory to additive Runge-Kutta methods [12, 13, 3].

The scheme (2.3) can be represented in matrix form as

$$(4.17a) \quad Y^{\{q\}} = \mathbf{1}_{\{q\} \times 1} \otimes y_n + h \sum_{m=1}^N \left(\mathbf{A}^{\{q,m\}} \otimes \mathbf{I}_{d \times d} \right) f^{\{m\}}(Y^{\{m\}}),$$

$$(4.17b) \quad y_{n+1} = y_n + h \sum_{q=1}^N \left(\mathbf{b}^{\{q\}^T} \otimes \mathbf{I}_{d \times d} \right) f^{\{q\}}(Y^{\{q\}}).$$

Using notation (4.1) and

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ (\mathbf{b})^T & \mathbf{0} \end{bmatrix}, \quad \hat{Y} = \begin{bmatrix} Y^{\{1\}} \\ \vdots \\ Y^{\{N\}} \\ y_{n+1} \end{bmatrix}, \quad \hat{f}(\hat{Y}) = \begin{bmatrix} f^{\{1\}}(Y^{\{1\}}) \\ \vdots \\ f^{\{N\}}(Y^{\{N\}}) \\ 0 \end{bmatrix},$$

equations (4.17) become

$$(4.18) \quad \hat{Y} = \mathbf{1}_{(s+1) \times 1} \otimes y_n + h \left(\hat{\mathbf{A}} \otimes \mathbf{I}_{d \times d} \right) \hat{f}(\hat{Y}).$$

The following definitions extend the corresponding ones from [12, 13].

DEFINITION 4.5 (Absolutely monotonic GARK). *Let $r^{\{1\}}, \dots, r^{\{N\}} > 0$ and*

$$(4.19) \quad r = \begin{bmatrix} r^{\{1\}} \\ \vdots \\ r^{\{N\}} \end{bmatrix}, \quad \hat{\mathbf{R}} = \text{diag} \left\{ r^{\{1\}} \mathbf{I}_{s^{\{1\}} \times s^{\{1\}}}, \dots, r^{\{N\}} \mathbf{I}_{s^{\{N\}} \times s^{\{N\}}}, 1 \right\}.$$

A GARK scheme (2.3) defined by $\hat{\mathbf{A}} \geq 0$ is called absolutely monotonic (a.m.) at $r \in \mathbb{R}^N$ if

$$(4.20a) \quad \alpha(r) = \left(\mathbf{I}_{\hat{s} \times \hat{s}} + \hat{\mathbf{A}} \hat{\mathbf{R}} \right)^{-1} \mathbf{1}_{\hat{s} \times 1} \geq 0, \quad \text{and}$$

$$(4.20b) \quad \beta(r) = \left(\mathbf{I}_{\hat{s} \times \hat{s}} + \hat{\mathbf{A}} \hat{\mathbf{R}} \right)^{-1} \hat{\mathbf{A}} \hat{\mathbf{R}} = \mathbf{I}_{\hat{s} \times \hat{s}} - \left(\mathbf{I}_{\hat{s} \times \hat{s}} + \hat{\mathbf{A}} \hat{\mathbf{R}} \right)^{-1} \geq 0,$$

where $\hat{s} = s + 1$. Here all the inequalities are taken component-wise.

DEFINITION 4.6 (Region of absolute monotonicity). *The region of absolute monotonicity of the GARK scheme (2.3) is*

$$(4.21) \quad \mathcal{R}(\hat{\mathbf{A}}) = \left\{ r \in \mathbb{R}_+^N : \hat{\mathbf{A}} \text{ is a.m. on } [0, r^{\{1\}}] \times \dots \times [0, r^{\{N\}}] \right\}.$$

THEOREM 4.7 (Monotonicity of solutions). *Consider the GARK scheme (2.3) defined by $\hat{\mathbf{A}}$ and a point in the interior of its absolute monotonicity region*

$$r \in \mathcal{R}(\hat{\mathbf{A}}), \quad r > \mathbf{0}_{N \times 1}.$$

For any step size obeying the restriction

$$(4.22) \quad h \leq \max_{q=1,\dots,N} \left\{ r^{\{q\}} \rho^{\{q\}} \right\}$$

the stage values and the solution of (2.3) are monotonic

$$(4.23a) \quad \|Y_i^{\{q\}}\| \leq \|y_n\|, \quad q = 1, \dots, N, \quad i = 1, \dots, s^{\{q\}},$$

$$(4.23b) \quad \|y_{n+1}\| \leq \|y_n\|.$$

In practice we are interested in the largest upper bound for the time step that ensures monotonicity.

Proof. The proof is a direct extension of the corresponding one for classical additively partitioned Runge-Kutta methods given in [12]. Construct the matrix $\hat{\mathbf{R}}$ as in (4.19). Add the same quantity $\hat{\mathbf{A}}\hat{\mathbf{R}} \otimes \mathbf{I}_{d \times d}$ to both sides of (4.18) to obtain

$$\begin{aligned} (\mathbf{I}_{\hat{s} \times d\hat{s}} + \hat{\mathbf{A}}\hat{\mathbf{R}} \otimes \mathbf{I}_{d \times d}) \hat{Y} &= \mathbf{1}_{\hat{s} \times 1} \otimes y_n \\ &+ (\hat{\mathbf{A}}\hat{\mathbf{R}} \otimes \mathbf{I}_{d \times d}) \left(\hat{Y} + (h \hat{\mathbf{R}}^{-1} \otimes \mathbf{I}_{d \times d}) \hat{f}(\hat{Y}) \right). \end{aligned}$$

Using the notation of (4.20) this relation can be written in the equivalent form

$$(4.24) \quad \hat{Y} = (\alpha(r) \otimes \mathbf{I}_{d \times d}) \cdot y_n + (\beta(r) \otimes \mathbf{I}_{d \times d}) \left(\hat{Y} + (h \hat{\mathbf{R}}^{-1} \otimes \mathbf{I}_{d \times d}) \hat{f}(\hat{Y}) \right).$$

Denote a vector of norms by

$$\llbracket \hat{Y} \rrbracket = [\|Y^{\{1\}}\|, \dots, \|Y^{\{N\}}\|, \|y_{n+1}\|]^T.$$

Since $r \in \mathcal{R}(\hat{\mathbf{A}})$ we have that $\alpha(r) \geq 0$ and $\beta(r) \geq 0$. Taking norms in (4.24) leads to

$$\llbracket \hat{Y} \rrbracket \leq \alpha(r) \|y_n\| + \beta(r) \llbracket \hat{Y} + (h \hat{\mathbf{R}}^{-1} \otimes \mathbf{I}_{d \times d}) \hat{f}(\hat{Y}) \rrbracket.$$

Under the step size restriction (4.22) we have $h (r^{\{q\}})^{-1} \leq \rho^{\{q\}}$ for any $q = 1, \dots, N$, and from (4.16)

$$\llbracket \hat{Y} + (h \hat{\mathbf{R}}^{-1} \otimes \mathbf{I}_{d \times d}) \hat{f}(\hat{Y}) \rrbracket \leq \llbracket \hat{Y} \rrbracket.$$

It follows that

$$\begin{aligned} \llbracket \hat{Y} \rrbracket &\leq \alpha(r) \|y_n\| + \beta(r) \llbracket \hat{Y} \rrbracket \\ &= (\mathbf{I}_{\hat{s} \times \hat{s}} + \hat{\mathbf{A}}\hat{\mathbf{R}})^{-1} \mathbf{1}_{\hat{s} \times 1} \|y_n\| + \left(\mathbf{I}_{\hat{s} \times \hat{s}} - (\mathbf{I}_{\hat{s} \times \hat{s}} + \hat{\mathbf{A}}\hat{\mathbf{R}})^{-1} \right) \llbracket \hat{Y} \rrbracket \end{aligned}$$

and

$$(\mathbf{I}_{\hat{s} \times \hat{s}} + \hat{\mathbf{A}}\hat{\mathbf{R}})^{-1} \llbracket \hat{Y} \rrbracket \leq (\mathbf{I}_{\hat{s} \times \hat{s}} + \hat{\mathbf{A}}\hat{\mathbf{R}})^{-1} \mathbf{1}_{\hat{s} \times 1} \|y_n\|.$$

Multiplication by the matrix $\mathbf{I}_{\hat{s} \times \hat{s}} + \hat{\mathbf{A}}\hat{\mathbf{R}} \geq 0$, whose entries are all non-negative, implies that

$$\llbracket \hat{Y} \rrbracket \leq \mathbf{1}_{\hat{s} \times 1} \otimes \|y_n\|,$$

and the monotonicity relation (4.23) follows. \square

EXAMPLE 5 (Monotonicity of classical IMEX RK).

For $N = 2$ we have

$$(4.25) \quad \hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A}^{\{1,1\}} & \mathbf{A}^{\{1,2\}} & 0 \\ \mathbf{A}^{\{2,1\}} & \mathbf{A}^{\{2,2\}} & 0 \\ (\mathbf{b}^{\{1\}})^T & (\mathbf{b}^{\{2\}})^T & 0 \end{bmatrix},$$

which we write in the equivalent form

$$(4.26) \quad \hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A}^{\{1,1\}} & 0 & \mathbf{A}^{\{1,2\}} & 0 \\ (\mathbf{b}^{\{1\}})^T & 0 & (\mathbf{b}^{\{2\}})^T & 0 \\ \mathbf{A}^{\{2,1\}} & 0 & \mathbf{A}^{\{2,2\}} & 0 \\ (\mathbf{b}^{\{1\}})^T & 0 & (\mathbf{b}^{\{2\}})^T & 0 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{A}}^{\{1,1\}} & \hat{\mathbf{A}}^{\{1,2\}} \\ \hat{\mathbf{A}}^{\{2,1\}} & \hat{\mathbf{A}}^{\{2,2\}} \end{bmatrix},$$

where the extra stage does not contribute to the final solution. In particular, for classical IMEX RK we have

$$\hat{\mathbf{A}} = \begin{bmatrix} \hat{\mathbf{A}}^{\{E,E\}} & \hat{\mathbf{A}}^{\{I,I\}} \\ \hat{\mathbf{A}}^{\{E,E\}} & \hat{\mathbf{A}}^{\{I,I\}} \end{bmatrix}.$$

Consequently

$$\mathbf{I} + \hat{\mathbf{A}}\hat{\mathbf{R}} = \begin{bmatrix} \mathbf{I}_{(s+1) \times (s+1)} + r^{\{E\}}\hat{\mathbf{A}}^{\{E,E\}} & r^{\{I\}}\hat{\mathbf{A}}^{\{I,I\}} \\ r^{\{E\}}\hat{\mathbf{A}}^{\{E,E\}} & \mathbf{I}_{(s+1) \times (s+1)} + r^{\{I\}}\hat{\mathbf{A}}^{\{I,I\}} \end{bmatrix}.$$

With

$$\mathbf{S} = \mathbf{I}_{(s+1) \times (s+1)} + r^{\{E\}}\hat{\mathbf{A}}^{\{E,E\}} + r^{\{I\}}\hat{\mathbf{A}}^{\{I,I\}}$$

we have

$$(4.27) \quad \alpha(r) = \left(\mathbf{I} + \hat{\mathbf{A}}\hat{\mathbf{R}} \right)^{-1} \mathbf{1}_{(2s+2) \times 1} = \begin{bmatrix} \mathbf{S}^{-1} \mathbf{1}_{(s+1) \times 1} \\ \mathbf{S}^{-1} \mathbf{1}_{(s+1) \times 1} \end{bmatrix},$$

and

$$\beta(r) = \left(\mathbf{I} + \hat{\mathbf{A}}\hat{\mathbf{R}} \right)^{-1} \hat{\mathbf{A}}\hat{\mathbf{R}} = \begin{bmatrix} r^{\{E\}} \mathbf{S}^{-1} \hat{\mathbf{A}}^{\{E,E\}} & r^{\{I\}} \mathbf{S}^{-1} \hat{\mathbf{A}}^{\{I,I\}} \\ r^{\{E\}} \mathbf{S}^{-1} \hat{\mathbf{A}}^{\{E,E\}} & r^{\{I\}} \mathbf{S}^{-1} \hat{\mathbf{A}}^{\{I,I\}} \end{bmatrix}.$$

GARK absolute monotonicity conditions (4.20) are equivalent to the absolute traditional additive RK monotonicity conditions obtained by Higueras [12]

$$\mathbf{S}^{-1} \cdot \mathbf{1}_{(s+1) \times 1} \geq 0, \quad \mathbf{S}^{-1} \cdot \hat{\mathbf{A}}^{\{E,E\}} \geq 0, \quad \mathbf{S}^{-1} \cdot \hat{\mathbf{A}}^{\{I,I\}} \geq 0.$$

EXAMPLE 6 (Monotonicity of classical-transposed IMEX RK).

Here we have

$$\hat{\mathbf{A}} = \begin{bmatrix} \hat{\mathbf{A}}^{\{E,E\}} & \hat{\mathbf{A}}^{\{E,E\}} \\ \hat{\mathbf{A}}^{\{I,I\}} & \hat{\mathbf{A}}^{\{I,I\}} \end{bmatrix}$$

and

$$\begin{aligned}
\alpha(r) &= \left(\mathbf{I} + \widehat{\mathbf{A}}\widehat{\mathbf{R}} \right)^{-1} \mathbf{1}_{(2s+2) \times 1} \\
&= \left[\begin{pmatrix} \mathbf{I}_{(s+1) \times (s+1)} - r^{\{I\}} \widehat{\mathbf{A}}^{\{I,I\}} + r^{\{E\}} \widehat{\mathbf{A}}^{\{E,E\}} \\ \mathbf{I}_{(s+1) \times (s+1)} - r^{\{E\}} \widehat{\mathbf{A}}^{\{E,E\}} + r^{\{I\}} \widehat{\mathbf{A}}^{\{I,I\}} \end{pmatrix} \mathbf{S}^{-1} \mathbf{1}_{(s+1) \times 1} \right] \\
&= \left[\begin{array}{c} \mathbf{1}_{(s+1) \times 1} - 2r^{\{I\}} \widehat{\mathbf{A}}^{\{I,I\}} \mathbf{S}^{-1} \mathbf{1}_{(s+1) \times 1} \\ \mathbf{1}_{(s+1) \times 1} - 2r^{\{E\}} \widehat{\mathbf{A}}^{\{E,E\}} \mathbf{S}^{-1} \mathbf{1}_{(s+1) \times 1} \end{array} \right], \\
\beta(r) &= \left(\mathbf{I} + \widehat{\mathbf{A}}\widehat{\mathbf{R}} \right)^{-1} \widehat{\mathbf{A}}\widehat{\mathbf{R}} = \begin{bmatrix} r^{\{E\}} \widehat{\mathbf{A}}^{\{E,E\}} \mathbf{S}^{-1} & r^{\{E\}} \widehat{\mathbf{A}}^{\{E,E\}} \mathbf{S}^{-1} \\ r^{\{I\}} \widehat{\mathbf{A}}^{\{I,I\}} \mathbf{S}^{-1} & r^{\{I\}} \widehat{\mathbf{A}}^{\{I,I\}} \mathbf{S}^{-1} \end{bmatrix}.
\end{aligned}$$

The GARK absolute monotonicity conditions (4.20) lead to

$$\begin{aligned}
\widehat{\mathbf{A}}^{\{I,I\}} \mathbf{S}^{-1} \cdot \mathbf{1}_{(s+1) \times 1} &\leq \frac{1}{2r^{\{I\}}} \mathbf{1}_{(s+1) \times 1}, \\
\widehat{\mathbf{A}}^{\{E,E\}} \mathbf{S}^{-1} \cdot \mathbf{1}_{(s+1) \times 1} &\leq \frac{1}{2r^{\{E\}}} \mathbf{1}_{(s+1) \times 1}, \\
\widehat{\mathbf{A}}^{\{I,I\}} \cdot \mathbf{S}^{-1} &\geq 0, \quad \text{and} \\
\widehat{\mathbf{A}}^{\{E,E\}} \cdot \mathbf{S}^{-1} &\geq 0.
\end{aligned}$$

EXAMPLE 7 (A monotonic IMEX-GARK scheme).

Consider the following second order IMEX-GARK scheme. The explicit method has order two, is strong stability preserving, and has an absolute stability radius $\mathcal{R}^{\{E,E\}} = 1$

$$(4.28a) \quad \mathbf{A}^{\{E,E\}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b}^{\{E\}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \mathbf{c}^{\{E,E\}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The implicit method has order two, is stiffly accurate, and has an absolute stability radius $\mathcal{R}^{\{I,I\}} = \infty$. The coefficients are $\gamma = (1 - \sqrt{2})/2$ and

$$(4.28b) \quad \mathbf{A}^{\{I,I\}} = \begin{bmatrix} \gamma & 0 \\ 1 - \gamma & \gamma \end{bmatrix}, \quad \mathbf{b}^{\{I\}} = \begin{bmatrix} 1 - \gamma \\ \gamma \end{bmatrix}, \quad \mathbf{c}^{\{I,I\}} = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}.$$

The two methods cannot be paired as a classical IMEX Runge-Kutta method. The following coupling terms

$$(4.28c) \quad \mathbf{A}^{\{E,I\}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{A}^{\{I,E\}} = \begin{bmatrix} \gamma & 0 \\ \alpha & 1 - \alpha \end{bmatrix},$$

ensure that the GARK scheme is second order. This can be seen from the fact that the internal consistency conditions (3.6) are satisfied.

The coupling has one free parameter α . For $\alpha = 1 - \gamma$ the IMEX scheme is of transposed-classical type. Different values of α lead to different regions of monotonicity, as illustrated in Figure 4.1. A numerical search has revealed that the largest region is obtained for $\alpha = 0.5$.

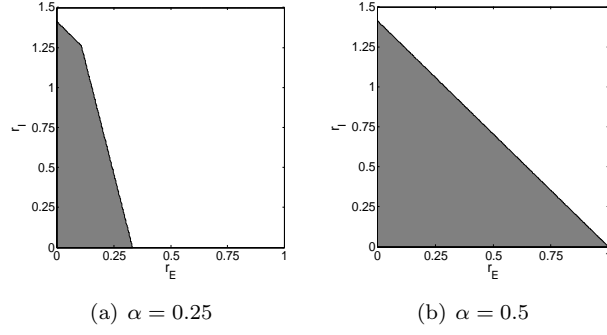


Fig. 4.1: Absolute monotonicity regions for the second order IMEX scheme (4.28).

The method has the following generalized Butcher tableau:

$$\begin{array}{c|c|c|c} \mathbf{c}^{\{E,E\}} & \mathbf{A}^{\{E,E\}} & \mathbf{A}^{\{E,I\}} & \mathbf{c}^{\{E,I\}} \\ \hline \mathbf{c}^{\{I,E\}} & \mathbf{A}^{\{I,E\}} & \mathbf{A}^{\{I,I\}} & \mathbf{c}^{\{I,I\}} \\ \hline & \mathbf{b}^{\{E\}} & \mathbf{b}^{\{I\}} & \end{array} := \begin{array}{c|cc|cc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ \hline \gamma & \gamma & 0 & \gamma & 0 & \gamma \\ 1 & \alpha & 1-\alpha & 1-\gamma & \gamma & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} & 1-\gamma & \gamma & \end{array}.$$

Monotonicity conditions for several multirate and partitioned explicit Runge-Kutta schemes are also discussed by Hundsdorfer, Mozartova, and Savcenko in a recent report [15].

5. Implicit-implicit GARK schemes. We now consider systems (1.1) with two way partitioned right hand sides where both components f and g are stiff. We apply a two way partitioned GARK method (2.3)

$$(5.1a) \quad Y_i = y_n + h \sum_{j=1}^i a_{i,j}^{\{1,1\}} f(Y_j) + h \sum_{j=1}^{i-1} a_{i,j}^{\{1,2\}} g(Z_j), \quad i = 1, \dots, s^{\{1\}},$$

$$(5.1b) \quad Z_i = y_n + h \sum_{j=1}^i a_{i,j}^{\{2,1\}} f(Y_j) + h \sum_{j=1}^i a_{i,j}^{\{2,2\}} g(Z_j), \quad i = 1, \dots, s^{\{2\}},$$

$$(5.1c) \quad y_{n+1} = y_n + h \sum_{i=1}^{s^{\{1\}}} b_i^{\{1\}} f(Y_i) + h \sum_{i=1}^{s^{\{2\}}} b_i^{\{2\}} g(Z_i).$$

The scheme (5.1) has the following characteristics:

- The discretization is implicit-implicit (IMIM); stages (5.1a) are implicit in Y_i , while stages (3.2b) are implicit in Z_i .
- One solves in succession nonlinear subsystems corresponding to each individual component.
- If each of the implicit schemes is algebraically stable, and the GARK scheme is stability decoupled, then the separation of subsystem solutions does not affect the algebraic stability of the overall method.

The theoretical investigations develop order conditions for the GARK family using the NB-series theory. We carry out linear and nonlinear stability analyses, extend the definition of algebraic stability to the new generalized family of schemes, and show that it is possible to construct stability-decoupled methods. We also perform a monotonicity analysis, extend the concept of absolute monotonicity to our new family, and prove monotonic behavior under step size restrictions.

We develop implicit-explicit GARK schemes in the new framework. We show that classical implicit-explicit Runge-Kutta methods are a particular subset, and develop a new set of transposed-classical schemes. A theoretical investigation of the stiff convergence motivates an extension of the stiff accuracy concept. We construct implicit-implicit GARK methods where the nonlinear system at each stage involves only one component of the system.

Future work will search for practical methods of high order, and will test their performance on relevant problems. Partitioned implicit-implicit methods (5.1) with the same coefficients on the diagonal of each method (SDIRK type) are desirable, as are orders higher than two and a partitioning of the right hand side into three or more components. Also stiffly accurate methods with $\mathbf{P}^{\{i,i\}} > 0$ and $\mathbf{P}^{\{i,j\}} = 0$ when $i \neq j$ are desirable for systems driven by multiple stiff physical processes. We are developing multirate schemes [6] as well as symplectic schemes [5] based on the generalized additive Runge-Kutta framework presented here.

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Appendix A. GARK order conditions.

The specific conditions for orders one to four are as follows.

Order 1:

$$(A.1a) \quad \sum_{i=1}^{s^{\{\sigma\}}} b_i^{\{\sigma\}} = 1, \quad \forall \sigma = 1, \dots, N.$$

Order 2:

$$(A.1b) \quad \sum_{i=1}^{s^{\{\sigma\}}} \sum_{j=1}^{s^{\{\nu\}}} b_i^{\{\sigma\}} a_{i,j}^{\{\sigma,\nu\}} = \frac{1}{2}, \quad \forall \sigma, \nu = 1, \dots, N.$$

Order 3:

$$(A.1c) \quad \sum_{i=1}^{s^{\{\sigma\}}} \sum_{j=1}^{s^{\{\nu\}}} \sum_{k=1}^{s^{\{\mu\}}} b_i^{\{\sigma\}} a_{i,j}^{\{\sigma,\nu\}} a_{i,k}^{\{\sigma,\mu\}} = \frac{1}{3},$$

$$(A.1d) \quad \sum_{i=1}^{s^{\{\sigma\}}} \sum_{j=1}^{s^{\{\nu\}}} \sum_{k=1}^{s^{\{\mu\}}} b_i^{\{\sigma\}} a_{i,j}^{\{\sigma,\nu\}} a_{j,k}^{\{\nu,\mu\}} = \frac{1}{6},$$

$$\forall \sigma, \nu, \mu = 1, \dots, N.$$

Order 4:

$$(A.1e) \quad \sum_{i=1}^{s^{\{\sigma\}}} \sum_{j=1}^{s^{\{\nu\}}} \sum_{\ell=1}^{s^{\{\lambda\}}} \sum_{m=1}^{s^{\{\mu\}}} b_i^{\{\sigma\}} a_{i,j}^{\{\sigma,\nu\}} a_{i,\ell}^{\{\sigma,\lambda\}} a_{i,m}^{\{\sigma,\mu\}} = \frac{1}{4},$$

$$(A.1f) \quad \sum_{i=1}^{s^{\{\sigma\}}} \sum_{j=1}^{s^{\{\nu\}}} \sum_{\ell=1}^{s^{\{\lambda\}}} \sum_{m=1}^{s^{\{\mu\}}} b_i^{\{\sigma\}} a_{i,j}^{\{\sigma,\nu\}} a_{j,\ell}^{\{\nu,\lambda\}} a_{i,m}^{\{\sigma,\mu\}} = \frac{1}{8},$$

$$(A.1g) \quad \sum_{i=1}^{s^{\{\sigma\}}} \sum_{j=1}^{s^{\{\nu\}}} \sum_{\ell=1}^{s^{\{\lambda\}}} \sum_{m=1}^{s^{\{\mu\}}} b_i^{\{\sigma\}} a_{i,j}^{\{\sigma,\nu\}} a_{j,\ell}^{\{\nu,\lambda\}} a_{j,m}^{\{\nu,\mu\}} = \frac{1}{12},$$

$$(A.1h) \quad \sum_{i=1}^{s^{\{\sigma\}}} \sum_{j=1}^{s^{\{\nu\}}} \sum_{\ell=1}^{s^{\{\lambda\}}} \sum_{m=1}^{s^{\{\mu\}}} b_i^{\{\sigma\}} a_{i,j}^{\{\sigma,\nu\}} a_{j,\ell}^{\{\nu,\lambda\}} a_{\ell,m}^{\{\lambda,\mu\}} = \frac{1}{24},$$

$$\forall \sigma, \nu, \lambda, \mu = 1, \dots, N.$$

Appendix B. GARK IMEX third order conditions.

Each of the implicit and explicit methods $(\mathbf{A}^{\{\sigma,\sigma\}}, \mathbf{b}^{\{\sigma\}})$ and $\mathbf{c}^{\{\sigma,\sigma\}} = \mathbf{A}^{\{\sigma,\sigma\}} \cdot \mathbf{1}^{\{\sigma\}}$ has to satisfy the corresponding order conditions for $\sigma \in \{E, I\}$. In addition, the following coupling conditions are required for third order accuracy.

The IMEX order two coupling conditions are:

$$(B.1a) \quad \mathbf{b}^{\{E\}^T} \cdot \mathbf{c}^{\{E,I\}} = \frac{1}{2},$$

$$(B.1b) \quad \mathbf{b}^{\{I\}^T} \cdot \mathbf{c}^{\{I,E\}} = \frac{1}{2}.$$

These are equivalent to the requirement that the coupling methods $(\mathbf{A}^{\{\sigma,\nu\}}, \mathbf{b}^{\{\sigma\}})$ for $\sigma \neq \nu$ are second order.

The IMEX order three coupling conditions read:

$$(B.2a) \quad \mathbf{b}^{\{E\}^T} \cdot (\mathbf{c}^{\{E,E\}} \mathbf{c}^{\{E,I\}}) = \frac{1}{3},$$

$$(B.2b) \quad \mathbf{b}^{\{E\}^T} \cdot \mathbf{A}^{\{E,E\}} \cdot \mathbf{c}^{\{E,I\}} = \frac{1}{6},$$

$$(B.2c) \quad \mathbf{b}^{\{E\}^T} \cdot (\mathbf{c}^{\{E,I\}} \mathbf{c}^{\{E,E\}}) = \frac{1}{3},$$

$$(B.2d) \quad \mathbf{b}^{\{E\}^T} \cdot \mathbf{A}^{\{E,I\}} \cdot \mathbf{c}^{\{I,E\}} = \frac{1}{6},$$

$$(B.2e) \quad \mathbf{b}^{\{E\}^T} \cdot (\mathbf{c}^{\{E,I\}} \mathbf{c}^{\{I,I\}}) = \frac{1}{3},$$

$$(B.2f) \quad \mathbf{b}^{\{E\}^T} \cdot \mathbf{A}^{\{E,I\}} \cdot \mathbf{c}^{\{I,I\}} = \frac{1}{6},$$

$$(B.2g) \quad \mathbf{b}^{\{I\}^T} \cdot (\mathbf{c}^{\{I,E\}} \mathbf{c}^{\{I,E\}}) = \frac{1}{3},$$

$$(B.2h) \quad \mathbf{b}^{\{I\}^T} \cdot \mathbf{A}^{\{I,E\}} \cdot \mathbf{c}^{\{E,E\}} = \frac{1}{6},$$

$$(B.2i) \quad \mathbf{b}^{\{I\}^T} \cdot (\mathbf{c}^{\{I,E\}} \mathbf{c}^{\{I,I\}}) = \frac{1}{3},$$

$$(B.2j) \quad \mathbf{b}^{\{I\}^T} \cdot \mathbf{A}^{\{I,E\}} \cdot \mathbf{c}^{\{E,I\}} = \frac{1}{6},$$

$$(B.2k) \quad \mathbf{b}^{\{I\}^T} \cdot (\mathbf{c}^{\{I,I\}} \mathbf{c}^{\{I,E\}}) = \frac{1}{3},$$

$$(B.2l) \quad \mathbf{b}^{\{I\}^T} \cdot \mathbf{A}^{\{I,I\}} \cdot \mathbf{c}^{\{I,E\}} = \frac{1}{6}.$$

Appendix C. GARK IMEX fourth order conditions.

The order four IMEX order conditions (3.7) under the simplifying assumption (3.6) are as follows. We have two compatibility relation between the implicit and the explicit methods:

$$\begin{aligned}\mathbf{b}^{\{E\}^T} \cdot \mathbf{A}^{\{E,E\}} \cdot \mathbf{A}^{\{I,I\}} \cdot \mathbf{c}^{\{I\}} &= \frac{1}{24}, \\ \mathbf{b}^{\{I\}^T} \cdot \mathbf{A}^{\{I,I\}} \cdot \mathbf{A}^{\{E,E\}} \cdot \mathbf{c}^{\{E\}} &= \frac{1}{24},\end{aligned}$$

and 16 conditions involving the coupling terms:

$$\begin{aligned}\mathbf{b}^{\{E\}^T} \cdot \mathbf{A}^{\{E,E\}} \cdot \mathbf{A}^{\{I,E\}} \cdot \mathbf{c}^{\{I\}} &= \frac{1}{24}, \\ \mathbf{b}^{\{E\}^T} \cdot \mathbf{A}^{\{E,E\}} \cdot \mathbf{A}^{\{E,I\}} \cdot \mathbf{c}^{\{E\}} &= \frac{1}{24}, \\ \left(\mathbf{b}^{\{E\}} \mathbf{c}^{\{E\}}\right)^T \cdot \mathbf{A}^{\{E,I\}} \cdot \mathbf{c}^{\{I\}} &= \frac{1}{8}, \\ \mathbf{b}^{\{E\}^T} \cdot \mathbf{A}^{\{E,I\}} \cdot \left(\mathbf{c}^{\{I\}} \mathbf{c}^{\{I\}}\right) &= \frac{1}{12}, \\ \\ \mathbf{b}^{\{E\}^T} \cdot \mathbf{A}^{\{E,I\}} \cdot \mathbf{A}^{\{E,E\}} \cdot \mathbf{c}^{\{E\}} &= \frac{1}{24}, \\ \mathbf{b}^{\{E\}^T} \cdot \mathbf{A}^{\{E,I\}} \cdot \mathbf{A}^{\{I,E\}} \cdot \mathbf{c}^{\{I\}} &= \frac{1}{24}, \\ \mathbf{b}^{\{E\}^T} \cdot \mathbf{A}^{\{E,I\}} \cdot \mathbf{A}^{\{E,I\}} \cdot \mathbf{c}^{\{E\}} &= \frac{1}{24}, \\ \mathbf{b}^{\{E\}^T} \cdot \mathbf{A}^{\{E,I\}} \cdot \mathbf{A}^{\{I,I\}} \cdot \mathbf{c}^{\{I\}} &= \frac{1}{24}, \\ \\ \left(\mathbf{b}^{\{I\}} \mathbf{c}^{\{I\}}\right)^T \cdot \mathbf{A}^{\{I,E\}} \cdot \mathbf{c}^{\{E\}} &= \frac{1}{8}, \\ \mathbf{b}^{\{I\}^T} \cdot \mathbf{A}^{\{I,E\}} \cdot \left(\mathbf{c}^{\{E\}} \mathbf{c}^{\{E\}}\right) &= \frac{1}{12}, \\ \mathbf{b}^{\{I\}^T} \cdot \mathbf{A}^{\{I,E\}} \cdot \mathbf{A}^{\{E,E\}} \cdot \mathbf{c}^{\{E\}} &= \frac{1}{24}, \\ \mathbf{b}^{\{I\}^T} \cdot \mathbf{A}^{\{I,E\}} \cdot \mathbf{A}^{\{I,E\}} \cdot \mathbf{c}^{\{I\}} &= \frac{1}{24}, \\ \\ \mathbf{b}^{\{I\}^T} \cdot \mathbf{A}^{\{I,E\}} \cdot \mathbf{A}^{\{E,I\}} \cdot \mathbf{c}^{\{E\}} &= \frac{1}{24}, \\ \mathbf{b}^{\{I\}^T} \cdot \mathbf{A}^{\{I,E\}} \cdot \mathbf{A}^{\{I,I\}} \cdot \mathbf{c}^{\{I\}} &= \frac{1}{24}, \\ \mathbf{b}^{\{I\}^T} \cdot \mathbf{A}^{\{I,I\}} \cdot \mathbf{A}^{\{I,E\}} \cdot \mathbf{c}^{\{I\}} &= \frac{1}{24}, \\ \mathbf{b}^{\{I\}^T} \cdot \mathbf{A}^{\{I,I\}} \cdot \mathbf{A}^{\{E,I\}} \cdot \mathbf{c}^{\{E\}} &= \frac{1}{24}.\end{aligned}$$

When both simplifying assumptions (3.6) and (3.8) are satisfied, each of the coupling methods $(\mathbf{A}^{\{I,E\}}, \mathbf{b}, \mathbf{c})$ and $(\mathbf{A}^{\{E,I\}}, \mathbf{b}, \mathbf{c})$ needs to be fourth order accurate in

its own right. In addition the following 12 coupling conditions are required:

$$\begin{aligned}
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{E}\}} \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{I}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{E}\}} \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{E}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{E}\}} \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{I}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{I}\}} \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{E}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{I}\}} \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{I}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{I}\}} \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{E}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{I}\}} \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{E}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{I}\}} \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{E}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{I}\}} \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{I}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{E}\}} \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{E}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{E}\}} \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{I}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{E}\}} \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{I}\}} \cdot \mathbf{c} &= \frac{1}{24}.
\end{aligned}$$

Moreover, if the two coupling terms are equal, $\mathbf{A}^{\{\mathbf{I}, \mathbf{E}\}} = \mathbf{A}^{\{\mathbf{E}, \mathbf{I}\}} = \mathbf{A}^{\{\text{cpl}\}}$, then $(\mathbf{A}^{\{\text{cpl}\}}, \mathbf{b}, \mathbf{c})$ needs to be fourth order accurate. The remaining six coupling conditions are

$$\begin{aligned}
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{E}\}} \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{I}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{E}\}} \cdot \mathbf{A}^{\{\text{cpl}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{I}\}} \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{E}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\mathbf{b}^T \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{I}\}} \cdot \mathbf{A}^{\{\text{cpl}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\mathbf{b}^T \cdot \mathbf{A}^{\{\text{cpl}\}} \cdot \mathbf{A}^{\{\mathbf{E}, \mathbf{E}\}} \cdot \mathbf{c} &= \frac{1}{24}, \\
\mathbf{b}^T \cdot \mathbf{A}^{\{\text{cpl}\}} \cdot \mathbf{A}^{\{\mathbf{I}, \mathbf{I}\}} \cdot \mathbf{c} &= \frac{1}{24}.
\end{aligned}$$

Appendix D. Extended Prothero-Robinson analysis.

We consider the modified Prothero-Robinson (PR) [19] test problem written as a split system (3.1)

$$(D.1) \quad \begin{bmatrix} y \\ w \end{bmatrix}' = \underbrace{\begin{bmatrix} \mu(y-w) \\ 0 \end{bmatrix}}_{g(t,y)} + \underbrace{\begin{bmatrix} \phi'(t) \\ \phi'(t) \end{bmatrix}}_{f(t,y)}, \quad \mu < 0, \quad \begin{bmatrix} y(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} \phi(0) \\ \phi(0) \end{bmatrix},$$

The method (3.2) applied to the scalar equation (D.1) reads

$$(D.2a) \quad Y = y_n \mathbf{1} + h \mathbf{A}^{\{E,E\}} \phi'^{\{E\}} + h \mu \mathbf{A}^{\{E,I\}} (Z - W),$$

$$(D.2b) \quad Z = y_n \mathbf{1} + h \mathbf{A}^{\{I,E\}} \phi'^{\{E\}} + h \mu \mathbf{A}^{\{I,I\}} (Z - W),$$

$$(D.2c) \quad W = w_n \mathbf{1} + h \mathbf{A}^{\{I,E\}} \phi'^{\{I\}},$$

$$(D.2d) \quad y_{n+1} = y_n + h \mathbf{b}^{\{E\}T} \phi'^{\{E\}} + h \mu \mathbf{b}^{\{I\}T} (Z - W),$$

$$(D.2e) \quad w_{n+1} = w_n + h \mathbf{b}^{\{E\}T} \phi'^{\{E\}}.$$

Here

$$\begin{aligned} \phi^{\{E\}} &= \phi \left(t_{n-1} + \mathbf{d}^{\{E\}} h \right) = \left[\phi(t_{n-1} + d_1^{\{E\}} h), \dots, \phi(t_{n-1} + d_{s^{\{E\}}}^{\{E\}} h) \right]^T, \\ \phi^{\{I\}} &= \phi \left(t_{n-1} + \mathbf{d}^{\{I\}} h \right) = \left[\phi(t_{n-1} + d_1^{\{I\}} h), \dots, \phi(t_{n-1} + d_{s^{\{I\}}}^{\{I\}} h) \right]^T, \end{aligned}$$

where $\mathbf{d}^{\{E\}}, \mathbf{d}^{\{I\}}$ are the stage approximation times. Due to the structure of the test problem (D.1), the method (3.2) uses an explicit approach for the time variable, therefore

$$(D.3) \quad \mathbf{d}^{\{E\}} = \mathbf{c}^{\{E,E\}}, \quad \mathbf{d}^{\{I\}} = \mathbf{c}^{\{I,E\}}.$$

The exact solution is expanded in Taylor series about t_n :

$$(D.4) \quad \begin{aligned} \phi(t_n + \mathbf{d}h) - \mathbf{1} \phi(t_n) &= \sum_{k=1}^{\infty} \frac{h^k \mathbf{d}^k}{k!} \phi^{(k)}(t_n), \\ h \phi'(t_n + \mathbf{d}h) &= \sum_{k=1}^{\infty} \frac{k h^k \mathbf{d}^{k-1}}{k!} \phi^{(k)}(t_n), \end{aligned}$$

where the vector power \mathbf{d}^k is taken componentwise.

Consider the global errors

$$e_n = y_n - \phi(t_n), \quad E_Y = Y - \phi^{\{E\}}, \quad E_Z = Z - \phi^{\{I\}}, \quad E_W = W - \phi^{\{I\}}.$$

From (D.2e) and the quadrature property of the explicit component method we infer that $w_n = \phi(t_n) + \mathcal{O}(h^p)$. From (D.2c) and (D.4)

$$W = \mathbf{1} \phi(t_n) + \mathbf{A}^{\{I,E\}} \sum_{k=1}^{\infty} \frac{k h^k (\mathbf{d}^{\{I\}})^{k-1}}{k!} \phi^{(k)}(t_n) + \mathcal{O}(h^p).$$

Write the stage equation (D.2b)

$$Z - W = y_n \mathbf{1} + h \mathbf{A}^{\{I,E\}} \phi'^{\{E\}} + h \mu \mathbf{A}^{\{I,I\}} (Z - W) - W$$

in terms of the exact solution and global errors, and use the Taylor expansions (3.15) to obtain

$$\begin{aligned} (\mathbf{I} - h\mu \mathbf{A}^{\{I,I\}}) (E_Z - E_W) &= e_n \mathbf{1} + \phi(t_n) \mathbf{1} + \mathbf{A}^{\{I,E\}} (h\phi'^{\{E\}}) - W \\ &= e_n \mathbf{1} + \delta_Z + \mathcal{O}(h^p), \\ \delta_Z &= \sum_{k=1}^{\infty} \left(k \mathbf{A}^{\{I,E\}} (\mathbf{d}^{\{E\}})^{k-1} - k \mathbf{A}^{\{I,E\}} (\mathbf{d}^{\{I\}})^{k-1} \right) \frac{h^k}{k!} \phi^{(k)}(t_n). \end{aligned}$$

Similarly, write the solution equation (3.2c) in terms of the exact solution and global errors:

$$\begin{aligned} e_{n+1} &= e_n + \phi(t_n) - \phi(t_{n+1}) + \mathbf{b}^{\{E\}T} h\phi'^{\{E\}} + h\mu \mathbf{b}^{\{I\}T} (E_Z - E_W) \\ &= R^{\{I,I\}}(h\mu) e_n + \sum_{k=1}^{\infty} \left(k \mathbf{b}^{\{E\}T} (\mathbf{d}^{\{E\}})^{k-1} - 1 \right) \frac{h^k}{k!} \phi^{(k)}(t_n) \\ &\quad + h\mu \mathbf{b}^{\{I\}T} (\mathbf{I} - h\mu \mathbf{A}^{\{I,I\}})^{-1} \sum_{k=1}^{\infty} \left(k \mathbf{A}^{\{I,E\}} (\mathbf{d}^{\{E\}})^{k-1} \right. \\ &\quad \left. - k \mathbf{A}^{\{I,E\}} (\mathbf{d}^{\{I\}})^{k-1} \right) \frac{h^k}{k!} \phi^{(k)}(t_n) + \mathcal{O}(h^p) \end{aligned}$$

The stability function of the implicit component method is

$$R^{\{I,I\}}(h\mu) = \left(1 + h\mu \mathbf{b}^{\{I\}T} (\mathbf{I} - h\mu \mathbf{A}^{\{I,I\}})^{-1} \mathbf{1} \right).$$

Since the explicit component method (by itself) has at least order p , it follows from (3.14) and the explicit order conditions that

$$(D.5) \quad k \cdot (\mathbf{b}^{\{E\}})^T (\mathbf{d}^{\{E\}})^{k-1} - 1 = 0 \quad \text{for } k = 1, \dots, p.$$

Consequently, the global error recurrence reads

$$\begin{aligned} e_{n+1} &= R^{\{I,I\}}(\infty) e_n + \\ &\quad - \mathbf{b}^{\{I\}T} \mathbf{A}^{\{I,I\}-1} \sum_{k=1}^{\infty} \left(k \mathbf{A}^{\{I,E\}} (\mathbf{d}^{\{E\}})^{k-1} \right. \\ &\quad \left. - k \mathbf{A}^{\{I,E\}} (\mathbf{d}^{\{I\}})^{k-1} \right) \frac{h^k}{k!} \phi^{(k)}(t_n) + \mathcal{O}(h^p) \end{aligned}$$